

Counting numerical semigroups of a given genus via even gaps

Matheus Bernardini¹
(Joint with Fernando Torres¹)

University of Campinas / Federal Institute of São Paulo

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Introduction

Let S be a numerical semigroup.

- $G(S) := \mathbb{N}_0 \setminus S$ - set of gaps of S ;
- $g(S) := \#G(S)$ - genus of S ;
- $n_g := \#\{S : g(S) = g\}$.

Examples

- $n_0 = 1$ - \mathbb{N}_0
- $n_1 = 1$ - $\mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2$ - $\mathbb{N}_0 \setminus \{1, 2\}$ and $\mathbb{N}_0 \setminus \{1, 3\}$
- $n_3 = 4$ - $\mathbb{N}_0 \setminus \{1, 2, 3\}$, $\mathbb{N}_0 \setminus \{1, 2, 4\}$, $\mathbb{N}_0 \setminus \{1, 2, 5\}$ and $\mathbb{N}_0 \setminus \{1, 3, 5\}$

Interest

Studying the behavior of n_g .

Main Goal (but still not solved)

$$n_g \leq n_{g+1}, \text{ for all } g.$$

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First Bound

If $g(S) = g$, then $2g + \mathbb{N}_0 \subset S$. Hence,

$$n_g \leq \binom{2g-1}{g}.$$

M. Bras-Amorós and A. de Mier - 2007

$$n_g \leq C_g = \frac{1}{g+1} \binom{2g}{g}.$$

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From now on, let $\varphi = \frac{1+\sqrt{5}}{2}$.

M. Bras-Amorós - 2006/2008 (Conjecture)

① $n_g + n_{g+1} \leq n_{g+2}$, for all g ;

② • $\lim_{g \rightarrow \infty} \frac{n_{g+1}}{n_g} = \varphi$;

• $\lim_{g \rightarrow \infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1$.

M. Bras-Amorós - 2009

Let $(F_n)_{n \geq 0} = (1, 1, 2, 3, 5, 8, 13, \dots)$ be the Fibonacci sequence. Then

$$2F_g \leq n_g \leq 1 + 3 \cdot 2^{g-3}, \forall g \geq 3.$$

S. Elizalde - 2010

$$a_g \leq n_g \leq c_g, \forall g \geq 1$$

where a_g and c_g are coefficients of some explicit generating functions.

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A. Zhai - 2011/2013

- 1 $\lim_{g \rightarrow \infty} \frac{n_{g+1}}{n_g} = \varphi;$
- 2 $\lim_{g \rightarrow \infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1.$

Remark

- Zhai's first item implies that $n_g < n_{g+1}$, for $g \gg 0$.
- Checking if $n_g \leq n_{g+1}$ for all g is still an open problem (weaker conjecture).

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- $m(S) := \min\{s \in S : s \neq 0\}$ - multiplicity of S ;
- $N(m, g) := \#\{S : g(S) = g \text{ and } m(S) = m\}$.

N. Kaplan - 2012

If $2g < 3m$, then

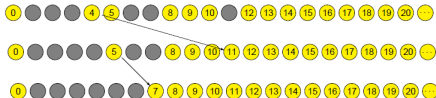
$$N(m, g) = N(m - 1, g - 1) + N(m - 1, g - 2).$$

g\m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	N(g)	
0	1																										1	
1	1																											1
2	1	1																										2
3	1	2	1																									4
4	1	2	3	1																								7
5	1	2	4	4	1																							12
6	1	3	6	7	5	1																						23
7	1	3	7	10	11	6	1																					39
8	1	3	9	13	17	16	7	1																				67
9	1	4	11	16	27	28	22	8	1																			118
10	1	4	13	22	37	44	44	29	9	1																		204
11	1	4	15	24	49	64	72	66	37	10	1																	343
12	1	5	18	32	66	85	116	116	95	46	11	1																592
13	1	5	20	35	85	112	172	188	182	132	56	12	1															1001
14	1	5	23	43	106	148	239	288	304	277	178	67	13	1														1693
15	1	6	26	51	133	191	325	409	492	486	409	234	79	14	1													2857
16	1	6	29	61	163	237	441	559	754	796	763	587	301	92	15	1												4806
17	1	6	32	68	196	301	573	750	1094	1246	1282	1172	821	380	106	16	1											8045
18	1	7	36	80	236	369	737	1015	1534	1841	2074	2045	1759	1122	472	121	17	1										13467
19	1	7	39	89	282	444	945	1334	2106	2601	3227	3356	3217	2580	1502	578	137	18	1									22464
20	1	7	43	104	330	541	1193	1737	2840	3561	4812	5301	5401	4976	3702	1974	699	154	19	1								37396
21	1	8	47	115	390	658	1490	2231	3793	4822	6939	8020	8721	8618	7556	5204	2552	836	172	20	1							62194
22	1	8	51	133	456	784	1847	2851	4967	6490	9752	11657	13643	14122	13594	11258	7178	3251	990	191	21	1						103246
23	1	8	55	143	525	940	2262	3582	6426	8638	13439	16375	20640	22364	22740	21150	16462	9730	4087	1162	211	22	1					170963
24	1	9	60	164	608	1114	2766	4475	8252	11355	18309	22518	30282	34252	36614	36334	32408	23640	12981	5077	1353	232	23	1				282828
25	1	9	64	181	697	1307	3353	5512	10519	14756	24663	30588	43234	50772	57340	59354	57484	48870	33370	17068	6239	1564	254	24	1			467224

Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity (smallest non-zero non-gap)
- Add the largest gap (the Frobenius number).



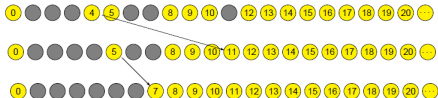
- The result is another numerical semigroup.
- The genus is kept constant in all the transforms.
- Repeating several times (:= ordinarization number) we obtain an ordinary semigroup.

- $r(S)$ - ordinarization number of S ;
- $n_{g,r} := \#\{S : g(S) = g \text{ and } r(S) = r\}$.

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M. Bras-Amorós - 2012

If $r > \max\{\frac{g}{3} + 1, \lfloor \frac{g+1}{2} \rfloor - 14\}$, then $n_{g,r} \leq n_{g+1,r}$.

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Our approach

- $\gamma(S)$: number of even gaps of S - $\#[G(S) \cap 2\mathbb{Z}]$;
- γ -hyperelliptic semigroup: numerical semigroup with γ even gaps;
- $N_\gamma(g) := \#\{S : g(S) = g \text{ and } \gamma(S) = \gamma\}$.

$$n_g = \sum_{\gamma=0}^g N_\gamma(g)$$

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$$n_g = \sum_{\gamma=0}^g N_\gamma(g)$$

Examples

- $n_0 = 1$ (\mathbb{N}_0) and

$$N_\gamma(0) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma \geq 1. \end{cases}$$

- $n_1 = 1$ ($\mathbb{N}_0 \setminus \{1\}$) and

$$N_\gamma(1) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 0, & \text{if } \gamma \geq 1. \end{cases}$$

- $n_2 = 2$ ($\mathbb{N}_0 \setminus \{1, 2\}$ and $\mathbb{N}_0 \setminus \{1, 3\}$) and

$$N_\gamma(2) = \begin{cases} 1, & \text{if } \gamma = 0 \\ 1, & \text{if } \gamma = 1 \\ 0, & \text{if } \gamma \geq 2. \end{cases}$$

F. Torres - 1997

If γ and g are the number of even gaps and the genus of a numerical semigroup S , respectively, then $3\gamma \leq 2g$.

Remark

If γ is even, then

$$\mathbb{N}_0 \setminus (\{2, 4, \dots, 2\gamma\} \cup \{1, 3, \dots, \gamma - 1\})$$

is a numerical semigroup with genus $g = \frac{3\gamma}{2}$.

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$$n_g = \sum_{\gamma=0}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g)$$

Theorem 1

Let γ be a positive integer and $g \geq 3\gamma$. Then

$$N_\gamma(g) = N_\gamma(3\gamma).$$

Thus, $N_\gamma(g) = N_\gamma(g + 1)$, for all $g \geq 3\gamma$.

Notice that

$$n_g = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2g}{3} \rfloor} N_\gamma(g).$$

$$n_{g+1} = \sum_{\gamma=0}^{\lfloor \frac{g}{3} \rfloor} N_\gamma(g+1) + \sum_{\gamma=\lfloor \frac{g}{3} \rfloor+1}^{\lfloor \frac{2(g+1)}{3} \rfloor} N_\gamma(g+1).$$

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Corollary

$$n_g \leq n_{g+1}$$

if, and only if,

$$\sum_{\gamma=\lfloor \frac{g}{3} \rfloor + 1}^{\lfloor \frac{2g}{3} \rfloor} N_{\gamma}(g) \leq \sum_{\gamma=\lfloor \frac{g}{3} \rfloor + 1}^{\lfloor \frac{2(g+1)}{3} \rfloor} N_{\gamma}(g+1).$$

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Let γ be a non-negative integer. Then

$$N_\gamma(g) \leq N_\gamma(g + 1), \forall g.$$

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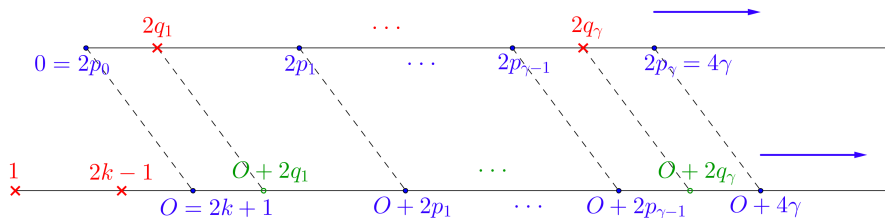
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Construction of a γ -hyperelliptic semigroup of genus g

- γ - positive integer
- $T = \mathbb{N}_0 \setminus \{q_1, \dots, q_\gamma\}$ numerical semigroup



- $S = 2 \cdot T \cup (2 \cdot \mathbb{N}_0 + 1) \setminus \{\text{"suitable choice" of } g - \gamma \text{ odd numbers}\}$



- “suitable choice” ensures that the final set is closed under addition.
- S is a γ -hyperelliptic semigroup: even gaps and even non-gaps are determined by T .
- S has genus g if, and only if, the number of green points chosen as gaps is $g - \gamma - k$.

Lemma

Let S be a γ -hyperelliptic semigroup of genus g and O the first odd number in S . Then

$$2g - 4\gamma + 1 \leq O \leq 2g - 2\gamma + 1.$$

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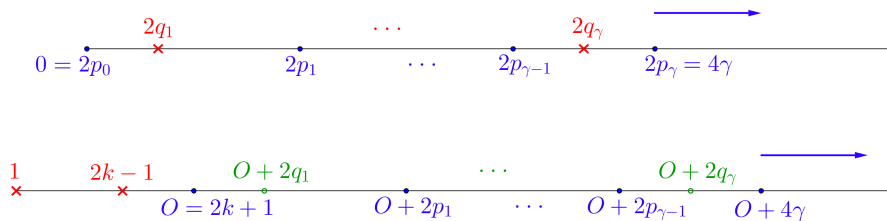
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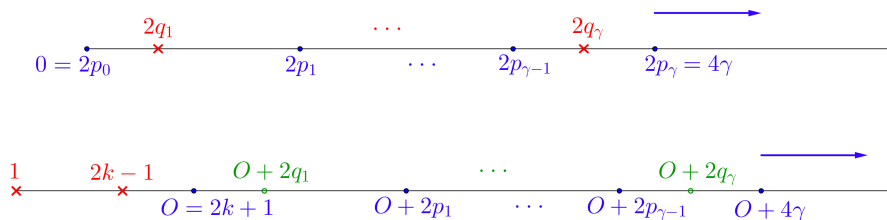
$$2g - 4\gamma + 1 \leq O \leq 2g - 2\gamma + 1.$$

PROOF (THM 1)



- Odd q_i must be gaps! (otherwise, $S \ni q_i + q_i = 2q_i \notin S$)
- $g \geq 3\gamma \implies O \geq 2\gamma + 1 > q_\gamma \geq q_i$, for all i .
 $O \geq 2g - 4\gamma + 1$

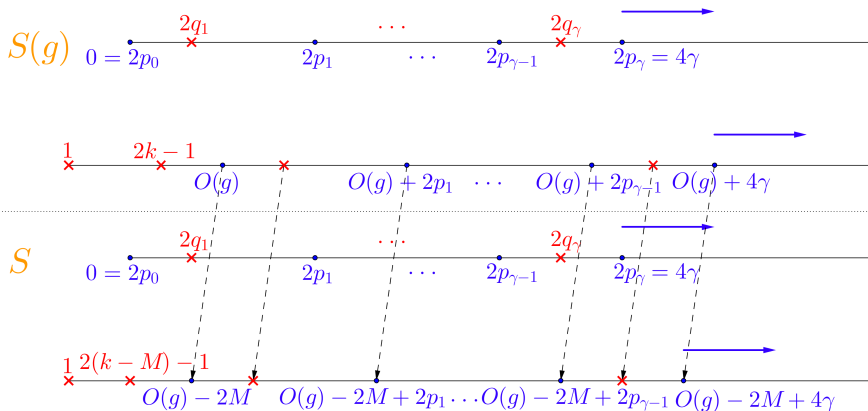
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PROOF (THM 1)

- $\mathcal{S}_\gamma(g) := \{S : g(S) = g \text{ and } \gamma(S) = \gamma\}$
- For a fixed $g \geq 3\gamma$, we find a bijection between $\mathcal{S}_\gamma(g)$ and $\mathcal{S}_\gamma(3\gamma)$



PROOF (THM 1)

- Given $S(g) \in S_\gamma(g)$, let $O(g)$ be the first odd number of $S(g)$
- Let $M := g - 3\gamma$. Making a translation by $-2M$ only on the odd numbers higher than or equal to $O(g)$, we obtain a NS S , such that $O(3\gamma) = O(g) - 2M \geq 2\gamma + 1$ (and this is the first odd number of S)
- The even gaps of $S(g)$ and S are the same, as the odd gaps of $S(g)$ and S lower than $O(3\gamma)$. The odd gaps of $S(g)$ and S higher than $O(3\gamma)$ are translated by $-2M$
- Under this construction, we have $g(S) = g - M = 3\gamma$. Hence, $S \in S_\gamma(3\gamma)$ and $\#S_\gamma(g) \leq \#S_\gamma(3\gamma)$
- Similarly (by making a translation by $+2M$), we can verify the other inequality and the result follows.

A related problem

- γ non-negative integer
- For $g \geq 3\gamma$, the sequence $N_\gamma(g)$ is constant and equal to $N_\gamma(3\gamma)$
- A natural task is about the behavior of $f_\gamma := N_\gamma(3\gamma)$

Lemma

Let γ be a non-negative integer and $M_\gamma := 2^\gamma \left(\frac{\gamma}{2} + 1\right) - 1$. Then

$$M_\gamma + (n_\gamma - \gamma) \cdot (\gamma + 1) \leq f_\gamma \leq M_\gamma + (n_\gamma - \gamma) \cdot 2^\gamma.$$

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Theorem 2

Let $\epsilon > 0$. Then

$$\lim_{\gamma \rightarrow \infty} \frac{f_\gamma}{(2\varphi + \epsilon)^\gamma} = 0$$

and

$$\lim_{\gamma \rightarrow \infty} \frac{f_\gamma}{2^\gamma} = \infty.$$

It suggests that the asymptotic behavior of f_γ is exponential of order β^γ , where $2 < \beta \leq 2\varphi$.

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g / γ	0	1	2	3	4	5	6	7	8	9	10	n_g
0	1											1
1	1											1
2	1	1										2
3	1	2	1									4
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5	1	2	6	3								12
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γ	f_γ	$f_\gamma/f_{\gamma-1}$	$n_{2\gamma}$	$f_\gamma/n_{2\gamma}$
0	1		1	1
1	2	2	2	1
2	7	3,5	7	1
3	23	3,285714	23	1
4	68	2,956522	67	1,015
5	200	2,941176	204	0,981
6	615	3,075	592	1,039
7	1764	2,868292	1693	1,042
8	5060	2,868480	4806	1,053
9	14626	2,890514	13467	1,086
10	41785	2,856899	37396	1,117
11	117573	2,813761	103246	1,139
12	332475	2,827818	282828	1,176
13	933891	2,808905	770832	1,212
14	2609832	2,794579	2091030	1,248

Conjecture

$$\lim_{\gamma \rightarrow \infty} \frac{f_\gamma}{f_{\gamma-1}} = \varphi^2 \approx 2,618$$

and

$$\lim_{\gamma \rightarrow \infty} \frac{f_\gamma}{n_{2\gamma}} = C,$$

where C is a constant.

Remark

There is a relation between the sequence f_γ and the conjecture proposed by M. Bras-Amorós (11). In fact, if f_γ is an increasing sequence, then the conjecture is also true.

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-  M. Bernardini and F. Torres, Counting numerical semigroups of a given genus via even gaps, *preprint*.
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THANK YOU!

FELIZ CUMPLEAÑOS, FERNANDO!
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