ON γ-HYPERELLIPTIC CURVES

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Abstract. These are notes for the corresponding talks at CAta 2018.

In this talks we are concern with the following matters:

(A) On the sequence (n_g), where n_g is the number of numerical semigroups of genus g.
(B) On the Weierstrass property of numerical semigroups.

Part A

A subset $S \subseteq \mathbb{N}_0$ is a *numerical semigroup* if $0 \in S$, S is closed under addition, and $G(S) := \mathbb{N}_0 \setminus S$, the set of *gaps* of S, is finite. The number $g(S) := \#G(S)$ is the *genus* of S. The elements of S are called *nongaps*.

For a fixed $g \in \mathbb{N}_0$ we let $\mathcal{S}_g$ denote the family of the numerical semigroups of genus g, and $n_g := \#\mathcal{S}_g$. Clearly $n_0 = 1$ ($G(S) = \emptyset$), $g(S) = 1$ iff $G(S) = \{1\}$ and hence $n_1 = 1$.

Lemma 1. *If $S \in \mathcal{S}_g$, then $G(S) \subseteq [1, 2g-1]$. In particular, $n_g \le \binom{2g-1}{g}$.*

Proof. We can assume $g \ge 2$. Suppose there is $\ell \in G(S)$ such that $\ell \ge 2g$. Then in $[1, 2g-1]$ there are at least g elements in S, say $h_1, \ldots, h_g$; thus S would have at least $g+1$ elements in $G(S)$, namely $\ell - h_g, \ldots, \ell - h_1, \ell$ which is a contradiction. □

For small g this lemma is useful to compute $G(S)$ (or equivalently S).

(1) For example if $g = 2$, $1 \in G(S) \subseteq [1,3]$. We see then that $G(S) \in \{\{1,2\},\{1,3\}\}$ and so $n_2 = 2$.
(2) Let $g = 3$ and so $1 \in G(S) \subseteq [1,5]$. Then $G(S) \in \{\{1,2,3\},\{1,3,5\},\{1,2,4\},\{1,2,5\}\}$ and $n_3 = 4$.
(3) Let $g = 4$ and so $1 \in G(S) \subseteq [1,7]$. Then $G(S) \in \{\{1,2,3,4\},\{1,3,5,7\},\{1,2,4,5\},\{1,2,4,7\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\}\}$ and $n_4 = 7$.
(4) Let $g = 5$ and so $1 \in G(S) \subseteq [1,9]$. Then $G(S) \in \{\{1,2,3,4,5\},\{1,3,5,7,9\},\{1,2,4,5,7\},\{1,2,4,5,8\},\{1,2,3,5,6\},\{1,2,3,5,7\},\{1,2,3,6,7\},\{1,2,3,5,9\},\{1,2,3,4,6\},\{1,2,3,4,7\},\{1,2,3,4,8\},\{1,2,3,4,9\}$. Thus $n_5 = 12$.

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It seems out of question producing a closed formula for n_g. Based on the values of n_g for $g \leq 100$ María Bras-Amorós amaizinly conjectured that $\lim_{g\to\infty} n_{g+1}/n_g = \varphi := (1+\sqrt{5})/2$ (the so-called golden number). This in fact is true and it follows from a result due to Zhai: $\lim_{g\to\infty} n_g/\varphi^g = \text{constante} \geq 3.78$.

In particular, for large g we have $n_{g+1} \geq n_g$. Is $n_{g+1} > n_g$ for any g (*)? So far this is an open problem. We approuch it by using *even gaps*.

Let $S \in \mathcal{S}_g$, $G_2(S) := \{\ell \in G(S) : \ell \equiv 0 \pmod 2\}$ be the set of even gaps of S snd $\gamma_2(S) := \#G_2(S)$. Clearly $g(S) \geq \gamma_2(S)$ and $g(S) = \gamma_2(S)$ iff $g(S) = \gamma_2(S) = 0$. By Lemma 1 $\#S \cap [1, 2g] = g$ and $2g \in S$. In particular, with $\gamma = \gamma_2(S)$,

- S has $g - \gamma$ even nongaps in $[1, 2g]$;
- S has γ odd nongaps in $[1, 2g]$, say $o_\gamma < \ldots < o_1$, with $o_i = o_i(S) \leq 2g - 2i + 1$ for any i.

As a way of terminology $S \in \mathcal{S}_g$ is said to be γ-*hyperelliptic* if $\gamma_2(S) = \gamma$. We set $\mathcal{S}_\gamma(g) := \{S \in \mathcal{S}_g : \gamma_2(S) - \gamma\}$ (the family of γ-hyperelliptic semigroups of genus g). We let $N_\gamma(g)$ stands for the cardinality of $\mathcal{S}_\gamma(g)$.

We have the following data from the computations of $\mathcal{S}_5$ above: $N_0(5) = 1$, $N_1(5) = 2$, $N_2(5) = 6$, $N_3(5) = 3$; hence in general we cannot expect that $N_\gamma(g) \leq N_{\gamma+1}(g)$ for a fixed g. Is it true that

$$N_\gamma(g) \leq N_\gamma(g+1)\,? \tag{1}$$

The key information on the elements of $G_2(S)$ is the following.

Lemma 2. *Let $S \in \mathcal{S}_\gamma(g)$ with $\gamma \geq 1$, and let ℓ be the biggest element of $G_2(S)$. Then $\ell \leq \min(4\gamma - 2, 4g - 4\gamma)$.*

Proof. Suppose that $\ell \geq 4\gamma$. Then in the interval $[2, 4\gamma - 2]$ there are at least γ even nongaps of S, say $h_1 < \ldots < h_\gamma$ and thus S would have at least $\gamma + 1$ even gaps, namely $\ell - h_\gamma, \ldots, \ell - h_1, \ell$, a contradiction.

Now if $4\gamma - 2 \leq 4g - 4\gamma$; i.e., $g \geq 2\gamma$, the proof follows. On the contrary let $I := 2\gamma - g$ so that $1 \leq I \leq \gamma$. We consider the odd nongaps $o_\gamma < \ldots < o_I$ of S, and we claim that $\ell < o_I$. This would imply $\ell \leq 2g - 2I = 4g - 4\gamma$.

Finally suppose on the contrary that $\ell > o_I$. Hence the odd numbers $\ell - o_i$, $i = I, \ldots, \gamma$ are odd gaps of S; so S would have $\gamma - I + 1 = g - \gamma + 1$ odd gaps which is a contradiction. □

Corollary 1. *For $S \in \mathcal{S}_\gamma(g)$, $2g \geq 3\gamma$. In particular, $\mathcal{S}_g$ is a disjoint union of the sets $\mathcal{S}_\gamma(g)$, $\gamma = 0, \ldots, \lfloor 2g/3 \rfloor$ and $n_g = \sum_{\gamma=0}^{\lfloor 2g/3 \rfloor} N_\gamma(g)$.*

Proof. The result is clear if $g \geq 2\gamma$. Otherwise, $4g - 4\gamma \leq 4\gamma - 2$ and Lemma 2 implies $G_2(S) \subseteq [2, 4g - 4\gamma]$, and the result follows. □

Corollary 2. *Let $S \in \mathcal{S}_\gamma(g)$ and o_γ the smallest odd nongap of S. Then $o_\gamma \geq |2g - 4\gamma| + 1$.*

Proof. Let $g \geq 2\gamma$ and so $G_2(S) \subseteq [2, 4\gamma - 2]$ by Lemma 2; thus S has γ even nongaps in $[2, 4\gamma]$ with $4\gamma \in S$. In particular, $o_\gamma + 4\gamma \in S$ and thus $o_\gamma + 4\gamma \geq 2g + 1$ as S has only γ odd nongaps in $[1, 2g - 1]$.

Let $g < 2\gamma$ and we have to show that $o_\gamma \geq 4\gamma - 2g + 1$. By Lemma 2 S has $2g - 3\gamma$ even nongaps in $[2, 4g - 4\gamma]$. By considerin the elements $2o_\gamma < \ldots < o_\gamma + o_{4\gamma-2g}$ we see that $o_\gamma + o_{4\gamma-2g} \geq 4g - 4\gamma + 2$; finally, as $o_{4\gamma-2g} \leq 2g - 2(4\gamma - 2g) + 1 = 6g - 8\gamma + 1$, the result follows. □

Example 1. Let γ be even and we look for a numerical semigroup in $\mathcal{S}_\gamma(g)$ with $2g = 3\gamma$. We must have $o_\gamma \geq \gamma + 1$ and since in $[\gamma + 1, 3\gamma - 1]$ there are exactly γ odd numbers, then the odd nongaps of S in $[1, 2g-1]$ must be $\gamma + 2i - 1$ with $i = 1, \ldots, \gamma$. On the other hand $G_2(S) \subseteq [2, 2\gamma]$ and hence the even nongaps in $[2, 2g]$ are $2\gamma + 2j$, $j = 1, \ldots, \gamma/2$.

If $\gamma = 2$ and $g = 3$, $S \cap [1, 6] = \{3, 5, 6\} = \langle 3, 5, 7\rangle$.

In general, S is generated by $\Sigma := \{2\gamma + 2i - 1 : i = 1, \ldots, \gamma + 1\}$; see [1, Ex. 2.9].

Example 2. ([1, Ex. 2.9]) Let γ be odd. In this case we look for $S \in \mathcal{S}_\gamma(g)$ with $2g = 3\gamma + 1$. There are $(\gamma + 1)/2$ elements $S \in S_\gamma(g)$ with $g = (3\gamma + 1)/2$.

For $S \in \mathcal{S}_\gamma(g)$ let $S/2 := \{h \in \mathbb{N}_0 : 2h \in S\}$. By Lemma 2 we see that $2\gamma + i$ for $i \in \mathbb{N}_0$. Moreover, $g(S/2) = \gamma$. Then we have a map

$$\mathbf{x} : \mathcal{S}_\gamma(g) \to \mathcal{S}_\gamma\,, \quad S \mapsto S/2\,.$$

Lemma 3. *$\mathbf{x}$ is surjective if and only if $g \geq 2\gamma$.*

Proof. Let $g \geq 2\gamma$ and $T \in \mathcal{S}_\gamma$. Let

$$S := 2T \cup \{2g - 2i + 1 : i = 1, \ldots, \gamma\} \cup \{2g + i : i \in \mathbb{N}_0\}\,.$$

We see that $S \supseteq 4\gamma, 4\gamma + 2, \ldots$ and $2(2g - 2\gamma + 1) \geq 4\gamma + 2$ and hence S is a numerical semigroup which by construction is in $\mathcal{S}_\gamma(g)$ and $S/2 = T$.

Now let $g \leq 2\gamma - 1$. Hence $2g + 2j \in S \in \mathcal{S}_g$ for $j = 0, \ldots, 2\gamma - 1 - g$. Then $g + j \in S/2$; i.e., $[g, 2\gamma - 1] \subseteq S/2$. In particular symmetric numerical semigroups in $\mathcal{S}_\gamma$ have no preimage under $\mathbf{x}$ and so this map is not surjective in this case. □

Problem 1. (A) Let $g \geq 2\gamma$ and $T \in \mathcal{S}_\gamma$, describe $\mathbf{x}^{-1}(T)$.

(B) Let $g \leq 2\gamma - 1$ and $T \in \mathcal{S}_\gamma$ such that $[g, 2\gamma - 1] \subseteq T$. Describe $\mathbf{x}^{-1}(T)$.

References

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