Exemplo 1:

Considere $\mathcal{C} \subseteq \mathbb{R}^2$ dado por $3x^2+2xy+3y^2-8=0$

PASSO 1: Eliminação do Termo xy (Diagonalizar a matriz $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ usando uma rotação)

(a) Na Forma Matricial:

$$(x\ y)\underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} - 8 = 0$$

$\det(A) > 0 \Rightarrow \mathcal{C}$ é elipse

(b) Autovalores de A

$$0 = \det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (\lambda-3)^2 - 1 = \lambda^2 - 6\lambda + 9 - 1 = \lambda^2 - 6\lambda + 8$$

$$= (\lambda-2)(\lambda-4) \quad \therefore \lambda_1 = 2,\ \lambda_2 = 4$$

(c) Autovetores de A

(Determine base ortonormal de Autovetores de A!)

(i) $\lambda_1 = 2$:

Resolve $(A-2I)V = 0$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

(variável livre)

$x = -y$

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x = -y \right\} = \left\{ \begin{pmatrix} -y \\ y \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\} = \left\{ \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$

$V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ é autovet. com autovalor 2

$\bar{V}_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ com $\|\bar{V}_1\| = 1$

(ii) $\lambda_2 = 4$:

Resolve $(A-4I)V = 0$

(variável livre)

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$S = \left\{ x\begin{pmatrix} 1 \\ 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$$

$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\bar{V}_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ é autovetor normalizado de A para $\lambda_2 = 4$

$\bar{\mathcal{B}} = \{\bar{V}_1, \bar{V}_2\} = \left\{ \frac{\sqrt{2}}{2}\begin{pmatrix}1\\-1\end{pmatrix}, \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix} \right\}$

é base ortonormal de autovetores de A

(d) Encontre matriz de rotação que representa a matriz de mudança de base

Seja $\mathcal{B} = \{E_1, E_2\}$ e $R = \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}$

Temos ~~de~~ que R é ortogonal.

Além disso $\det(R) = \frac{2}{4}\det\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix} = \frac{1}{2}\cdot 2 = 1$

Lembre-se: Se o determinante desta matriz é -1, então troque $\bar{V}_1 \Leftrightarrow \bar{V}_2$, $\lambda_1 \Leftrightarrow \lambda_2$!

Observe que $R = I_{\mathcal{B}}^{\bar{\mathcal{B}}}$!

(e) Eq. no sistema rodado $\bar{x}-\bar{y}$

$$(\bar{x}\ \bar{y})_{\bar{\mathcal{B}}}\begin{pmatrix}2 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}_{\bar{\mathcal{B}}} - 8 = 0$$

$$\Leftrightarrow 2\bar{x}^2 + 4\bar{y}^2 - 8 = 0$$

$$\Leftrightarrow \frac{\bar{x}^2}{4} + \frac{\bar{y}^2}{2} = 1$$

Vértices no eixo focal em $\bar{x}-\bar{y}$: $\begin{pmatrix}\pm 2\\0\end{pmatrix}_{\bar{\mathcal{B}}}$

Vértices no eixo menor em $\bar{x}-\bar{y}$: $\begin{pmatrix}0\\\pm\sqrt{2}\end{pmatrix}_{\bar{\mathcal{B}}}$

Vertices no eixo focal em x-y:

$$I_{\mathcal{B}}^{\bar{\mathcal{B}}}\begin{pmatrix}\pm 2\\ 0\end{pmatrix}_{\bar{\mathcal{B}}} = R\begin{pmatrix}\pm 2\\ 0\end{pmatrix}_{\bar{\mathcal{B}}}$$

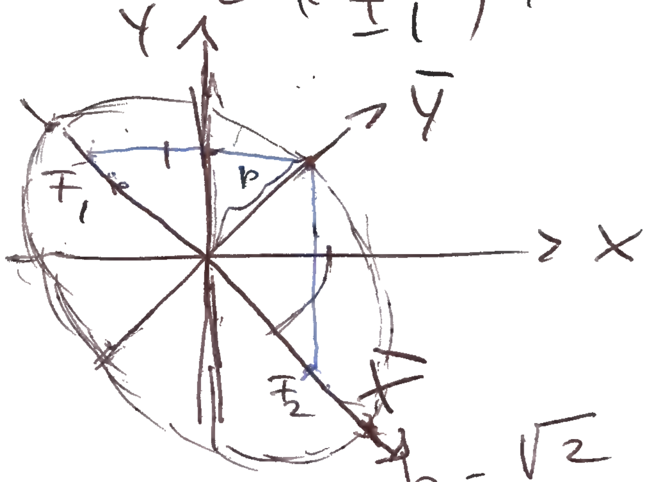
$$= \frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\pm 2\\ 0\end{pmatrix} = \begin{pmatrix}\pm\sqrt{2}\\ \mp\sqrt{2}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\pm 1\\ \mp 1\end{pmatrix}$$

i.e. $\sqrt{2}\begin{pmatrix}1\\ -1\end{pmatrix}$ e $\sqrt{2}\begin{pmatrix}-1\\ 1\end{pmatrix}$

Vertices no eixo menor em x-y:

$$I_{\mathcal{B}}^{\bar{\mathcal{B}}}\begin{pmatrix}0\\ \pm\sqrt{2}\end{pmatrix} = \frac{\cancel{\sqrt{2}}\,2}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0\\ \pm\cancel{\sqrt{2}}\end{pmatrix}$$

$$= \cancel{\tfrac{1}{2}}\begin{pmatrix}\pm 1\\ \pm 1\end{pmatrix}, \text{ i.e. } \cancel{\tfrac{1}{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \text{ e } \cancel{\tfrac{1}{2}}\begin{pmatrix}-1\\ -1\end{pmatrix}$$



excentricidade

$$e = \frac{c}{a} = \frac{\sqrt{2}}{2}$$

Note que $a = 2$ e $b = \sqrt{2}$

$a^2 = b^2 + c^2 \Rightarrow c^2 = a^2 - b^2 = 4 - 2 = 2$

$\Rightarrow c = \sqrt{2}$

Focos F_1, F_2 em $\begin{pmatrix}-\sqrt{2}\\ 0\end{pmatrix}_{\bar{\mathcal{B}}}$ e $\begin{pmatrix}\sqrt{2}\\ 0\end{pmatrix}_{\bar{\mathcal{B}}}$ $\quad \begin{pmatrix}-1\\ 1\end{pmatrix}$ e $\begin{pmatrix}1\\ -1\end{pmatrix}$ $\parallel$

em x-y: ~~$F_{1,2} = \pm\sqrt{2}$~~ $F_{1,2} = I_{\mathcal{B}}^{\bar{\mathcal{B}}}\begin{pmatrix}\mp\sqrt{2}\\ 0\end{pmatrix}_{\bar{\mathcal{B}}} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\mp 1\\ 0\end{pmatrix}$

Passo 2. Completar Quadrados para Determinar a Translação i.e. para Colocar ℓ na forma canônica

Desnecessário aqui

Exemplo 2:

Identifique $\ell = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, 3x^2 + 2xy + 3y^2 - \sqrt{2}x = 0 \right\}$

Resultado do Passo 1:

$$(\bar{x} \ \bar{y})_{\bar{\mathcal{B}}} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} + \underbrace{(-\sqrt{2} \ \ 0)\, I_{\mathcal{B}}^{\bar{\mathcal{B}}} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}}}_{\begin{pmatrix} x \\ y \end{pmatrix}} = 0$$

Onde $I_{\mathcal{B}}^{\bar{\mathcal{B}}} = R = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\Leftrightarrow (\bar{x} \ \bar{y})_{\bar{\mathcal{B}}} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}_{\bar{\mathcal{B}}} + \underbrace{(-1 \ \ 0) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{(-1 \ -1)} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = 0$$

$$\Leftrightarrow 2\bar{x}^2 + 4\bar{y}^2 - \bar{x} - \bar{y} = 0$$

Considere $2\bar{x}^2 - \bar{x} = 2\left(\bar{x}^2 - \frac{1}{2}\bar{x}\right)$

Temos $\bar{x}^2 - \frac{1}{2}\bar{x} = \bar{x}^2 - 2 \cdot \frac{1}{4}\bar{x} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2$

$= \left(\bar{x} - \frac{1}{4}\right)^2 - \frac{1}{16} \Rightarrow 2\bar{x}^2 - \bar{x} = 2\left(\bar{x} - \frac{1}{4}\right)^2 - \dots$

Considere $4\bar{y}^2 - \bar{y} = 4\left(\bar{y}^2 - \frac{1}{4}\bar{y}\right)$

$$\bar{y}^2 - \frac{1}{4}\bar{y} = \bar{y}^2 - 2\cdot\frac{1}{8}\bar{y} + \left(\frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2$$

$$= \left(\bar{y} - \frac{1}{8}\right)^2 - \frac{1}{64}$$

$$\Rightarrow 4\bar{y}^2 - \bar{y} = 4\left[\left(\bar{y} - \frac{1}{8}\right)^2 - \frac{1}{64}\right]$$

$$= 4\left(\bar{y} - \frac{1}{8}\right)^2 - \frac{1}{16}$$

Resumindo, temos

$$2\left(\bar{x} - \frac{1}{4}\right)^2 - \frac{1}{8} + 4\left(\bar{y} - \frac{1}{8}\right)^2 - \frac{1}{16} = 0$$

$$\Leftrightarrow 2\left(\bar{x} - \frac{1}{4}\right)^2 + 4\left(\bar{y} - \frac{1}{8}\right)^2 - \frac{3}{16} = 0$$

$$\Leftrightarrow 2\left(\bar{x} - \frac{1}{4}\right)^2 + 4\left(\bar{y} - \frac{1}{8}\right)^2 = \frac{3}{16}$$

$$\Leftrightarrow \frac{\left(\bar{x} - \frac{1}{4}\right)^2}{2} + \left(\bar{y} - \frac{1}{8}\right)^2 = \frac{3}{64}$$

$$\Leftrightarrow \frac{\left(\bar{x} - \frac{1}{4}\right)^2}{\left(\frac{3}{32}\right)} + \frac{\left(\bar{y} - \frac{1}{8}\right)^2}{\left(\frac{3}{64}\right)} = 1$$

Centro da elipse em $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{8} \end{pmatrix}_{\bar{\mathcal{B}}}$ no sitema $\bar{x}-\bar{y}$

em x, y o cento está em

$$I_{\mathcal{B}}^{\bar{\mathcal{B}}} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{8} \end{pmatrix}_{\bar{\mathcal{B}}} = \frac{\sqrt{2}}{16}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{16}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

O centro da elipse é $O_{\hat{x}-\hat{y}}$ do sistema rodado e transladado

Vertices em $\hat{x}-\hat{y}$ no eixo focal:

$(\hat{x} = \bar{x} - \frac{1}{4}, \hat{y} = \bar{y} - \frac{1}{8})$: $\begin{pmatrix} \pm\sqrt{\frac{3}{32}} \\ 0 \end{pmatrix}_{\hat{x}-\hat{y}} = \begin{pmatrix} \pm\frac{\sqrt{3}}{4\sqrt{2}} \\ 0 \end{pmatrix}_{\hat{x}-\hat{y}}$

Vertices em $\hat{x}-\hat{y}$ no eixo menor:

$$\begin{pmatrix} 0 \\ \pm\frac{\sqrt{3}}{8} \end{pmatrix}_{\hat{x}-\hat{y}}$$

Note que $a = \frac{\sqrt{3}}{4\sqrt{2}}$ e $b = \frac{\sqrt{3}}{8}$

$$a^2 = b^2 + c^2 \Rightarrow c^2 = \frac{3}{32} - \frac{3}{64} = \frac{3}{64} \Rightarrow c = \frac{\sqrt{3}}{8}$$

$$e = \frac{c}{a} = \frac{\cancel{\sqrt{3}}}{8} \cdot \frac{4\sqrt{2}}{\cancel{\sqrt{3}}} = \frac{\sqrt{2}}{2}$$

Vertices em $\bar{x},\bar{y}$: $\begin{pmatrix} \pm\frac{\sqrt{3}}{4\sqrt{2}} + \frac{1}{4} \\ \frac{1}{8} \end{pmatrix}_{\bar{\mathcal{B}}}$ e $\begin{pmatrix} \frac{1}{4} \\ \pm\frac{\sqrt{3}}{8} + \frac{1}{8} \end{pmatrix}_{\bar{\mathcal{B}}}$

Focos em $\hat{x},\hat{y}$:

$$\begin{pmatrix} \pm\frac{\sqrt{3}}{8} \\ 0 \end{pmatrix}_{\hat{x},\hat{y}}$$

Focos em $\bar{x}-\bar{y}$:

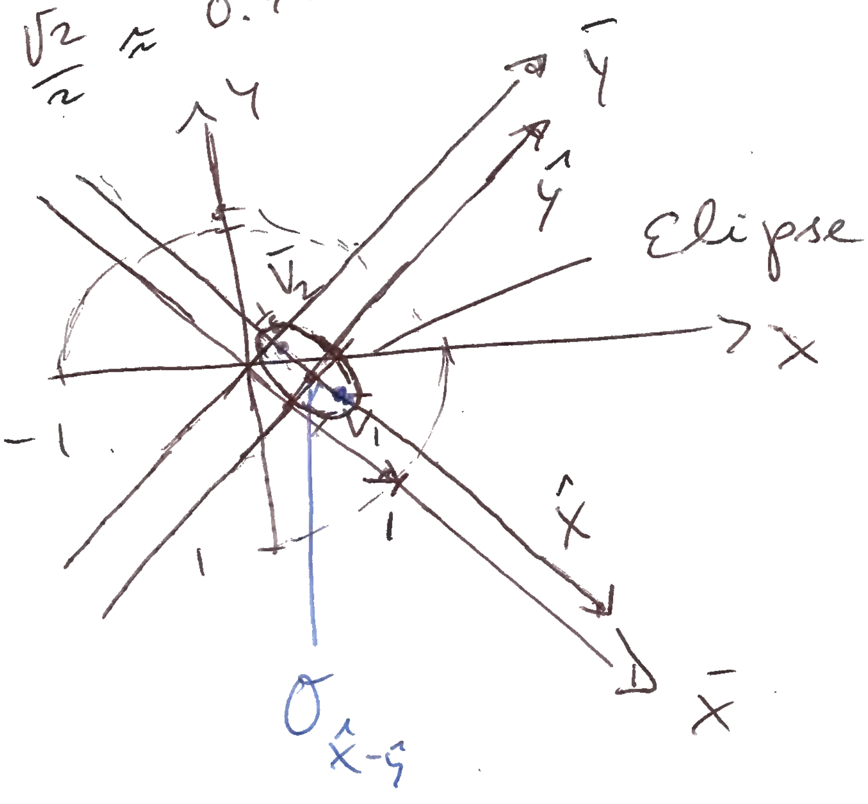
$$\begin{pmatrix} \pm\frac{\sqrt{3}}{8} + \frac{1}{4} \\ \frac{1}{8} \end{pmatrix}_{\bar{\mathcal{B}}}$$

Se quiser calcular os vertices em x-y: Aplique $I_{\mathcal{B}}^{\bar{\mathcal{B}}}$ aos coordenadas em $\bar{\mathcal{B}}$.

Esboço: $\frac{\sqrt{3}}{4\sqrt{2}} \approx 0.3062$

$\frac{\sqrt{3}}{8} \approx 0.2165$

$\frac{\sqrt{2}}{2} \approx 0.7071$



Sugestão de Exercicio:

Identifique $\mathcal{C}$ dado por

$$\frac{7}{2}x^2 + 3xy - \frac{1}{2}y^2 + \sqrt{10}\,x + \sqrt{10}\,y = 0$$