

Solving Maxwell's Equations using Interpolating Wavelets

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Abstract— This paper describes the use of interpolating wavelets for the discretization of Maxwell equations. Because these wavelets interpolate, it is possible to maintain the physical space representation, now based on nonuniformly sampled fields, and to handle boundary conditions in a simple way. Furthermore, the computation of products and sums is not difficult, and approximations to the derivatives can be obtained based on finite differences. Maxwell equations can be discretized using two meshes, corresponding to the electric and magnetic fields. The meshes can be either staggered or non-staggered, and adapt in space and in time to the nature of the fields. A comparison of the performance of the staggered and non-staggered meshes is presented, and conclusions are obtained as far as dispersion is concerned. In order to show the feasibility of the method, 1D example and comparisons with FDTD are described.

I. INTRODUCTION

Computational electromagnetics is a very active area, depending on a rich variety of numerical methods. One of the most often used is FDTD (Finite Difference Time-Domain), in which the electric and magnetic fields are described by vectors, and both time and space are discretized. In fact, each component of the electric and magnetic field vectors describe its value in a particular cell in space, and at a given time. The partial derivatives that occur in the Maxwell's equations are approximated by centered finite differences.

FDTD is of course related to the physical space representations of the fields. In physical space, differentiation and multiplication are straightforward, and boundary conditions can be enforced in a simple way. These are important advantages, but disadvantages also exist: the vectors are dense and one is forced to use uniformly dense grids that might oversample in certain points of space, leading to long computation times.

However, the fields may also be represented in the context of multiresolution analysis, leading to the MRTD methods (Multi-Resolution Time-Domain). The method known as S-MRTD is characterized by using scaling functions only, whereas W-MRTD uses both scaling functions and wavelets.

These techniques have been considerably developed, by several authors, mainly due to their advantages over FDTD, leading to a variety of specific methods that explore distinct

families of wavelets, such as Battle-Lemarié [1]; [2]; [3]; Daubechies [4]; [5] and Haar [6]; [7].

These methods tend to improve the dispersion when compared with FDTD, allowing discretizations at densities close to the Nyquist density, therefore cutting down the computational requirements and improving the performance.

However, when using wavelets, boundary conditions can be difficult to enforce. This is the case when dealing with wavelets of infinite support, but even compactly supported wavelets lead to similar problems, since the length of the support usually exceeds the mesh width. This means that the boundary condition can only be satisfied by a superposition of basis functions, rather than by a single one. It is possible to design wavelets suited to certain given boundary conditions, but as a rule this leads to a orthonormal set containing boundary functions, and renders the computation of the derivative matrix more complex.

For these reasons, in this paper we explore second-generation, biorthogonal interpolatory wavelets. The inner product of a basis function that interpolates with a given data function can be obtained by evaluating the data function at a certain point. As a result, the representation in terms of that basis function and the physical space representation coincide. In other words, and in the case of electromagnetic problems, the transform coefficients are samples of the fields. The coefficients can be obtained in a recursive fashion, leading to fast algorithms that explore the underlying multiresolution representation, and allow for sparse representations (that is, representations in which the number of coefficients N_s is substantially smaller than the original number of data samples, N).

The computation of nonlinear functions of the data (such as products and squares, for example) are not straightforward with conventional wavelet representations, but are quite easily obtained with interpolatory wavelets, since the coefficients are samples of the fields. For the same reasons, enforcing boundary conditions is a simple matter.

To deal with differential equations a method for generating approximations to the derivative of the data is needed. The transform coefficients are samples, but sparse, and in general nonuniformly distributed. Nevertheless, it is possible to

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compute the derivative starting from the sparse set of points, a crucial factor to speed up the computations.

The methods used lead to nonuniformly sampled data, the density of which is a function of the rate of variation of the fields. If the field is slowly-varying in a region of space, but fast-varying in another region, the decomposition procedure will lead to comparatively few samples in the former region, and to more samples in the latter.

II. MULTIREOLUTION ANALYSIS AND WAVELETS FOR PDE'S

Let $f \in V_J$. Then we have

$$f = \sum_k c_{J,k} \phi_{J,k} \quad (1)$$

with

$$c_{J,k} = \langle f, \phi_{J,k} \rangle$$

Such representations in terms of dilated and translated ϕ are useful, but usually for sparse representations we need both scaling functions and wavelets. Introducing the subspaces W_J such that $V_J = V_{J-1} \oplus W_{J-1}$ leads to the expression

$$V_J = V_0 \oplus W_0 \oplus W_1 \cdots \oplus W_{J-2} \oplus W_{J-1} \quad (2)$$

and to the related scaling function and wavelet representation:

$$f = \sum_k c_{0,k} \phi_{0,k} + \sum_{j=0}^{J-1} \sum_k d_{j,k} \psi_{j,k} \quad (3)$$

where, of course,

$$c_{0,k} = \langle f, \phi_{0,k} \rangle, \quad d_{j,k} = \langle f, \psi_{j,k} \rangle$$

The projection of f at a given resolution level can be regarded as arising from the projection at a coarsest resolution level, corrected by a term that describes the details, that is, the information missing at the coarsest level. With interpolatory wavelets, the detail coefficients $d_{j,k}$ are given by the difference between the values of the data and the predicted (interpolated) values. To obtain data compression and sparse representations, the value of $d_{j,k}$ is compared with a threshold ε . If it is smaller than the threshold, it is discarded. Data that can be efficiently predicted from coarsest level, that is, require few detail coefficients, are data that can be efficiently represented by this scheme. In that case we end up with N_s samples, out of the initial N , with $N_s \ll N$.

The thresholding and the nonuniformly spaced data to which it leads can be repeated in time, as the fields evolve, leading to an adaptive nonuniform grid, that adapts both in space and in time to the fields. Our aim is to keep the size of this grid, N_s , small when compared with N , and perform all the computations necessary to track the fields (by discretizing the Maxwell's equations) in a time proportional to N_s , rather than N .

To compute the sum of two functions expressed by two sparse representations, involving independent grids, it is necessary to interpolate both representations to a common

grid. The union of the two original grids is perhaps the most natural candidate. The derivatives can be computed by applying a centered stencil to each sparse representation sample, with an adequate mesh size, and if necessary interpolating any missing points from the coarsest levels, recursively. Boundary conditions are handled just as in the case of uniform (dense) grids.

III. SOLVING MAXWELL'S EQUATIONS

Consider a TM wave propagating along the x axis, in a lossless medium. Maxwell's equations, assuming that $H_x=0$ at $t=0$, are

$$\begin{aligned} \frac{\partial H}{\partial t} &= \frac{1}{\mu} \left(\frac{\partial E}{\partial x} \right) \\ \frac{\partial E}{\partial t} &= \frac{1}{\varepsilon} \left(\frac{\partial H}{\partial x} \right) \end{aligned} \quad (6)$$

where ε is the permittivity and μ is the permeability of the medium. To discretize the equations, one starts by representing the electric and magnetic fields by vectors, which can be thought of as field samples on a spatial grid. The grid may be common to the electric and magnetic fields, but this is not strictly necessary: It is possible to adopt independent grids for each field (for example, two staggered grids, as in the FDTD case). When dealing with adaptive grids, the grids are, as a rule, non-staggered.

A. Dispersion

To avoid dispersion, the phase constant ξ must be a linear function of ω , the angular frequency. However, the numerical methods lead to dispersion (nonlinear dependency in ω) the degree of which depends on the model adopted.

In Figure 1 we plot the dispersion curves $\xi(\omega)$, for both staggered and non-staggered grids. It clearly shows that, from the viewpoint of dispersion, the staggered grid is preferable.

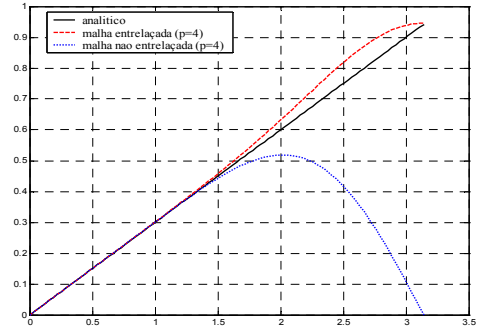


Fig. 1. The dispersion curve for numerical simulation using staggered and non-staggered grids.

B. An example

In order to demonstrate the advantages of interpolating wavelets for the solution of Maxwell's equations, we considered an example involving the inhomogeneous

medium illustrated in Figure 2. It is in fact an otherwise homogeneous material, in which two segments of a different media have been embedded.

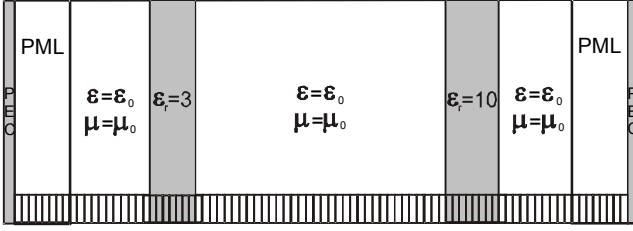


Fig. 2. The inhomogeneous medium used for the numerical simulations. Note the two sections made out of different media.

The fields at $t=0$ are assumed to be

$$E_z(x) = e^{-150(x-\frac{1}{2})^2}, \quad H_y(x) = 0 \quad (7)$$

The initial grids have N cells. The algorithm starts by converting the fields at $t=0$ to sparse representations, using recursive interpolation and thresholding, leading to N_s nonuniformly sampled points. To make the comparison with FDTD easier the grids used were staggered. The results obtained are shown in Figure 3. The plots refer to the electric field at five different time instants: 11, 768, 1067, 1569 and 2392 ps.

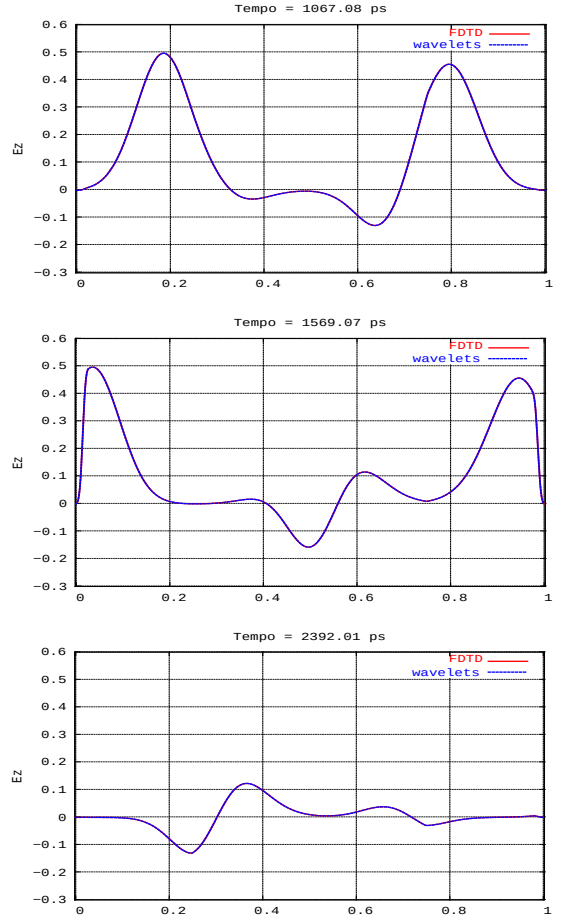
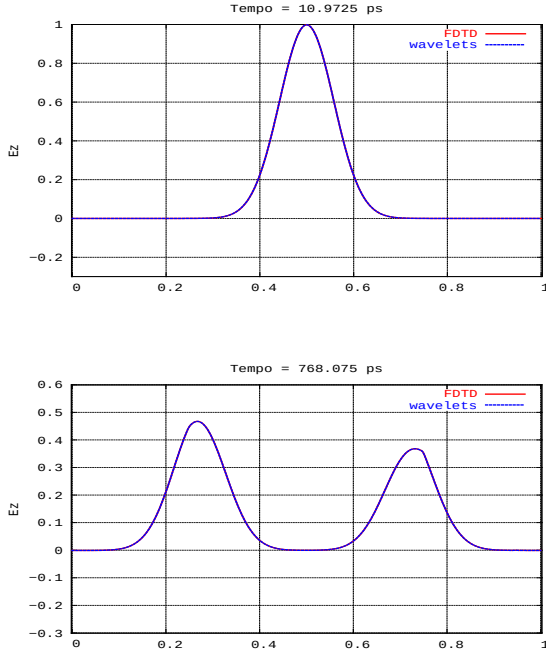


Fig. 3. The results of the numerical simulations, using FDTD and interpolating wavelets and thresholding over staggered grids, for several values of time.

The electric field distribution was obtained with a threshold $\varepsilon=10^{-6}$ and the magnetic field distribution was obtained with a threshold $\varepsilon=10^{-9}$.

The fields predicted by the two methods were virtually identical, over space and time.

As time increases and the fields evolve, the number of samples N_s kept in the sparse representations varies. Figure 4 demonstrates this. It shows the ratio N_s/N as a percentage, for both the electric and magnetic fields.

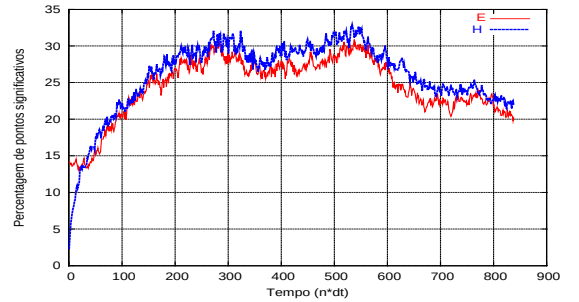


Fig. 4. The evolution of N_s , the number of samples used, as a function of time, expressed as a percentage of the total number of points in the grid, N .

IV. CONCLUSIONS

This paper describes an application of interpolating wavelets and recursive interpolation schemes with thresholding, aiming at the representation of the electric and magnetic fields in nonuniform, adaptive grids. Applied to Maxwell's equations, the method leads to sparse grids that adapt in space to the local smoothness of the fields, and at the same time track the evolution of the fields over time. In general, the number of points in the grids, N_s , is well below the maximum number of points, N . It is possible to control N_s , trading off representation accuracy by data compression, and therefore speed. When $N_s \ll N$ there are substantial gains in memory and speed, two important advantages over FDTD, which deals with dense grids that as a rule oversample the fields in space.

The representation in terms of interpolating wavelets leads to a transform in which the coefficients are nonuniformly distributed field samples. This tends to increase the complexity of the code that computes the derivatives, which can nevertheless be made to work in time proportional to N_s . However, at the same time, it renders the implementation of the boundary conditions quite straightforward. Also, the implementation of operations such as products and squares becomes much simpler than when using compactly supported, first generation wavelets.

Implementations based on staggered and non-staggered grids were made, and tested from the viewpoint of dispersion. The results show that the staggered grids, from this perspective, are clearly preferable. The comparisons made with FDTD using staggered adaptive grids show that similar results can be obtained using interpolating wavelets when only a fraction N_s/N of the grid points are used, confirming the memory and computational savings. The fields predicted by the two methods were virtually identical, over space and time.

The use of interpolating wavelets, and the associated sparse representations and adaptive grids, appear to have a very good potential in reference to computational electromagnetics in general and the solution of Maxwell's equations in particular.

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