

High Order Finite-Difference Schemes for Maxwell's Equations with Adaptive Wavelet Grid Refinement

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Abstract

In the present paper we apply the Sparse-Point-Representation method to the solution of bidimensional Maxwell's equations. This method is formulated in two parts: operational and representation parts. In the operational part, as reference scheme, we consider a fourth order finite difference formula for the discretization of the electrical and magnetic fields in the same uniform grid. In the representation part, a local regularity indicator, defined by interpolatory wavelet coefficients, is used to represent the fields on dynamically sparse adaptive grids. We present a careful theoretical analysis of stability and numerical dispersion comparing the reference scheme with Yee's scheme. We observed that the fourth order scheme in non-staggered grids can be used with larger time steps to achieve a prescribed accuracy level with a certain number of cells per wavelength. The simulations of our adaptive scheme lead to sparse grids that adapt in space to the local smoothness of the fields, and, simultaneously, track the evolution of the fields over time. In our experiments, we also verify that, for a given accuracy, the adaptive scheme requires less computational effort than the reference scheme in uniform grids.

1 Introduction

Computational simulation of electromagnetic waves is important for a wide class of applications. For the numerical solution of such problems, Yee's Finite-Difference Time-Domain (FDTD) scheme has become a quite popular technique. Over the last decade, several FDTD schemes have been proposed to improve the performance of the original Yee's scheme. For instance, higher order difference schemes may be applied to reduce the number of cells per wavelength required to achieve a certain degree of accuracy [1]. However, in the presence of local irregularities, grid density needs to be increased, even for higher order methods. In this direction, this paper is concerned with high order finite difference schemes for Maxwell's equations combined with an wavelet adaptive strategy based on interpolating techniques. More precisely, our purpose is to apply the SPR method, Sparse-Point-Representation, for solving Maxwell's equations in 2D.

The SPR method is an adaptive finite difference strategy introduced in [2] to calculate numerical solutions of partial differential equations constrained by initial and boundary conditions. The method combines the simplicity and accuracy of traditional finite difference schemes with the ability of wavelet coefficients in the characterization of local regularity of functions. Typically, few points are found in each time step, the grid being coarse in smooth regions, and refined close to irregularities. In this sense, the method can be formulated as formed by two basic parts: the operational part, and the representation part. In the operational part, the background context is that of an accurate and stable finite difference scheme. In the representation part, there are wavelet tools for the organization of the discrete information of a function by levels of

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resolution. Inter-level transformations are defined by means of an interpolation subdivision scheme. At each level ℓ , the wavelet coefficients d^ℓ are defined in terms of local interpolation errors at the new points of the current level of discretization.

In the construction of the SPR grids, the idea is to use the wavelet coefficients as indicators of local smoothness of the solution. This is the main ingredient in the definition of the thresholding operator: only those points corresponding to wavelet coefficients of magnitude greater than a given threshold should be kept in the SPR grid. For simulations of partial differential equations, since the regions of smoothness or irregularities of the solution may change with time, before evolving the fields in time, the SPR representation of the solution should be extended (refinement) by the inclusion of new interpolated point values in a security zone. Then, for points in the extended grid, spatial derivatives are approximated by uniform finite differences, using step size proportional to each point directional local scale. It may occur that required neighboring stencils are not present in the grid. In such case, the corresponding point values are approximated from coarser scales using the interpolating subdivision scheme. Finally, a thresholding operation (coarsening) is applied, in order to eliminate those points that are unnecessary for an accurate representation of the solution at the new time level.

In a previous paper [3], two classes of SPR methods are described for the solution of Maxwell's equations in 1D. In one class, the operational part is performed by high order finite difference schemes on staggered grids in the time-space domain for the components of magnetic and electrical fields. In the second class, high order finite difference schemes are designed for grids that coincide for both fields. The main conclusions of the analysis in [3] are:

- The stability region decreases for increasing approximation orders, but it is less restrictive with non-staggered grids than with staggered ones.
- From the viewpoint of numerical dispersion, staggering the grids is helpful, specially for low order schemes and grid densities. However, as the grid density and order of accuracy increase, the effect becomes less noticeable.

- For accuracy comparison with respect of the grid-ding effect, it is also important to take into account the effect of the CFL (Courant-Friedrics-Lewy) parameter. It is important to evaluate and compare the CFL numbers required for the schemes to achieve a prescribed accuracy level with a certain number of cells per wavelength. With this respect, it is observed that high order schemes in non-staggered grids can be used with larger CFL parameters (i.e. larger time steps).
- The use of non-staggered grids facilitates the implementation of adaptive strategies.

Following these directions, our purpose is to analyze the performance of the SPR method in non-staggered grids for the solution of the bidimensional TE mode Maxwell's equations. For such problem, first we describe a fourth order (in space) finite difference model in non-staggered grids, which is used in the representation part of the SPR method. Detailed stability and dispersion analyzes, and comparisons with the FDTD scheme, are presented. Based on such reference model, we present some results of numerical simulations of the SPR scheme. In general, the number of points in the SPR grids is well below the maximum number of points in the finest scale level. It is possible to reduce the number of SPR points, trading off representation accuracy by data compression, and therefore speed. The comparison made with the finite difference scheme in uniform grids shows that the SPR method achieves a given accuracy with reduced computational effort.

The present paper is organized as follows. In Section 2, we recall the main wavelet tools required for the representation part of the SPR method. In Section 3, we describe the reference finite difference model and perform the stability and dispersion analyzes. In Section 4, we present some results of numerical simulations. Some concluding remarks are included in Section 5.

2 Wavelet tools

In a multiscale framework, functions are represented in different scale levels, and the main tools are appropriate transformations relating the information at the finest scale level j to the lower ones, and vice versa.

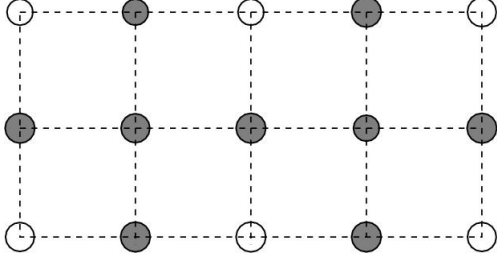


Figure 1: Dyadic Grid: $\circ$ - points in X^ℓ , $\bullet$ - new points in $X^{\ell+1} \setminus X^\ell$

In the context of point values, the interpolating multiresolution approach was introduced by D. Donoho [4]. The input data f^j contains the sample values at a grid in finest scale level, and after a transformation WT^j , the output contains the information f^0 in the coarsest level ℓ , and d^ℓ that keeps the details between a scale level ℓ and the next upper level $\ell + 1$. Schematically, we have

$$f^j \xrightarrow{WT^j} f_{MR}^j = (f^0, d^0 \dots d^{j-1}). \quad (1)$$

In a uniform setting, each scale level ℓ corresponds to a uniform grid X^ℓ . Starting from X^0 , the grid at the coarsest level with spacings Δ_x^0 in the x -direction and Δ_y^0 in the y direction, the hierarchy of uniform grids X^ℓ , $\ell \geq 1$ is defined by

$$X^\ell = \{\nu = (k\Delta_x^\ell, m\Delta_y^\ell), 0 \leq k \leq N_x^\ell, 0 \leq m \leq N_y^\ell\},$$

where $\Delta_x^\ell = \Delta_x^0 2^{-\ell}$ and $\Delta_y^\ell = \Delta_y^0 2^{-\ell}$. As indicated in Figure 1, its dyadic refinement $X^{\ell+1}$ is obtained by the inclusion of new midpoints

$$\begin{aligned} \nu_1 &= (2k\Delta_x^{\ell+1}, (2m+1)\Delta_y^{\ell+1}), \\ \nu_2 &= ((2k+1)\Delta_x^{\ell+1}, 2m\Delta_y^{\ell+1}), \\ \nu_3 &= ((2k+1)\Delta_x^{\ell+1}, (2m+1)\Delta_y^{\ell+1}). \end{aligned}$$

Let $f^j = (f_\nu^j)$ be the set of sample values at points $\nu = (k\Delta_x^j, m\Delta_y^j) \in X^j$. As described in the following algorithms, for the definition of WT^j and its inverse $(WT^j)^{-1}$, the main ingredient is an interpolating prediction operator $P_\ell^{\ell+1} : s^\ell \rightarrow s^{\ell+1}$, where s^ℓ and $s^{\ell+1}$ are sequences labeled according to the points $\nu = (k\Delta_x^\ell, m\Delta_y^\ell) \in X^\ell$ and $(k\Delta_x^{\ell+1}, m\Delta_y^{\ell+1}) \in X^{\ell+1}$, respectively.

Analysis - $f^j \xrightarrow{WT^j} (f^0, d^0, \dots, d^{j-1})$

- For $\ell = j-1, \dots, 0$
For $\nu \in X^\ell$, define $f_\nu^\ell = f_\nu^{\ell+1}$
For $\nu \in X^{\ell+1} \setminus X^\ell$, calculate $\tilde{f}_\nu^{\ell+1} = (P_\ell^{\ell+1} f^\ell)(\nu)$,
and set $d_\nu^\ell = f_\nu^{\ell+1} - \tilde{f}_\nu^{\ell+1}$

Synthesis - $(f^0, d^0, \dots, d^{j-1}) \xrightarrow{(WT^j)^{-1}} f^j$

- For $\ell = 0, \dots, j-1$
For $\nu \in X^\ell$, define $f_\nu^{\ell+1} = f_\nu^\ell$
For $\nu \in X^{\ell+1} \setminus X^\ell$, calculate $\tilde{f}_\nu^{\ell+1} = (P_\ell^{\ell+1} f^\ell)(\nu)$,
and set $f_\nu^{\ell+1} = d_\nu^\ell + \tilde{f}_\nu^{\ell+1}$.

Usually, the interpolating prediction scheme is defined in terms of 1D central polynomial Lagrange interpolation of degree $M-1$, where $M = 2p \geq 2$ is an even integer.

Interpolating prediction operator $P_\ell^{\ell+1}$

- At the existing points $\nu = (2k\Delta_x^{\ell+1}, 2m\Delta_y^{\ell+1})$ in X^ℓ , define $s_{2k,2m}^{\ell+1} = s_{k,m}^\ell$.
- At the new points in $X^{\ell+1} \setminus X^\ell$, use the formulae

$$\begin{aligned} s_{2k,2m+1}^{\ell+1} &= \sum_r h(2r+1) s_{k,m-r}^\ell, \text{ if } \mu = \nu_1 \\ s_{2k+1,2m}^{\ell+1} &= \sum_r h(2r+1) s_{k-r,m}^\ell, \text{ if } \mu = \nu_3 \\ s_{2k+1,2m+1}^{\ell+1} &= \sum_r h(2r+1) s_{2k,2m-2r+1}^{\ell+1} \text{ if } \mu = \nu_3. \end{aligned}$$

The interpolation coefficients $h(k)$ are symmetric $h(-k) = h(k)$ and vanish for all integers such that $|k| \geq M$ (see [5]). For the case of cubic interpolation ($M = 4$), the non vanishing coefficients are $h(1) = h(-1) = 9/16$ and $h(3) = h(-3) = -1/16$.

2.1 Adaptive SPR grids

As described in the previous section, the result of the application of the wavelet transform WT^j on the input data f^j is organized by levels of resolution as $f_{MR}^j = (f^0, d^0 \dots d^{j-1})$. At each level ℓ , the wavelet coefficients d_ν^ℓ are defined in terms of local interpolation errors at the new points of the current level of

discretization. As interpolation errors, the wavelet coefficients can be used as indicators of local smoothness. Indeed, suppose that a connected region $\mathcal{N}$, with diameter $\lesssim 2^{-\ell}$, contains $\nu \in \Lambda^\ell$ and the corresponding interpolating points required for the computation of d_ν^ℓ . Using the fact that the wavelet coefficients vanish for polynomial with degree less or equal $M - 1$, if the function f presents a regularity σ in a Sobolev space $\mathcal{W}^{\sigma,\infty}(\mathcal{N})$, $1 \leq \sigma \leq M$, then Whitney's Theorem implies that [6]

$$\begin{aligned} |d_\nu^\ell| &\leq \inf_{g \in \mathcal{P}_{\sigma-1}} \max_{\gamma \in \mathcal{N}} |f(\gamma) - g(\gamma)| \\ &\lesssim 2^{-\ell\sigma} \|f\|_{\mathcal{W}^{\sigma,\infty}(\mathcal{N})}. \end{aligned}$$

where $\mathcal{P}_{\sigma-1}$ is the space of the polynomial with degree less or equal $\sigma - 1$. This fact leads to the definition of a point index $i_\varepsilon(\gamma)$, where the idea is to use the wavelet coefficients as indicators of local smoothness. It is based on the significance of the wavelet coefficients associated to γ , as compared to a certain given threshold ε . Therefore, according to these principles, the point index i_ε is defined by

$$i_\varepsilon(\gamma) = \begin{cases} 0 & \text{if } |d_\gamma^\ell| \leq \varepsilon \text{ for } \gamma \in X^{\ell+1} \setminus X^\ell \\ 1 & \text{otherwise.} \end{cases}$$

Based on this point regularity indicator, the construction of the adapted SPR grids $\Gamma_{SPR}^j = \Gamma_{SPR,\varepsilon}^j$ uses a thresholding operator $\mathcal{T}_\varepsilon$.

Thresholding operator $\mathcal{T}_\varepsilon$

- For $\gamma \in X^0$, keep γ in the adapted SPR grid Γ_{SPR}^j
- For $\ell = 0, \dots, j - 1$
For $\nu \in X^{\ell+1} \setminus X^\ell$
If $i_\varepsilon(\gamma) = 1$, keep γ in the adapted SPR grid Γ_{SPR}^j
If $i_\varepsilon(\gamma) = 0$, remove it .

This procedure led to a sparse representations

$$f_{SPR}^j = (f_\gamma^j), \gamma \in \Gamma_{SPR}^j,$$

where only the sample values required to reconstruct f in X^j with error less than ε are retained.

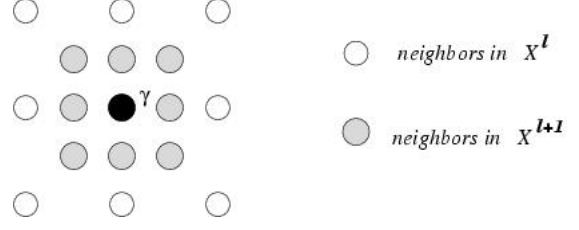


Figure 2: Illustration of the refinement process

2.2 Discretization of derivatives in the SPR context

The computation of a differential operator in the SPR context is performed in two steps. The first one is a refinement procedure to represent the function in a extended grid. Then, for points in the extended grid, the derivatives are approximated by an adaptive finite difference scheme. In the following, we describe these two operations.

Refinement

Consider an adaptive sparse point representation f_{SPR}^j of a function in a grid Γ_{SPR}^j . Let us assume that we need to approximate the action of a differential operator L on f using just the information contained in this SPR representation. Although Γ_{SPR}^j is a good mesh for representing f , it may not be appropriate for the representation of Lf . Therefore, some kind of refinement must be performed consisting in the inclusion of some of the points $\mu \notin \Gamma_{SPR}^j$ in a security zone, in order to get an extended grid Γ_{SPR}^{j+} more appropriated for the representation of Lf . In the applications of the present paper, we use the following refinement procedure: for each point ν in Γ_{SPR}^j corresponding to a significant wavelet coefficient, i.e. $\nu \in \Lambda^\ell$, $0 \leq \ell \leq j - 1$, its closest eight neighbors in X^ℓ are included in Γ_{SPR}^{j+} . Furthermore, if $\ell < j - 1$, then its closest eight neighbors in $X^{\ell+1}$ are also included in Γ_{SPR}^{j+} . This procedure is illustrated in Figure 2.

Adaptive differentiation

Our interest is on discretizations for partial derivatives $Lf = \frac{\partial}{\partial r} f$, $r \in \{x, y\}$, by finite difference schemes of

the form

$$(\partial^x f)(x, y) = \frac{1}{\Delta_x} \sum_{k'} f(x - k' \Delta_x, y) \beta(k'), \quad (2)$$

$$(\partial^y f)(x, y) = \frac{1}{\Delta_y} \sum_{m'} f(x, y - m' \Delta_y) \beta(m'). \quad (3)$$

M. Holmstrom [2] introduced the idea of implementing the finite difference schemes (2)–(3) in an adaptive fashion, using spacing adapted to each point local scale. For each point $\nu = (x, y) \in \Gamma_{SPR}^{j+}$, we define the local directional scales $a_r(\nu)$ as the minimum distance between ν and the other points $\gamma \neq \nu$ in Γ_{SPR}^{j+} sharing with ν the coordinate in the complementary direction

$$\begin{aligned} a_x(\nu) &= \min\{|x - x'|, \gamma = (x', y) \in \Gamma_{SPR}^{j+}, x' \neq x\} \\ a_y(\nu) &= \min\{|y - y'|, \gamma = (x, y') \in \Gamma_{SPR}^{j+}, y' \neq y\}. \end{aligned}$$

To compute the adaptive partial derivative in the $r \in \{x, y\}$ direction at a point $\nu \in \Gamma_{SPR}^{j+}$, consider $\Delta_r = a_r(\nu)$, and define

$$[\partial_a^r f](\nu) = [\partial^r \tilde{f}](\nu), \quad (4)$$

where $\tilde{f}$ represents the interpolation operator: $\tilde{f}(\gamma) = f(\gamma)$, if $\gamma \in \Gamma_{SPR}^{j+}$, and $\tilde{f}(\gamma)$ is obtained by the iterative application of the subdivision scheme, if $\gamma \in X^j \setminus \Gamma_{SPR}^{j+}$. This means that the required stencils for the computation of the finite difference, which may not be present in Γ_{SPR}^{j+} , are interpolated from coarser scales.

Figure 3 shows the function $f(x, y) = e^{-3500(x-\frac{1}{2})^2} e^{-3500(y-\frac{1}{2})^2}$, $(x, y) \in [0, 1] \times [0, 1]$ (top) and its SPR adaptive grid Γ_{SPR}^7 (bottom left) for the representation with $\varepsilon = 10^{-4}$ and $j = 7$. For this example, the finest uniform grid X^7 has equal spacing in both directions $\Delta_x^7 = \Delta_y^7 = 2^{-7}$ with $129 \times 129 = 16641$ total number of points, and the coarsest level is X^4 . The extended grid Γ_{SPR}^{7+} obtained by the refinement process is presented in Figure 3 (bottom right). Table 1 shows the number discarded points and the percentage of points kept in each grid. The maximum error in computing the adaptive spatial derivative in the x direction is indicated in the last column, for both Γ_{SPR}^7 and Γ_{SPR}^{7+} , showing the gain in accuracy in using the extended grid. The error is defined as the difference between the finite difference

Grids	# discarded points	% used points	error
Γ_{SPR}^7	16156	2.9	0.01705
Γ_{SPR}^{7+}	15128	9.1	0.00541

Table 1: Maximum error in the adaptive spatial derivative $\partial_a^x f(\nu)$ without and with refinement: full grid: X^7 , (129×129) , $\epsilon = 10^{-4}$.

operator $\partial^x f$ applied in the finest uniform grid X^7 and the interpolation up to X^7 of the adaptive operator $\partial_a^x f$ computed at Γ_{SPR}^7 and Γ_{SPR}^{7+} .

3 Numerical Finite Difference Scheme for Maxwell's Equations

We consider the TE mode system

$$\mu \frac{\partial H^z}{\partial t} = \frac{\partial E^x}{\partial y} - \frac{\partial E^y}{\partial x}, \quad (5)$$

$$\epsilon \frac{\partial E^x}{\partial t} = \frac{\partial H^z}{\partial y}, \quad (6)$$

$$\epsilon \frac{\partial E^y}{\partial t} = -\frac{\partial H^z}{\partial x}, \quad (7)$$

where $\mathbf{E} = (E^x, E^y, 0)$ and $\mathbf{H} = (0, 0, H^z)$ are the electrical and magnetic fields, ϵ is the electrical permittivity and μ is the magnetic permeability. The fields are sampled at the same uniform grid with spacing Δ_x and Δ_y in x and y directions $H_{k,m}^{z,n} = H^z(k\Delta_x, m\Delta_y, n\Delta t)$, $E_{k,m}^{x,n} = E^x(k\Delta_x, m\Delta_y, n\Delta t)$, and $E_{k,m}^{y,n} = E^y(k\Delta_x, m\Delta_y, n\Delta t)$. Based on these discrete values, applying the finite difference operators (2–3), and integrating in time using a first order Euler scheme, we obtain the following set of discrete equations

$$\begin{aligned} H_{k,m}^{z,n+1} &= H_{k,m}^{z,n} + \frac{\Delta t}{\mu} (\partial^y E^{x,n})_{k,m} \\ &\quad - \frac{\Delta t}{\mu} (\partial^x E^{y,n})_{k,m}, \end{aligned} \quad (8)$$

$$E_{k,m}^{x,n+1} = E_{k,m}^{x,n} + \frac{\Delta t}{\epsilon} (\partial^y H^{z,n+1})_{k,m}, \quad (9)$$

$$E_{k,m}^{y,n+1} = E_{k,m}^{y,n} - \frac{\Delta t}{\epsilon} (\partial^x H^{z,n+1})_{k,m}. \quad (10)$$

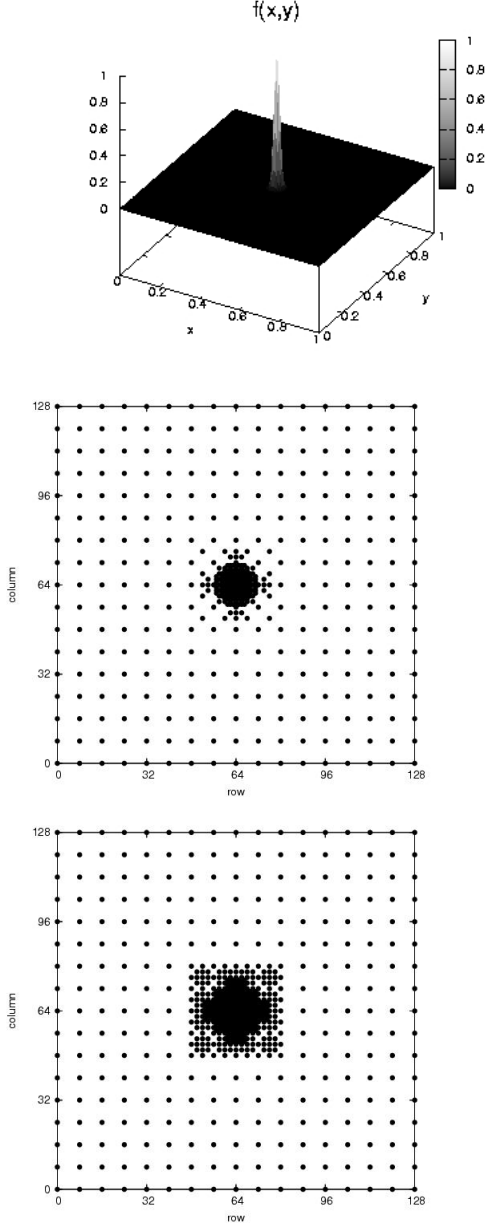


Figure 3: SPR grid (center) and its extended version (bottom) for the Gaussian function $f(x,y) = e^{-3500(x-\frac{1}{2})^2} e^{-3500(y-\frac{1}{2})^2}$ (top), for $\epsilon = 10^{-4}$ and $j = 7$.

For the applications of the present paper, we shall be concerned with the standard fourth order central difference scheme, where the coefficients are anti-symmetric ($\beta(-k) = -\beta(k)$), $\beta(1) = 2/3$, $\beta(2) = -1/12$, and $\beta(k) = 0$, $k > 2$. It is well known that this scheme can be obtained by a collocation formulation in shift invariant approximating spaces spanned by the Deslaurliers-Dubuc scaling functions associated with the cubic subdivision scheme (see [3] and references therein).

Note that, during the time step $t^n \leq t \leq t^{n+1}$, the spatial derivatives of the components of the electrical field are approximated by their values at the initial time level t^n to obtain the equation (8), and the spatial derivatives of the component of the magnetic field are approximated by their values at the final time level t^{n+1} to obtain the equations (9) and (10).

Stability and Numerical Dispersion

Let us consider plane wave solutions in the form

$$H^z(x, y, t) = H^z e^{i(\kappa_x x + \kappa_y y - \omega t)}, \quad (11)$$

$$E^x(x, y, t) = E^x e^{i(\kappa_x x + \kappa_y y - \omega t)}, \quad (12)$$

$$E^y(x, y, t) = E^y e^{i(\kappa_x x + \kappa_y y - \omega t)}, \quad (13)$$

where ω is the frequency in time and κ_x and κ_y are the wave numbers in space. Replacing these expressions in the Maxwell's system of equations, the conclusion is that plane waves solutions are admitted if the phase frequency and the wave numbers verify the dispersion relation $\omega^2 = c^2(\kappa_x^2 + \kappa_y^2)$, where $c = \frac{1}{\sqrt{\mu\epsilon}}$. Replacing the plane wave solutions in the discrete equations of these models, the numerical dispersion relation becomes

$$\frac{\sin^2(\omega\Delta t/2)}{(c\Delta t)^2} = \left(\frac{\tilde{\beta}(\kappa_x\Delta x)}{\Delta x} \right)^2 + \left(\frac{\tilde{\beta}(\kappa_y\Delta y)}{\Delta y} \right)^2, \quad (14)$$

where $\tilde{\beta}(\eta) = \sum_{l \geq 1} \beta(l) \sin(l\eta)$. Considering uniform step size $\Delta = \Delta_x = \Delta_y$ in both directions, and the normalized variables $\Omega = \omega\Delta t$, $S_x = \kappa_x\Delta$, $S_y = \kappa_y\Delta$, this dispersion relation reads

$$\sin^2\left(\frac{\Omega}{2}\right) = s^2[(\tilde{\beta}(S_x))^2 + (\tilde{\beta}(S_y))^2], \quad (15)$$

where $s = \frac{c\Delta t}{\Delta}$ is the CFL parameter. This dispersion

relation means that $s^2[(\tilde{\beta}(S_x))^2 + (\tilde{\beta}(S_y))^2] \leq 1$, implying the stability condition $s \leq s_{max} = \frac{1}{\sqrt{2}K_\beta}$, for $K_\beta = \max_\eta |\tilde{\beta}(\eta)|$. Since $K_\beta^{-1} = 1.4575$ for the fourth order central difference scheme [3], then $s_{max} = 1.0306$. Recall that for the FDTD scheme $s_{FDTD,max} = \frac{1}{\sqrt{2}}$.

To analyze the effect of the angle of incidence in the numerical dispersion, let us express the wave numbers in polar coordinates $\kappa_x = r \cos \theta$, $\kappa_y = r \sin \theta$. Consider the normalized velocity $\bar{v} = v_p/c$, where $v_p = \omega/r$, and define $\delta = \omega\Delta/c2\pi$. Note that, if $\lambda = cT$ is the wavelength, where $T = 2\pi/\omega$ is the period, then for $\delta = 1/N$, $N = \lambda/\Delta$ is the number of cells per wavelength. This means that δ measures the relative grid density. The analytical dispersion relation implies that $\bar{v} = 1$. Since $\omega\Delta t = s2\pi\delta$, $\kappa_x\Delta = \frac{2\pi\delta\cos\theta}{\bar{v}}$, $\kappa_y\Delta = \frac{2\pi\delta\sin\theta}{\bar{v}}$ and $\omega\Delta t = 2\pi\delta s$, the numerical dispersion relation can be expressed in terms of the parameters $\bar{v}$, δ and θ by the following implicit equation

$$\sin(\delta\pi s) = s\sqrt{\left[\beta\left(\frac{2\pi\delta\cos\theta}{\bar{v}}\right)\right]^2 + \left[\beta\left(\frac{2\pi\delta\sin\theta}{\bar{v}}\right)\right]^2}. \quad (16)$$

The plots in Figure 4 (bottom) show the variations of numerical phase velocity error $|1-\bar{v}|$ versus the angle of incidence θ , with CFL parameters $s = 0.3$ and $s = 0.7$, and for grid densities $N = 14, 16, 18, 20, 22$ and 24 . For comparison, in Figure 4 (top) curves of the error for the FDTD scheme are displayed for the same parameters. Because of symmetry, the plots are only for $0 \leq \theta \leq \frac{\pi}{4}$. For $s = 0.7$, it seems that for the non-staggered grid scheme the effect of the angle of incidence diminishes for high grid densities. It can also be observed that the maximum error occurs at $\theta = \pi/4$. For $s = 0.3$ this property only holds for $N \geq 16$; for $N = 14$ the error is maximum at $\theta = 0$.

Figure 5 indicates the angles where the error is maximum for this non-staggered grid model with $0.2 \leq s \leq 1.0$ and $5 \leq N \leq 25$. Observe that, for most of the combinations of N and s , the maximum error occurs at $\theta = \frac{\pi}{4}$, but for low values of these two parameters, $\theta = 0$ is the angle of maximum error. This kind of behavior differs from the FDTD case, where the maximum error always occurs at $\theta = 0$.

Another important difference between these two schemes is the effect of the CFL parameter in the error

value. For instance, comparing the plots for $s = 0.3$ and $s = 0.7$ in Figure 4, we note that for the FDTD scheme the error tends to decrease with increasing s and for the non-staggered scheme the tendency is in the opposite direction.

To analyze the effect of the CFL parameter, curves of the maximum numerical dispersion error, in the plane $N \times s$, are presented in Figure 6 for the FDTD method (solid lines) and the non-staggered grid model (dotted lines), for four different levels of accuracy 0.25%, 0.5%, 0.75% and 1%. Note that for the non-staggered grid model the curves have an inflexion point, corresponding to the transition between the angle for which this maximum error occurs. The lower part of the curves means that the maximum error has occurred at $\theta = 0$, and the upper part means that the maximum error has occurred at $\theta = \frac{\pi}{4}$.

In order to increase the accuracy, it is possible to observe that, for a given s , N must be increased, for both schemes. For a given N , the increase of accuracy depends on the values of N and s . For low values of N and s , s must be increased. However, in the most relevant region, corresponding to higher values of N and s , the situation is dependent upon the used scheme. For the FDTD scheme, s must be increased, and for the non-staggered grid model, s must be decreased. Note also that, in such region, for each given level of accuracy, the curve of the FDTD scheme is placed under the corresponding curve for the non-staggered model. This means that, for the same grid density, the CFL parameter s (i.e. Δt) for the non-staggered grid scheme can be greater than the one used in the FDTD scheme.

4 SPR numerical simulations

Our purpose in the present section is to analyze the performance of the SPR method for the numerical solution of the TE mode system of Maxwell's equations. In the operational part, we shall use the finite difference scheme described in Section 3, namely, the fourth order finite difference scheme in non-staggered grids. The representation corresponds to the multiresolution analysis for cubic interpolation, as detailed in Section 2.

We consider an example where the fields at $t = 0$ are

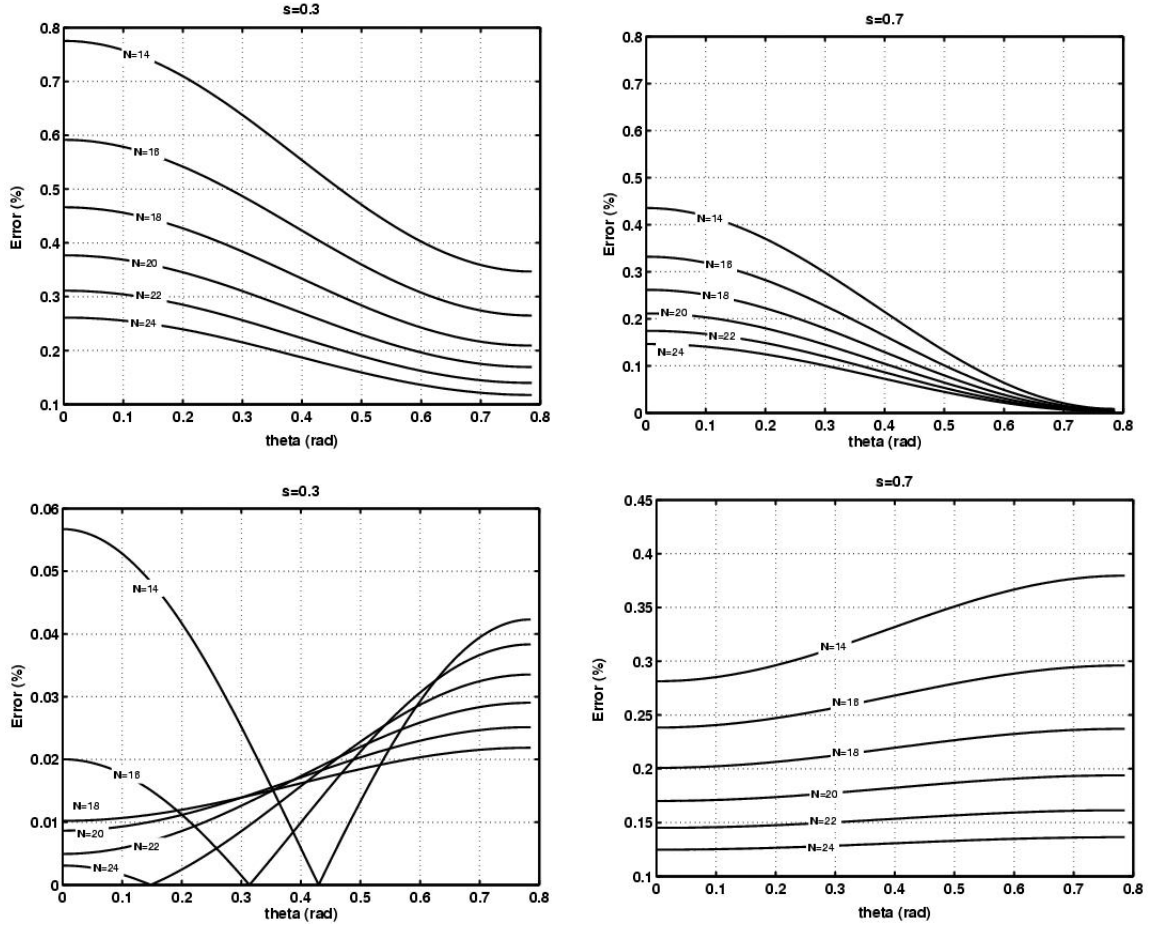


Figure 4: The effect of the angle of incidence in the numerical dispersion for the FDTD scheme (top) and for the fourth order scheme in non-staggered grid (bottom) using $s = 0.3$ and $s = 0.7$.

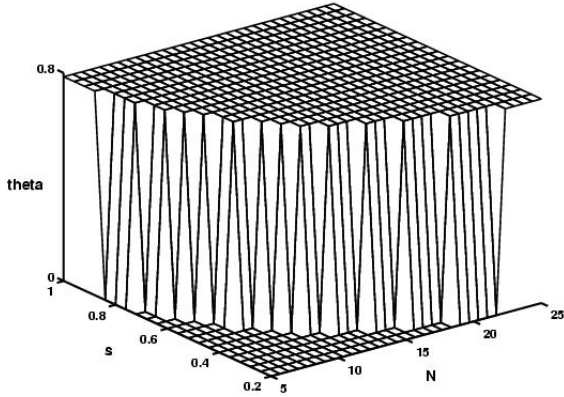


Figure 5: Angle of maximum error.

assumed to be $E^x(x, y) = E^y(x, y) = 0$ and $H^z(x, y) = e^{-3500(x-\frac{1}{2})^2} e^{-3500(y-\frac{1}{2})^2}$. We restrict the problem to the computational domain $R = [0, 1] \times [0, 1]$. We assume PEC boundary conditions, meaning that tangential components of the electrical field vanish at the boundary. This means that $E^y(0, y, t) = E^y(1, y, t) = 0$ and $E^x(x, 0, t) = E^x(x, 1, t) = 0$ for all $0 \leq x, y \leq 1$ and $t > 0$. Since some extra virtual finite difference stencils outside the boundary are required, we use the anti-symmetry properties for E^y at $x = 0$ and $x = 1$, and for E^x at $y = 0$ and $y = 1$. Consequently, symmetry properties are assumed for the magnetic field at all boundary faces.

We assume that the uniform grids X^j have the same spacing in both directions $\Delta_x^j = \Delta_y^j = 2^{-j}$. In the SPR simulations, the results are always for CFL parameter $s = 0.7$ and the coarsest grid is X^4 , having 17×17 points. If ε_H is the threshold for the magnetic field, then the threshold for the electrical fields is $\varepsilon_E = 100 \times$

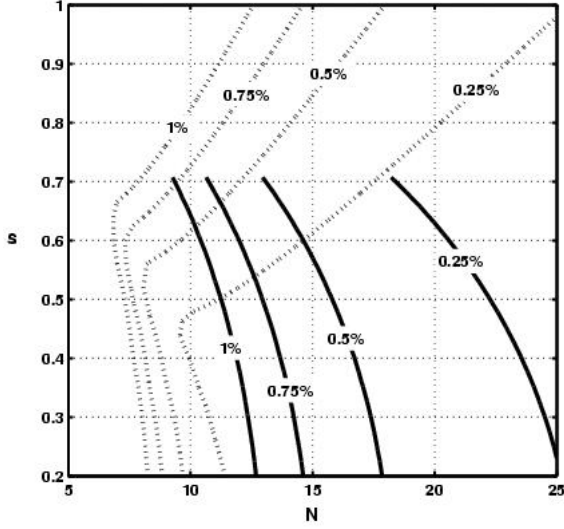


Figure 6: The effect of the CFL parameter: for the non-staggered grid model (dotted lines) and the FDTD scheme (solid lines).

ε_H . Since the time evolution of the electrical fields requires the partial derivatives of the magnetic field, and vice versa, the time stepping in the SPR scheme is performed on a nonuniform sparse grid formed by the union of the extended SPR grids of the three fields.

For $j = 7$, corresponding to the finest uniform grid containing 129×129 points, and $\varepsilon_H = 10^{-4}$, the SPR simulation produces the plots of the magnetic field H_z presented in Figure 7, for $t = 0.2, 0.6, 1.0$, and 1.4 ns. As expected, we can observe the SPR grids tracking the wave propagation, from a concentrated initial pulse to a circular wave front moving towards the boundary. We can also observe the dispersion effect in the diagonal propagation directions $\theta = \pm \frac{\pi}{4}$, as predicted by the theoretical analysis in Figure 4. Note that the SPR grids automatically become denser in these directions of irregularities.

In Figures 8, similar results are shown for simulations with $j = 8$, corresponding to the finest uniform grid of 257×257 points. Now, since more resolution is allowed, more grid points are used in the SPR grids to better resolve the wave front. Consequently, the dispersion effect becomes less noticeable. It is possible to observe that, as the wave front approaches the boundary of the computational region, the SPR grid becomes automatically more sparse in central parts, where there are less variation of fields.

The curves in Figure 9 describe the time evolution of the number $\#N_{SPR}(\%)$ of points used in the SPR simulations (i.e. the total number of points in the union of the extended SPR grids of the electrical and magnetic fields) as a percentage of the total number of points in the uniform finest grids X^j , $j = 7, 8, 9$ and 10 (corresponding to 129×129 , 257×257 , 513×513 and 1025×1025 points, respectively). For this example, using the threshold $\varepsilon_H = 10^{-3}$, the percentage of points in the SPR grids, at the final time $t = 1.0$ ns, ranges from 50% ($j = 7$) down to less than 20% ($j = 10$). However, decreasing the threshold to $\varepsilon_H = 10^{-4}$, we can observe that allowing more than 8 scale levels does not affect significantly the percentage of points in the adaptive grids, which is kept around 40% for $t = 1.0$ ns.

The plots in Figure 10 show the computational effort versus the error in the magnetic field at $t = 1.0$. The curves are plotted for simulations using $CFL = 0.7$ and different scale levels $j = 7, 8, 9$ and 10 , for both uniform and SPR grids. The computational effort is defined as the average number of point values used during the time period of simulations. In each case, the error is evaluated at X^j and corresponds to the maximum absolute value of the difference between the obtained solution (or its interpolation up to X^j) and a reference solution, which is produced by the higher resolution simulation in the uniform grid X^{11} (with 2049×2049 points), with CFL parameter $s = 0.35$. As expected, the error decreases with increasing computational effort, both for uniform and SPR simulations. In comparison with the simulations in uniform grids, the plots show that, to achieve a given accuracy, less computational effort is required in the SPR scheme.

5 Concluding Remarks

This paper describes the application of the SPR method for solving Maxwell's equations in 2D. In the operational part, we consider a fourth order finite difference scheme in non-staggered grids in the time-space domain for the components of magnetic and electrical fields. In the representation part, fields are dynamically represented in SPR adaptive grids in the context of fourth order interpolatory multiresolution analysis. From our results, we highlight the following aspects.

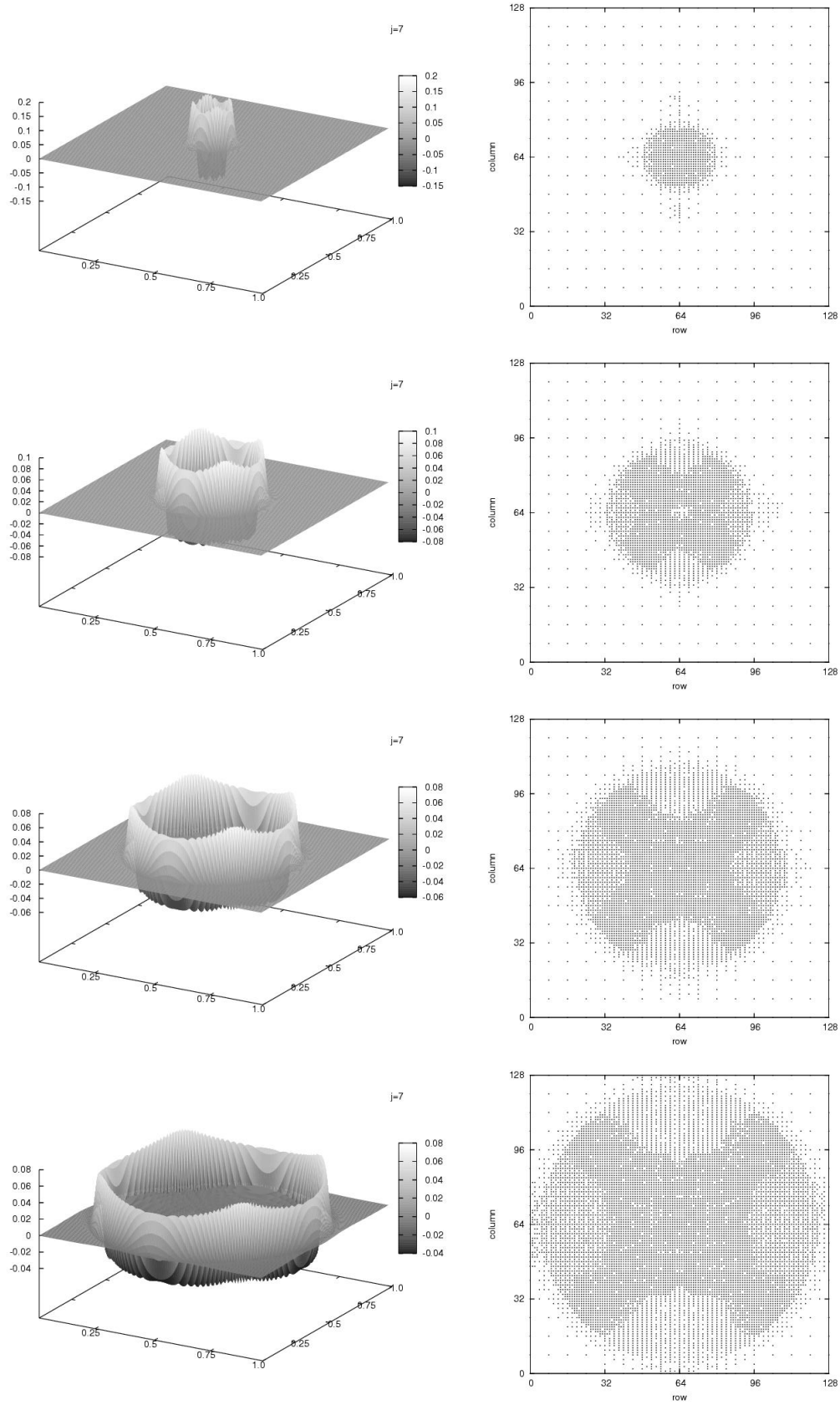


Figure 7: Evolution of H_z , and the corresponding SPR grids for $t = 0.2, 0.6, 1.0$ and $1.4 ns$. Finest uniform grid: 129×129 .

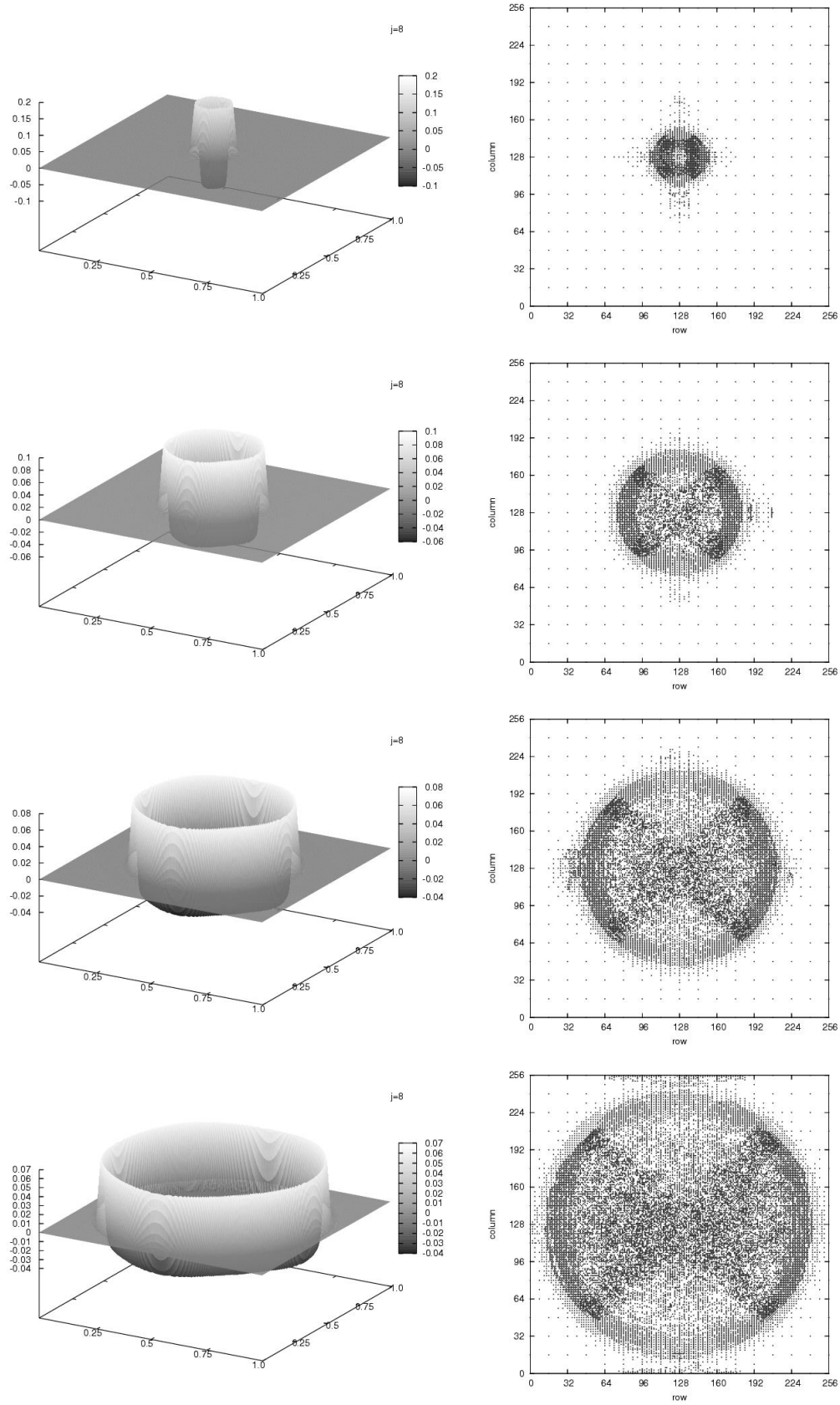


Figure 8: Evolution of H_z , and the corresponding SPR grids for $t = 0.2, 0.6, 1.0, \text{ and } 1.4 \text{ ns}$. Finest uniform grid: 257×257 .

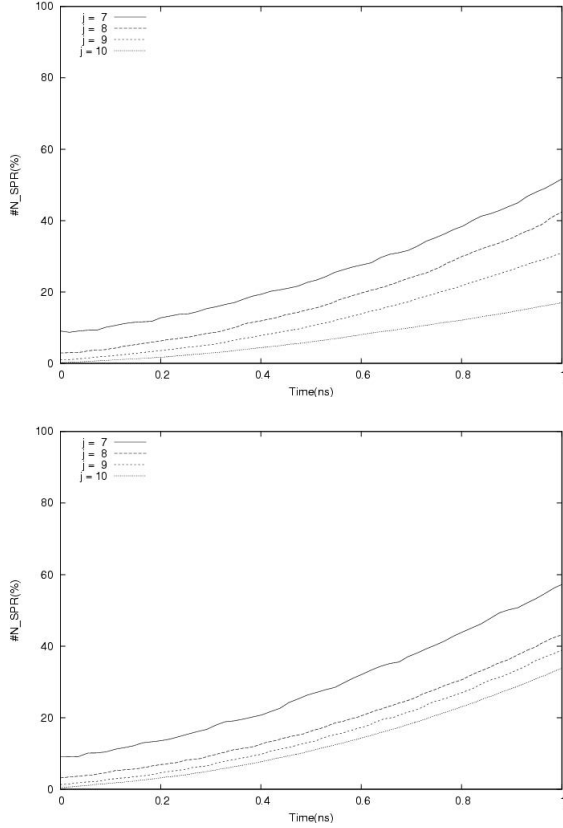


Figure 9: Percentage of grid points used in SPR simulations: $\epsilon_H = 10^{-3}$ (top) and $\epsilon_H = 10^{-4}$ (bottom).

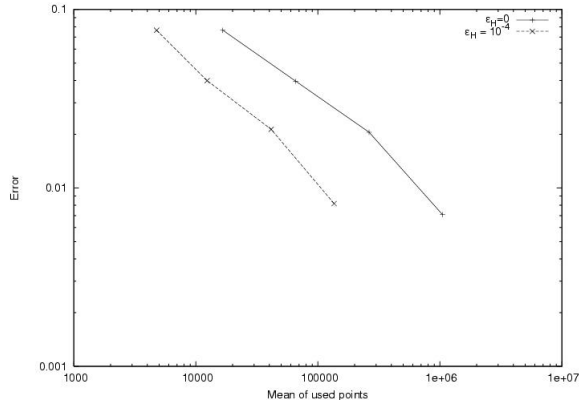


Figure 10: Error versus computational effort.

- Concerning numerical dispersion, and for accuracy comparison with respect of the FDTD scheme, we take into account the CFL effect. To evaluate and compare the CFL numbers required for the two schemes to achieve a prescribed accuracy level with a certain number of cells per wavelength, we observed that the fourth order scheme in non-staggered grids can be used with greater CFL parameters.
- The SPR method leads to nonuniformly sampled data, and at the same time tracks the evolution of the fields over time. If the field is slowly-varying in a region of space, but fast-varying in another region, the SPR procedure will lead to comparatively few samples in the former region and more samples in the latter.
- In general, the number of points in the SPR grids is well below the maximum number of points in the finest scale level. It is possible to reduce the number of SPR points, trading off representation accuracy by data compression, and therefore speed.
- The comparison made with the finite difference scheme in uniform grids shows that the SPR method achieves a given accuracy with reduced computational effort.

Acknowledgments: The research of this paper is part of a scientific international cooperation between GRICES (Portugal) and CAPES (Brazil) (contract 142/05). This research has also been partially supported by CNPq (contract 301667/203-9) and FAPESP (Brazil) (contract 2004/06880-3). We are thankful to J. Stolfi for fruitful discussions and helpful assistance.

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