ON THE WEIGHT OF NUMERICAL SEMIGROUPS

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ABSTRACT. We investigate the weight of a family of numerical semigroups by means of even gaps and the Weierstrass property for such a family. Our motivation comes from results on double coverings of curves.

References:

(1) G. Oliveira, F. Torres and J. Villanueva, "On the weight of numerical semigroups", J. Pure Appl. Algebra, to appear.

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1. Introduction

Let $\mathcal{X}$ be a (non-singular, projective, irreducible) curve of genus $g \geq 2$ defined over an algebraically closed field $\mathbf{K}$ of characteristic zero. The *Weierstrass semigroup* (or the *non-gaps*) at a point P is the set $H(P)$ of poles of regular functions in $\mathcal{X} \setminus \{P\}$. The elements of $G(P) := \mathbb{N}_0 \setminus H(P) = \{\ell_1 < \cdots < \ell_g\}$ ($\ell_i = \ell_i(P)$) are the *gaps* at P. The Weierstrass gap theorem asserts that $\ell_g \leq 2g-1$, see e.g. [4]. Let $\mathcal{X}$ be a double covering of a curve $\tilde{\mathcal{X}}$ of genus γ; i.e., there exists a morphism $\pi : \mathcal{X} \to \tilde{X}$ of degree two. Assume that $P \in \mathcal{X}$ is ramified (thus $g \geq 2\gamma$) and set $\tilde{P} = \pi(P)$. If g is large enough with respect to γ, properties of $H(P)$ characterize the morphism π (see Theorem 1). To be more precise let us first recall that $2h \in H(P)$ iff $h \in H(\tilde{P})$ ([9]). Then $H(P)$ has exactly γ even gaps which are contained in $[2, 4\gamma]$ and thus γ odd non-gaps in $[3, 2g-1]$, say $u_\gamma < \cdots < u_1$ ($u_i = u_i(P)$). Set $2\tilde{H}(\tilde{P}) := \{2h : h \in \tilde{H}(\tilde{P})\}$. Therefore the semigroup $H(P)$ is of the form

$$H(P) = 2\tilde{H}(\tilde{P}) \cup \{u_\gamma, \ldots, u_1\} \cup \{2g + i : i \in \mathbb{N}_0\}\,. \tag{1}$$

The *Weierstrass weight* at P is $w(P) := \sum_{i=1}^{\gamma}(\ell_i - i)$. The following result is our starting point (see Problems 1, 2, 3). Throughout this paper let $F(\gamma)$ $(*)$ be the function defined by $F(0) = 2$, $F(1) = 11$, $F(2) = 23$, $F(3) = 34$, $F(4) = 44$, $F(5) = 56$, $F(6) = 65$ and $F(\gamma) = \gamma^2 + 4\gamma + 3$ for $\gamma \geq 7$.

Theorem 1. ([10], [9], [8], [5], [24]) *Let $\gamma \geq 0$ be an integer and $\mathcal{X}$ a curve of genus $g \geq F(\gamma)$. The following statements are equivalents*:

(I) *There exists a point P in $\mathcal{X}$ such that the Weierstrass semigroup $H(P)$ is of the form* (1);

(II) *There exists a point P in $\mathcal{X}$ such that the Weierstrass weight satisfies*

$$\binom{g-2\gamma}{2} \leq w(P) \leq \binom{g-2\gamma}{2} + 2\gamma^2\,; \tag{2}$$

(III) *The curve $\mathcal{X}$ is a double covering of a curve of genus γ* (*for short, we say that $\mathcal{X}$ is γ-hyperelliptic*).

The following problem arises.

Problem 1. Find the true values of $w(P)$ in (2).

A point $P \in \mathcal{X}$ is called *Weierstrass* if $w(P) > 0$. This concept has an important role in the study of the geometry of curves (see [2] for a beautiful exposition on several topics on Weierstrass points). Let W denote the set of Weierstrass points. By means of Hürwitz's Wronskian method [6] it can be shown that W is the support of a divisor $\mathcal{W}$ on $\mathcal{X}$ and that $w(P)$ is the multiplicity of P in $\mathcal{W}$; moreover $\sum_P w(P) = (g-1)g(g+1)$. Hürwitz, among other things, was concerned about the following matters:

(1) On the number of Weierstrass points of $\mathcal{X}$ (which have to do with bounds on weights);

(2) Suppose that $\mathcal{X} \subseteq \mathbb{P}^{g-1}(\mathbf{K})$ is non-hyperelliptic and let $P \in \mathcal{X}$; since $(\ell_1(P)-1,\ldots,\ell_g(P)-1)$ is the sequence of all multiplicity of intersections of hyperplanes and $\mathcal{X}$ at P, it is natural to look for the sequence that can be appear in this way. This is equivalent to ask on arbitrary semigroups H that occur as a Weierstrass semigroups; i.e. whether $H = H(P)$ for some $P \in \mathcal{X}$ (for short, we say that H satisfies the *Weierstrass property*).

Counting Weierstrass points is important e.g. in getting bounds on the number of automorphisms of curves [6] or on the study of constellations of curves [22]. Question two was raised approximately in 1893; further historical accounts can be read in [3]. Long after that, in 1980 Büchweitz [1] showed that not every semigroup satisfies the Weierstrass property (see also [13] and [24]).

In Section 2 we investigate weights of arbitrary numerical semigroup by means of even gaps. In this context the analogue of Theorem 1 is Theorem 2; Problem 1 is related then to Problems 2, 3. Theorem 3 subsume several values of weights. In Section 3 we investigate the Weierstrass property of the semigroups arising in Section 2; here we are mainly concern with the cases $\gamma = 0,1,2,3,4$. A typical application of our results is for example the non-existence of a 2-hyperelliptic curve of genus $g \geq 23$ with a ramification point of weight $\binom{g-4}{2} + 6$ (Example 3).

2. On Problem 1

In this section we consider Problem 1 above in the context of Numerical Semigroup Theory only (as was already mentioned in the introduction there are numerical semigroups that cannot be realized as Weierstrass semigroups).

Let $H = \{0 = m_0 < m_1 < m_2 < \cdots\}$ ($m_i = m_i(H)$) be an arbitrary (numerical) semigroup; i.e., H is a subsemigroup of the nonnegative integers $\mathbb{N}_0$ such that its complement is finite. Let $G(H) := \mathbb{N}_0 \setminus H = \{\ell_1 < \cdots < \ell_g\}$ ($\ell_i = \ell_i(H)$). Following geometrical settings we say that m_1, the elements of H, the elements of $G(H)$ and $g = g(H)$ are respectively the *multiplicity*, the *non-gaps*, the *gaps* and the *genus* of H. The 'Weierstrass gap theorem' is also true here; in particular, $m_{g+i} = 2g + i$ for all $i \in \mathbb{N}_0$ ([1], [18]). The number

$$w(H) := \sum_{i=1}^{g}(\ell_i - i) = \frac{1}{2}(3g^2 + g) - \sum_{i=1}^{g} m_i \tag{3}$$

is the *weight* of H. We say that H is *γ-hyperelliptic* if it has exactly γ even gaps. In this case, for $g \geq 2\gamma$, H satisfies property (1), [7] (cf. [2]), [23]; i.e., there exists a unique semigroup $\tilde{H}$ of genus γ and γ odd numbers $3 \leq u_\gamma < \cdots < u_1 \leq 2g - 1$ ($u_i = u_i(H)$) such that

$$H = 2\tilde{H} \cup \{u_\gamma, \ldots, u_1\} \cup \{2g + i : i \in \mathbb{N}_0\}\,.$$

As a matter of fact, $\tilde{H} = \{h/2 : h \in H\,, h \equiv 0 \pmod 2\}$. We say that H is a *double covering* of $\tilde{H}$. Let $\tilde{H} = \{0 = \tilde{m}_0 < \tilde{m}_1 < \tilde{m}_2 < \cdots\}$. After some computations from (3) we have the following.

Lemma 1. *Notation as above. For $g \geq 2\gamma$, the weight of a γ-hyperelliptic semigroup H can be computed by any of the formulas below:*

$$\begin{aligned} w(H) &= \binom{g-2\gamma}{2} + (2g + 2\gamma + 1)\gamma - \sum_{i=1}^{\gamma}(2\tilde{m}_i + u_i) \\ &= \binom{g-2\gamma}{2} + (2g - \gamma)\gamma - \sum_{i=1}^{\gamma} u_i + 2w(\tilde{H}). \end{aligned} \tag{4}$$

Corollary 1. *Let* $\gamma \geq 0$ *be an integer and* H_1 *and* H_2 *semigroups of genus* $g \geq 2\gamma$ *which are double coverings of a semigroup* $\tilde{H}$ *of genus* γ*. Then*

$$w(H_1) = w(H_2) \quad \textit{iff} \quad \sum_{i=1}^{\gamma} u_i(H_1) = \sum_{i=1}^{\gamma} u_i(H_2)\,.$$

From (4) we have:

Lemma 2. ([25]) *For a* γ*-hyperelliptic semigroup* H *of genus* $g \geq 2\gamma$, $w(H) \equiv \binom{g-2\gamma}{2}$ (mod 2).

Lemma 3. ([23], [26]) *Notation as above.*

1. *For each* $i = 1, \ldots, \gamma$, $2\tilde{m}_i + u_i \geq 2g + 1$, *especially we get* $u_\gamma \geq 2g - 4\gamma + 1$*. Thus if* $g \geq 4\gamma$, *then* $m_i = 2\tilde{m}_i$ *for* $i = 1, \ldots, \gamma$.
2. *Let* $2\tilde{m}_1 \geq 4$*. Then* $u_1 + 2\tilde{m}_1 = 2g + 1$ *iff* $u_1 = 2g - 3$ *and* $2\tilde{m}_1 = 4$; *in this case, for each* $i = 1, \ldots, \gamma$, $u_i + 2\tilde{m}_i = 2g + 1$.
3. *Let* $u_\gamma = 2g - 2\gamma - 1$*. If* $u_1 = 2g - 3$, *then for* $i = 1, \ldots, \gamma$, $u_i = 2g - 2i - 1$*. If* $u_1 = 2g-1$, *let* $k := \max\{i \in \{1, \ldots, \gamma-1\} : u_i = 2g-2i+1\}$*. Then* $u_i = 2g-2i+1$ *for* $i = 1, \ldots, k$ *and* $u_i = 2g - 2i - 1$ *for* $i = k + 1, \ldots, \gamma$.
4. *Let* $u_\gamma = 2g - 2\gamma - 3$ *and* $u_1 = 2g - 1$*. Let* $k := \max\{i \in \{1, \ldots, \gamma - 1\} : u_i = 2g - 2i + 1\}$ *and* $s := \min\{i \in \{k + 1, \ldots, \gamma\} : u_i = 2g - 2i - 3\}$ (*thus* $s \geq k + 1$)*. Then* $u_i = 2g - 2i + 1$ *for* $i = 1, \ldots, k$; $u_i = 2g - 2i - 1$ *for* $i = k + 1, \ldots, s - 1$ *and* $u_i = 2g - 2i - 3$ *for* $i = s, \ldots, \gamma$.

Proof. (1) It follows from the fact that all the numbers in the sequence $u_i < u_i + 2\tilde{m}_1 < \cdots < u_i + 2\tilde{m}_i$ are odd non-gaps of H.

(2) Here the odd numbers $u_\gamma < u_\gamma + 2\tilde{m}_1 < \cdots < u_2 + 2\tilde{m}_1$ are the odd non-gaps of H. Therefore $u_{\gamma-1} = u_\gamma + 2\tilde{m}_1, \ldots, u_1 = u_2 + 2\tilde{m}_1$, so that $u_\gamma + (\gamma - 1)2\tilde{m}_1 = u_1$. We have $u_1 \leq 2g - 1$. If $\tilde{m}_1 \geq 3$, $u_\gamma \leq 2g - 6\gamma - 5$ which is a contradiction according to Item (1). Then the result follows.

(3) If $u_1 = 2g - 3$, clearly $u_i = 2g - 2i - 1$ for $i = 1, \ldots, \gamma$. Let $u_1 = 2g - 1$ and k the number defined as above. Then $u_{k+1} \leq 2g - 2k - 1$. We claim that $u_{k+1} \leq 2g - 2k - 3$,

otherwise $u_{k+1} = 2g-2k-1 = 2g-2(k+1)+1$ a contradiction with the definition of k. Now in the interval $[2g-2\gamma-1, 2g-2k-3]$ there are $\gamma - k$ odd numbers as well as $\gamma - k$ odd non-gaps. The result now follows.

(4) Let k and s be the numbers defined above. Arguing as in Item (3), $u_{k+1} \leq 2g-2k-3$. In the interval $[u_\gamma, u_s]$ there are $\gamma + 1 - s$ odd numbers and the same number of odd non-gaps. Thus $u_i = 2g-2i-3$ for $i = s, \ldots, \gamma$. If $s = k+1$ the result is clear; otherwise, $u_{s-1} \geq 2g-2s-1$ and thus $u_{s-1} \geq 2g-2s+1$ by the definition of s. In the interval $[2g-2s+1, 2g-2k-3]$ we have both $s-(k+1)$ odd numbers and odd non-gaps and the proof is complete. □

The analogous of Theorem 1 is formulated as follows. Recall that $F(\gamma)$ is the function defined by $(*)$ in the introduction. Let $\gamma \geq 0$ and g be integers. Let H be a semigroup of genus $g \geq 2\gamma$. Consider the following statements:

(I) H is a γ-hyperelliptic;

(II) The weight $w(H)$ satisfies

$$\binom{g-2\gamma}{2} \leq w(H) \leq \binom{g-2\gamma}{2} + 2\gamma^2 . \tag{5}$$

Then (I) implies (II) which follows from Lemmas 1 and 3(1).

Problem 2. Find the true values of weights of γ-hyperelliptic semigroups H (of genus $g \geq 2\gamma$) in (5).

Concerning the converse we have the following.

Theorem 2. ([23]) (II) *implies* (I) *provided that* $g \geq F(\gamma)$.

Thus if we impose the condition $g \geq F(\gamma)$ in Problem 2, then all the possible value attained in (5) will be from γ-hyperelliptic semigroups only.

Remark 1. The bound $F(\gamma)$ on g is sharp [5], [23].

Problem 3. Which semigroups satisfying (5) are Weierstrass?

The border cases in (5) follow from Lemma 3 (1) (2).

Proposition 1. ([23]) *Let* $\gamma \geq 0$ *be an integer and* H *a semigroup of genus* $g \geq 2\gamma$. *Then*:

1. $w(H) = \binom{g}{2}$ *iff* $2 \in H$;
2. *Let* $\gamma \geq 1$. *Then* $w(H) = \binom{g-2\gamma}{2}$ *iff* $m_i = 2\gamma + 2i$ *and* $u_i = 2g - 2i + 1$ *for* $i = 1, \ldots, \gamma$;
3. *Let* $\gamma \geq 1$. *Then* $w(H) = \binom{g-2\gamma}{2} + 2\gamma^2$ *iff* $m_1 = 4$ *and* $u_1 = 2g - 3$. *In this case,* $H = \langle 4, 4\gamma + 2, 2g - 4\gamma + 1\rangle$.

Next we compute the weights of semigroups of multiplicity 4 and 6.

Proposition 2. ([26]) *Let* $\gamma \geq 1$ *be an integer and* H *a* γ*-hyperelliptic semigroup of genus* $g \geq 3\gamma$ *of multiplicity* $m_1 = 4$. *Then*

1. $w(H) \in \left\{\binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k : k = 1, \ldots, \gamma + 1\right\}$;
2. $w(H) = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k$ *iff* $H = \langle 4, 4\gamma + 2, 2g - 2\gamma - 2k + 3, 2g - 2\gamma + 2k + 1\rangle$ $(k = 1, \ldots, \gamma + 1)$.

Proof. (1) By Proposition 1 (3) we shall assume $k \leq \gamma$. We have to show that $w(H) \notin \left\{\binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k : k = 1, \ldots, \gamma\right\}$ gives a contradiction. We have that $u_1 \geq 2g - 3$ (as $m_1 = 4$) and thus the aforementioned result allows to define the integer $j := \max\{i \in \{1, \ldots, \gamma - 1\} : u_i = 2g - 2i + 1\}$. Now, the weight of a hyperelliptic semigroup of genus γ is $\gamma^2 - \gamma$; then $w(H) \geq \binom{g-2\gamma}{2} + \gamma^2 - \gamma$ by (4). It follows that

$$w(H) = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k + 2n$$

for some integer $n \in \{1, \ldots, k - 1\}$; finally Lemma 1 implies

$$j^2 - (2\gamma + 1)j + (\gamma^2 + \gamma - k^2 + k - 2n) = 0\,,$$

which contradicts the fact that j is an integer.

(2) The case $k = \gamma + 1$ is just Proposition 1 (3). Let now $k \in \{1, \ldots, \gamma\}$ and $w(H) = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k$. From Proposition 1 (3) there is $j = \max\{i \in \{1, \ldots, \gamma\} : u_i =$

$2g-2i+1\}$, therefore if $j<\gamma$ we have $u_{j+l}=2g-2j-4l+1$ for each $l=1,\dots,\gamma-j$. And so from Lemma 1 we have

$$j^2-(2\gamma+1)j+(\gamma^2+\gamma-k^2+k)=0\,.$$

Therefore, $j=\gamma-k+1$. In particular, $u_\gamma=2g-2\gamma-2k+3$ and $u_{\gamma-k}=2g-2\gamma+2k+1$. So $H_k:=\langle 4,4\gamma+2,2g-2\gamma-2k+3,2g-2\gamma+2k+1\rangle\subseteq H$. Since $g\geq 3\gamma$ it follows that H_k is a γ-hyperelliptic semigroup. Let $4e_2+2,4e_1+1,4e_3+3$ be the smallest integers in H_k which are congruent respectively to $2,1,3 \pmod 4$. We have that $e_1+e_2+e_3=g(H_k)$ and $e_2=\gamma$ is the number of even gaps of H_k. Since $4e_1+1,4e_3+3\in\{2g-2\gamma-2k+3,2g-2\gamma+2k+1\}$ we have $g(H_k)=g$ and so $H_k=H$. The converse is clear. □

A semigroup $\tilde{H}$ of genus γ of multiplicity $\tilde{m}_1=3$ has the following form and weight [8]. For each $k=0,1,\dots,\lfloor\gamma/3\rfloor$,

(1) If $\gamma\equiv 0 \pmod 3$, $\tilde{H}=\tilde{H}_k=\{3i:i=1,\dots,2\gamma/3\}\cup\{\gamma-2+3k+3s:s=1,\dots,\gamma/3-k\}\cup\{2\gamma-1+3s-3k:s=1,\dots,k\}\cup\{2\gamma+i:i\geq 0\}$ and $w(\tilde{H})=\gamma(\gamma-1)/3+3k^2-k\gamma-k$;

(2) If $\gamma\equiv 1 \pmod 3$, $\tilde{H}=\tilde{H}_k=\{3i:i=1,\dots,(2\gamma-2)/3\}\cup\{\gamma-2+3k+3s:s=1,\dots,(\gamma+2)/3-k\}\cup\{2\gamma-1+3s-3k:s=1,\dots,k\}\cup\{2\gamma+i:i\geq 0\}$ and $w(\tilde{H})=\gamma(\gamma-1)/3+3k^2-k\gamma-k$;

(3) If $\gamma\equiv 2 \pmod 3$, $\tilde{H}=\tilde{H}_k=\{3i:i=1,\dots,(2\gamma-1)/3\}\cup\{\gamma-1+3k+3s:s=1,\dots,(\gamma+1)/3-k\}\cup\{2\gamma-2+3s-3k:s=1,\dots,k\}\cup\{2\gamma+i:i\geq 0\}$ and $w(\tilde{H})=\gamma(\gamma-2)/3+3k^2-k\gamma+k$.

Therefore a direct computation via (4) shows the following.

Proposition 3. *Let $\gamma\geq 1$ be an integer and H a γ-hyperelliptic semigroup of genus $g\geq 2\gamma$ of multiplicity $m_1=6$. Let $u_\gamma<\cdots<u_1$ be the odd non-gaps of H. Then*

$$w(H)=\binom{g-2\gamma}{2}+2\gamma g-\sum_{i=1}^{\gamma}u_i+I_k(\gamma)\,,$$

where $k=0,1,\dots,\lfloor\gamma/3\rfloor$, $I_k(\gamma)=2(3k^2-k\gamma-k)-\gamma(\gamma+2)/3$ if $\gamma\equiv 0,1 \pmod 3$ and $I_k(\gamma)=2(3k^2-k\gamma+k)-\gamma(\gamma+4)/3$ otherwise.

We obtain an improvement on (5), namely:

Proposition 4. ([26]) *Let $\gamma \geq 1$ be an integer and H a γ-hyperelliptic semigroup of genus g.*

(1) *If $g \geq 3\gamma$ and $m_1 = 4$, then either $w(H) = \binom{g-2\gamma}{2} + 2\gamma^2$, $w(H) = \binom{g-2\gamma}{2}$ or*
$$\binom{g-2\gamma}{2} + \gamma^2 - \gamma \leq w(H) \leq \binom{g-2\gamma}{2} + 2(\gamma^2 - \gamma)\,;$$

(2) *If $g \geq 2\gamma$ and $m_1 \geq 6$, then $w(H) \leq \binom{g-2\gamma}{2} + 2\gamma^2 - (2\gamma - 4)$.*

Proof. (1) It follows from Proposition 2.

(2) We have $\tilde{m}_i \geq 2i+1$ for $i = 1, \ldots, \gamma-2$, $\tilde{m}_{\gamma-1} \geq 2(\gamma-1)$ ([18]) and $u_i \geq 2g - 4i + 1$ for $i = 1 \ldots, \gamma$ ([23]). Thus $\sum_{i=1}^{\gamma}(2\tilde{m}_i + u_i) \geq 2g\gamma + 3\gamma - 4$. Now the inequality follows from (4). □

Proposition 5. *Let H and $\tilde{H}$ be semigroups of genus g and γ respectively with $g \geq 2\gamma$. Suppose that H is a double covering of $\tilde{H}$.*

(1) *If $u_\gamma = u_\gamma(H) = 2g - 2\gamma + 1$, then $w(H) = \binom{g-2\gamma}{2} + 2w(\tilde{H})$.*
(2) *If $u_\gamma = u_\gamma(H) = 2g - 4\gamma + 1$, then $w(H) = \binom{g-2\gamma}{2} + 2\gamma + 4w(\tilde{H})$.*

Proof. In both cases the odd non-gaps are determined by u_γ. We have respectively u_γ, $u_i = u_\gamma + 2\tilde{m}_i$ $(i = 1, \ldots, \gamma-1)$ and $u_i = 2g - 2i + 1$ $(i = 1, \ldots \gamma)$; now the result follows from (4). □

Proposition 6. *Let H be a γ-hyperelliptic semigroup of genus $g \geq 2\gamma$ and multiplicity $m_1 = 2\gamma + 2$. Suppose in addition that $u_\gamma = 2g - 2\gamma - 1$. Let k be the integer defined in Lemma* 3 (3). (*For the case $u_1 = 2g - 3$, we let $k = 0$*). *Then*
$$w(H) = \binom{g-2\gamma}{2} + 2(\gamma - k)\,.$$

Proof. H is a double covering of a semigroup of multiplicity $\gamma + 1$ and the result follows from (4) and Lemma 3 (3). □

In a similar way we obtain:

Proposition 7. *Let H be a γ-hyperelliptic semigroup of genus $g \geq 2\gamma$ and multiplicity $m_1 = 2\gamma+2$. Suppose in addition that $u_\gamma = 2g-2\gamma-3$. Let k and s be the numbers defined in Lemma* 3 (4)*. Then*

$$w(H) = \binom{g-2\gamma}{2} + 2(2\gamma - s - k + 1)\,.$$

Proof. H is a double covering of a semigroup of multiplicity $\gamma+1$ and $m_1 + u_\gamma = 2g-1$. In particular $w(\tilde{H}) = 0$ and $u_1 = 2g-1$. Therefore from (4) and Lemma 3 (4) we have

$$\begin{aligned} w(H) &= \binom{g-2\gamma}{2} + (2g-\gamma)\gamma - \sum_{i=1}^{\gamma} u_i \\ &= \binom{g-2\gamma}{2} + (2g-\gamma)\gamma - \sum_{i=1}^{\gamma}(2g-2i) - (2s+2k-3\gamma-2) \\ &= \binom{g-2\gamma}{2} + 2(2\gamma - s - k + 1)\,. \end{aligned}$$

□

We subsume next some values in (5) obtained so far. For H a γ-hyperelliptic semigroup of genus $g \geq 2\gamma$ set $D(H) := w(H) - \binom{g-2\gamma}{2}$.

Theorem 3. *Notation as above. The function $D(H)$ attains the following values*:

1. *$D(H) = 2k$ for $k = 0, 1, \ldots, 2\gamma-2$;*
2. *$D(H) = 2\gamma + 4k$ for $k = 1, \ldots, \gamma-1$;*
3. *If further $g \geq 3\gamma$, $D(H) = 2\gamma^2 - (\gamma+k)(\gamma+1-k)$ for $k = 1, \ldots, \gamma+1$.*

Proof. Let k be an integer with $0 \leq k \leq \gamma-1$. Let $\tilde{H}_k$ be the semigroup of genus γ with gaps $1, \ldots, \gamma-1$ and $\gamma+k$ so that $w(\tilde{H}_k) = k$. Thus Proposition 5 (or Proposition 6) implies Item (1) for $k = 0, \ldots, \gamma$. From these H_k's, Item (2) follows. Item (1) for $k = \gamma+1, \ldots, 2\gamma-2$ follows from Proposition 7. Item (3) follows from Proposition 2. □

3. On the Weierstrass property

In this section we will find out semigroups H satisfying both (5) and the Weierstrass property. Let $\gamma \geq 0$ and $g \geq 2\gamma$ be integers. For such semigroups recall that $D(H) = w(H) - \binom{g-2\gamma}{2}$ and that $F(\gamma)$ is the function defined in the introduction via $(*)$. Let $S(H) := \{2g + i : i \in \mathbb{N}_0\}$.

Example 1. Let $\gamma = 0$. Then $D(H) = 0$. This case is only possible if $2 \in H$ and the semigroup is Weierstrass for $g \geq F(0)$; see e.g. [4].

Example 2. Let $\gamma = 1$. From Lemma 2 $D(H) = 0$ or $D(H) = 2$. By Proposition 1 the former case occurs iff $m_1 = 4$ and $u_1 = 2g - 1$ and the second iff $m_1 = 4$ and $u_1 = 2g - 3$. In both cases H is Weierstrass [17].

Example 3. Let $\gamma = 2$ and assume $g \geq F(2)$. From Lemma 2, $D(H) \in \{2i : i = 0, 1, 2, 3, 4\}$. We shall show that $D(H) \neq 6$. Here we have $m_1 \in \{4, 6\}$. If $m_1 = 4$, $D(H) = 8$ or $D(H) \leq 4$ by Proposition 4(1). Let $m_1 = 6$. Then H is a double covering of the semigroup $\tilde{H} = \{0, 3, 4, 5, \ldots\}$ and hence $D(H) = 4g - 4 - u_2 - u_1$ by (4). We know that $u_2 \geq 2g - 7$ (Lemma 3 (1)). Thus $u_2 + u_1 \geq 4g - 8$ and $D(H) \leq 4$.

All the values $D(H) = 0, 2, 4, 8$ occur only if H is 2-hyperelliptic (Theorem 2). As a matter of fact all these semigroups are Weierstrass [17], [5], [19]. We observe that if $\tilde{H} = \{0, 3, 4, 5, \ldots\}$, $H_1 := 2\tilde{H} \cup \{2g-7, 2g-1\} \cup S(H)$ and $H_2 := 2\tilde{H} \cup \{2g-5, 2g-3\} \cup S(H)$ then $w(H_1) = w(H_2)$ by Corollary 1.

Example 4. Any semigroup of multiplicity 4 is Weierstrass [17]. Thus all the values in Theorem 3(3) arise from Weierstrass semigroups.

Example 5. If $m_1 = 2\gamma + 2$ and $u_\gamma = 2g - 2\gamma + 1$, then $w(H) = \binom{g-2\gamma}{2}$ (Proposition 5 (1)). By using the theory of Fuchsian groups and Lewittes' theorem on fixed points of automorphisms, it can be shown that H is Weierstrass [21]. Next we give a direct proof for $\gamma = 3$. We use ideas from [19]. Thus at least for $g \geq 11$ we have to show that $H = 2\langle 4, 5, 6, 7\rangle \cup \{2g-5, 2g-3, 2g-1\} \cup S(H)$ is Weierstrass.

Let $g \geq 11$ and $0 \leq r \leq 7$ be integers such that $g+r$ is odd. Let $\alpha(x)$ be a degree four polynomial in $\mathbf{K}[x]$ and let b be a constant such that the roots of $a(x) := \alpha^2(x) - b^2$, say $a_1, \ldots, a_8$, are pairwise different. Let $b_1, \ldots, b_{g-10} \in \mathbf{K} \backslash \{a_1, \ldots, a_8\}$ pairwise different.

We consider the curves $\mathcal{X}$ and $\mathcal{X}_r$ defined respectively by

$$y^2 = a(x)\,, \qquad z^4 = a(x)(x-a_1)^2 \cdots (x-a_r)^2 (x-b_1)^2 \cdots (x-b_{g-10})^2\,.$$

Then $\mathcal{X}_r$ is a double covering of $\mathcal{X}$. There are two points $R_\infty, R'_\infty \in \mathcal{X}$ over $x = \infty$; since $\gcd(4, 2r+2g-12) = 2$, there exist just two points S_∞ and S'_∞ in $\mathcal{X}_r$ over R_∞ and R'_∞ respectively. We shall show that $H(S_\infty) = H$. Observe that $H(R_\infty) = \langle 4,5,6,7 \rangle$ and thus the even non-gaps of $H(S_\infty)$ are $\{0, 8, 10, 12, 14 \ldots\}$.

Claim 1. By applying the Riemann-Hurwitz formula it follows that the genus of $\mathcal{X}_r$ is g.

Claim 2. The numbers $2g-5, 2g-3, 2g-1$ belong to $H(S_\infty)$.

Proof. (Claim 2) We compute some divisors (cf. [19]). Let $P_i \in \mathcal{X}_r$ be the unique point over $x = a_i$ and Q_i, Q_i' the two points in $\mathcal{X}_r$ over $x = b_i$. We permute the roots of $a(x)$ such way that $\alpha(a_j) = b$ for $j = 1,2,3,4$ and we write $f := y - \alpha(x) + b$. Then

- $\mathrm{div}(x-a_i) = 4P_i - 2S_\infty - 2S'_\infty$;
- $\mathrm{div}(x-b_i) = 2Q_i + 2Q'_i - 2S_\infty - 2S'_\infty$;
- $\mathrm{div}(y) = 2P_1 + \cdots + 2P_8 - 8S_\infty - 8S'_\infty$;
- $\mathrm{div}(z) = 3P_1 + \cdots + 3P_r + P_{r+1} + \cdots + P_8 + (Q_1 + Q'_1) + \cdots + (Q_{g-10} + Q'_{g-10}) - (g+r-6)(S_\infty + S'_\infty)$;
- $\mathrm{div}(y + \alpha(x)) = 8S'_\infty - 8S_\infty$;
- $\mathrm{div}(f) = 2P_1 + \cdots + 2P_4 - 8S'_\infty$.

We choose r such that $g + r \equiv 7 \pmod 8$. Let $h := z(y+\alpha(x))^n$ where $n = (g+r+1)/8$ if $r \leq 4$ and $n = (g+r-7)/8$ if $r \geq 5$. Then $\mathrm{div}_\infty(h) = (g+r-6+8n)S_\infty$ iff $8n - (g+r-6) \geq 0$. For each such integer r we shall give in the table bellow functions h_1, h_2 and h_3 such that their pole divisors are respectively $(2g-5)S_\infty$, $(2g-3)S_\infty$ and $(2g-1)S_\infty$.

r	h_1	h_2	h_3
0	h	$(x-a_1)h$	$(x-a_1)^2h$
1	$fh/(x-a_1)$	h	$(x-a_1)h$
2	$fh/(x-a_1)(x-a_2)$	$(x-a_2)h_1$	h
3	$fh/(x-a_1)(x-a_2)(x-a_3)$	$(x-a_3)h_1$	$(x-a_2)h_2$
4	$fh/(x-a_1)(x-a_2)(x-a_3)(x-a_4)$	$(x-a_4)h_1$	$(x-a_3)h_2$
5	$(x-a_6)(x-a_7)(x-a_8)h/y$	h/f	$(x-a_5)h_2$
6	$(x-a_7)(x-a_8)h/y$	$(x-a_6)h_1$	h/f
7	$(x-a_8)h/y$	$(x-a_7)h_1$	$(x-a_6)h_2$

$\square$

Example 6. For $g \geq 9$ an odd integer we show that the semigroup $H = 2\langle 3, 4\rangle \cup \{2g-7, 2g-3, 2g-1\} \cup S(H)$ is Weierstrass. The method of the proof follows as the example above. Let $a_1, \ldots, a_4, b_1, \ldots, b_{(g-9)/2} \in \mathbf{K}$ be pairwise different. Let $1 \leq r \leq 4$ be an integer. We consider the curves $\mathcal{X}$ and $\mathcal{X}_r$ defined by

$$\mathcal{X} : y^4 = a(x) := (x-a_1)(x-a_2)(x-a_3)(x-a_4)\,, \qquad \mathcal{X}_r : z^8 = a(x)A(x)^4\,,$$

where $A(x) := (x-a_1)\cdots(x-a_r)(x-b_1)\cdots(x-b_{(g-9)/2})$. Then $\mathcal{X}_r$ is a double covering of $\mathcal{X}$ and let D_∞ be the pole divisor of the function x on $\mathcal{X}_r$. Let $P_i \in \mathcal{X}_r$ be the unique point over $x = a_i$ and Q_i, Q_i', Q_i'', Q_i''' the four different points over $x = b_i$. We shall show that $H(P_1) = H$. As a matter of fact the even non-gaps of $H(P_1)$ are precisely the set $2\langle 3, 4\rangle$, because the unique point p_1 of $\mathcal{X}$ over $x = a_1$ is a hyperflex point on the non-singular curve $\mathcal{X}$. It is not difficult to see that:

- $\operatorname{div}(x-a_i) = 8P_i - D_\infty$;
- $\operatorname{div}(y) = 2(P_1+P_2+P_3+P_4) - D_\infty$;
- $\operatorname{div}(z) = 5P_1 + \cdots + 5P_r + P_{r+1} + \cdots + P_4 + (Q_1+Q_1'+Q_1''+Q_1''') + \cdots + (Q_{(g-9)/2} + Q'_{(g-9)/2} + Q''_{(g-9)/2} + Q'''_{(g-9)/2}) - \frac{g+2r-7}{4}D_\infty$.

Let $F_1 := z/(x-a_1)^n$. Then $\operatorname{div}_\infty(F_1) = (8n-5)P_1$ iff $2n - \frac{g+2r-7}{2} \geq 0$. First, we consider the case $g \equiv 1 \pmod 4$. We set $n = (g-1)/4$. Here $r = 2$ implies $2g-7 \in H(P_1)$. With

$F_2 = (x - a_3)(x - a_4)F_1/y^2$ and $F_3 = yF_1(x - a_1)$ we find that $2g - 3$ and $2g - 1$ belong to $H(P_1)$ respectively. In the case $g \equiv 3 \pmod 4$ we set $n = (g - 3)/4$. Then the proof with $r = 1$ similar to that of the case $g \equiv 1 \pmod 4$ with $r = 2$ works well. Notice that (4) implies $D(H) = 6$.

Example 7. Let us consider a γ-hyperelliptic semigroup of genus g with $n := u_\gamma = 2g - 4\gamma + 1$. Suppose that H is the double covering of $\tilde{H} = \{0, \tilde{m}_1, \ldots, \tilde{m}_\gamma = 2\gamma, 2\gamma + 1, \ldots\}$. Thus $u_{\gamma-1}(H) = n + 2\tilde{m}_1, \ldots, u_1(H) = n + 2\tilde{m}_{\gamma-1}$ so that $H = 2\tilde{H} + n\mathbb{N}_0$. Let $c(\tilde{H})$ denote the conductor of $\tilde{H}$; i.e., the least integer c such that $c + h \in \tilde{H}$ for all $h \in \tilde{\mathbb{N}}_0$. We look for the hypotheses in [11, Thm. 2.2], namely $n \geq 2c(\tilde{H}) - 1$ and $n \neq 2\tilde{m}_1 - 1$. Since $c(\tilde{H}) \leq 2\gamma$, both conditions are satisfied for $g \geq 4\gamma$. Therefore if $\tilde{H}$ is Weierstrass so H does, loc. cit. There are several sufficient conditions in order that $\tilde{H}$ be Weierstrass; e.g.,

- $m_1(\tilde{H}) \leq 5$ ([20], [17], [15]);
- Either $g(\tilde{H}) \leq 8$ ([14], [12]);
- Either $w(\tilde{H}) \leq g(\tilde{H})/2$ or $g(\tilde{H})/2 < w(\tilde{H}) \leq g(\tilde{H}) - 1$ and $2m_1(\tilde{H}) > \ell_g(\tilde{H})$ ([3], [16]).

Example 8. Let us consider further values of $D(H)$ in Theorem 2. We let $\tilde{H}$ be a ρ-hyperelliptic of genus γ. For example for $\rho = 1$, $\gamma \geq 3$, let $w(\tilde{H}) = \binom{g-2}{2}, \binom{g-2}{2} + 2$; thus the example above and Proposition 5 (2) show that there are Weierstrass semigroups H with $D(H) = 2\gamma^2 - (8\gamma - 12), 2\gamma^2 - (8\gamma - 20)$.

Example 9. Let us consider the case $\gamma = 3$, $g \geq F(3)$. Arguing as in Example 3 ($\gamma = 2$) we can rule out the possibility $D(H) = 16$. Thus by Lemma 2 $D(H) \in \{2i : i = 0, \ldots, 9\} \setminus \{16\}$. We have the following table:

H	D(H)	Reference	Weierstrass
$\langle 4, 14, 2g-11\rangle$	18	Prop. 1 (3)	yes ([17])
$2\langle 3, 4\rangle + (2g-11)\mathbb{N}_0$	14	Prop. 5 (2)	yes (Ex. 7)
$\langle 4, 14, 2g-9\rangle$	12	Prop. 2	yes ([17])
$2\langle 3, 5, 7\rangle + (2g-11)\mathbb{N}_0$	10	Prop. 5 (2)	yes (Ex. 7)
$\langle 4, 14, 2g-7, 2g-1\rangle$	8	Prop. 2	yes ([17])
$2\langle 4, 5, 6, 7\rangle + (2g-11)\mathbb{N}_0$	6	Prop. 5 (2)	yes (Ex. 7)
$2\langle 3, 4\rangle \cup \{2g-7, 2g-3, 2g-1\} \cup S(H)$	6	Ex. 6 (g odd)	yes
$2\langle 3, 4\rangle \cup \{h : h \geq 2g-5\}$	4	Prop. 5 (1)	??
$2\langle 3, 5\rangle \cup \{h : h \geq 2g-5\}$	2	Prop. 5 (1)	??
$2\langle 4, 5, 6, 7\rangle \cup \{2g-5, 2g-3, 2g-1\} \cup S(H)$	0	Example 5	yes ([21])

Example 10. Let $\gamma = 4$ and $g \geq F(4)$. From Lemma 2 and Proposition 4 $D(H) \in \{2i : i = 0, 1, \ldots, 16\} \setminus \{30\}$.

(1) For $k = 1, 2, 3, 4, 5$, $D(H) = 32-(4+k)(5-k) = 32, 24, 18, 14, 12$ by using a Weierstrass semigroup of multiplicity 4.

(2) Let $u_4 = 2g-15$ and $\tilde{H}$ a semigroup of genus 4 (which is always Weierstrass). Thus $H = 2\tilde{H} + u_4\mathbb{N}$ is Weierstrass (cf. Example 7) and $D(H) = 8 + 4w(\tilde{H})$ by Proposition 5 (2). Now $w(\tilde{H}) \leq 4(4-1)/3$ ([8]) and in fact there exists $\tilde{H}$ such that $w(\tilde{H}) = 0, 1, 2, 3, 4$. Thus $D(H) = 24, 20, 16, 12, 8$ occur for H a Weierstrass semigroup.

(3) If $u_4 = 2g-7$, via the $\tilde{H}$'s in (2), Proposition 5 (1) gives $D(H) = 8, 6, 4, 2, 0$.

(4) If $m_1 = 10$ and $u_4 = 2g-11$, Proposition 7 gives $D(H) = 12, 10, 8, 6, 4, 2, 0$.

In contrast to the case $\gamma = 3$, the following questions remain open: (a) Is there exists H such that $D(H) = 28, 26, 22$? (b) Is any semigroup in Item 3 above Weierstrass?

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