1.(a) (i) $0 \in \mathcal{U}$

(ii) $\forall\, u, v \in \mathcal{U}$ tem-se $u + v \in \mathcal{U}$

(iii) $\forall\, \alpha \in \mathbb{F}, \forall u \in \mathcal{U}$ -u- $\alpha u \in \mathcal{U}$

(b) $\mathcal{U} + \mathcal{W} = \{ u + w \mid u \in \mathcal{U}, w \in \mathcal{W} \}$

$\mathcal{U} + \mathcal{W}$ é denotado por $\mathcal{U} \oplus \mathcal{W}$ se $\mathcal{U} \cap \mathcal{W} = \{0\}$.

(c) $\mathcal{U} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid y = x \right\} = \left\{ \begin{pmatrix} x \\ x \\ z \end{pmatrix} \mid x, z \in \mathbb{R} \right\}$

$= \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right],$

quer dizer o subesp. gerado por $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Portanto $\mathcal{U}$ é subespaço de $\mathbb{R}^3$

$\mathcal{W} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid y = -x, z = 0 \right\} = \left\{ x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

$= \left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right]$ é subespaço de $\mathbb{R}^3$

$\mathcal{U} + \mathcal{W} = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right]$

Note que $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ é um conjunto ortogonal

$\left(\begin{array}{l} 0 = \langle (1,1,0)^T, (0,0,1)^T \rangle = \\ = \langle (1,1,0)^T, (1,-1,0)^T \rangle = \langle (0,0,1)^T, (1,-1,0)^T \rangle \end{array} \right)$

$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ é L.I.

$\Rightarrow \dim(\mathcal{U} + \mathcal{W}) = 3$ e $\dim(\mathcal{U}) = \cancel{1}\,2$

Portanto

$$3 = \dim(\mathcal{U}+\mathcal{W}) = \dim(\mathcal{U}) + \dim(\mathcal{W}) - \dim(\mathcal{U}\cap\mathcal{W})$$
$$= 2 + 1 - \dim(\mathcal{U}\cap\mathcal{W})$$

implica que $\dim(\mathcal{U}\cap\mathcal{W}) = 0$

$$\Rightarrow \mathcal{U}\cap\mathcal{W} = \{0\}$$

$\Rightarrow$ a soma $\mathcal{U}+\mathcal{W}$ é direta

2.(a) Seja $S \subseteq \cancel{\mathcal{W}}\ \mathcal{V}$

Temos $S = \{V_i \mid i \in I\}$ para algum conjunto de índices I

(i) S é L.I. se e somente se:

$\sum_{j\in J} \alpha_j V_j = 0$ para $J \subseteq I$ com $|J| < \infty$ implica que $\alpha_j = 0 \ \forall j \in J$

(ii) $[S] = \mathcal{V}$ se e somente se

$$\mathcal{V} = \left\{ \sum_{j\in J} \alpha_j V_j \mid \alpha_j \in \mathbb{F}\ \forall j \in J \subseteq I \text{ com } |J| < \infty \right\}$$

(iii) S é base de $\cancel{\mathcal{W}}\ \mathcal{V} \Longleftrightarrow$

S é L.I. e $[S] = \cancel{\mathcal{W}}\ \mathcal{V}$

2(b) $\dim(\mathcal{P}_6) = 7$ porque
$\mathcal{E} = \{1, x, x^2, \ldots, x^6\}$ é uma base de $\mathcal{P}_6$

(c) Seja $S = \{1+x, 1+x^2, \ldots, 1+x^6, x^6\}$

$$= \left\{ \begin{pmatrix} 1\\1\\0\\0\\0\\0\\0 \end{pmatrix}_{\mathcal{E}}, \begin{pmatrix} 1\\0\\1\\0\\0\\0\\0 \end{pmatrix}_{\mathcal{E}}, \ldots, \begin{pmatrix} 1\\0\\0\\0\\0\\0\\1 \end{pmatrix}_{\mathcal{E}}, \begin{pmatrix} 0\\0\\0\\0\\0\\0\\1 \end{pmatrix}_{\mathcal{E}} \right\}$$

Considere

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \\ \text{e } \sum_{i=0}^{5} \alpha_i = 0 \\ \text{e } \alpha_5 + \alpha_6 = 0 \end{cases}$$

$\Rightarrow \alpha_0 = \alpha_1 = \ldots = \alpha_5 = \alpha_6 = 0$

$\Rightarrow$ S é L.I. $\Rightarrow \dim([S]) = 7$

$[S]$ é ~~sd~~ subesp. de $\mathcal{P}_6$ com $\dim([S]) = 7 \Rightarrow [S] = \mathcal{P}_6$

2.d) $\{1+x, \dots, 1+x^5\}$ é subconjunto do conjunto L.I. S de c)

$\Rightarrow \{1+x, \dots, 1+x^5\}$ é LI também

e $\dim(W) = 5$

3.(a) $\langle\ ,\ \rangle : \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ é um produto interno se

(i) $\langle\ , V\rangle$ é linear $\forall V\in\mathcal{V}$

quer dizer $\forall u, V\in\mathcal{V}$ e $\forall\alpha\in\mathbb{C}$ tem

$$\langle u+w, V\rangle = \langle u, V\rangle + \langle w, V\rangle$$

$$\langle \alpha u, V\rangle = \alpha\langle u, V\rangle$$

(ii) $\langle\ ,\ \rangle$ é hermitiano

$$\langle u, V\rangle = \overline{\langle V, u\rangle} \quad \forall u, V\in\mathcal{V}$$

(iii) $\langle\ ,\ \rangle$ é positivo definido:

$$\langle V, V\rangle > 0 \quad \forall V\neq 0\in\mathcal{V}$$

(b)

Seja $\{V_1, V_2, V_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Passo 1:

$$u_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

x4

$W_2 =$

Passo 2:

$$W_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}}\, 3\, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$= \langle V_2, u_1 \rangle u_1$

$= \frac{\langle Q_2, w_1 \rangle}{\langle w_1, w_1 \rangle} \langle V_2, \frac{w_1}{\|w_1\|} \rangle \frac{w_1}{\|w_1\|}$

$\mathrm{proj}_{u_1}(V_2) + w_2 = V_2$

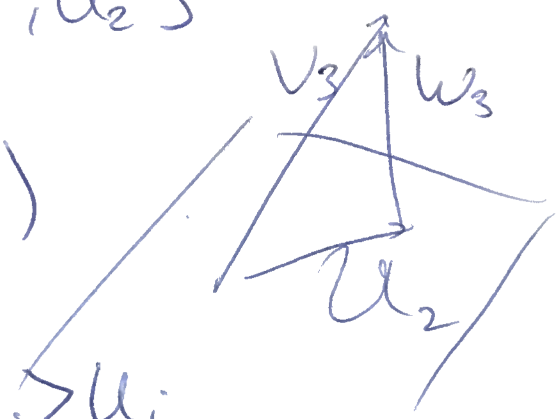
$$= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$w_2 = V_2 - \mathrm{proj}_{u_1}(V_2)$

$$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow u_2 = \frac{W_2}{\|W_2\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$\frac{3}{4}$

Passo 3: Seja $\mathcal{U}_2 = [u_1, u_2]$



$$w_3 = v_3 - \text{proj}_{\mathcal{U}_2}(v_3)$$

$$= v_3 - \sum_{i=1}^{2} \langle v_3, u_i \rangle u_i$$

$$= \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \left[\frac{1}{2}\, 2 \cdot \begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{2} \cdot 0 \cdot \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\right]$$

$$= \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$\Rightarrow \cancel{w_3 =} \; u_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{2}} \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$\cancel{\frac{1}{2}}$

$\therefore \{u_1, u_2, u_3\}$ é base ortonormal de $[v_1, v_2, v_3$

4.(a) VERDADEIRO:

Seja $\mathcal{U} = \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right] = \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right] \oplus \left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right]$

e $\mathcal{W} = \left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right] = \left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right] \oplus \left[\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right]$

$\Rightarrow \dim(\mathcal{U}) = \dim(\mathcal{W}) = 2$

$\mathcal{U} \cap \mathcal{W} = \left[\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right]$ e $\dim(\mathcal{U} \cap \mathcal{W}) = 1$

(b) $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$

$\Leftrightarrow \alpha_1 + \alpha_2 x^2 + \alpha_3 e^x = 0 \quad \forall x \in \mathbb{R}$

Em particular $\begin{cases} \alpha_1 + \alpha_2 \cdot 0^2 + \alpha_3 e^0 = 0 \\ \alpha_1 + \alpha_2 \cdot 1^2 + \alpha_3 e^1 = 0 \\ \alpha_1 + \alpha_2 (-1)^2 + \alpha_3 e^{-1} = 0 \end{cases}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & e \\ 1 & 1 & e^{-1} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & e \\ 0 & 0 & e^{-1}-e \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & e-1 \\ 0 & 0 & e-e^{-1} \end{pmatrix}$$

$\therefore$ solução única $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow \{f_1, f_2, f_3\}$ é L.I.

VERDADEIRO

4.(c) Seja $\chi_{\{i\}}(x) = \begin{cases} 1 & \text{se } x = i \\ 0 & \text{c.c.} \end{cases}$

$\forall i \in \mathbb{N}$

Seja $J \subseteq \mathbb{N}$ com $|J| < \infty$

Considere $\sum_{j \in J} \alpha_j \chi_{\{i\}} = 0$

$\Rightarrow \sum_{j \in J} \alpha_j \chi_{\{i\}}(i) = 0 \quad \forall i \in \mathbb{N}$

$\Rightarrow \alpha_j = 0 \quad \forall j \in J$

$\therefore \mathcal{X} := \{\chi_{\{i\}} \mid i \in \mathbb{N}\}$ é L.I. e subconj. de $\mathcal{F}(\mathbb{R})$

$[\mathcal{X}]$ é subesp. de $\mathcal{F}(\mathbb{R})$ com

$\dim([\mathcal{X}]) = \infty \Rightarrow \dim(\mathcal{F}(\mathbb{R})) = \infty$

FALSO

(d) $\mathcal{V} = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$

$\Rightarrow \dim(\mathcal{V}) = 2$

Temos $4 = \dim(\mathbb{R}^4) = \dim(\mathcal{V} \oplus \mathcal{V}^\perp)$

$= \dim(\mathcal{V}) + \dim(\mathcal{V}^\perp)$

$= 2 + \dim(\mathcal{V}^\perp) \Rightarrow \dim(\mathcal{V}^\perp) = 2 \neq 1$

FALSO!