LIMIT CYCLES FOR A MECHANICAL SYSTEM COMING FROM THE PERTURBATION OF A 4–DIMENSIONAL CENTER

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ABSTRACT. We study the bifurcation of limit cycles from the periodic orbits of a 4–dimensional center in a class of differential systems coming from the Mechanics. The tool for proving these results is the averaging theory.

1. INTRODUCTION

The goal of this paper is to study the existence of limit cycles of the mechanical system of the form

$$\ddot{x} + x = \varepsilon f(x,y), \qquad \ddot{y} + y = \varepsilon g(x,y),$$

for small values of the parameter ε. Of course, this system is equivalent to the 4–dimensional system

$$\begin{array}{rcl} \dot{x} & = & u, \\ \dot{u} & = & -x + \varepsilon f(x,y), \\ \dot{y} & = & v, \\ \dot{v} & = & -y + \varepsilon g(x,y), \end{array} \tag{1}$$

formed by differential equations of first order.

For $\varepsilon = 0$ system (1) becomes

$$\dot{x} = u, \quad \dot{u} = -x, \quad \dot{y} = v, \quad \dot{v} = -y. \tag{2}$$

We note that the origin of $\mathbb{R}^4$ is a *global isochronous center* for system (2), i.e. all orbits different from the origin are periodic with the same period, i.e. 2π.

A *limit cycle* of system (1) is an isolated periodic orbit in the set of all periodic orbits of (1). The *Poincaré map* (or, equivalently, the

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displacement map) is a good tool for studying the limit cycles of autonomous systems (for more details see [3, 5]), see also the end part of Section 2. We recall that a limit cycle of a system corresponds to an isolated zero of its displacement function.

Our main result is the following.

Theorem 1. *System* (1)*, with f and g arbitrary polynomials of degree* 3 *in the variables x and y, has at most* 2 *limit cycles bifurcating from the periodic orbits of system* (2)*, up to first order expansion of the displacement function of* (1) *with respect to the small parameter ε. Moreover, there are systems* (1) *having exactly* 2 *limit cycles.*

We use the averaging theory for proving Theorem 1, see Sections 2 and 3. For additional results in the use of the averaging theory for computing periodic orbits see [1, 8]. In general, it is not easy to find a change of variables to pass a given differential system to the normal form for applying the averaging method for finding periodic orbits. In particular, it is not easy to apply the averaging method for studying the limit cycles bifurcating from the periodic orbits of a center; for 2–dimensional system see [11, 8], for higher dimensional systems see [6, 2]. The general idea is to relate this change of variables to the first integrals of the center.

The next result shows that, if instead of polynomials of degree 3 the functions f and g are substituted by convenient analytic functions, then the number of limit cycles bifurcating from the center (2) becomes unbounded for $|\varepsilon| \neq 0$ sufficiently small. This result is not surprising because in $\mathbb{R}^2$ already occurs.

Proposition 2. *For an arbitrary positive integer n the analytic differential system*

$$\dot{x} = u, \quad \dot{u} = -x + \varepsilon \sin u, \quad \dot{y} = v, \quad \dot{v} = -y + \varepsilon \sin v, \tag{3}$$

has at least n limit cycles in the ball centered at the origin of radius $(n+1)\pi$, for $|\varepsilon| \neq 0$ sufficiently small.

Proposition 2 follows easily from [12], or [9]. But since its proof, using averaging, is so short we provide it in Section 4.

Systems (1) and (3) are perturbation of two independent harmonic oscillators. In fact, many models in nonlinear oscillations are expressed as perturbations of the unforced harmonic oscillator or of the simple pendulum. In many applications the main problem consists to know whether such systems possess limit cycles, as in the classical van der Pol equation $\ddot{x} + \varepsilon(1 - x^2)\dot{x} + x = 0$, which has a unique limit cycle for $\varepsilon > 0$ small. On the other hand, Krylov and Bogoliulov in [7] studied important classes of differential equations in dimension n which can be reduced to the standard form $\ddot{x} + kx = \varepsilon f(x, \dot{x})$, where $x \in \mathbb{R}^n$ and $k \in \mathbb{R}^n$ is a constant vector. The averaging method is a powerful tool to analyze such systems.

2. First order averaging method for periodic orbits

We consider the differential system

$$\dot{x}(t) = \varepsilon F(t, x(t)) + \varepsilon^2 R(t, x(t), \varepsilon), \tag{4}$$

with $x \in D \subset \mathbb{R}^n$, D a bounded domain and $t \geq 0$. Moreover, we assume that $F(t, x)$ and $R(t, x, \varepsilon)$ are T–periodic in t.

The averaged system associated to system (4) is defined by

$$\dot{y}(t) = \varepsilon f(y(t)), \tag{5}$$

where

$$f(y) = \frac{1}{T} \int_0^T F(s, y) ds. \tag{6}$$

The next theorem says us under which conditions the singular points of the averaged system (5) provide T–periodic orbits of system (4). For a proof see Theorem 2.6.1 of [10], Theorems 11.5 and 11.6 of [11], and Theorem 4.1.1 of [5].

Theorem 3. *We consider system* (4) *and assume that the vector functions F, R, D_xF_1, $D_x^2F_1$ and D_xR are continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times D$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Moreover, we suppose that F and R are T–periodic in t, with T independent of ε.*

(a) *If $a \in D$ is a singular point of the averaged system* (5) *such that $\det(D_x f(a)) \neq 0$ then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x_\varepsilon(t)$ of system* (4) *such that $x_\varepsilon(t) \to a$ as $\varepsilon \to 0$.*

(b) *If the singular point $y = a$ of the averaged system* (5) *is hyperbolic then, for $|\varepsilon| > 0$ sufficiently small, the corresponding periodic solution $x_\varepsilon(t)$ of system* (4) *is unique, hyperbolic and of the same stability type as a.*

In [1] is proved the conclusion of statement (a) of Theorem 3 without needing the differentiable assumptions on the functions F and R, but using the Brouwer degree of a as a zero of $f(y)$.

For each $z \in D$ we denote by $x(\cdot, z, \varepsilon)$ the solution of (4) with the initial condition $x(0, z, \varepsilon) = z$. We consider also the function $\zeta : D \times$

$(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ defined by

$$\zeta(z,\varepsilon) = \int_0^T \Big[\varepsilon F(t, x(t,z,\varepsilon)) + \varepsilon^2 R(t, x(t,z,\varepsilon), \varepsilon)\Big]\, dt. \tag{7}$$

From (4) it follows for every $z \in D$ that

$$\zeta(z,\varepsilon) = x(T,z,\varepsilon) - x(0,z,\varepsilon). \tag{8}$$

The function ζ can be written in the form

$$\zeta(z,\varepsilon) = \varepsilon f(z) + \varepsilon^2 O(1), \tag{9}$$

where f is given by (6) and the symbol $O(1)$ denotes a bounded function on every compact subset of $D \times (-\varepsilon_0, \varepsilon_0)$. Moreover, for $|\varepsilon|$ sufficiently small, $z = x_\varepsilon(0)$ is an isolated zero of $\zeta(\cdot, \varepsilon)$. Of course, due to (8) the function ζ is a *displacement* function for system (4), and its fixed points are the T–periodic solutions of (4).

3. Proof of Theorem 1

In this section f and g will be arbitrary polynomials of degree 3 in the variables x and y, more precisely

$$\begin{aligned} f &= a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3, \\ g &= b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2 + b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3. \end{aligned}$$

Following ideas of [2] we do the change of variables

$$x = r\cos\theta, \quad u = r\sin\theta, \quad y = s\cos(\theta+\alpha), \quad v = s\sin(\theta+\alpha). \tag{10}$$

We note that r, s, α are three independent first integrals of system (2), and that this change restricted to the plane (x,u) is a change to polar coordinates. Then, system (1) becomes

$$\begin{array}{rcl} \dot r &=& \varepsilon F_1(\theta, r, s, \alpha), \\ \dot\theta &=& -1 + \varepsilon G(\theta, r, s, \alpha), \\ \dot s &=& \varepsilon F_2(\theta, r, s, \alpha), \\ \dot v &=& \varepsilon F_3(\theta, r, s, \alpha), \end{array} \tag{11}$$

where

$$
\begin{aligned}
F_1 &= \sin\theta \Big[a_1 r\cos\theta + a_2 s\cos(\theta+\alpha) + a_3 r^2\cos^2\theta + \\
&\quad a_4 rs\cos\theta\cos(\theta+\alpha) + a_5 s^2\cos^2(\theta+\alpha) + a_6 r^3\cos^3\theta + \\
&\quad a_7 r^2 s\cos^2\theta\cos(\theta+\alpha) + a_8 rs^2\cos\theta\cos^2(\theta+\alpha) + a_9 s^3\cos^3(\theta+\alpha)\Big], \\
G &= \frac{1}{r}\cos\theta \Big[a_1 r\cos\theta + a_2 s\cos(\theta+\alpha) + a_3 r^2\cos^2\theta + \\
&\quad a_4 rs\cos\theta\cos(\theta+\alpha) + a_5 s^2\cos^2(\theta+\alpha) + a_6 r^3\cos^3\theta + \\
&\quad a_7 r^2 s\cos^2\theta\cos(\theta+\alpha) + a_8 rs^2\cos\theta\cos^2(\theta+\alpha) + a_9 s^3\cos^3(\theta+\alpha)\Big], \\
F_2 &= \sin(\theta+\alpha) \Big[b_1 r\cos\theta + b_2 s\cos(\theta+\alpha) + b_3 r^2\cos^2\theta + \\
&\quad b_4 rs\cos\theta\cos(\theta+\alpha) + b_5 s^2\cos^2(\theta+\alpha) + b_6 r^3\cos^3\theta + \\
&\quad b_7 r^2 s\cos^2\theta\cos(\theta+\alpha) + b_8 rs^2\cos\theta\cos^2(\theta+\alpha) + b_9 s^3\cos^3(\theta+\alpha)\Big], \\
F_3 &= -\frac{1}{rs} \Big[a_1 rs\cos^2\theta + a_3 r^2 s\cos^3\theta + a_6 r^3 s\cos^4\theta + \\
&\quad (a_2 s^2 - b_1 r^2)\cos\theta\cos(\theta+\alpha) + (a_4 s^2 - b_3 r^2) r\cos^2\theta\cos(\theta+\alpha) + \\
&\quad (a_7 r^2 s^2 - b_6 r^4)\cos^3\theta\cos(\theta+\alpha) - b_2 rs\cos^2(\theta+\alpha) + \\
&\quad (a_5 s^2 - b_4 r^2) s\cos\theta\cos^2(\theta+\alpha) + (a_8 s^2 - b_7 r^2) rs\cos^2\theta\cos^2(\theta+\alpha) - \\
&\quad b_5 rs^2\cos^3(\theta+\alpha) + (a_9 s^2 - b_8 r^2) s^2\cos\theta\cos^3(\theta+\alpha) - \\
&\quad b_9 rs^3\cos^4(\theta+\alpha)\Big].
\end{aligned}
$$

Therefore, taking θ as the independent variable, system (11) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= -\varepsilon F_1(\theta, r, s, \alpha) + \varepsilon^2 R_1(\theta, r, s, \alpha, \varepsilon), \\
\frac{ds}{d\theta} &= -\varepsilon F_2(\theta, r, s, \alpha) + \varepsilon^2 R_2(\theta, r, s, \alpha, \varepsilon), \\
\frac{dv}{d\theta} &= -\varepsilon F_3(\theta, r, s, \alpha) + \varepsilon^2 R_3(\theta, r, s, \alpha, \varepsilon).
\end{aligned}
\tag{12}
$$

This system already is into the normal form (4) for applying the averaging theory. We note that system (12) satisfies the assumptions of Theorem 3 in a big ball D centered at the origin.

If we denote by

$$
(f_1, f_2, f_3)(r, s, \alpha) = \frac{1}{2\pi}\int_0^{2\pi} (F_1, F_2, F_3)(\theta, r, s, \alpha)\, d\theta,
$$

then

$$
\begin{aligned}
f_1 &= \frac{1}{8}\, s\sin\alpha\left(4a_2+a_7r^2+3a_9s^2+2a_8rs\cos\alpha\right),\\
f_2 &= -\frac{1}{8}\, r\sin\alpha\left(4b_1+3b_6r^2+b_8s^2+2b_7rs\cos\alpha\right),\\
f_3 &= -\frac{1}{8rs}\Big[\left(-4a_1+4b_2-3a_6r^2+2b_7r^2-2a_8s^2+3b_9s^2\right)rs+\\
&\quad \left(4b_1r^2+3b_6r^4-4a_2s^2-3a_7r^2s^2+3b_8r^2s^2-3a_9s^4\right)\cos\alpha+\\
&\quad \left(b_7r^2-a_8s^2\right)rs\cos(2\alpha)\Big].
\end{aligned}
$$

By Theorem 3, the zeros (r_0,s_0,α_0) of

$$(f_1,f_2,f_3)(r,s,\alpha)=(0,0,0), \tag{13}$$

such that

$$\det\begin{pmatrix} \partial f_1/\partial r & \partial f_1/\partial s & \partial f_1/\partial\alpha\\ \partial f_2/\partial r & \partial f_2/\partial s & \partial f_2/\partial\alpha\\ \partial f_3/\partial r & \partial f_3/\partial s & \partial f_3/\partial\alpha\end{pmatrix}(r_0,s_0,\alpha_0)\neq 0, \tag{14}$$

provide periodic solutions of system (1), where f and g are the cubic polynomials defined at the beginning of this section. So, in particular a zero of (13) must be isolated in the set of all zeros of (13).

If there are solutions of (13) with $r=0$, then from the third equation of (13), we get that $a_2=a_9=0$. Then, it is easy to show that the zeros of (13) cannot be isolated. So, we are not interested in the zeros of (13) with $r=0$. Similar arguments show that the zeros of (13) with $s=0$ are also non–isolated. Again if $\sin\alpha=0$, then the corresponding zeros are again non–isolated. Hence, in the rest of this section we assume that $r>0$, $s>0$ and $\sin\alpha\neq 0$, and consequently we can restrict to look for the zeros of

$$(g_1,g_2,g_3)(r,s,\alpha)=(0,0,0), \tag{15}$$

satisfying (14), where

$$g_1=\frac{8f_1}{s\sin\alpha},\quad g_2=-\frac{8f_2}{r\sin\alpha},\quad g_3=-8rsf_3.$$

The rest of the proof of Theorem 1 is divided into cases and subcases.
Case 1: Suppose $a_8\neq 0$. Then, from $g_1(r,s,\alpha)=0$, we get

$$\cos\alpha=-\frac{4a_2+a_7r^2+3a_9s^2}{2a_8rs}.$$

Now, substituting $\cos\alpha$ in $g_2(r,s,\alpha)=0$ we obtain

$$4a_8b_1-4a_2b_7+3a_8b_6r^2-a_7b_7r^2-3a_9b_7s^2+a_8b_8s^2=0. \tag{16}$$

Subcase 1.1: Assume $a_7b_7 - 3a_8b_6 \neq 0$. Then, solving equation (16) with respect to r we have

$$r = \sqrt{\frac{4a_8b_1 - 4a_2b_7 - 3a_9b_7s^2 + a_8b_8s^2}{a_7b_7 - 3a_8b_6}}.$$

Now, substituting $\cos\alpha$ and r in $g_3(r, s, \alpha) = 0$ we get an equation of the form $a + bs^2 = 0$, which can have at most one positive solution for s. Therefore, in this subcase we get a unique value for r, s and $\cos\alpha$. But this provides at most two solutions for α. Hence, assuming that in these two solutions the determinant (14) is not zero, by Theorem 3, it follows that in this subcase we have at most two periodic solutions of system (1).

Subcase 1.2: Suppose $a_7b_7 - 3a_8b_6 = 0$. That is, $b_6 = a_7b_7/(3a_8)$. Then, we need to consider two additional subcases.

Subcase 1.2.1: Suppose $a_8b_8 - 3a_9b_7 \neq 0$. Therefore, from (16) we get that

$$s = 2\sqrt{\frac{a_2b_7 - a_8b_1}{a_8b_8 - 3a_9b_7}}.$$

We must consider that $a_2b_7 - a_8b_1 \neq 0$, otherwise $s = 0$ and we cannot get periodic solutions. We substitute $\cos\alpha$ and s in $g_3(r, s, \alpha) = 0$, and we get an equation of the form $r(a + br^2) = 0$. Since r must be positive, again in this subcase we get a unique value for r, s and $\cos\alpha$; and consequently at most two periodic solutions of system (1).

Subcase 1.2.2: Assume $a_8b_8 - 3a_9b_7 = 0$. Then, from (16) we must have that $a_2b_7 - a_8b_1 = 0$, otherwise we do not have solutions. That is, $b_1 = a_2b_7/a_8$. Substituting now $\cos\alpha$ in $g_3(r, s, \alpha) = 0$, we get a continuum of solutions for r and s. So, in this case we cannot apply Theorem 3.

Case 2: Assume $a_8 = 0$. Again we need to consider subcases.

Subcase 2.1: Suppose $a_7 \neq 0$. Then, from $g_1(r, s, \alpha) = 0$ we obtain that

$$r = \sqrt{-\frac{4a_2 + 3a_9s^2}{a_7}}.$$

Of course we suppose $4a_2 + 3a_9s^2 \neq 0$, otherwise $r = 0$. Now, we substitute r in $g_2(r, s, \alpha) = 0$.

Subcase 2.1.1: Suppose $b_7 \neq 0$. Then, from $g_2 = 0$ we get that

$$\cos\alpha = -\frac{4a_7b_1 - 12a_2b_6 + (a_7b_8 - 9a_9b_6)s^2}{2b_7s\sqrt{-a_7(4a_2 + 3a_9s^2)}}.$$

Now, substituting r and $\cos\alpha$ in $g_3(r,s,\alpha)=0$, we obtain an equation of the form
$$s\sqrt{-\frac{4a_2+3a_9s^2}{a_7}}(a+bs^2)=0.$$
Since the first two factors cannot be zero, as in the previous subcases we can get at most two periodic solutions of system (1).

Subcase 2.1.2: Suppose $b_7=0$.

Subcase 2.1.2.1: $a_7b_8-9a_9b_6\neq 0$. Then, from $g_2=0$ we get
$$s=2\sqrt{\frac{3a_2b_6-a_7b_1}{a_7b_8-9a_9b_6}}.$$
Substituting r and s in $g_3(r,s,\alpha)=0$, we obtain an equation of the form $a+b\cos\alpha=0$. So, again we get at most one solution for r, s and $\cos\alpha$, consequently at most two periodic solutions for system (1).

Subcase 2.1.2.2: $a_7b_8-9a_9b_6=0$. That is, $b_8=9a_9b_6/a_7$. Now, from $g_2=0$ it follows that $a_7b_1-3a_2b_6=0$, otherwise we do not have solutions. Therefore, $b_1=3a_2b_6/a_7$. Substituting r in $g_3(r,s,\alpha)=0$, we obtain a continuum of solutions. So, again we are not in the assumptions of Theorem 3.

Subcase 2.2: Suppose $a_7=0$. Looking at equation $g_1(r,s,\alpha)=0$, we see that a_9 cannot be zero, otherwise $g_1=0$ reduces to $a_2=0$, and either we do not have solutions or we have a continuum of solutions. Therefore, from $g_1=0$ we obtain
$$s=2\sqrt{-\frac{a_2}{3a_9}}.$$
We substitute s in $g_2(r,s,\alpha)=0$.

Subcase 2.2.1: Assume $b_7\neq 0$. Then, from $g_2=0$, it follows that
$$\cos\alpha=\frac{12a_9b_1-4a_2b_8+9a_9b_6r^2}{4b_7\sqrt{-3a_2a_9}r}.$$
Substituting s and $\cos\alpha$ in $g_3(r,s,\alpha)=0$, we obtain an equation of the form $r(a+br^2)=0$. Hence, as in previous subcases system (1) has at most two periodic solutions.

Subcase 2.2.1: Assume $b_7=0$. Then, equation $g_2=0$ is of the form $a+br^2=0$, so at most one positive solution for r. Moreover, substituting s and r in $g_3(r,s,\alpha)=0$ we get an equation of the form $c+d\cos\alpha=0$. Therefore, at most one solution for $\cos\alpha$. In short, at most one solution for s, r and $\cos\alpha$. So, at most two periodic solutions for system (1).

Summarizing we have proved that using Theorem 3 we can obtain at most two periodic solutions of system (1) when f and g are arbitrary polynomials of degree 3 in the variables x and y.

We claim that the displacement function of system (4) for the transversal section $u = 0$, written in the coordinates (10), has the form

$$\varepsilon(f_1, f_2, f_3)(r, s, \alpha) + \varepsilon^2 O(1).$$

Hence, this claim completes the proof of the first part of Theorem 1.

In order to prove the claim we focus on the function ζ given by (7) and (9), where the notations correspond to system (1) written in the form (4). This function ζ is the displacement function of system (1) associated to the transversal section $u = 0$, because it applies to every initial vector $(r(0), s(0), \alpha(0))$, the vector

$$(r(2\pi) - r(0), s(2\pi) - s(0), \alpha(2\pi) - \alpha(0)),$$

and the points of the transversal section are also characterized by $\theta = 0$ (mod 2π).

In order to complete the proof of Theorem 1 we must provide a system (1) with given polynomials f and g of degree 3, for which Theorem 3 provides exactly two limit cycles.

We consider the system of differential equations

$$\begin{array}{rcl} \dot{x} &=& u, \\ \dot{u} &=& -x + \varepsilon\Big(-\dfrac{1}{2}y + \dfrac{2}{3}x^3 - 3x^2y + 2xy^2 + \dfrac{1}{2}y^3\Big), \\ \dot{y} &=& v, \\ \dot{v} &=& -y + \varepsilon\Big(-\dfrac{3}{8}y - \dfrac{1}{3}x^3 + x^2y\Big). \end{array} \tag{17}$$

Then, it is easy to check that $(r, s, \alpha) = (2, 2, \pm\pi/3)$ are zeros of system (13) with determinant (14) equal to $\mp 3\sqrt{3}/8$, respectively. So, this system has two periodic solutions coming from periodic orbits of the center (2).

4. Proof of Proposition 2

Note that system (3) is formed by two independent systems, one formed by the two first variables, and other formed by the two last variables. Then, the proof of Proposition 2 follows immediately from the following result due to Zhifen Zhang [12], but the proof that we give here is strongly easier than the one of [12] and appeared in [9].

Proposition 4. *For an arbitrary positive integer n the planar analytic differential system*

$$\dot{x} = u, \quad \dot{u} = -x + \varepsilon \sin u, \tag{18}$$

has at least n limit cycles in the disc centered at the origin of radius $(n+1)\pi$, for $|\varepsilon| \neq 0$ sufficiently small.

Proof: We take a disc D centered at the origin of radius $(n+1)\pi$. Writing system (18) in the polar coordinates (r, θ), given by $x = r\sin\theta$ and $u = r\cos\theta$, we get

$$\dot{r} = \varepsilon \cos\theta \sin(r\cos\theta), \quad \dot{\theta} = 1 - \frac{\varepsilon}{r}\sin\theta\sin(r\cos\theta). \tag{19}$$

Note that the change to polar coordinates in this section is different to the previous one.

Instead of working with system (19) we consider the equivalent system

$$\frac{dr}{d\theta} = \varepsilon\cos\theta\sin(r\cos\theta) + \varepsilon^2 R(\theta, r, \varepsilon). \tag{20}$$

Clearly this system satisfies the assumptions of Theorem 3 in the disc D.

We note that

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\sin(r\cos\theta)d\theta = J_1(r),$$

where $J_1(r)$ is the Bessel function of first kind, see [4]. Hence, the averaged equation of (20) is

$$\frac{dr}{d\theta} = \varepsilon J_1(r).$$

It is known that the function $J_1(r)$ has exactly n simple zeros in the interval $[0, (n+1)\pi]$. Therefore, by Theorem 3, it follows that system (18) has at least n limit cycles in D for $|\varepsilon| \neq 0$ sufficiently small. ∎

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