The Turán number for $C_4$

Michael Tait

University of California-San Diego

mtait@math.ucsd.edu

Joint work with Craig Timmons
Preliminaries

Question
How many edges can an $n$-vertex graph have without containing a copy of $C_4$?

Definition
The maximum number of edges possible is denoted by $\text{ex}(n; C_4)$ and is called the Turán number for $C_4$. 
History

Theorem

Upper bound by Kővari, Sós, Turán in 1954. Lower bound by Erdős, Rényi, Sós in 1966:

\[ \text{ex}(n; C_4) \leq \frac{1}{2} n^{3/2} + \frac{1}{2} n. \]

\[ \text{ex}(q^2 + q + 1; C_4) \geq \frac{1}{2} q(q + 1)^2 \]

where \( q \) is a prime power.

Theorem

Füredi, 1980s and 1996 For \( q \geq 15 \) a prime power

\[ \text{ex}(q^2 + q + 1; C_4) = \frac{1}{2} q(q + 1)^2. \]

Furthermore, all extremal graphs are orthogonal polarity graphs of some projective plane.
$n \neq q^2 + q + 1$

**Technique for lower bounds:**

- Take an extremal graph on $q^2 + q + 1 > n$ vertices and remove a subgraph of it.
- Denser subgraphs yield better lower bounds.
- Densest subgraph problem.
- This technique has been used to study $\text{ex}(n; C_4)$ for

$$n = q^q + q$$
$$q^2 - \sqrt{q}$$
$$q^2 - 1$$
$$q^2 - q - 2$$
some others...
Future work

- To date, all of the best lower bounds come somehow from an orthogonal polarity of a classical projective plane (giving rise to a graph called $ER_q$. Perhaps studying other non-classical planes could yield graphs with denser subgraphs, which would give better lower bounds.
- Other algebraic constructions? Several constructions using Cayley sum graphs of dense Sidon sets turn out to be equivalent to studying $ER_q$.
- Computational data. It is not clear what the correct answer should be for most values of $n$.
- Upper bounds seem to be harder.