1. $(A|B)=\left(\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & a & b & 3 \\ 1 & 1 & b & 3 \end{array}\right)$

$\underset{L_1=L_1-2L_3}{\overset{L_2=L_2-L_3}{\rightsquigarrow}}\left(\begin{array}{ccc|c} 0 & 0 & 2-2b & -4 \\ 0 & a-1 & 0 & 0 \\ 1 & 1 & b & 3 \end{array}\right)$ det. (-1)

$\rightsquigarrow\left(\begin{array}{ccc|c} 1 & 1 & b & 3 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & 2(1-b) & -4 \end{array}\right)$

~~Única~~ Solução única $\Leftrightarrow$

(a) $\det(A) = -2(a-1)(1-b) \neq 0$

$\Leftrightarrow a \neq 1$ e $b \neq 1$

(b) Se $a=1$, $b\neq 1$

$\rightsquigarrow\left(\begin{array}{ccc|c} 1 & 1 & b & 3 \\ 0 & 0 & b-1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $z=\frac{2}{b-1}$

variável livre

infinito de soluções

(c) Se $b=1$

$\rightsquigarrow\left(\begin{array}{ccc|c} 1 & 1 & b & 3 \\ 0 & a-1 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right)$ $0x+0y+0z=-4$ ↯

$S=\emptyset$

(d) (i) $a,b\neq 1$

$z=\frac{2}{b-1}$

$y=0$

$x+\frac{2b}{b-1}=3 \Leftrightarrow x = 3-\frac{2b}{b-1}$

$S=\left\{\begin{pmatrix} -\frac{2b}{b-1}+3 \\ 0 \\ \frac{2}{b-1} \end{pmatrix}\right\}$

(ii) $a=1$, $b\neq 1$

$z=\frac{2}{b-1}$

$x+y+\frac{2b}{b-1}=3$

$\Leftrightarrow x = 3-y-\frac{2b}{b-1}$

$S=\left\{\begin{pmatrix} 3-\frac{2b}{b-1} \\ 0 \\ \frac{2}{b-1} \end{pmatrix}+y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} : y\in\mathbb{R}\right\}$

2. $\det\left(\begin{pmatrix} -a & d+a & 2g \\ -b & e+b & 2h \\ -c & f+c & 2i \end{pmatrix}\right) \underset{\substack{\uparrow \\ \det(A)=\det(A^T) \\ \forall A \in \mathbb{R}^{n\times n}}}{=}$

$= \det\left(\begin{pmatrix} -a & -b & -c \\ d+a & e+b & f+c \\ 2g & 2h & 2i \end{pmatrix}\right)$

$= \det\left(\begin{pmatrix} +a & +b & +c \\ d+a & e+b & f+c \\ g & h & i \end{pmatrix}\right) \cdot (-1) \cdot 2$

$= -2 \cdot \det\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right)$

$\underset{\substack{\uparrow \\ \det(A)=\det(A^T) \\ \forall A \in \mathbb{R}^{n\times n}}}{=} -2 \cdot \det\left(\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\right) = (-2) \cdot 2 = \underline{\underline{-4}}$

3. $\det(C) = 1 \cdot (1-1) + 0 \cdot (-1) \cdot \det\left(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right)$
$+ 0 \cdot \det\left(\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right)$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}^{*} \quad = 0$

Então $\nexists\, C^{-1}$

4. (a) Verdadeiro

$\det(A) \neq 0$ e $\det(A) = \det(A^T) \Rightarrow \det(A^T) \neq 0$

(b) $A^3 - 2A^2 + I_n = 0$ Verdadeiro

$\Leftrightarrow A \cdot (A^2 - 2A) = -I_n$

$\Leftrightarrow A \cdot (2A - A^2) = I_n$

Então A é invertível e $A^{-1} = 2A - A^2$

(c) Falso:

Contra-exemplo: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

quer dizer para $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ temos $A^2 = I_2$

mas $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(d) Verdadeiro:

$A(\alpha U + \beta V) = A(\alpha U) + A \cdot (\beta V)$

$= \alpha A U + \beta A V = \alpha \cdot 0 + \beta \cdot 0 = 0$

Então $\alpha U + \beta V$ é solução de $AX = 0$

(e) $\forall U$ tal que $AU = 0$ tem-se

Se $AV = B$ então

$A(U + V) = AU + AV = 0 + B = B$

Então $U + V$ é solução de $AX = B$