A New Algorithm for Designing Θ-Fuzzy Associative Memories Based on Subsethood Measures

Peter Sussner
Estevão Esmi
João Florindo

University of Campinas
Department of Applied Mathematics
Campinas, Brazil

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Topic of this Presentation

- Θ-Fuzzy Associative Memories (Θ-FAMs) associate elements of an arbitrary bounded lattice $\mathbb{L}$ with fuzzy sets.
- A Θ-FAM is determined by functions $\Theta^{\xi} : \mathbb{L} \to [0, 1]$ and its weights.
- The new approach towards Θ-FAMs presented here consists of
  1. generating a fundamental memory set (set of associations) $\mathcal{M} \subset \mathbb{L}$ from a given training set $\mathcal{T}$,
  2. using $\mathcal{M}$ to design $\Theta^{\xi} : \mathbb{L} \to [0, 1]$, where $\xi = 1, \ldots, |\mathcal{M}|$, in terms of a subsethood measure.
- The resulting Θ-FAM differs from previous Θ-FAMs since
  - generally, $\mathcal{M} \not\subseteq \mathcal{T}$;
  - it combines Subsethood- and Dual Subsethood-FAMs.

Organization

Partially Ordered Sets

Poset and Dual Poset:

- A pair $(P, \leq)$ consisting of a set $P \neq \emptyset$ and a reflexive, antisymmetric, and transitive relation "$\leq$" is called a partially ordered set or poset.
- The symbols $\bigwedge X$ and $\bigvee X$ denote respectively the infimum and the supremum of $X \subseteq P$ if they exist in P.
- The dual partial order relation is the relation $\geq$ defined by $x \geq y \Leftrightarrow y \leq x \ \forall x, y \in P$.
- $y, z \in X \subseteq P$ are resp. said to be a minimal element and a maximal element of X if, $\forall x \in X$, we have that $x \leq y$ implies that $x = y$ and $z \leq x$ implies that $x = z$, resp..
- $x \in X \subseteq P$ is a maximal element of X in the poset $(P, \leq)$ if and only if x is a minimal element of X in the dual poset $(P, \geq)$.

Bounded and Complete Lattices

Lattices:

- A lattice is a poset $\mathbb{L} \neq \emptyset$ such that every set $\{x, y\} \subseteq \mathbb{L}$ has an infimum, denoted $x \wedge y$ and a supremum, denoted $x \vee y$ in $\mathbb{L}$.
  - $\mathbb{L}$ is called totally ordered or a chain if $x \leq y$ or $y \leq x$ $\forall x, y \in \mathbb{L}$.
  - $\mathbb{L}$ is bounded if $0_{\mathbb{L}} := \bigwedge \mathbb{L}$ and $1_{\mathbb{L}} := \bigvee \mathbb{L}$ exist in $\mathbb{L}$.
  - $\mathbb{L}$ is complete if $\bigwedge X$ and $\bigvee X$ exist in $\mathbb{L}$ for every $X \subseteq \mathbb{L}$.
- If $\mathbb{L}$ is a lattice, then $\mathbb{L}^n$ is a lattice with the partial order

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \Leftrightarrow a_i \leq b_i, \; i = 1, \ldots, n.$$

and $\mathbb{L}^U = \{f : U \to \mathbb{L}\}$ is a lattice with

$$f \leq g \Leftrightarrow f(u) \leq g(u) \, \forall \, u \in U.$$

If $\mathbb{L}$ is bounded or complete, then $\mathbb{L}^n$ and $\mathbb{L}^U$ are also bounded or complete, resp..

Isomorphisms

Definition:

Let $\mathbb{L}$ and $\mathbb{M}$ be (bounded or complete) lattices.

- A lattice isomorphism is a bijection $\phi : \mathbb{L} \to \mathbb{M}$ that satisfies

$$x \leq y \Leftrightarrow \phi(x) \leq \phi(y) \, \forall \, x, y \in \mathbb{L}.$$

- $\mathbb{L}$ and $\mathbb{M}$ are said to be isomorphic if there is an isomorphism $\phi : \mathbb{L} \to \mathbb{M}$. In this case, one writes $\mathbb{L} \simeq \mathbb{M}$.

Examples:

- If $|U| = n$, then $\mathbb{L}^U \simeq \mathbb{L}^n$. In particular, the complete lattice $[0,1]^n$ is isomorphic to the complete lattice $[0,1]^U$.
- $(\mathcal{F}(U), \subseteq)$, where $\mathcal{F}(U)$ denotes the class of fuzzy sets on U, is a complete lattice that is isomorphic to $([0,1]^U, \leq)$. If $|U| = n$, then $\mathcal{F}(U) \simeq [0,1]^n$.

Subsethood Measures on Bounded Lattices

Definition:

Let $\mathbb{L}$ be a bounded lattice. A function $S : \mathbb{L} \times \mathbb{L} \to [0, 1]$ that has the following properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{L}$ is called a subsethood measure on $\mathbb{L}$.

1. If $\mathbf{x} \leq \mathbf{y}$, then $S(\mathbf{x}, \mathbf{y}) = 1$;
2. $S(1_{\mathbb{L}}, 0_{\mathbb{L}}) = 0$;
3. If $\mathbf{x} \leq \mathbf{y} \leq \mathbf{z}$ then $S(\mathbf{z}, \mathbf{x}) \leq S(\mathbf{y}, \mathbf{x})$ and $S(\mathbf{z}, \mathbf{x}) \leq S(\mathbf{z}, \mathbf{y})$.

If $\mathbb{L} = \mathcal{F}(U)$ for some $U \neq \emptyset$, then we speak of a fuzzy subsethood measure.

An Example of a Fuzzy Subsethood Measure

If I is a fuzzy implication such that $I(a,b) = 1 \ \forall \ a \leq b \in [0,1]$ and $\upsilon : \mathcal{F}(U) \to [0,1]$ is an increasing function such that $\upsilon(\emptyset) = 0$ and $\upsilon(U) = 1$, then a fuzzy subsethood measure is given by

$$S^{\cup}(A,B) = I(\upsilon(A \cup B), \upsilon(B)).$$

If $U = \{u^1, \ldots, u^k\}$ for some $k \in \mathbb{N}$, $p \in (0, +\infty)$ and

$$\upsilon_p(A) = \sum_{i=1}^{k} \frac{1 - \cos(\pi \cdot [\mu_A(u^i)]^p)}{2k}, \quad \forall A \in \mathcal{F}(U),$$

then $\upsilon_p : \mathcal{F}(U) \to [0,1]$ is increasing, $\upsilon_p(\emptyset) = 0$ and $\upsilon_p(U) = 1$.

Θ-FAMs on Bounded Lattices

Let $\mathbb{L}$ be a bounded lattice. The purpose of a Θ-FAM is to store a set of associations, also called fundamental memory set, of the form $\mathcal{M} = \{(\mathbf{x}^\xi, B^\xi) \in \mathbb{L} \times \mathcal{F}(Y) \,|\, \xi \in P\}$, where $P = \{1, \ldots, p\}$.

Definition:

Given $\Theta^\xi : \mathbb{L} \to [0,1]$ satisfying $\Theta^\xi(\mathbf{x}^\xi) = 1 \; \forall \, \xi \in P$, a vector $\mathbf{v} \in \mathbb{R}^p$, $F : \mathbb{R}^p \to [0,1]^p$ and a t-norm, the following mapping $\mathcal{O} : \mathbb{L} \to \mathcal{F}(Y)$ yields a Θ-fuzzy associative memory (Θ-FAM):

$$\mathcal{O}(\mathbf{x}) = R \circ_t F(v_1 \Theta^1(\mathbf{x}), \ldots, v_p \Theta^p(\mathbf{x})). \tag{1}$$

Here, $R \in \mathcal{F}(Y \times P)$ is given by $R(y, \xi) = B^\xi(y) \; \forall y \in Y, \xi \in P$.

Outputs and Topology of a Θ-FAM

From now on, let $F : \mathbb{R}^p \to \{0,1\}^p \subset [0,1]^p$ be such that

$$F((a_1,\ldots,a_p)^T)_i = \begin{cases} 1 & \text{if } a_i = \bigvee_{j=1}^p a_j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } i = 1,\ldots,p.$$

For each $\mathbf{x} \in \mathbb{L}$, the Θ-FAM produces the following output:

$$\mathcal{O}(\mathbf{x}) = \bigcup_{j \in I_{\mathbf{v}}(\mathbf{x})} B^j, \qquad (2)$$

where

$$I_{\mathbf{v}}(\mathbf{x}) = \left\{ j \in P : v_j \Theta^j(\mathbf{x}) = \bigvee_{\xi=1,\ldots,p} v_\xi \Theta^\xi(\mathbf{x}) \right\}.$$

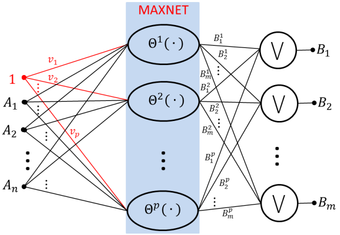
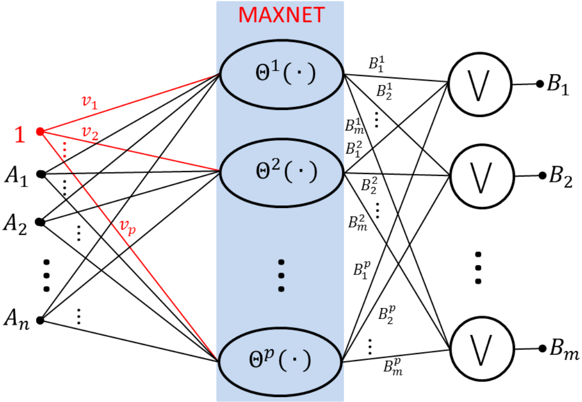


Figure: Topology of a Θ-FAM.

Topology of a Θ-FAM

The following displays the topology of a Θ-FAM for inputs **x** in a product of n bounded lattices and $Y = \{y_1, \ldots, y_m\}$



The Fundamental Memory Set of a Θ-FAM

- Previously, the fundamental memory set $\mathcal{M}$ was obtained by
  - setting $\mathcal{M} = \mathcal{T}$;
  - extracting a subset $\mathcal{M}$ of $\mathcal{T}$;
  - using a certain algorithm to generate a set $\mathcal{M} \subset \mathbb{I}_{\mathbb{L}}$, where $\mathbb{I}_{\mathbb{L}}$ is the class of closed subintervals of $\mathbb{L}$.
- In classification problems, we associate a pattern key $\mathbf{x}^{\xi} \in \mathbb{L}$ whose class label is l^{ξ} with the following fuzzy set $B^{\xi} \in \mathcal{F}(Y)$:

$$\mu_{B^{\xi}}(i) = \begin{cases} 1, & \text{if } i = l^{\xi} \\ 0, & \text{otherwise}. \end{cases}$$

General Design Scheme

Let $\mathbb{L}$ be a product of bounded lattices, $\mathcal{C} = \{l_1, \ldots, l_m\}$ (class labels).

Steps for constructing $\mathcal{M}$ and Θ^ξ:

Given an arbitrary subsethood measure $S : \mathbb{L}^2 \to [0,1]$ and a training set $\mathcal{T} = \{(\mathbf{x}^\xi, l^\xi) \in \mathbb{L} \times \mathcal{C} \mid \xi = 1, \ldots, s\}$, perform the following steps:

1. Apply the construction algorithm in $(\mathbb{L}, \leq)$ in order to obtain

$$\check{\mathcal{M}} = \{(\check{\mathbf{x}}^\xi, l^\xi) \in [0,1]^{\mathbb{L}} \times \mathcal{C} \mid \xi = 1, \ldots, \check{p}\},$$

   and define $\Theta^\xi(\,\cdot\,) = S(\,\cdot\,, \check{\mathbf{x}}^\xi)$ for $\xi = 1, \ldots, \check{p}$.

2. Apply the construction algorithm in $(\mathbb{L}, \geq)$ in order to obtain

$$\hat{\mathcal{M}} = \{(\hat{\mathbf{x}}^\xi, l^\xi) \in [0,1]^{\mathbb{L}} \times \mathcal{C} \mid \xi = 1, \ldots, \hat{p}\},$$

   and define $\hat{\Theta}^\xi(\,\cdot\,) = S(\hat{\mathbf{x}}^\xi, \cdot)$ for $\xi = 1, \ldots, \hat{p}$.

3. Set $\mathcal{M} = \check{\mathcal{M}} \cup \hat{\mathcal{M}}$ and $\Theta^{\xi + \check{p}} = \hat{\Theta}^\xi$ for $\xi = 1, \ldots, \hat{p}$.

Example of a Decision Surface Produced by this Θ-FAM

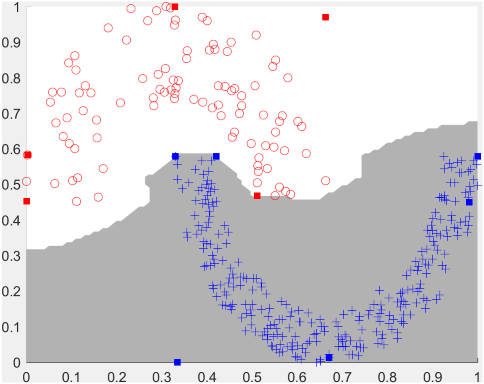
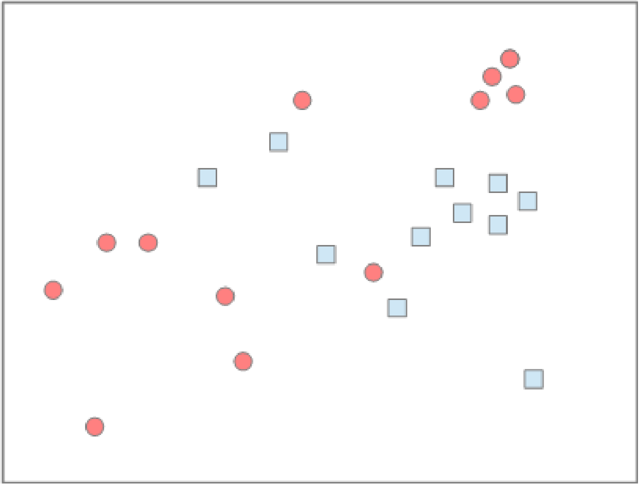


Figure: Decision surface produced by the Θ-FAM in an application to the Jain dataset. Points in the white region are classified as "○" and points in the gray region are classified as "+". The solid red and blue squares correspond resp. to the fundamental memories that are generated for the "○" and "+" classes.

An Example to Illustrate the Construction of $\mathcal{M}$



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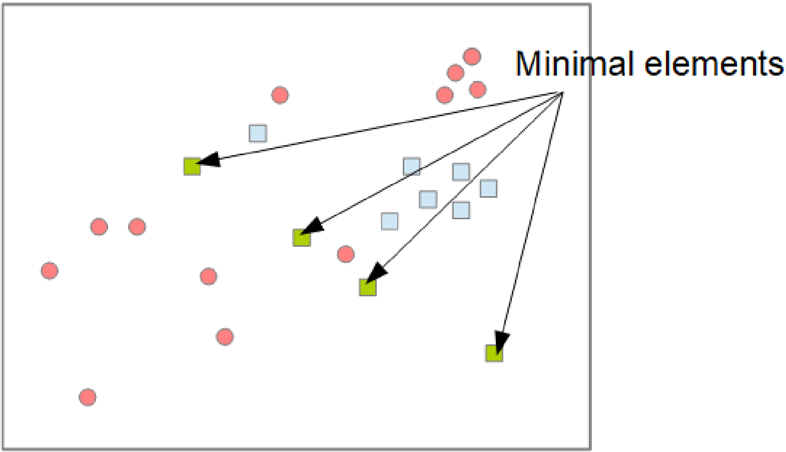

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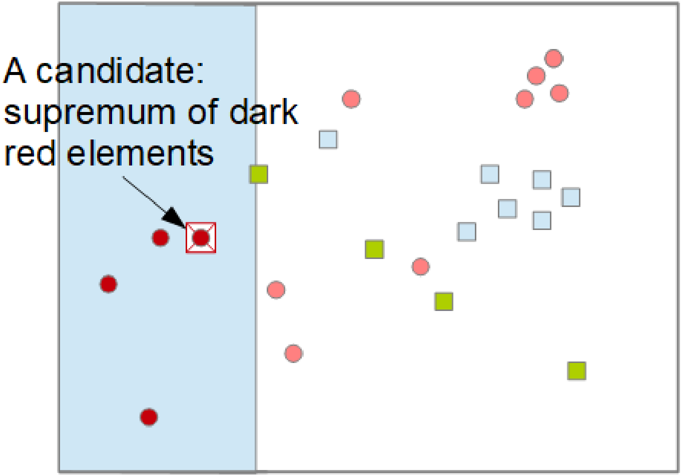

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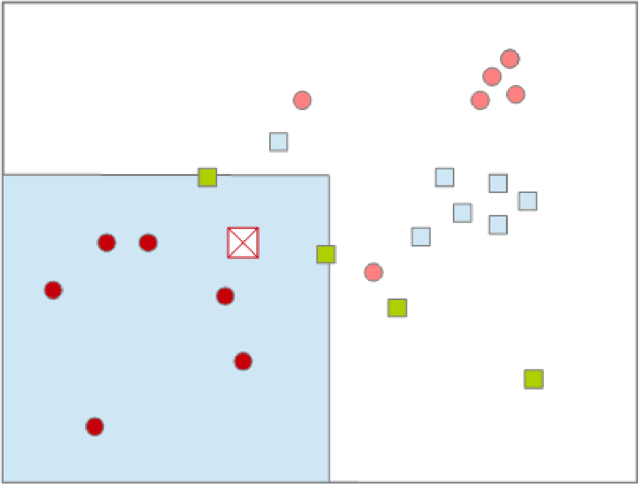

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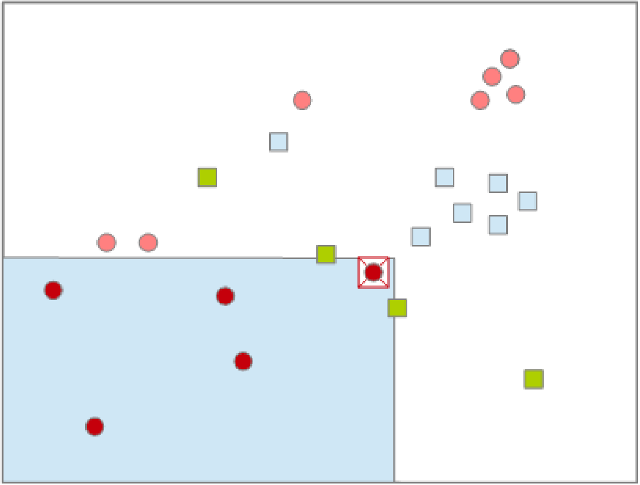

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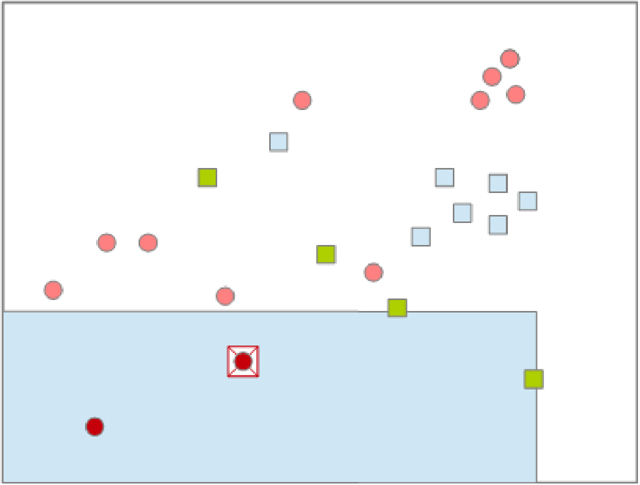

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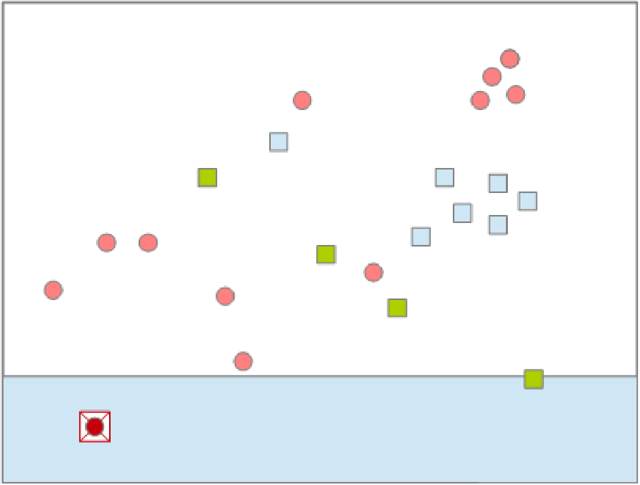

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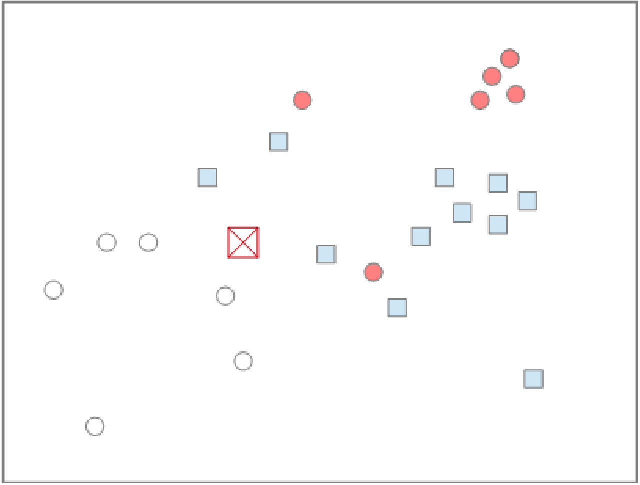

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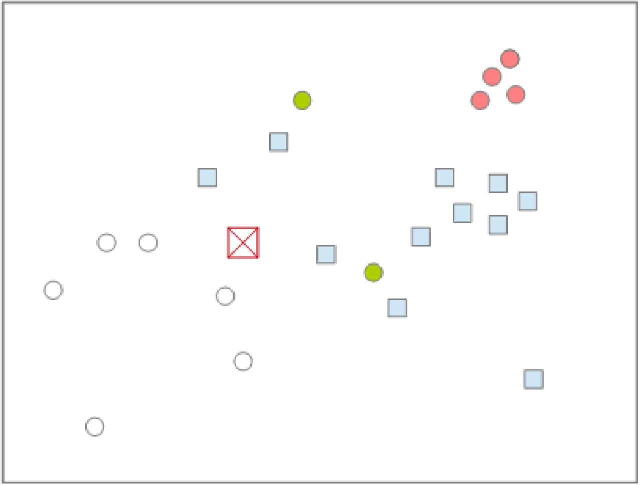

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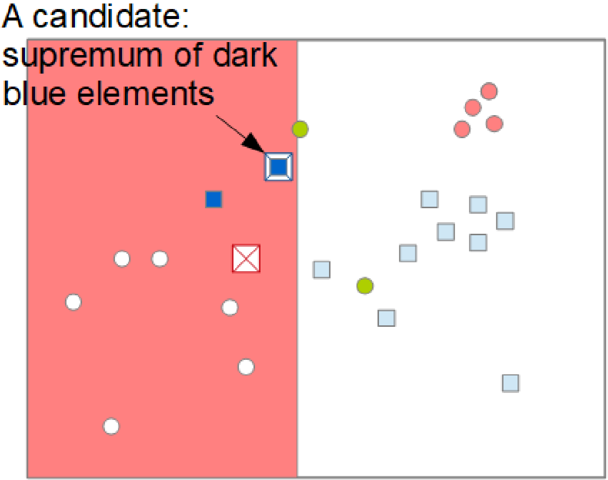

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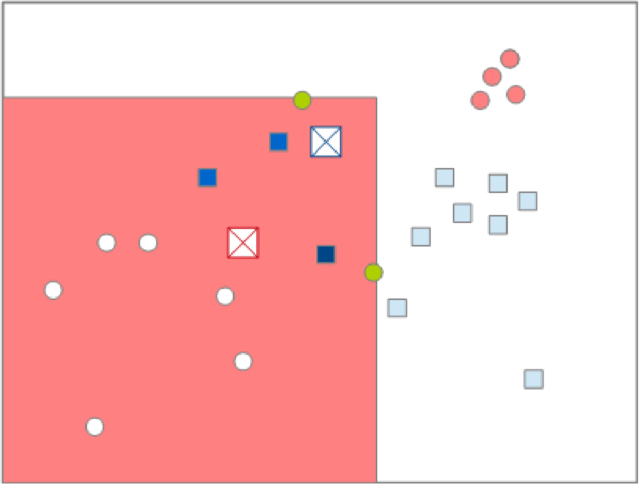

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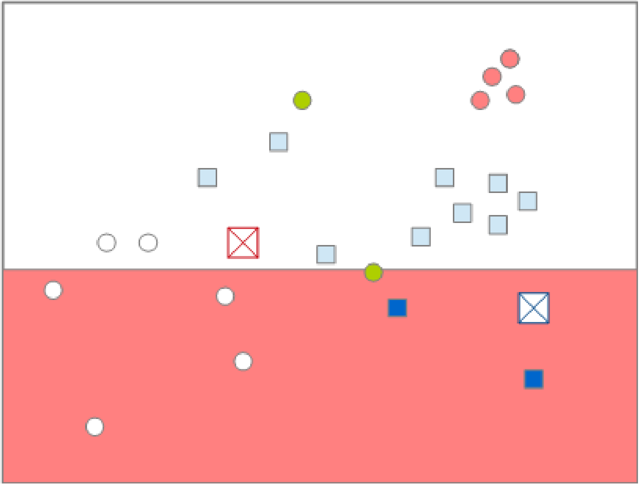

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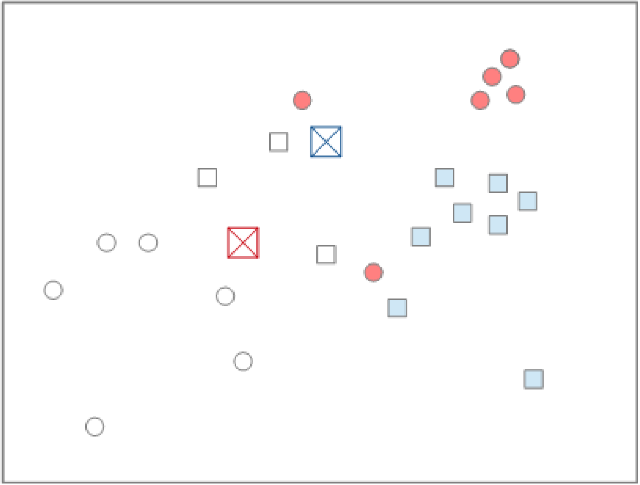

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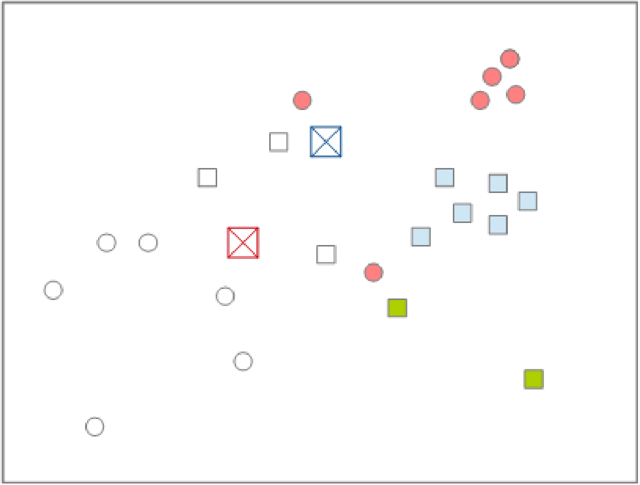

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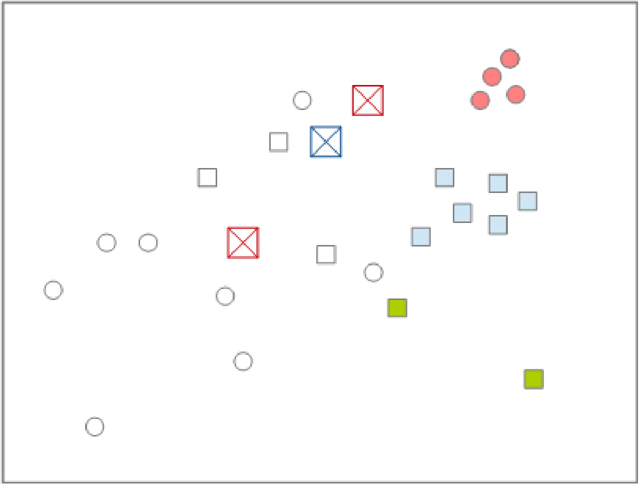

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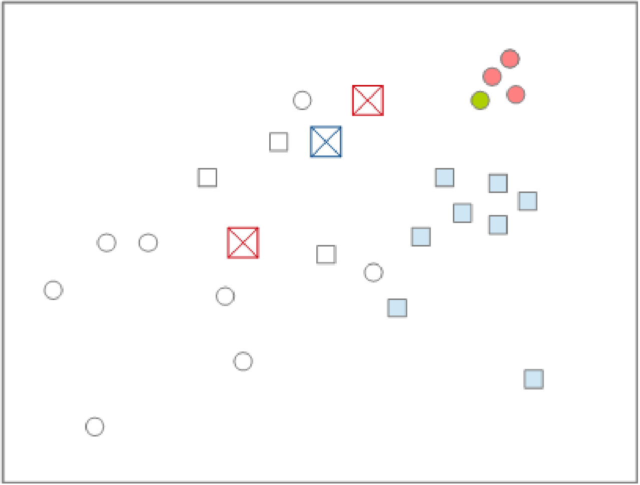

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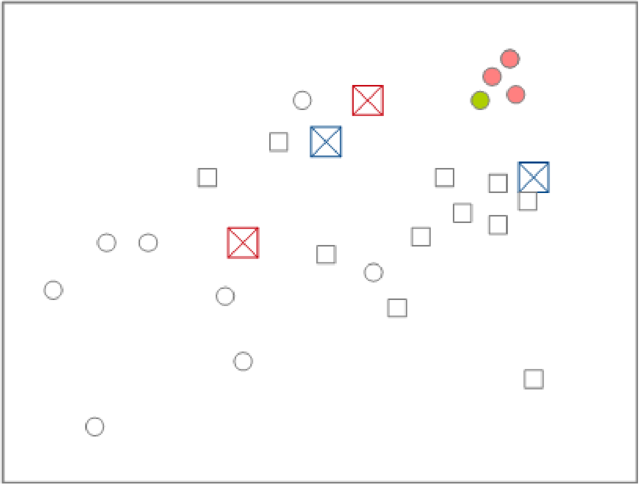

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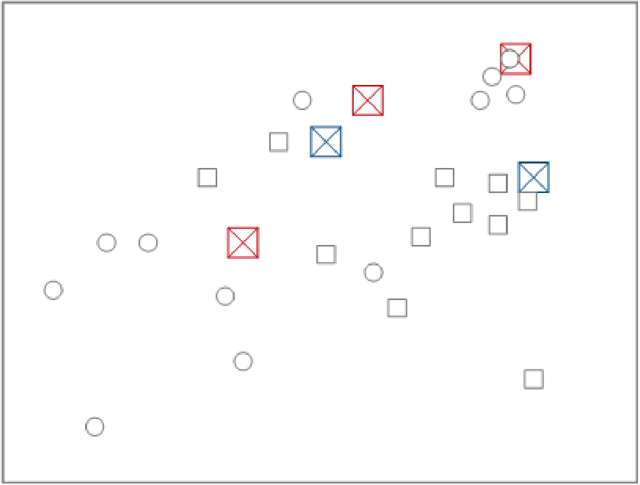

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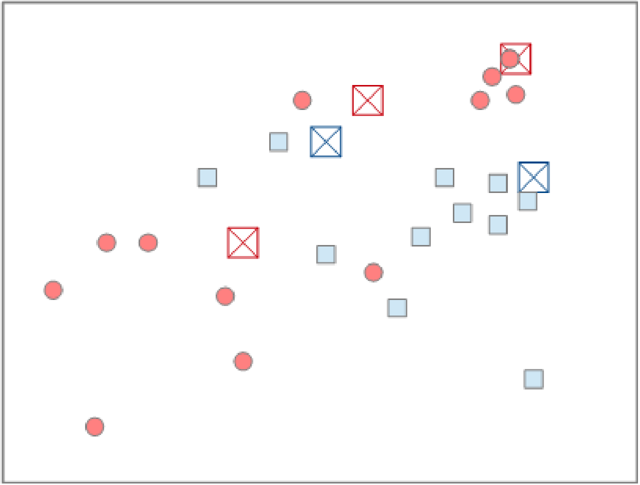

Experimental Setup

- We considered datasets consisting of a finite subset of $I_1 \times \ldots I_n \times \mathcal{C}$, where $I_i = [a_i, b_i] \subset \mathbb{R}$ for $i = 1, \ldots, n$ and Cc is the set of class labels.
- We normalized the data in the range $[0, 1]^n$ ($\simeq \mathcal{F}(X)$, where $X = \{1, \ldots, n\}$) so as to facilitate the use of a fuzzy subsethood measure

We employed the subsethood measure

$$S^{\cup}(A, B) = I_L(\upsilon_1(A \cup B), \upsilon_1(B)) \tag{3}$$

where I_L stands for the Łukasiewicz implication and υ_1

$$\upsilon_1(A) = \sum_{i=1}^{n} \frac{1 - \cos(\pi \cdot [\mu_A(i)])}{2n}, \quad \forall A \in [0, 1]^n. \tag{4}$$

10 Datasets From Knowledge Extraction Based on Evolutionary Learning (KEEL)-Dataset Repository

Classification Rates Achieved by FARC-HD, Original Θ-FAM, TE-FAM, $\mathcal{G}_{XX}$, and the proposed Θ-FAM.

	FARC-HD	Original Θ-FAM	TE-FAM (Reduction # of Θ^{ξ})	$\mathcal{G}_{XX}$	Proposed Θ-FAM (Reduction # of Θ^{ξ})
Clevel.	55.2	51.2	61.0 (21.3%)	45.8	55.9 (35.9%)
Ecoli	82.2	76.8	81.0 (34.9%)	77.1	82.2 (64.3%)
Glass	70.2	70.5	63.9 (31.9%)	71.2	66.2 (60.2%)
Heart	84.4	78.1	82.6 (31.8%)	68.9	77.8 (71.3 %)
Monks	99.8	98.6	97.7 (12.8%)	91.9	100 (97.3%)
Movem.	76.7	84.7	69.2 (35.1%)	88.1	77.2(63.1%)
Pima	75.7	67.4	76.2 (26.2%)	71.4	74.4 (74.6%)
Spect.	79.8	81.3	81.7 (21.6%)	80.5	79.1 (43.5%)
Vowel	71.8	97.1	80.0 (64.4%)	98.7	91.2 (68.7%)
Wdbc	95.3	96.1	93.7 (53.2%)	86.5	95.1 (95.2%)
Average	79.1	80.2	78.6 (33.3%)	78.0	79.9 (67.4%)

A two-sample Student's *t*-test indicates that the proposed model is competitive.

Texture Recognition Problem: KTHTIPS-2b Database

- This challenging database comprises textures that are grouped according to the respective material such as wood, linen, cork, etc.
- We followed the recommendations of the literature and used one sample for training and three samples for testing.

Figure: KTHTIPS-2b exemplar images: one image for each material.

Comparison with Approaches from the Recent Literature

We employed deep convolutional neural networks (CNN) to extract features and the proposed Θ-FAM to classify them.

Table: Classification rates and standard deviations for KTHTIPS-2b dataset.

Method	Accuracy (%)
VZ-Joint	53.3
ELBP	58.1
OBIFs INNC	66.26
SIFT + IFV	69.3±1.0
FV-CNN VGGVD	81.8±2.5
CNN + FC	75.4±1.5
CNN + LDA	73.9±1.4
CNN + SVM	71.7±5.9
CNN + Proposed Θ-FAM	72.9±2.5

Concluding Remarks and Future Work

- We presented a method for designing a Θ-FAM model based on a subsethood measure on a bounded lattice.
- This method automatically determines a set of fundamental memories (associations) and hidden layer functions Θ^{ξ}.
- Despite a significant reduction in the number of fundamental memories and Θ^{ξ}s, the classification performance in some benchmark classification problems was not affected.
- The combination of this approach with a deep convolutional network yielded satisfactory results in a challenging image texture classification problem.

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Thanks for your interest! Any questions?