

Adaptive Method in Space for the Resolution of Maxwell's Equations

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Abstract— In this paper a scheme to obtain an adaptive method in space for the resolution of Maxwell's equations is presented. Using interpolating wavelets it is possible to obtain an adaptive grid allowing a reduction of the computation time and an economy of the computational resources. Using the staggered grid model the dispersion proprieties are improved with the increase of the interpolating polynomial, despite of a more restricted stability factor.

Index Terms — Multiresolution, FDTD, interpolating wavelets, numerical dispersion, numerical stability.

I. INTRODUCTION

The computation electromagnetic became in the last years an extremely powerful tool and one important technique in the resolution of the most diverse electromagnetic problems. There are a variety of methods and algorithms for the resolution of a particular equation, some in the frequency domain and others in the time domain. One of the most often used is FDTD (*Finite Difference Time-Domain*) [1], in which the electric and magnetic fields are described by vectors, both discretized in time and space. In fact, each component of the electric and magnetic field vectors describe its value in a particular cell in space, at a given time. The partial derivatives that occur in the Maxwell's equations are approximated by centered finite differences.

FDTD is of course related to the physical space representations of the fields. In physical space, the fields are discretized and represented as vector, in a grid usually uniform. Differentiation and multiplication are straightforward, and boundary conditions can be enforced in a simple way. These are important advantages, but disadvantages also exist: the vectors are dense and one is forced to use uniformly dense grids that might oversample in certain regions of space, leading to long computation times. The main disadvantage of this representation is therefore the possible increase of the computation time to simulate such grid.

For this reason, it is important to study adaptive numerical methods that use a refined grid only in certain regions of the space, and a less refined grid in other regions where the variation of the fields is smoother. The objective is to obtain an adaptive mesh as a function of time that allows to an

economy of resources and a relatively short or acceptable time of simulation.

The method used to obtain an adaptive grid is based on the wavelet analysis using the second generation wavelets (biorthogonal interpolating wavelets) [2].

When this type of wavelets is used the physical representation and the scaling representation are essentially the same. The scaling coefficients and the sampling values coincide. Since the coefficients are directly obtained from the physical representation it is possible to implement non-linear operators (such as products and squares) in an easy way. The implementation of the derivative is more complex and takes advantage of the reduction of the dimensionality allowed by the multiresolution representation.

In the case of electromagnetic problems, the coefficients are samples of the fields. In a multilevel setting, a wavelet basis function is also interpolatory and it is associated to one of the new points between two consecutive levels. The corresponding wavelet coefficient is then given by the difference between the exact sample value at such point and its approximation, obtained by interpolation from samples at the coarser level. These coefficients can be obtained in a recursive fashion, leading to fast algorithms for the transformation between one-level sample values and the multilevel representation.

In this method, the mesh structure presents a heterogeneous composition: sparse in smoothness regions and dense in regions of accented variation. Using this technique it is possible to use a number N_s of points, less than the N points of the original representation. This leads to the so called sparse point representation (SPR) [3].

The basic principle is to represent the function only through those points values indicated by the significant wavelet coefficients. Typically, a SPR representation has a number N_s of samples which is significantly smaller than the original number of data samples N . The grid is therefore coarse on smooth regions and refined close to irregularities. Figure 1 shows the representation of a function using the SPR concept. It also shows the representation of the samples for the several levels.

In the present paper we demonstrate some of the potential of interpolating wavelets for the resolution of Maxwell equations, through the results of a numerical simulation in one dimension.

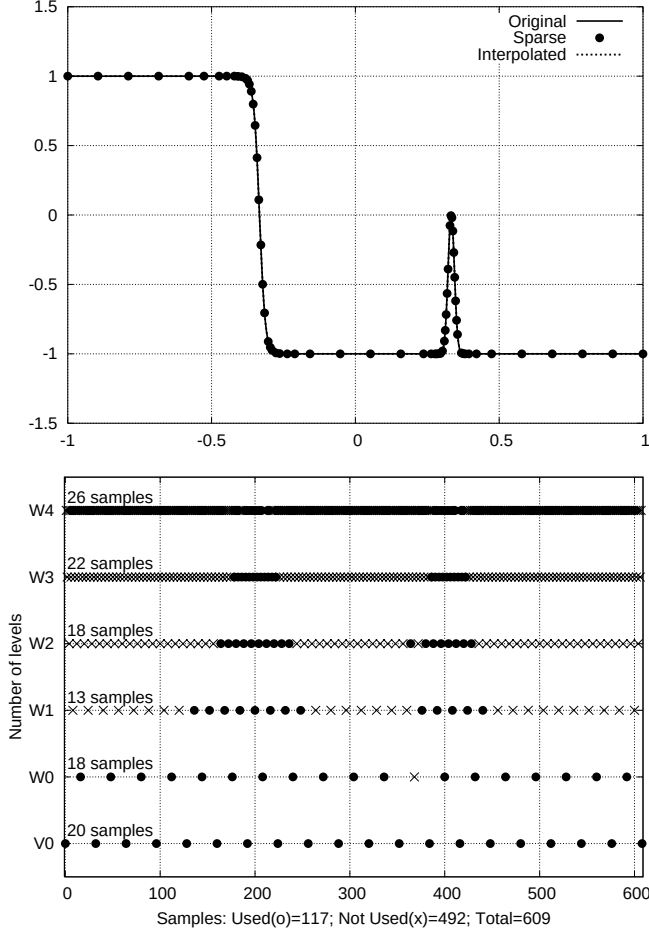


Fig 1 Sparse representation of a generic function

II. MULTIREOLUTION REPRESENTATION OF A FUNCTION

We call scaling representation of $f \in V_J$ to the representation of the form

$$f(x) = \sum_k c_{J,k} \phi_{J,k}(x) \quad (1)$$

$$c_{J,k} = \langle f, \phi_{J,k} \rangle$$

The scaling functions $\phi_{J,k}$ are such that $\phi_{J,k}(m2^{-J}) = \delta_{k,m}$. Therefore, the coefficients are given by node values $c_{J,k} = f(k2^{-J})$.

Introducing the subspaces W_j such that $V_J = V_{J-1} \oplus W_{J-1}$ leads to the expression: $V_J = V_0 \oplus W_0 \oplus W_1 \cdots \oplus W_{J-1}$ and produces the following wavelet representation:

$$f(x) = \sum_k c_{0,k} \phi_{0,k}(x) + \sum_{j=0}^{J-1} \sum_k d_{j,k} \psi_{j,k}(x) \quad (2)$$

where:

$$c_{0,k} = \langle f, \phi_{0,k} \rangle \text{ e } d_{j,k} = \langle f, \psi_{j,k} \rangle$$

Now $c_{0,k} = f(k)$ and $\psi_{j,k}(x) = \phi_{j+1,2k+1}(x)$ are the

wavelets associated to new points $x_{2k+1}^{j+1} = (2k+1)2^{-j-1}$.

The projection of f at a given resolution level can be regarded as arising from the projection at a coarsest resolution level, corrected by a term that describes the details, that is, the information missing at the coarsest level. With interpolatory wavelets, the detail coefficients $d_{j,k}$ are given by the difference between the values of the data and the predicted (interpolated) values. To obtain data compression and sparse representations, the value of $d_{j,k}$ is compared with a threshold $\mathcal{E}$. By this method, we end up with N_s samples, associated with the significant wavelet coefficients, with $N_s \ll N$.

One of the motivations for the use of wavelets for solving PDEs is its compression capability. For many types of signals, a small fraction of the coefficients

$$c_{0,k}, d_{0,k}, d_{1,k}, \dots, d_{J-1,k}$$

is enough to achieve a good approximation to the function $f(x)$. It is acceptable to assume that this sparse representation could lead to efficient algorithms in terms of computer memory and the number of arithmetic operations. In order to illustrate the application of interpolating wavelets, Fig. 1 shows a generic function and its sparse representation.

III. MAXWELL EQUATIONS IN 1D

Here we describe a 1-D example in order to demonstrate the method described above. Let us consider a TEM wave propagating along the x axis, in the interface of two homogeneous medium with different characteristic impedance Z . The electric field is directed along the z axis and the magnetic field is directed along the y axis being equal to zero at the initial instant. The equations to be solved are:

$$\frac{\partial E}{\partial x} - \mu \frac{\partial H}{\partial t} = 0 \quad ; \quad \frac{\partial H}{\partial x} - \varepsilon \frac{\partial E}{\partial t} = 0 \quad (3)$$

where ε is the permittivity and μ is the permeability of the medium.

The solutions of (3) are obtained using interpolating wavelets with an initial grid of N points.

At $t=0$, the initial fields distributions are replaced by their sparse representations, i. e., only a number $N_s < N$ of the initial grid remain, since all the others can be obtained from these N_s points by interpolation with a predefined error $\mathcal{E}$.

Each component of the field has one independent grid, as in the FDTD case (two staggered grids). However, the grid may be common for both the electric and magnetic fields. When dealing with adaptive methods, the grids are usually non-staggered. In the present paper, the results are for an adaptive scheme designed for staggered grids because we want to compare the performance of this method with the FDTD.

A. Staggered grid model

As in the classic FDTD scheme, we use Yee's mesh for the discretization in space. This corresponds to take the following samples of the components of H and E :

$$E_k^n \approx E(k\Delta x, n\Delta t)$$

$$H_{k+1/2}^{n+1/2} \approx H((k+1/2)\Delta x, (n+1/2)\Delta t)$$

Based on these discrete values, approximations for the components of E and H at any point are defined by the following interpolation operators,

$$E(x, n\Delta t) \approx \sum_k E_k^n \phi(\Delta x^{-1}x - k)$$

$$H((x+1/2), (n+1/2)\Delta t) \approx \sum_k E_{k+1/2}^{n+1/2} \phi(\Delta x^{-1}x - k - 1/2)$$

The basic function $\phi(x)$ is chosen in the family of Dubuc-Delauniers scaling functions [4], which are the basis for the construction of interpolating multi resolution analyses. These scaling functions are identified by a parameter p associated to the interpolating subdivision schemes, based on central Lagrange polynomial interpolation of degree $2p-1$, $p \geq 1$.

Based on the interpolation of E and H, we consider the following approximations:

$$\frac{\partial E}{\partial x}(j\Delta x, n\Delta t) \approx \frac{1}{\Delta x} \sum_k E_k^n \phi'(j-k-1/2)$$

$$= \frac{1}{\Delta x} \sum_k E_k^n \alpha(j-k)$$

$$\frac{\partial H}{\partial x}((j+1/2)\Delta x, (n+1/2)\Delta t) \approx \frac{1}{\Delta x} \sum_k H_{k+1/2}^{n+1/2} \phi'(j-k+1/2)$$

$$= \frac{1}{\Delta x} \sum_k H_{k+1/2}^{n+1/2} \alpha(j-k)$$

where $\alpha(u) = \phi'(u-1/2)$

Combining the last expressions and, considering for the temporal derivative the central difference formula, we obtain the following discrete equation system:

$$E_j^{n+1} = E_j^n + \frac{\Delta t}{\varepsilon \Delta x} \sum_k H_{k+1/2}^{n+1/2} \alpha(j-k+1)$$

$$H_{j+1/2}^{n+1/2} = E_{j+1/2}^{n-1/2} + \frac{\Delta t}{\mu \Delta x} \sum_k E_k^n \alpha(j-k)$$

The next table show the coefficients $\alpha(u)$ for $p=1$ to $p=4$.

k	p=1	p=2	p=3	p=4
1	1	472/384	1.29181	1.3110340773
2		-36/384	-0.137134	-0.1560100710
3		4/384	0.0287618	0.0419957460
4			-0.00347014	-0.0086543236
5			8.02654e-6	0.0008308695
6				0.0000108999
7				-0.0000000041

B. Dispersion

To avoid dispersion, the phase constant ξ must be a linear function of the angular frequency ω . However, the numerical methods lead to dispersion (non-linear dependency in ω) the

level of which depends on the adopted model.

Figure 2 shows the dispersion curves $\xi(\omega)$, for the staggered grid model. It clearly shows that, from the viewpoint of dispersion, when the order of model increases the dispersion relation is better.

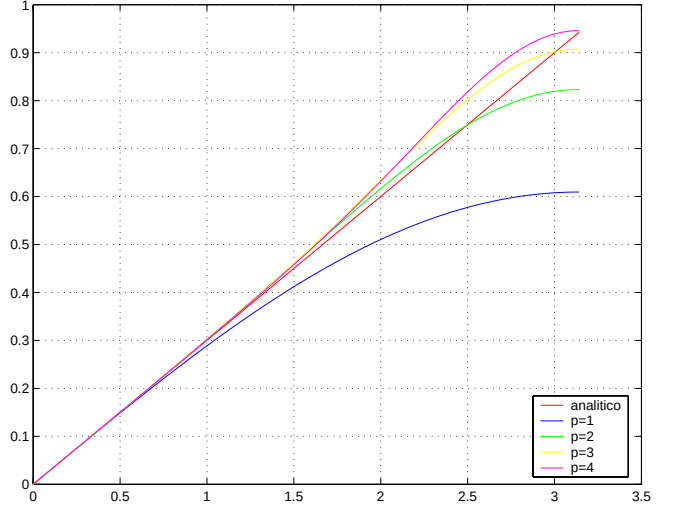


Fig 2 Dispersion properties of the model

C. Stability

The stability condition is also obtained for different interpolation degrees, and it is possible to prove that in staggered grids the CFL numbers are smaller for higher orders of the interpolation scheme as we can see in the next table.

Table 1 CFL numbers for the staggered grid model

Model	CFL
P=1	1
P=2	0.75
P=3	0.6843
P=4	0.6585

In order to demonstrate the advantages of interpolating wavelets for the solution of Maxwell's equations, we considered an example involving the inhomogeneous medium illustrated in Figure 3.

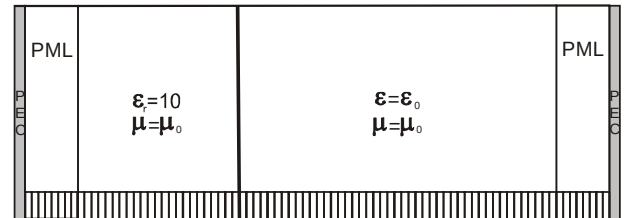


Fig 3 Propagation in an inhomogeneous medium

D. Simulated results

The algorithm starts by converting the fields at $t=0$ to sparse representations, on staggered grids, using recursive cubic polynomial interpolation and thresholding, leading to N_s

nonuniformly sampled points. For the discretization of the spatial derivatives, at each point of the sparse grids, a fourth order finite difference scheme is applied with uniform step size 2^{-J} corresponding to the finest level. If some required stencil point is not present, it is obtained by using the recursive interpolation scheme. For comparison, we also present results for the FDTD scheme on the staggered uniform grids of the finest level corresponding to $J = 5$.

The results obtained are shown in Figure 4. The plots refer to the electric field at 3 different time instants.

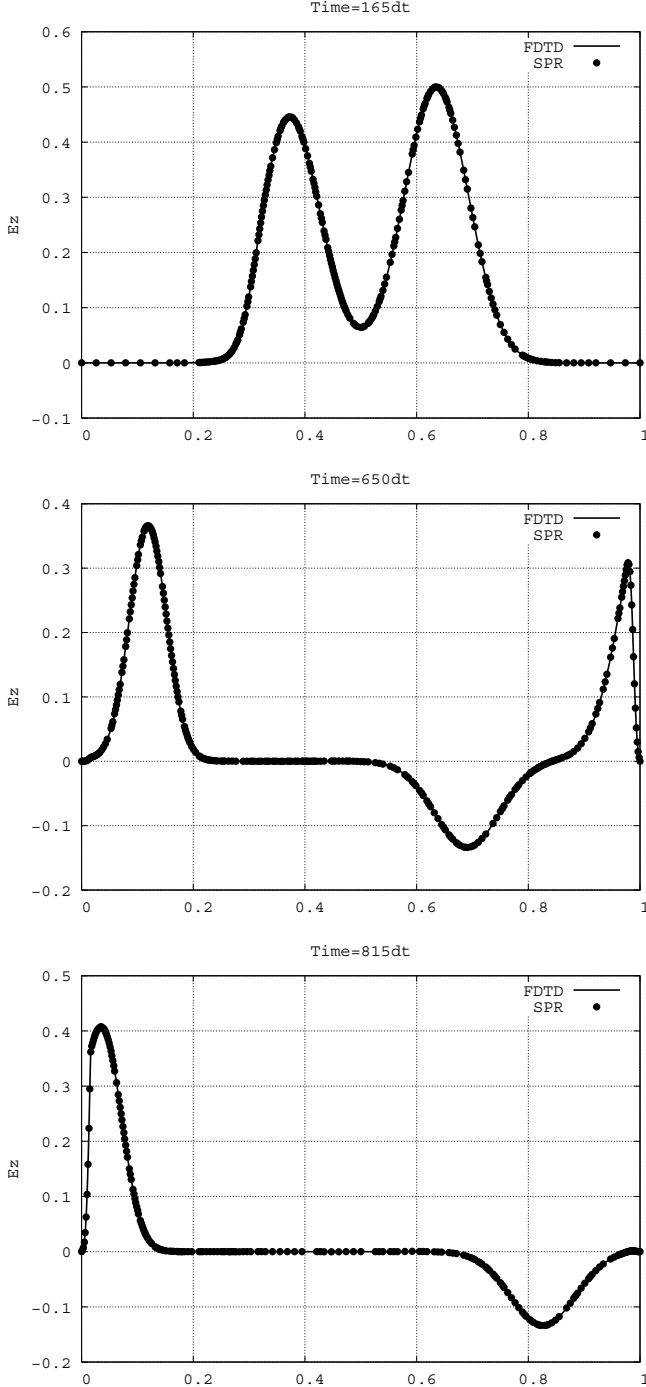


Figure 4 Time evolution of SPR representation

IV. CONCLUSIONS

This paper describes an application of interpolating wavelets and recursive interpolation schemes with thresholding, aiming the representation of the electric and magnetic fields in nonuniform, adaptive grids. Applied to Maxwell's equations, the method leads to sparse grids that adapt in space to the local smoothness of the fields, and at the same time track the evolution of the fields over time. In general, the number of points in the grids, N_s , is well below the maximum number of points, N . It is possible to control N_s , by trading off representation accuracy and data compression, and therefore speed. When $N_s \ll N$ there are substantial gains in memory and speed, two important advantages over FDTD, which deals with dense grids that as a rule oversample the fields in space.

REFERENCES

- [1] A. Taflov. *Computational Electrodynamics: The Finite Difference Time-Domain Method*. Norwood, MA, 1996.
- [2] G. Beylkin, R. Coifman, e V. Rokhlin. Fast wavelet transforms and numerical algorithms. I *Commun. Pure Appl. Math.*, 44:141--183, 1991.
- [3] L. Jameson. A wavelet-optimized very high order adaptive grid and order numerical method. *SIAM J. Sci. Comput.*, 19(6):1980--2013, Nov 1998.
- [4] G. Deslauriers, S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.*, 5:49--68, 1989.