On (N, r)-Galois-Weierstrass semigroups

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Introduction

A numerical semigroup is a subset $H \subseteq \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$, contains 0, that is closed under addition $\mathbb{N}$ and has finite complement $\mathbb{N} \backslash H$.

$$H = \{0 < h_1 < h_2 < h_3 < ...\}$$

$$\text{Gap}(H) := \mathbb{N} \backslash H = \{\ell_1 < \ell_2 < ... < \ell_g\}$$

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- ℓ_g is the Frobenius number of H;
- g is the genus of H;
- $c(=\ell_g+1)$ is the conductor of H (is the smallest positive integer in H tal que $c+n \in H$, $\forall n \in \mathbb{N}_0$).

Weierstrass semigroup

Let $\mathcal{X}$ a curve and $P \in \mathcal{X}$ a point, the Weierstrass semigroup of P is the set:

$$H(P) := \{n \in \mathbb{N}_0; \exists f \in k(\mathcal{X}) \text{ with } (f)_\infty = nP\}.$$

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Buchweitz (1980):

$$H = \{0, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 26, \rightarrow\}$$

$$\text{Gap}(H) = \{1, 2, ..., 12, 19, 21, 24, 25\}$$

(N, r)-Galois-Weierstrass semigroups

Let N be a positive integer number and let $\ell_1, ..., \ell_{N-1}$ be non-negative integers. Consider the polynomial

$$f(x, y) = y^N - \prod_{i=1}^{N-1} \prod_{j=1}^{\ell_i} (x - a_{ij})^i,$$

with a_{ij}'s $\in k$, all different ("big enough"), of characteristic $p \geq 0$ relatively prime to N. Then we have the curve $f(x, y) = 0$.

(N, r)-Galois-Weierstrass semigroups

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1. $P_\infty|(0:1)$ is a unique extension of $(0:1)$ in $\mathcal{X}$;
2. The Weierstrass semigroups of P_∞ is:

$$H(P_\infty) = \langle N, L_1, L_2, ..., L_{N-1} \rangle,$$

where

$$L_i = \sum_{j=1}^{N-1} \left(ij - \left\lfloor \frac{ij}{N} \right\rfloor N \right) \ell_j.$$

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Corollary

Let H be a numerical semigroup and $N \in H$. Then H is a (N, r)-Galois-Weierstrass semigroup iff the following linear system has a non-negative integer solution:

$$(\pi(ij))_{(N-1)\times(N-1)}(\ell_j)_{(N-1)\times 1} = (w(\pi(ir)))_{(N-1)\times 1},$$

where $w(i) = \min\{h \in H; h \equiv i \mod N\}$.

A verification criterion

Teorema

Let $H = \langle N, w(1), w(2), ..., w(N-1) \rangle$ *be a numerical semigroup. If* H *is* (N, r)*-Galois-Weierstrass, then*

$$w(\pi(r)) + w(\pi((N-1)r)) = w(\pi(qr)) + w(\pi((N-q)r)),$$

for all $q \in \{1, ..., N-1\}$ *such that* $\gcd(q, N) = 1$.

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- For all $q \in \{1, ..., N-1\}$ relatively prime with N,

$$w(\pi(qr)) + w(\pi((N - q)r)) = 2N;$$

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1. $q \in H \Rightarrow H$ não é $(N, 1)$-Galois-Weierstrass;
2. $q \notin H$, nada podemos afirmar sobre H ser ou não $(N, 1)$-Galois-Weierstrass.

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- $H = \{0, 6, 8, 12, 13, 14, 16, 18, 19, 20, 21, 22, 24, \rightarrow\}$ is $(6,1)$-G-W but it is not $(8,1)$-G-W;

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- $H = \{0,6,8,12,14,16,18,20,22,23,24,26,28,29,30,31,32,34,\rightarrow\}$ is $(8,1)$-G-W but it is not $(6,1)$-G-W;

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- (Arf and sparse numerical semigroups) For $t \in \mathbb{Z}$, the sparse semigroup $H = \langle 2j+1; j \in \mathbb{N}, t \leq j \leq 2t-1 \rangle \cup H_{6t+1}$ is not $(N,1)$-G-W, since N be odd, greater than 7 and $N < h_{2t+4}$.

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- (Doble covering) H' numerical semigroups, $H = 2H' + n\mathbb{N}_0$. Classification of all numerical semigroups H of this type, of multiplicity $h_1 = 6, 8$, that are $(h_1, 1)$-G-W from the numerical properties of n and the Apèry set of the multiplicity of H';

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$$\mathcal{X}_b : y^b - \prod_{j=1}^{a} (x - c_j)^s, \text{ with } as \equiv 1 \mod b \text{ and } \operatorname{char}(k) \nmid b.$$

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thanks