The Turán number for C_4

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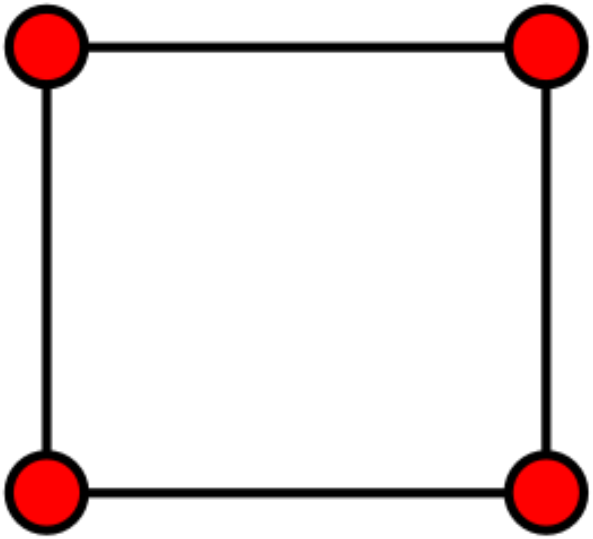
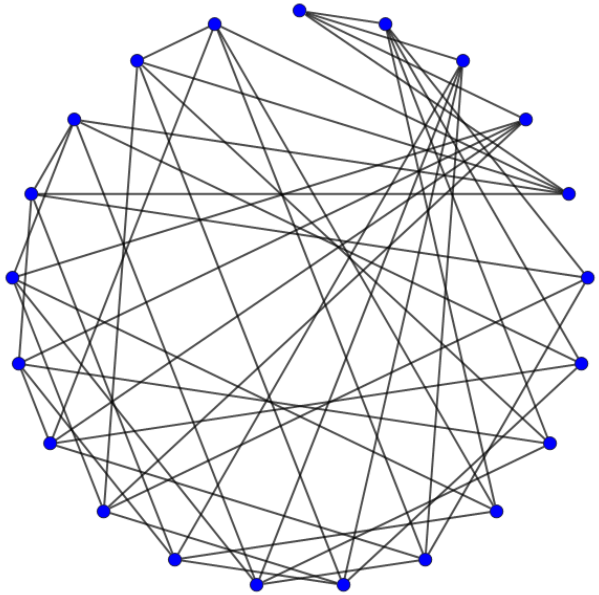
Preliminaries





Question

How many edges can an n-vertex graph have without containing a copy of C_4?

Definition

The maximum number of edges possible is denoted by $\mathrm{ex}(n; C_4)$ and is called the *Turán number for* C_4.

History

Theorem

Upper bound by Kővari, Sós, Turán in 1954. Lower bound by Erdős, Rényi, Sós in 1966:

$$\mathrm{ex}(n; C_4) \leq \frac{1}{2}n^{3/2} + \frac{1}{2}n.$$

$$\mathrm{ex}(q^2 + q + 1; C_4) \geq \frac{1}{2}q(q+1)^2$$

where q is a prime power.

Theorem

Füredi, 1980s and 1996 For $q \geq 15$ a prime power

$$\mathrm{ex}(q^2 + q + 1; C_4) = \frac{1}{2}q(q+1)^2.$$

Furthermore, all extremal graphs are orthogonal polarity graphs of some projective plane.

$n \neq q^2 + q + 1$

Technique for lower bounds:

- Take an extremal graph on $q^2 + q + 1 > n$ vertices and remove a subgraph of it.
- Denser subgraphs yield better lower bounds.
- Densest subgraph problem.
- This technique has been used to study $\mathrm{ex}(n; C_4)$ for

$$
\begin{aligned}
n = & \; q^q + q \\
& \; q^2 - \sqrt{q} \\
& \; q^2 - 1 \\
& \; q^2 - q - 2 \\
& \; \text{some others...}
\end{aligned}
$$

Future work

- To date, all of the best lower bounds come somehow from an orthogonal polarity of a classical projective plane (giving rise to a graph called ER_q. Perhaps studying other non-classical planes could yield graphs with denser subgraphs, which would give better lower bounds.
- Other algebraic constructions? Several constructions using Cayley sum graphs of dense Sidon sets turn out to be equivalent to studying ER_q.
- Computational data. It is not clear what the correct answer should be for most values of n.
- Upper bounds seem to be harder.