1.(a) $\varphi : V \to W$

é transf. lin se

$\varphi(v+w) = \varphi(v) + \varphi(w) \quad \forall v, w \in V$ $\frac{1}{4}$

$\cancel{\alpha}\ \varphi(\alpha v) = \alpha \varphi(v) \quad \forall \alpha \in \mathbb{F}, \forall v \in V$ $\frac{1}{4}$

(b) (i) $\varphi\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}-1\\0\\0\end{pmatrix} \neq 0 \in \mathbb{R}^3$ $\frac{1}{4}$

$\Rightarrow$ φ não é transf. li.

(ii) $\varphi\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x-y\\x+y\end{pmatrix}$

$$\varphi\left(\alpha\begin{pmatrix}x\\y\end{pmatrix}\right) = \varphi\begin{pmatrix}\alpha x\\\alpha y\end{pmatrix} = \begin{pmatrix}\alpha x - \alpha y\\\alpha x + \alpha y\end{pmatrix}$$

$$= \begin{pmatrix}\alpha(x-y)\\\alpha(x+y)\end{pmatrix} = \alpha\begin{pmatrix}x-y\\x+y\end{pmatrix} = \alpha\varphi\begin{pmatrix}x\\y\end{pmatrix}$$

$\forall \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2\ \forall \alpha \in \mathbb{R}$ $\frac{1}{4}$

$$\varphi\left(\begin{pmatrix}u\\v\end{pmatrix} + \begin{pmatrix}x\\y\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}u+x\\v+y\end{pmatrix}\right) = \begin{pmatrix}(u+x)-(v+y)\\(u+x)+(v+y)\end{pmatrix}$$

$$= \begin{pmatrix}u-v+x-y\\u+v+x+y\end{pmatrix} = \begin{pmatrix}u-v\\u+v\end{pmatrix} + \begin{pmatrix}x-y\\x+y\end{pmatrix} = \varphi\begin{pmatrix}u\\v\end{pmatrix} + \varphi\begin{pmatrix}x\\y\end{pmatrix}$$ $\frac{1}{4}$

$\forall \begin{pmatrix}u\\v\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$

(iii) $\varphi\left(2\begin{pmatrix}1\\0\end{pmatrix}\right) = \varphi\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}4\\0\\0\end{pmatrix} \neq 2\begin{pmatrix}1\\0\\0\end{pmatrix} = 2\varphi\begin{pmatrix}1\\0\end{pmatrix}$

$\therefore$ φ não é transf. li. $\frac{1}{4}$

5. (2.0 pt.) Responda falso ou verdadero (justificando):

(a) Se $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ é a matriz associada à uma transformação linear $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ referente a bases $\mathcal{B}$ e $\mathcal{C}$ de $\mathbb{R}^2$, então $\varphi = id$ ($id(V) = V$ $\forall V \in \mathbb{R}^2$).

(b) Existe uma transformação linear $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ tal que $\mathcal{N}(\varphi) = \{(1,1)^T\}$.

(c) A transformação linear derivada $\delta : \mathcal{P}_3 \to \mathcal{P}_3$, dada por $\delta(p) = p'$, é injetiva.

(d) O posto da matriz $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ é 2.

Incluir na prova, por favor, **todas** as "contas" feitas nas resoluções. Respostas não acompanhadas de argumentos que as justifiquem não serão consideradas.

Boa Prova!

1.(c) $\varphi\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right)$

$= \varphi\left(-2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ $\frac{1}{4}$

$= (-2)\,\varphi\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) + 3\,\varphi\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= (-2)\begin{pmatrix} 0 \\ -1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}.$

$\frac{1}{4}$

2. (a) $N(\varphi) = \{ v \in V \mid \varphi(v) = 0 \in W \}$ $\frac{1}{4}$

$\mathrm{Im}(\varphi) = \{ \varphi(v) \mid v \in V \}$

φ é um isomorfismo $\Leftrightarrow N(\varphi) = \{0\}$ e $\mathrm{Im}(\varphi) = W$ $\frac{1}{4}$

(b)

$$A^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} \text{coord's de } \varphi(v_1) \text{ em termos de } \mathcal{C} & \cdots & \text{coord's de } \varphi(v_n) \text{ em termos de } \mathcal{C} \end{pmatrix} = \begin{pmatrix} \varphi(v_1)_{\mathcal{C}} & \varphi(\ldots \end{pmatrix}$$ $\frac{1}{2}$

3.

$$\varphi : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$$
$$M \mapsto M - M^T$$

(a) Seja $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right.$

base de $\mathbb{R}^{2\times 2}$

$$A^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} \mathbb{R}^{2\times 2} & \longrightarrow & \mathbb{R}^{2\times 2} \\ \Phi_{\mathcal{B}} \uparrow \cong & & \Phi_{\mathcal{B}} \uparrow \cong \\ \mathbb{R}^4 & \xrightarrow{A^{\mathcal{B}}_{\mathcal{B}}} & \mathbb{R}^4 \end{array}$$

$$\mathrm{Im}(A^{\mathcal{B}}_{\mathcal{B}}) = \left[\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right] \Rightarrow \mathrm{Im}(\varphi) = \left[\Phi_{\mathcal{B}} \left(\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right) \right.$$

$$= \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right].$$ $\frac{1}{4}$

base para $\mathrm{Im}(\varphi) = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ $\frac{1}{4}$

$\Rightarrow \dim(\mathrm{Im}(\varphi)) = 1$

Temos $4 = \dim(\mathbb{R}^{2\times 2}) = \dim(N(\varphi)) + \dim(\mathrm{Im}(\varphi))$

$= \dim(N(\varphi)) + 1$ $\frac{1}{4}$

$\Rightarrow \dim(N(\varphi)) = 3$ $\frac{1}{4}$

(b) $\dim(\mathrm{Im}(\varphi)) = 1 < 4 = \dim(\mathbb{R}^{2\times 2})$

$\Rightarrow$ φ não é sobrejetiva $\frac{1}{4}$

$\dim(N(\varphi)) = 3 \neq 0 \Rightarrow$ φ não é injetiva $\frac{1}{4}$

(c) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 0$

$\mathcal{S} = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \beta \\ \delta \end{pmatrix} \,\middle|\, \alpha, \beta, \cancel{\gamma}, \delta \in \mathbb{R} \right\} \Rightarrow N(A_{\mathcal{B}}^{\mathcal{B}}) =$

$= \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \cancel{A}$ $\frac{1}{4}$

$$\Rightarrow N(\varphi) = \left[\varphi_B\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \varphi_B\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \varphi_B\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right]$$

$$= \left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \right]$$

$\left\{ \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \right\}$ é base de $N(\varphi)$. $\frac{1}{4}$

4. (a) $A_e^e = \begin{pmatrix}0 & 1 & 1\\1 & 0 & 0\\-1 & 0 & 2\end{pmatrix}$ $\frac{1}{2}$

(b) $A_B^B = ?$ $\quad B = \left\{ \underset{v_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\1\\0\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \right\}$

$$\varphi(v_1) = \begin{pmatrix}0\\1\\-1\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}2\\1\\1\end{pmatrix}_B = \left(1 \cdot v_2 + 1 \cdot v_3\right)_B = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\varphi(v_2) = \begin{pmatrix}0\\1\\-1\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\end{pmatrix} = \alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\1\end{pmatrix} + \gamma\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\alpha+\beta+\gamma\\\alpha+\beta\\\alpha+\gamma\end{pmatrix}$$

$$\alpha + \beta = 1 \quad \Rightarrow \quad \beta = \gamma$$
$$-\alpha - \gamma = 1$$

$$\cancel{2+1}\ \alpha + 2\beta = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 \end{array}\right)$$

$$\Rightarrow \gamma = 0,\ \beta = 2$$

$$\alpha = -1$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}_{\mathcal{B}} = \left(-\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)_{\mathcal{B}} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \quad \frac{1}{4}$$

$$\varphi(V_3) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \frac{1}{4}$$

$$\therefore A^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}_{\mathcal{B}}$$

$$A^{\mathcal{B}}_{\mathcal{B}} = I^{\mathcal{B}}_{e} \cdot A^{e}_{e} \circ I^{e}_{\mathcal{B}}$$

(c)

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "I^{\mathcal{B}}_{e}"] \arrow[rrr, bend left, "A^{\mathcal{B}}_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[r, "A^{e}_{e}"] & \mathbb{R}^3 \arrow[r, "I^{e}_{\mathcal{B}}"] & \mathbb{R}^3 \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "A^{e}_{e}"] & \mathbb{R}^3 \\ \mathbb{R}^3 \arrow[u, "I^{\mathcal{B}}_{e}"] \arrow[r, "A^{\mathcal{B}}_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[u, "I^{\mathcal{B}}_{e}"'] \end{tikzcd}$$

$$I^{e}_{B} = \left(\begin{pmatrix}1\\0\\0\end{pmatrix}_B \begin{pmatrix}0\\1\\0\end{pmatrix}_B \begin{pmatrix}0\\0\\1\end{pmatrix}_B \right)$$

$$\mp \begin{pmatrix}1\\0\\0\end{pmatrix} = -\left[\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} \right]$$

$$= (-1)\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= (-1)V_1 + (1)V_2 + (1)V_3$$

$$I^{e}_{B} = \Big(\quad \begin{pmatrix}0\\1\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&1&1&0\\1&1&0&1\\1&0&1&0\end{array}\right) \sim \cancel{(0\ 0\ 1)} \sim \left(\begin{array}{ccc|c}1&1&1&0\\0&0&-1&1\\0&-1&0&0\end{array}\right)$$

$$\Rightarrow \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}_B \qquad \gamma = -1$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}_B \qquad I^{e}_{B} = \begin{pmatrix}-1&1&1\\1&0&-1\\1&-1&0\end{pmatrix} \quad +4$$

$$I^{B}_{e} = \begin{pmatrix}1&1&1\\1&1&0\\1&0&1\end{pmatrix} \quad +4$$

$P = I^{B}_{e}$ $\quad$ $P^{-1} = I^{e}_{B}$ $\quad$ +4

$A = A^{e}_{e}$ $\quad$ P^{-1} $\quad$ P

$A^{B}_{B} = I^{e}_{B} A^{e}_{e} I^{B}_{e}$

4. d) $A_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

A matriz de $\varphi \circ \varphi$ referida $\mathcal{B}$ é

$$\frac{1}{4} \quad \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix} \quad \frac{1}{4}$$

(e) $\exists \varphi^{-1}$?

sim os vetores coluna de A_e^e são L.I

p A_e^e $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

posto $(A_e^e) = 3$ $\quad \frac{1}{2}$

$\Rightarrow \varphi$ é injetiva

ou $\det(A_e^e) = -2 \neq 0$

$\Rightarrow \varphi$ invertível

5. (2.0 pt) Responda falso ou verdadero (justificando):

(a) Se $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ é a matriz associada à uma transformação linear $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ referente a bases $\mathcal{B}$ e $\mathcal{C}$ de $\mathbb{R}^2$, então $\varphi = id$ $(id(V) = V\ \forall V \in \mathbb{R}^2)$.

(b) Existe uma transformação linear $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ tal que $\mathcal{N}(\varphi) = \{(1,1)^T\}$.

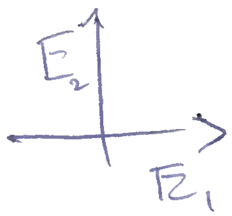
(c) A transformação linear derivada $\delta : \mathcal{P}_3 \to \mathcal{P}_3$, dada por $\delta(p) = p'$, é injetiva.

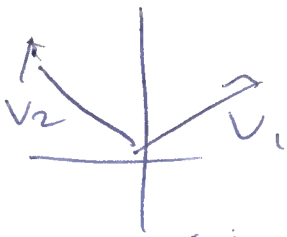
(d) O posto da matriz $\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 1\\ 1 & 1 & 2\end{pmatrix}$ é 2.

Incluir na prova, por favor, **todas** as "contas" feitas nas resoluções. Respostas não acompanhadas de argumentos que as justifiquem não serão consideradas.

Boa Prova!

5. (a) FALSO





Seja $\mathcal{B} = \{E_1, E_2\}$ $\quad \mathcal{C} = \{V_1 = \binom{1}{1}, V_2 = \binom{-1}{1}\}$

$\varphi(E_i) = V_i \quad i = 1,2 \qquad A^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

mas $\varphi \neq id$

(b) FALSO

Se $\binom{1}{1} \in N(\varphi) \Rightarrow \binom{2}{2} \in N(\varphi) \cancel{\Rightarrow N(\varphi)}$

$N(\varphi)$ sempre é subesp. de $\mathbb{R}^2$

(c) FALSO: $\delta(1) = 0$

$\Rightarrow 1 \in N(\delta) \Rightarrow N(\delta) \neq \{0\}$

$\Rightarrow \delta$ não é injetiva

$$\text{posto}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \dim\left[\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}3\\1\\2\end{pmatrix}\right]$$

comb. lin. de $\begin{pmatrix}1\\0\\1\end{pmatrix}$ e $\begin{pmatrix}2\\1\\1\end{pmatrix}$

$$= \dim\left[\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix}\right] = 2$$

pq. $\begin{pmatrix}1\\0\\1\end{pmatrix}$ não é mult de $\begin{pmatrix}2\\1\\1\end{pmatrix}$

e $\begin{pmatrix}2\\1\\1\end{pmatrix}$ " " $\begin{pmatrix}1\\0\\1\end{pmatrix}$

VERDADEIRO.