Sparse Point Representation for Functions and Derivatives

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Main aspects of the SPR method

The SPR (*Sparse Point Representation*) method is an adaptive finite difference strategy introduced in [1] to calculate numerical solutions of partial differential equations constrained by initial and boundary conditions. The basic principle is to use wavelet coefficients as regularity indicators to form the SPR adaptive grids. Typically, few points are found in each time step, the grid being coarse in smooth regions, and refined close to irregularities. In the SPR method the partial derivatives are approximated by uniform finite differences. Eventually, required neighboring stencils are not present in the grid. In such case, the correspondig point values are approximated from coarser scales using an interpolating subdivision scheme. Our purpose is to present a consistency analysis of the SPR method. Using cubic interpolation subdivision scheme and fourth order finite difference operators, we obtain some conditions under which the SPR approximations of derivatives using (a) the step size of the finest scale level, and (b) each point local scale coincide. Therefore, the amount of interpolations may be reduced by chosing the step size proportional to each point local scale without affecting the accuracy.

Basic principles in wavelet analysis

- Information is organized in a hierarchy of scale levels:
$\mathcal{X}^\ell = \{(x_k^\ell, y_m^\ell) = (kh_x^\ell, mh_y^\ell) \in I \times I\}$
$\mathcal{X}^\ell \subset \mathcal{X}^{\ell+1}$: $(x_k^\ell, y_m^\ell) \in \mathcal{X}^\ell \Rightarrow (x_{2k}^{\ell+1}, y_{2m}^{\ell+1}) \in \mathcal{X}^{\ell+1}$.
- The samples $\mathbf{f}^j$ at the finest scale level is obtained from the samples the coarsest level $\mathbf{f}^0$ and the details $\mathbf{d}^\ell$ between two consecutive levels (wavelet coefficients).
- The multilevel transformations are fast and stable

$$\mathbf{f}^j \overset{1:1}{\longleftrightarrow} \{\mathbf{f}^0, \mathbf{d}^0, \cdots \mathbf{d}^{j-1}\}$$

- Wavelet coefficients are polynomial interpolation errors at the new points in $\Lambda^\ell = \mathcal{X}^{\ell+1} \setminus \mathcal{X}^\ell$ using the closest samples from $\mathcal{X}^\ell$.

$$d_{k,m}^{(1)\,\ell} = f_{2k,2m+1}^{\ell+1} - \tilde{f}_{2k,2m+1}^{\ell+1}$$

$$d_{k,m}^{(2)\,\ell} = f_{2k+1,2m}^{\ell+1} - \tilde{f}_{2k+1,2m}^{\ell+1}$$

$$d_{k,m}^{(3)\,\ell} = f_{2k+1,2m+1}^{\ell+1} - \tilde{f}_{2k+1,2m+1}^{\ell+1}$$

They are significant in the presence of strong gradients and insignificant in smooth regions. Thus, they can be used as local regularity indicators.

- Only the points corresponding to significant wavelet coefficients $|d_{k,m}^{(\alpha)}| > \epsilon$ are kept in the adaptive SPR grids.

SPR discretization of derivatives

The focus is on dicretizations of the partial derivative $L^x f = \frac{\partial}{\partial x} f$. The analysis for $L^y f$ is similar. Consider a central difference operator in uniform grids, with coefficients $\beta(k)$. Given the representation of a function in a SPR grid $\Gamma \subset \mathcal{X}^j$, we shall consider two schemes for $L^x f(\mu)$:

1. FD discretization using the step size h_x^j of the highest level:

$$(L_{SPR}^x f)(\mu) = \frac{1}{h_x^j} \sum_k \tilde{f}(\mu_x - kh_x^j, \mu_y)\beta(k). \qquad (1)$$

2. FD discretization using as step size the point local scale in the x direction $a_{x,\mu}$

$$(L_{SPR}^x f)(\mu) = \frac{1}{a_{x,\mu}} \sum_k \tilde{f}(\mu_x - ka_{x,\mu}, \mu_y)\beta(k). \qquad (2)$$

In both cases, $\tilde{f}$ represents interpolated values of possible missing stencils.

In terms of operation count, the formulation (2) is more economic than (1) since, for those points $\mu \in \Gamma$ having big local scales $a_{x,\mu} > h_x^j$, it does not require the calculation of the stencils up to the most refined level.
For the definition of the directional local scale, one possibility is to take

$$a_{x,\mu} = \min\{|\nu_x - \mu_x|, \nu \in \Gamma, \nu \neq \mu\}. \qquad (3)$$

We shall also consider the directional local scale that takes into account the point level if $\mu \in \Lambda^\ell$:

$$a_{x,\mu} = \min\{h_x^\ell, |\nu_x - \mu_x|, \nu \in \Gamma, \nu \neq \mu.\} \qquad (4)$$

Tree condition

To study the difference in such SPR discretizations of derivatives, we consider SPR grids satisfying the tree condition, such that:

- a) If a point $\mu \in \Lambda^\ell$ is of the form $\mu = ((2k+1)h_x^{\ell+1}, 2mh_y^{\ell+1})$, then the two neighbors in X^ℓ are in Γ, as indicated in Figure 1, left.
- b) If a point $\mu \in \Lambda^\ell$ is of the form $\mu = (2kh_x^{\ell+1}, (2m+1)h_y^{\ell+1})$, then the two neighbors in X^ℓ are in Γ, as indicated in Figure 1, center.
- c) If a point $\mu \in \Lambda^\ell$ is of the form $\mu = ((2k+1)h_x^{\ell+1}, (2m+1)h_y^{\ell+1})$, then its four closest neighbors in X^ℓ are in Γ, as indicated in Figure 1, right .

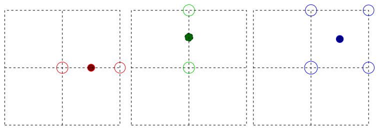

Figure 1: Neighbour points (circles) of μ (bullet)

Theorem: *Consider that Γ satisfies the tree condition, and that the directional local scale is defined as in (4). Using cubic interpolating subdivision scheme and the standard fourth order FD operator ($\beta(1) = 2/3$, $\beta(2) = -1/12$, $\beta(n) = 0$, $n > 2$, and $\beta(k) = \beta(-k)$), then both SPR formulations (1) and (2) coincide: $(L_{SPR}^x f)(\mu) = (L_{SPRa}^x f(\mu), \mu \in \Gamma$.*

Numerical Results

As illustrating example, we consider magnetic fields obtained by solving Maxwell's equations with the FD scheme in the finest uniform grid, as shown in Figure 2. To form the SPR representation of such fields, the wavelet coefficients are compared with the truncation parameter $\epsilon = 10^{-4}$. It can be observed that the SPR grids are non uniformly sampled, tracking the evolution of the fields.
Next tables contain information of the error of the SPR derivatives $|(L_{SPR}^x f)(\mu) - (L_{SPRa}^x f(\mu)|$. In the first table, the error is computed after the enforcement of the grid tree structure. As expected, the error when using the directional local scale (4) is practicaly zero, in accordance with the statement of the Theorem. However, the error is significant when using the directional local scale (3).

time	error using (3)	error using (4)	% of used points
t_1	2.07294×10^{-1}	15.6880×10^{-17}	4.669
t_2	1.36269×10^{-1}	335.240×10^{-17}	9.429
t_3	0.93914×10^{-1}	366.373×10^{-17}	16.568
t_4	1.85313×10^{-1}	277.572×10^{-17}	23.346

time	error using (3)	error using (4)	% of used points
t_1	1.38777×10^{-17}	1.38777×10^{-17}	8.924
t_2	2.77555×10^{-17}	2.77555×10^{-17}	18.737
t_3	49.6004×10^{-17}	49.9601×10^{-17}	33.363
t_4	58.2867×10^{-17}	58.2867×10^{-17}	49.522

For adaptive SPR simulations [2], since the regions of smoothness of the solution may change with time, before evolving the fields in time, the SPR representation of the solution should be extended to a refined grid $\hat{\Gamma}$ by the inclusion of new interpolated point values in a security zone.

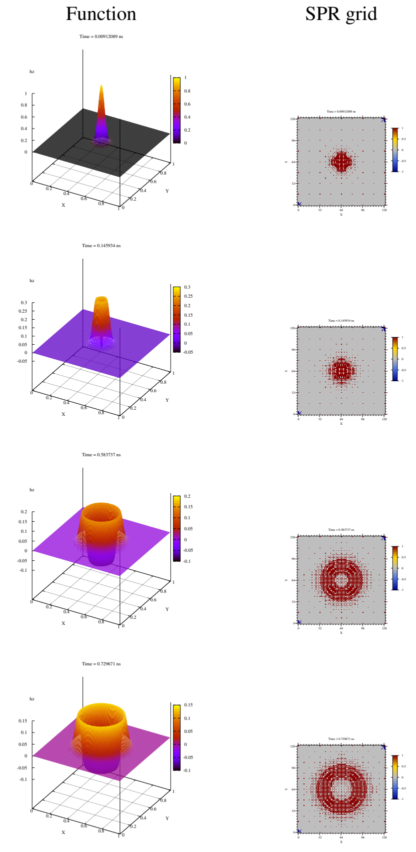

Figure 2: Time evolution of a magnetic field

We consider a refinement procedure that, for each point ν in Γ corresponding to a significant wavelet coefficient, its closest eight neighbors in the current and in the next finer level are included in $\hat{\Gamma}$. It is interesting to observe in the second table that, eventhought the extended grid may not satisfy the tree structure condition, the errors in the SPR discretization of the x partial derivative is practicaly zero, independently of the kind of directional local scale used as step size.

Conclusions and extensions

To preserve accuracy in SPR discretization of derivatives, the grid stucture and the form of the directonal local scale are important. To pove the Theorem, the grid tree structure guaranties that for $\mu \in \Gamma$ the directional local scale (4) satisfies $a_{x,\mu} = h_r^q$ for some $q \leq j$. Furthermore, in such case, $\mu \in \mathcal{X}^q$. To explain the results in the second table, our conjecture is that the structure of the extended grids $\hat{\Gamma}$ is also sufficient to hold such properties for both directional local scales (3) and (4), without enforcing the tree sctructure.

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