1.(a) $\cos \measuredangle (R^u_v, R^u_w)$

$$= |\cos \measuredangle (v,w)| = \frac{|\langle v,w\rangle|}{\|v\|\cdot\|w\|} = \frac{|4+2-4|}{\sqrt{24}\,\sqrt{6}}$$

$$= \frac{6}{\sqrt{4\cdot 6}\,\sqrt{6}} = \frac{6}{2\cdot 6} = \frac{1}{2} \qquad \pi \simeq 180^\circ$$

$$\Rightarrow \measuredangle (R^u_v, R^u_w) = \frac{\pi}{3} = \underline{\underline{60^\circ}}$$

Note que $30^\circ = \frac{60^\circ}{2}$, então se

já que $60^\circ = \measuredangle(v,-w)$

porque $\cos \measuredangle (v,-w) = \frac{\langle v,-w\rangle}{2\cdot 6} = \frac{1}{2}$

podemos escolher z entre v e w como vetor direto de R

(Note que $\|\bar{v}\| = \|-w\| = \sqrt{6}$ onde $\bar{v} = \begin{pmatrix}-1\\1\\2\end{pmatrix} = \frac{1}{2}v$

e que $\|-2w\| = \|v\|$

podemos escolher $z = \frac{1}{2}v + \frac{1}{2}(-2w)$

$$= \frac{1}{2}v - w = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}-3\\0\\3\end{pmatrix}$$

Seja $R = R^u_z$ $\qquad = (-3,0,3)^t$

$$\cos \measuredangle (R^u_v, R^u_z) = |\cos \measuredangle (v,z)| =$$

$$= \frac{|\langle v,z\rangle|}{\|v\|\,\|z\|} = \frac{|+6+12|}{\sqrt{24}\cdot\sqrt{2\cdot 9}} = \frac{18}{\sqrt{4\cdot 6}\cdot 3\sqrt{2}} = \frac{2\cdot 3\cdot 3}{2\cdot 3}$$

$$= \frac{\cancel{2}\cdot 3\cdot 3}{2\cancel{\cdot 3}\sqrt{2\cdot 3}\cdot\sqrt{2}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \measuredangle\left(R^u_{v_1}, R^u_z\right) = 30^\circ$$

$$\cos\measuredangle\left(R^u_{w_1}, R^u_z\right) = \left|\cos\measuredangle(w, z)\right|$$

$$= \frac{|\langle w, z\rangle|}{\|w\|\cdot\|z\|} = \frac{|-6-3|}{\sqrt{6}\cdot 3\sqrt{2}} = \frac{9}{2\cdot 3\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \measuredangle\left(R^u_{v_1}, R^u_z\right) = 30^\circ$$

(b) $R = R^u_z = \left\{ \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \right\}$. Sei $P \in R$

$$P = \begin{pmatrix} 3+\alpha(-3) \\ -3 \\ 3\alpha \end{pmatrix}$$

$$P_{\mathcal{E}} = \begin{pmatrix} x_q \\ y_q \\ z \end{pmatrix} \in \mathcal{E}$$

$$\Leftrightarrow \cancel{3} - 3\alpha - \cancel{3} - 3\alpha = 1 \Leftrightarrow -6\alpha = 1$$

$$\Leftrightarrow \alpha = -\frac{1}{6}$$

$$\therefore R \cap P = \left\{ \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 3{,}5 \\ -3 \\ -0{,}5 \end{pmatrix} \right\}$$

$3{,}5 - 3 + 0{,}5 = 1$ ✓ o.k.

1. (e) Solução alternativa:

Vamos encontrar $\mathcal{R} = \mathcal{R}^u_Z \subseteq \mathcal{P} = \mathcal{P}^u_{V,W}$

tal que $\measuredangle(\mathcal{R}, \mathcal{R}^u_V) = \measuredangle(\mathcal{R}, \mathcal{R}^u_W) = 30°$

Note que Z é da forma $Z = \alpha V + \beta W$

para alguns $\alpha, \beta \in \mathbb{R}$

Podemos escolher Z t.q. $\cos \measuredangle(V, Z) > 0$

Já que $\cos \measuredangle(-W, V) = \dfrac{\langle -W, V \rangle}{\|-W\| \, \|V\|} > 0$

podemos escolher Z t.q.

$\cos \measuredangle(-W, Z) > 0$

porque $\measuredangle(\mathcal{R}, \mathcal{R}^u_W) = \measuredangle(\mathcal{R}, \mathcal{R}^u_{-W})$

Além disso $\measuredangle(\mathcal{R}, \mathcal{R}^u_V) = \measuredangle(\mathcal{R}, \mathcal{R}^u_{\bar{V}})$

onde $\bar{V} = \frac{1}{2} V$. Seja $\bar{W} = -W$

É suficiente encontrar Z tal que $Z = \bar{\alpha} \bar{V} + \bar{\beta} \bar{W}$

$\measuredangle(Z, \bar{V}) = \measuredangle(Z, \bar{W}) = 30°$

$$\Rightarrow \frac{\langle Z, \bar{V} \rangle}{\|\bar{V}\| \, \|Z\|} = \frac{\langle Z, \bar{W} \rangle}{\|Z\| \, \|\bar{W}\|}$$

$$\Leftrightarrow \frac{\langle Z, \bar{V} \rangle}{6} = \frac{\langle Z, \bar{W} \rangle}{6}$$

$$\Leftrightarrow \langle \bar{\alpha} \bar{V} + \bar{\beta} \bar{W}, \bar{V} \rangle = \langle \bar{\alpha} \bar{V} + \bar{\beta} \bar{W}, \bar{W} \rangle$$

$\Leftrightarrow \bar{\alpha} \|\bar{v}\|^2 + \bar{\beta} \langle \bar{w}, \bar{v} \rangle = \bar{\alpha} \langle \bar{v}, \bar{w} \rangle + \bar{\beta} \|\bar{w}\|^2$

$\Leftrightarrow 6\bar{\alpha} + \bar{\beta} \left\langle \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\rangle = \bar{\alpha} \left\langle \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\rangle + \bar{\beta} \cdot 6$

$\Leftrightarrow 6\bar{\alpha} + 3\bar{\beta} = 3\bar{\alpha} + 6\bar{\beta}$

$\Leftrightarrow 2\bar{\alpha} + \bar{\beta} = \bar{\alpha} + 2\bar{\beta}$

$\Leftrightarrow \bar{\alpha} = \bar{\beta}$

Seja $z = 1\bar{v} + 1\bar{w} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$

Temos $\cos \measuredangle(\bar{v}, z) = \cos \measuredangle(\bar{w}, z)$

$= \dfrac{9}{\sqrt{6}\sqrt{2 \cdot 9}} = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \measuredangle(\bar{v}, z) = \measuredangle(\bar{w}, z) = 30°$

(Isto já é garantido porque $\measuredangle(\bar{v}, z) = \measuredangle(\bar{w}, z)$ e $z = \bar{\alpha}\bar{v} + \bar{\beta}\bar{w}$)

$\Rightarrow \measuredangle(R, R^u_v) = \measuredangle(R, R^u_w) = 30°$

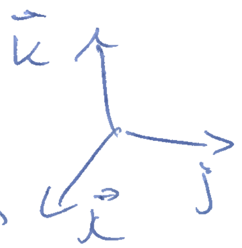
2 (a) $\|V\|^2 + \|w\| - \|V - w\|^2$

$$= \sum_{i=1}^{3} v_i^2 + \sum_{i=1}^{3} w_i^2 - \sum_{i=1}^{3} (v_i - w_i)^2$$

$$= v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - \left[(v_1^2 - 2v_1w_1 + w_1^2) + (v_2^2 - 2v_2w_2 + w_2^2) + (v_3^2 - 2v_3w_3 + w_3^2) \right]$$

$$= 2v_1w_1 + 2v_2w_2 + 2v_3w_3 = 2\langle V, w \rangle$$

VERDADEIRO



(b) Seja $u = v = \vec{i}$ e $w = j$

$$u \times (v \times w) = i \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$

$$\neq \vec{0} = 0 \times j = (\vec{i} \times \vec{i}) \times \vec{j}$$

3. $$\text{Volume} = |\langle u, v \times w \rangle| = \left| \det \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{pmatrix} \right|$$

$$= |2(-4+1) - (-2)| = |2 \cdot (-3) + 2| = |-6+2| = 4$$

4.(a) Vamos encontrar $P^* = \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}$ tal que

$P^* = \bar{P} + \alpha^* N$ e tal que

$$\|\overrightarrow{\bar{P}P^*}\| = \sqrt{5}$$

$$\Leftrightarrow \|\overrightarrow{\bar{P}P^*}\|^2 = 5$$

$$\Leftrightarrow (x^* - 1)^2 + (y^* - 1)^2 + (z^* - 1)^2 = 5$$

$$\Leftrightarrow (\alpha^*)^2 + (\alpha^*)^2 + (\alpha^*)^2 = 5$$

$$\Leftrightarrow (\alpha^*)^2 = \frac{5}{3} \Leftrightarrow \alpha^* = \pm\sqrt{\frac{5}{3}}$$

$\therefore \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \pm \sqrt{\frac{5}{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ são 2 pts que distam $\sqrt{5}$ de $\bar{P}$

Podemos defini. $\mathcal{P} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z + d = 0 \right\}$

onde $d = -3\left(1 + \sqrt{\frac{5}{3}}\right) = -3 - \sqrt{15}$

$$Q = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z - 3 + \sqrt{15} = 0 \right\}$$

Alternativamente, observe que $\forall P_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in \mathcal{P}$:

$$\sqrt{5} = \text{dist}(\bar{P}, \mathcal{P}) = \left| \text{proj}_N (\overrightarrow{P_0 \bar{P}}) \right|$$

$$= \frac{|\langle \overrightarrow{P_0 \bar{P}}, N \rangle|}{\|N\|} = \frac{|\langle \bar{P} - P_0, N \rangle|}{\|N\|}$$

$$= \frac{|\langle \bar{P}, N \rangle - \langle P_0, N \rangle|}{\|N\|} = \frac{|\langle \bar{P}, N \rangle - (x_0 + y_0 +}{\|N\|}$$

$$= \frac{|3+d|}{\sqrt{3}}$$

$$\Leftrightarrow 5 = \frac{(3+d)}{3}$$

$$\Leftrightarrow (3+d)^2 = \frac{5}{3}$$

$$\Leftrightarrow 3+d = \pm\sqrt{\frac{5}{3}} \Leftrightarrow d = -3 \pm \frac{\sqrt{5}}{\sqrt{3}}$$

$$\Leftrightarrow d = -3 \pm \sqrt{15}$$

Portanto, $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x+y+z-3-\sqrt{15} = 0 \right\}$

$Q = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x+y+z-3+\sqrt{15} = 0 \right\}$

4.(b) (i) Vamos encontrar $\cancel{P^*}$ α^* t.q.

$\exists \cancel{\alpha}^*$ $P^* = P_1 + \alpha^* \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in P$

$$\Leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha^* \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in P$$

$$\Leftrightarrow 1+\alpha^* + \alpha^* + 1 + \alpha^* - 3 - \sqrt{15} = 0$$

$$\Leftrightarrow 3\alpha^* = 1 + \sqrt{15} \Leftrightarrow \alpha^* = \frac{1}{3} + \frac{\sqrt{15}}{3}$$

$$\therefore P^* = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left(\frac{1}{3} + \frac{\sqrt{15}}{3}\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4+\sqrt{15} \\ 1+\sqrt{15} \\ 4+\sqrt{15} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + \frac{\sqrt{15}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(ii) Vamos encontrar α^* t.q.

$$Q^* = P_1 + \alpha^* \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in Q$$

$$\Leftrightarrow 3\alpha^* + 2 - 3 + \sqrt{15} = 0$$

$$\Leftrightarrow 3\alpha^* = 1 - \sqrt{15}$$

$$\Leftrightarrow \alpha^* = \frac{1}{3}(1-\sqrt{15})$$

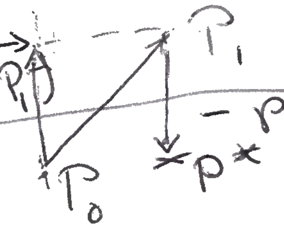
$$\therefore Q^* = \frac{1}{3}(1-\sqrt{15})\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}4-\sqrt{15}\\1-\sqrt{15}\\4-\sqrt{15}\end{pmatrix} = \frac{1}{3}\left[\begin{pmatrix}4\\1\\4\end{pmatrix} - \sqrt{15}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]$$

Alternativamente, escolhe P_0 e P_1, p. ex. $P_0 = \begin{pmatrix}3+\sqrt{15}\\0\\0\end{pmatrix}$

e note que

$$P^* = P_1 - \mathrm{proj}_N(\overrightarrow{P_0P_1})$$



$$= P_1 - \frac{\langle \overrightarrow{P_0P_1}, N\rangle}{\|N\|^2} \cdot N$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{\left\langle \begin{pmatrix}-2-\sqrt{15}\\0\\+1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\rangle}{3} N$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{3}(-1-\sqrt{15})\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{3}(3+\sqrt{15})\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{1}{3}\left[\begin{pmatrix}4\\1\\4\end{pmatrix} + \sqrt{15}\begin{pmatrix}1\\1\\1\end{pmatrix}\right]$$

Similarmente, escolhe $Q_0 \in \mathcal{Q}$

Por ex. $Q_0 = \begin{pmatrix} 3-\sqrt{15} \\ 0 \\ 0 \end{pmatrix} \in \mathcal{Q}$

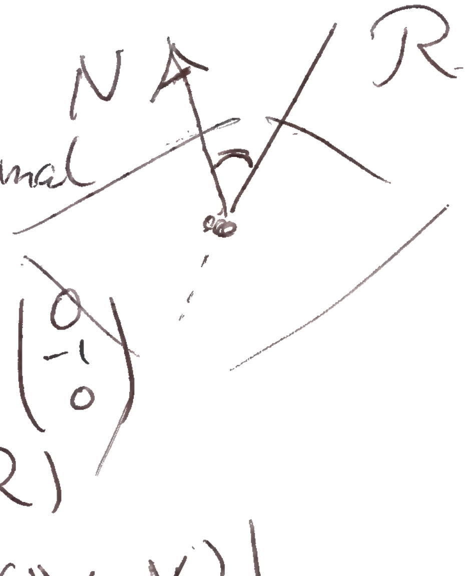
$$Q^* = \cancel{Q_0}\, P_1 - \text{proj}_N \frac{\langle \overrightarrow{Q_0P_1}, N\rangle}{\|N\|^2}\cdot N$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{3}\left\langle \begin{pmatrix}-2+\sqrt{15}\\0\\1\end{pmatrix}, N\right\rangle N$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{3}(-1+\sqrt{15})\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{1}{3}(\cancel{\sqrt{15}-1}\ 1-\sqrt{15})\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \frac{1}{3}\left[\begin{pmatrix}4\\1\\4\end{pmatrix} - \sqrt{15}\begin{pmatrix}1\\1\\1\end{pmatrix}\right].$$



4.(c) $N = \begin{pmatrix}1\\1\\1\end{pmatrix}$ é vetor normal de $\mathcal{P}$ e de $\mathcal{Q}$

$\mathcal{R} = \mathcal{R}_V^{\bar{P}}$, onde $V = \overrightarrow{\bar{P}P_1} = \begin{pmatrix}0\\-1\\0\end{pmatrix}$

$\measuredangle(\mathcal{P},\mathcal{R}) = 90^\circ - \measuredangle(\mathcal{R}_N^{\bar{P}}, \mathcal{R})$

$\cos\measuredangle(\mathcal{R}_N^{\bar{P}},\mathcal{R}) = |\cos\measuredangle(N,V)|$

$$= \frac{|\langle N,V\rangle|}{\|N\|\,\|V\|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow$$

$$\Rightarrow \measuredangle(\mathcal{P},\mathcal{R}) = \measuredangle(\mathcal{Q},\mathcal{R}) = 90^\circ - \arccos\left(\frac{\sqrt{3}}{3}\right)$$

$$= \arcsin\left(\frac{\sqrt{3}}{3}\right)$$