

SPARSE POINT REPRESENTATION FOR FUNCTIONS AND DERIVATIVES: CONSISTENCY ANALYSIS *

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Abstract. In the Sparse Point Representation method the principle is to represent functions only through those point values indicated by significant wavelet coefficients, which are defined as interpolation errors by means of an interpolating subdivision scheme. Typically, a SPR grid is coarse in smooth regions, and refined close to irregularities. Furthermore, the computation of partial derivatives of a function from the information of its SPR representation is performed in two steps. The first one is a refinement procedure to extend the SPR representation by the inclusion of new interpolated point values in a security zone. Then, for points in the refined grid, such derivatives are approximated by uniform finite differences, using step size proportional to each point local scale. Eventually, required neighboring stencils are not present in the grid. In such case, the corresponding missing point values are approximated from coarser scales using the interpolating subdivision scheme. We show that, using cubic interpolation subdivision scheme, such wavelet adaptive finite differences can be formulated in terms of a collocation scheme for an adapted wavelet expansion of the function. For this purpose, we prove some results concerning the local behavior of the adaptive wavelet reconstruction operators, which stand for SPR grids having appropriate structures. The conclusion of our consistency analysis indicates that, in order to keep the truncation error under a certain given precision, some care must be taken concerning two aspects of the adaptive grids: the grid structure and the refinement strategy. Illustrating numerical results, arising in the numerical simulation of Maxwell's equations are presented.

Key words. wavelets, multiresolution analysis, adaptivity, sparse grids, finite differences, consistency analysis

AMS subject classifications. 42C40, 65M06, 65M50

1. Introduction. The SPR method, for *Sparse Point Representation*, is an adaptive finite difference strategy introduced in [1] to calculate numerical solutions of partial differential equations constrained by initial and boundary conditions. The method combines the simplicity and accuracy of traditional finite difference schemes with the ability of wavelet coefficients in the characterization of local regularity of functions. Typically, few points are found in each time step, the grid being coarse in smooth regions, and refined close to irregularities.

The method has two basic parts: the operational part, and the representation part. In the operational part, the background context is of an accurate and stable finite difference scheme L_{FD}^j in a uniform grid X^j . In the representation part, the tools come from interpolating wavelet analysis. The data is organized by levels of resolution $0 \leq \ell \leq j$, j being the finest scale level. In such multilevel context, the key role is played by interpolating subdivision schemes, which are the basis for the definition of reconstruction operators, wavelet coefficients being precisely the interpolation errors at the new points occurring when going from one level of discretization to the next finer one.

In the construction of SPR grids, the idea is to use the wavelet coefficients as indicators of local smoothness of the solution. This is the main ingredient in the definition of the thresholding operator: only those points corresponding to wavelet coefficients of magnitude greater than a prescribed threshold ϵ should be kept in the adaptive grid $\Gamma_\epsilon^j \subset X^j$. A reduced number of points are found in Γ_ϵ^j , the grid being less refined in regions where the variation of the function is smooth.

For PDE simulations in SPR grids, the scheme considers an adaptive finite difference version L_a^j . Eventhough the grid spacing now is not uniform, at each point of the SPR grid the FD formula is applied using a uniform step size. Since required neighboring stencils may not be present in the grid, such missing point values are approximated from coarser

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scales by the interpolating subdivision scheme of the same order. Two aspects of this SPR scheme should be emphasized. Firstly, the rule is to choose the step size proportional to each point local scale. The goal is to be more economic, in terms of operation count, than the procedure of calculating the stencils up to the most refined level. In fact, for those points in smooth regions with wide local scales, the number of stencil interpolations required for the application of L_a^j may be significantly reduced. Secondly, the SPR grid of a function may not be convenient for the SPR representation after the application of the finite difference operator. Therefore, before the application of L_a^j , the SPR grid should be refined by the inclusion of a security zone of new (interpolated) point values.

For this adaptive finite difference strategy, a generalization of the concept of truncation error, which is the familiar basis of the analysis of difference schemes, is presented in [2] for unidimensional domains and for schemes of arbitrary orders. In the present paper, we extend the consistency analysis in [2] to bidimensional domains, for the particular case of the cubic interpolating subdivision scheme, and of fourth order FD discretizations of partial derivatives $L^r f = \frac{\partial f}{\partial r}$, $r \in \{x, y\}$. The purpose is to analyse the effect of using the directional local scale in the SPR scheme $L_a^{r,j}$ instead of computing the FD formula with the spacing of the finest scale level.

The sections of the present paper are organized as follows. In Section 2, we describe the main wavelet tools required for the representation part of the SPR method. In Section 3 we prove some results concerning the local behavior of the reconstruction operators, which stand for grids holding appropriate directional local scales. Section 4 is dedicated to the study of SPR adaptive finite difference schemes $L_a^{r,j}$. Firstly, it is shown that they can be formulated in terms of a collocation scheme for an adapted wavelet expansion of the solution. This fact, in combination with properties of local behavior of the reconstruction operators, implies that the SPR scheme can give the same result either using the point directional local scale as step size or using the spacing of the finest scale level. Concluding, the consistency analysis indicates that, in order to keep the truncation error under a certain given precision, some care must be taken concerning two aspect of the adaptive grids: in addition to an appropriate grid structure, a reliable refinement strategy is also necessary. In Section 5, we present some illustrating examples of SPR representations for magnetic fields arising in the numerical simulation of Maxwell's equations. Concluding remarks are included in Section 6.

2. Multiresolution analysis on irregular grids. As explained before, our goal in this paper is to show how the the truncation error can be controlled, regarding possible perturbations coming from the application of an adaptive SPR scheme. Before going to this point in the following sections, we shall describe here the required tools from wavelet analysis in more detail.

★ Discrete aspect

In a multiscale framework, functions are represented in different scale levels, and the main tools are appropriate transformations relating the information at the finest scale level j to the lower ones, and vice versa. In the context of interpolating multiresolution analyses, the input data f^j contains the sample values at a grid in finest scale level. After a transformation WT^j , the output contains the information f^0 in the coarsest level, and d^ℓ that keeps the details between a scale level ℓ and the next upper level $\ell + 1$. Schematically, we have

$$f^j \xrightarrow{WT^j} f_{MR}^j = (f^0, d^0, \dots, d^{j-1}).$$

In a uniform setting in 2D, each scale level ℓ corresponds to a uniform grid X^ℓ . Starting from X^0 , the grid at the coarsest level with spacings Δ_x^0 in the x -direction and Δ_y^0 in the y direction, the hierarchy of uniform grids X^ℓ , $\ell \geq 1$ is

$$X^\ell = \{\gamma = (k\Delta_x^\ell, m\Delta_y^\ell), 0 \leq k \leq N_x^\ell, 0 \leq m \leq N_y^\ell\},$$

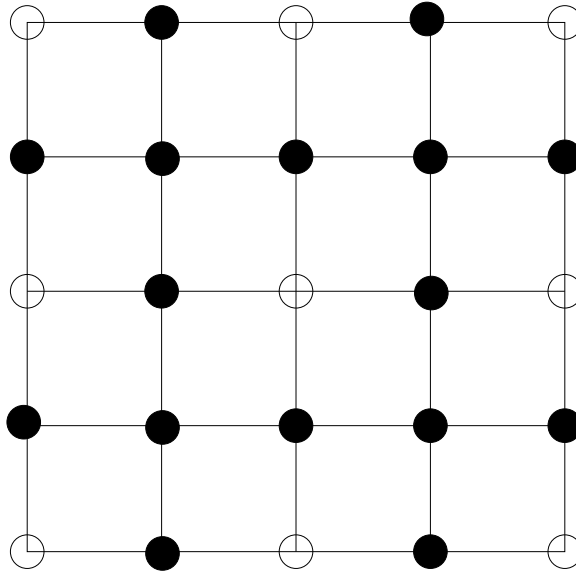


FIG. 2.1. Uniform dyadic grids: ○- points in X^ℓ , ●- points in $X^{\ell+1} \setminus X^\ell$

where $\Delta_x^\ell = 2^{-\ell}\Delta_x^0$ and $\Delta_y^\ell = 2^{-\ell}\Delta_y^0$. As indicated in Figure 2.1, its dyadic refinement $X^{\ell+1}$,

$$X^{\ell+1} = \{\gamma = (k\Delta_x^{\ell+1}, m\Delta_y^{\ell+1}), 0 \leq k \leq 2N_x^\ell, 0 \leq m \leq 2N_y^\ell\},$$

is obtained by the inclusion of new midpoints $\gamma = (2k\Delta_x^{\ell+1}, (2m+1)\Delta_y^{\ell+1})$, $\gamma = ((2k+1)\Delta_x^{\ell+1}, 2m\Delta_y^{\ell+1})$, and $\gamma = ((2k+1)\Delta_x^{\ell+1}, (2m+1)\Delta_y^{\ell+1})$.

In a non-uniform setting, the principle is also to have the information data hierarchically organized by different levels of resolution. On this matter, we are concerned with a sequence of embedded grids $\Gamma^\ell \subset \Gamma^{\ell+1}$. It shall be assumed that $\Gamma^0 = X^0$, and $\Gamma^\ell \subset X^\ell$, for $\ell > 0$, is constructed from $\Gamma^{\ell-1}$ by including some new points of $X^\ell \setminus X^{\ell-1}$.

Let $f^\ell = (f_\gamma^\ell)$ be the set of sample values labeled by the points $\gamma \in \Gamma^\ell$. As described in the following algorithms, for the definition of WT^j and its inverse $(WT^j)^{-1}$, the main ingredients are reconstruction interpolating operators $\mathcal{R}^\ell(\nu; f^\ell)$ that predict $f(\nu)$ at any location ν from the knowledge of its discrete sample values f^ℓ at Γ^ℓ . The wavelet coefficients d_γ^ℓ are defined for points $\gamma \in \Lambda^\ell = \Gamma^{\ell+1} \setminus \Gamma^\ell$ as the errors in this prediction.

Analysis - $f^j \xrightarrow{WT^j} (f^0, d^0, \dots, d^{j-1})$.

- For $\ell = j-1, \dots, 0$

Define $f_\gamma^\ell = f_\gamma^{\ell+1}$ if $\gamma \in \Gamma^\ell$

If $\gamma \in \Lambda^\ell$, calculate $\tilde{f}_\gamma^{\ell+1} = \mathcal{R}^\ell(\gamma; f^\ell)$, and set $d_\gamma^\ell = f_\gamma^{\ell+1} - \tilde{f}_\gamma^{\ell+1}$

Synthesis - $(f^0, d^0, \dots, d^{j-1}) \xrightarrow{(WT^j)^{-1}} f^j$.

- For $\ell = 0, \dots, j-1$

Define $f_\gamma^{\ell+1} = f_\gamma^\ell$ if $\gamma \in \Gamma^\ell$

If $\gamma \in \Lambda^\ell$, calculate $\tilde{f}_\gamma^{\ell+1} = \mathcal{R}^\ell(\gamma; f^\ell)$ and set $f_\gamma^{\ell+1} = d_\gamma^\ell + \tilde{f}_\gamma^{\ell+1}$.

★ The reconstruction operators

For input data f^ℓ sampled at uniform grids X^ℓ , the definition of the interpolation operator $\mathcal{R}^\ell(\nu, f^\ell)$, at any point ν , uses the classical interpolation subdivision scheme [3], which is done iteratively. On the first step, we set $\mathcal{R}^\ell(\nu; f^\ell) = f_\nu^\ell$, $\nu \in X^\ell$. Having $\mathcal{R}^\ell(\nu; f^\ell)$ being defined for all $\nu \in X^q$, $q \geq \ell$, then the definition is refined to X^{q+1} by means of a prediction interpolation scheme P_q^{q+1} . For other non dyadic points, the definition is set by density arguments. Thus, the interpolating

prediction operator $P_q^{q+1} : s^q \rightarrow s^{q+1}$ is the crucial tool, where $s^q = (s_{k,m}^q)$ and $s^{q+1} = (s_{k,m}^{q+1})$ are sequences labeled according to the points $\nu = (k\Delta_x^q, m\Delta_y^q) \in X^q$ and $\nu = (k\Delta_x^{q+1}, m\Delta_y^{q+1}) \in X^{q+1}$, respectively. In the wavelet analysis context, the most popular interpolating prediction schemes use central polynomial Lagrange interpolation of degree $M-1$, where $M = 2p$ is an even integer. These are the basis for different factorizations for the construction of the well known orthogonal Daubechies filters and biorthogonal spline filters [7].

Interpolating prediction operator P_q^{q+1} .

- At the old points $\nu = (2k\Delta_x^{q+1}, 2m\Delta_y^{q+1}) \in X^q$, define $s_{2k,2m}^{q+1} = s_{k,m}^q$.
- At the new points $\nu \in X^{q+1} \setminus X^q$, use the formulae

$$\begin{aligned} s_{2k,2m+1}^{q+1} &= \sum_r h(2r+1) s_{k,m-r}^q, \\ s_{2k+1,2m}^{q+1} &= \sum_r h(2r+1) s_{k-r,m}^q, \\ s_{2k+1,2m+1}^{q+1} &= \sum_r h(2r+1) s_{2k,2m-2r+1}^{q+1}. \end{aligned}$$

The interpolation coefficients $h(k)$ are symmetric $h(-k) = h(k)$, $h(0) = 1$, and vanish for even integers $k \neq 0$ and all integers such that $|k| \geq M$. For the case of cubic interpolation ($M = 4$), the non vanishing coefficients for odd indices are $h(1) = h(-1) = 9/16$ and $h(3) = h(-3) = -1/16$.

For non-uniform grids $\Gamma^\ell \subsetneq X^\ell$, the definition of the interpolation operators $\mathcal{R}^\ell(\nu; f^\ell)$ uses a simple modification of the classical interpolation subdivision scheme [5]. On the first step, we set $\mathcal{R}^\ell(\nu; f^\ell) = f_\nu^0$, $\nu \in X^0$. For $0 \leq q < \ell$, having $\mathcal{R}^\ell(\nu; f^\ell)$ being defined for all $\nu \in X^q$, then the definition is extended to $X^{q+1} \setminus \Gamma^\ell$ by means of the prediction interpolation scheme P_q^{q+1} , and it is set $\mathcal{R}^\ell(\nu; f^\ell) = f_\nu^\ell$, if $\nu \in \Gamma^\ell \cap X^{q+1}$. For points $\nu \in X^q$, $q > \ell$, the definition is set as in the classical subdivision scheme.

★ Functional aspect

Since the reconstruction operator $\mathcal{R}^\ell(\nu; f^\ell)$ is linear, it can be expressed as

$$\mathcal{R}^\ell(\nu; f^\ell) = \sum_{\gamma \in \Gamma^\ell} f_\gamma^\ell \Phi_\gamma^\ell(\nu),$$

where the function $\Phi_\gamma^\ell(\nu)$ is the reconstruction $\mathcal{R}^\ell(\nu; \delta^\ell)$ of the delta-sequence $\delta_\gamma^\ell = \delta(\eta - \gamma)$, $\eta \in \Gamma^\ell$. Therefore, Φ_γ^ℓ satisfies the interpolatory property $\Phi_\gamma^\ell(\eta) = \delta(\eta - \gamma)$, $\eta \in \Gamma^\ell$. Consequently, the scale relation holds

$$\Phi_\gamma^\ell(\nu) = \sum_{\eta \in \Gamma^{\ell+1}} \Phi_\gamma^\ell(\eta) \Phi_\eta^{\ell+1}(\nu).$$

Then it is clear that $\mathcal{V}^\ell \subset \mathcal{V}^{\ell+1}$, where we define

$$\mathcal{V}^\ell = \text{span}\{\Phi_\gamma^\ell : \gamma \in \Gamma^\ell\}.$$

Let us introduce the notation $\phi_\gamma^\ell(\nu)$ and V^ℓ for the case of uniform grid $\Gamma^\ell = X^\ell$. From its definition, the interpolation operators $\mathcal{R}^\ell(\nu; f^\ell)$ can be expressed as

$$(2.1) \quad \mathcal{R}^\ell(\nu; f^\ell) = \sum_{\gamma \in X^j} \tilde{f}_\gamma^\ell \phi_\gamma^\ell(\nu),$$

where $\tilde{f}_\gamma^\ell = f_\gamma$ if $\gamma \in \Gamma^\ell$, and $\tilde{f}_\gamma^\ell = \mathcal{R}^\ell(\gamma; f^\ell)$ if $\gamma \in X^j \setminus \Gamma^\ell$. It is known that there is a basic function $\varphi(x)$, so that for

$\nu = (\nu_x, \nu_y)$, and $\gamma = (k\Delta_x^\ell, m\Delta_y^\ell)$, $\phi_\gamma^\ell(\nu) = \varphi(\frac{\nu_x}{\Delta_x^\ell} - k)\varphi(\frac{\nu_y}{\Delta_y^\ell} - m)$). The scaling function φ of Deslauriers-Dubuc [3] satisfies the interpolation condition $\varphi(k) = \delta(k - 0)$, and the refinement equation $\varphi(x) = \sum_n h(n)\varphi(2x - n)$, where $h(n)$ are the coefficients associated to the prediction operator. For $M = 2$, $\varphi(x)$ is the hat function. For $M \geq 4$, $\varphi(x)$ is a symmetric function supported on the interval $[-M + 1, M - 1]$ having smoothness at least in $C^{1+\alpha}$, $0 < \alpha < 1$. Therefore, it follows immediately that $\mathcal{V}^\ell \subset \mathcal{V}^{\ell-1}$, and the functions in $\mathcal{V}^\ell$ inherit all regularity properties verified by the classical interpolatory scaling function $\varphi(x)$.

Let $\gamma \in \Lambda^{\ell-1}$. By definition, $\Phi_\gamma^\ell(\nu) = 0$ for all $\nu \in \Gamma^\ell \setminus \{\gamma\}$. It is easy to conclude from the iterative interpolation algorithm that $\Phi_\gamma^\ell(\nu)$ also vanishes at all $\nu \in X^\ell \setminus \{\gamma\}$. Therefore, for points γ in $\Lambda^{\ell-1}$, the basic functions $\Phi_\gamma^\ell(x)$, of the irregular case, and $\phi_\gamma^\ell(x)$, of the uniform case, coincide.

Consider the wavelet subspace

$$\mathcal{W}^{\ell-1} = \text{span}\{\psi_\gamma^{j\ell-1}(\nu); \gamma \in \Lambda^{\ell-1}\},$$

where $\psi_\gamma^{\ell-1}(x) = \phi_\gamma^\ell(x)$. Note that $\mathcal{W}^{\ell-1}$ is formed by those functions in $\mathcal{V}^j$ that vanish on $\Gamma^{\ell-1}$. Therefore, $\mathcal{V}^\ell$ is the direct sum of the spaces $\mathcal{V}^{\ell-1}$ and $\mathcal{W}^{\ell-1}$,

$$\mathcal{V}^\ell = \mathcal{V}^{\ell-1} \oplus \mathcal{W}^{\ell-1}.$$

Consequently, the following two-level representation holds

$$(2.2) \quad \sum_{\gamma \in \Gamma^\ell} f_\gamma^\ell \Phi_\gamma^\ell(\nu) = \sum_{\gamma \in \Gamma^{\ell-1}} f_\gamma^{\ell-1} \Phi_\gamma^{\ell-1}(\nu) + \sum_{\gamma \in \Lambda^{\ell-1}} d_\gamma^{\ell-1} \psi_\gamma^{\ell-1}(\nu).$$

Therefore, the difference of information between two consecutive levels is expressed by the wavelet terms

$$\sum_{\gamma \in \Lambda^{\ell-1}} d_\gamma^{\ell-1} \psi_\gamma^{\ell-1}(\nu) = \mathcal{R}^\ell(\nu; f^\ell) - \mathcal{R}^{\ell-1}(\nu; f^{\ell-1}).$$

Starting from the finest scale level, and iteratively applying (2.2), we obtain the multi-level decomposition

$$(2.3) \quad \sum_{\gamma \in \Gamma^j} f_\gamma^j \Phi_\gamma^j(\nu) = \sum_{\gamma \in X^{j0}} f_\gamma^0 \phi_\gamma^0(\nu) + \sum_{\ell=0}^{j-1} \sum_{\gamma \in \Lambda^\ell} d_\gamma^\ell \psi_\gamma^\ell(\nu).$$

Under this functional point of view, the transformations $f^j \leftrightarrow (f^0, d^0, \dots, d^{j-1})$ correspond to the change between the basis $\{\Phi_\gamma^j, \gamma \in \Gamma^j\}$ and $\{\phi_\gamma^0, \gamma \in X^0\} \cup \{\psi_\gamma^0, \gamma \in \Lambda^0\} \cup \dots \cup \{\psi_\gamma^{j-1}, \gamma \in \Lambda^{j-1}\}$ of $\mathcal{V}^j$. In the case of regular grids X^j , this interpolating wavelet approach was introduced by D. Donoho [4]. In the context of irregular grids, it was first considered in [5], and then extended in [6] to discretizations by cell averages on the interval. Note that the basic functions appearing on the right hand side of (2.3) correspond to the multiresolution analysis based on the uniform grid. This means that $\{\Phi_\gamma^j(x), \gamma \in \Gamma^j\}$ forms a basis for the subspace $\mathcal{V}^j \subset V^j$ that is spanned by $\{\phi_\gamma^0(x), \gamma \in X^0\}$ and the wavelets $\psi_\gamma^\ell(x)$ associated with the points $\gamma \in \Gamma^{\ell+1} \setminus \Gamma^\ell$. It should be also noted that, for functions $f \notin \mathcal{V}^j$, the wavelet coefficients $d_\gamma^\ell, \gamma \in \Gamma^{\ell+1} \setminus \Gamma^\ell$ appearing on the right hand side of (2.3), which are defined by means of the refinement scheme adapted to Γ^j , may be different from the corresponding wavelet coefficients produced by the multiresolution algorithm based on the full grid X^j .

2.1. SPR Representation. In the above discussion, multiresolution analyses for point values on a pre-established irregular grid have been considered. The connection between the grid and the particular function under analysis has not been covered. However, in wavelet analysis this kind of irregular grid appears after the thresholding of wavelet coefficients. Therefore, different grids are usually associated to distinct functions. The terminology *Sparse Point Representation* introduced in [1] is related to this kind of mapping between functions and grid point values. The main operations involved are described next.

As interpolation errors, the wavelet coefficients can be used as indicators of local smoothness. Indeed, suppose that a connected region $\mathcal{N}$, with diameter $\lesssim 2^{-\ell}$, contains $\gamma \in \Lambda^\ell$ and the corresponding interpolating points required for the computation of d_γ^ℓ . Suppose also that the function f presents a regularity σ in a Sobolev space $\mathcal{W}^{\sigma,\infty}(\mathcal{N})$, $1 \leq \sigma \leq 2p$. Then,

$$\begin{aligned} |d_\gamma^\ell| &\leq \inf_{g \in \mathcal{P}_{\sigma-1}} \max_{\nu \in \mathcal{N}} |f(\nu) - g(\nu)| \\ &\lesssim 2^{-\ell\sigma} \|f\|_{\mathcal{W}^{\sigma,\infty}(\mathcal{N})}. \end{aligned}$$

where $\mathcal{P}_{\sigma-1}$ is the space of the polynomial with degree less or equal $\sigma - 1$. This fact leads to the definition of a point index $i_\varepsilon(\gamma)$ which is based on the significance of the wavelet coefficients associated to γ , as compared to a certain given threshold ε . Therefore, according to these principles, the point index i_ε is defined by

$$i_\varepsilon(\gamma) = \begin{cases} 0 & \text{if } |d_\gamma^\ell| \leq \varepsilon \text{ for } \gamma \in \Lambda^\ell \\ 1 & \text{otherwise.} \end{cases}$$

This is the main ingredient in the definition of the thresholding operator $\mathcal{T}_\varepsilon$.

Thresholding operator (grid coarsening) $\Gamma^j \xrightarrow{\mathcal{T}_\varepsilon} \Gamma_\varepsilon^j$.

- If $i_\varepsilon(\gamma) = 1$, keep γ in the adapted grid Γ_ε^j .
- If $i_\varepsilon(\gamma) = 0$, remove γ from the adapted grid Γ_ε^j .

This procedure led to sparse representations $f_{SPR,\varepsilon}^j = (f_\gamma^j, \gamma \in \Gamma_\varepsilon^j)$ where only the sample values required to reconstruct f^j with error less than ε are retained.

Grid refinement. There is another useful converse operation. From the representation of f by node values on a grid Γ^j , we may want another representation on an refined grid $\hat{\Gamma}^j$, containing Γ^j . Since, in principle, we may not have access to the exact node value at a new point $\gamma \in \hat{\Gamma}^j \setminus \Gamma^j$, it is set by the application of the reconstruction operator $\mathcal{R}^j(\gamma; f^j)$ associated to Γ^j .

For instance, in PDE applications, the SPR grid of a function may not be convenient for the SPR representation after the application of the finite difference operator. Therefore, before the application of the differential operator, the SPR grid should be refined by the inclusion a security zone of new (interpolated) point values. For the applications of the present paper, for each point γ in Γ^j corresponding to a significant wavelet coefficient, i.e. $\gamma \in \Lambda^{\ell-1}$, $0 \leq \ell \leq j-1$, its closest eight neighbors in X^ℓ are included to form a refined grid $\hat{\Gamma}^j$. Furthermore, if $\ell < j$, then its closest eight neighbors in $X^{\ell+1}$ are also included in $\hat{\Gamma}^j$.

This refinement procedure is illustrated in Figure 2.2, where the grid Γ is formed by the coarsest grid X^0 and one point $\gamma \in \Lambda^2$. To form the refined grid $\hat{\Gamma}$, the sixteen neighbors of this point γ are added to Γ , which are eight in X^3 and eight in X^4 .

★ **A note about the notation**

In applications of this paper, there are several grids involved, with discretizations and reconstructions operators associated to them, and thus several multiresolution analyses. It is useful to keep a unified notation for most of the operations involved, independently of which sparse grid the discretization and reconstruction operators are associated with. For instance, we have already used the convenient notation $\tilde{f}_\gamma^j = \mathcal{R}^j(\gamma; f^j)$, $\gamma \notin \Gamma^j$. Similarly, the multiresolution transform, and the thresholding operators are persistently referred to as WT^j and $\mathcal{T}_\varepsilon$, independently of the multiresolution analysis at hand. It is clear that allways $\mathcal{T}_\varepsilon f^j = \mathcal{T}_\varepsilon \tilde{f}^j$.

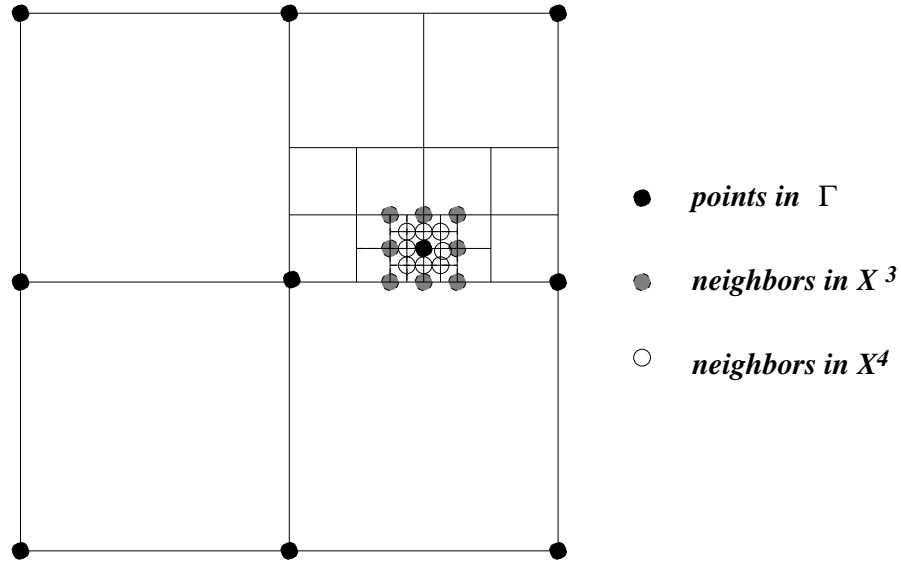


FIG. 2.2. Illustration of the refinement process: Γ is formed by the coarsest grid X^0 and one point $\gamma \in \Lambda^2$. The refined grid $\hat{\Gamma}$ is formed by adding to Γ the sixteen neighbors of this point γ , which are eight in X^3 and eight in X^4

3. Local behavior of the reconstruction operators. For the definition of SPR discretizations of partial derivatives, we need the concept of directional local scales $a_r^j(\gamma) > 0, \gamma \in \Gamma^j$, where $r \in \{x, y\}$. We have particular interest in directional local scales satisfying the following conditions:

$$(3.1) \quad \text{There is some } 0 \leq q \leq j \text{ such that } a_r^j(\gamma) = \Delta_r^q.$$

$$(3.2) \quad \text{If } a_r^j(\gamma) = \Delta_r^q, \text{ then } \gamma \in \Gamma^q.$$

$$(3.3) \quad \text{If } \gamma \in \Lambda^\ell, \text{ then } a_r^j(\gamma) \leq \Delta_r^{\ell+1}, \quad 0 \leq \ell \leq j-1.$$

Under such conditions, we shall prove the following local behavior of the reconstructions $\mathcal{R}^j(\nu; f^j)$ close to a point $\gamma \in \Gamma^j$, which is the basis for the consistency analysis of the next section.

Lemma 3.1. Suppose that for points in Γ^j we have defined directional local scales $a_r^j(\gamma)$ satisfying conditions (3.1-3.3). Let $\gamma = (\gamma_x, \gamma_y) \in \Gamma^j$ such that $a_r^j(\gamma) = \Delta_r^q$, $r \in \{x, y\}$, where $q = q(\gamma, r) \leq j$. Then, for reconstructions based on cubic interpolating prediction ($M = 4$),

$$\mathcal{R}^q(\gamma; f^q) = \mathcal{R}^j(\gamma; f^j), \quad \forall \gamma \in S_r^\ell(\gamma), \quad q \leq \ell \leq j,$$

where $S_x^\ell(\gamma) = \{\gamma = (\gamma_x + k\Delta_x^\ell, \gamma_y), \quad |k| \leq 2\}$ and $S_y^\ell(\gamma) = \{\gamma = (\gamma_x, \gamma_y + m\Delta_y^\ell), \quad |m| \leq 2\}$.

Proof. By construction, $\mathcal{R}^j(\nu; f^j)$ and $\mathcal{R}^q(\nu; f^q)$ coincide on X^q . By condition (3.2), we know that $\gamma \in \Gamma^q$. Then $S_x^q(\gamma) \subset X^q$, and thus the assertion is true for $\ell = q$. The proof advances by induction for $\ell \geq q+1$. Let $r = x$. For $\gamma = (\gamma_x \pm 2\Delta_x^\ell, \gamma_y) \in S_x^{\ell-1}(\gamma)$, we already know that the result is true from the previous step. The points $\gamma = (\gamma_x \pm \Delta_x^\ell, \gamma_y) \in \Lambda^{\ell-1}$ are not in Γ^j (otherwise, $a_x^j(\gamma) \leq \Delta_x^\ell < \Delta_x^q$, which is a contradiction). Consequently, $\mathcal{R}^j(\gamma; f^j)$ and $\mathcal{R}^q(\gamma; f^q)$ are obtained by cubic interpolation from their corresponding values at the four points in $X^{\ell-1}$ closest to γ in the x -direction. For $\gamma = (\gamma_x - \Delta_x^\ell, \gamma_y)$, the interpolation points are $(\gamma_x - 2\Delta_x^{\ell-1}, \gamma_y)$, $(\gamma_x - \Delta_x^{\ell-1}, \gamma_y)$, (γ_x, γ_y) and $(\gamma_x + \Delta_x^{\ell-1}, \gamma_y)$. For $\gamma = (\gamma_x + \Delta_x^\ell, \gamma_y)$, the interpolation points are $(\gamma_x - \Delta_x^{\ell-1}, \gamma_y)$, (γ_x, γ_y) , $(\gamma_x + \Delta_x^{\ell-1}, \gamma_y)$ and $(\gamma_x + 2\Delta_x^{\ell-1}, \gamma_y)$. In both cases, the result of the previous induction step implies that $\mathcal{R}^j(\cdot; f^j)$ and $\mathcal{R}^q(\cdot; f^q)$ coincide at the interpolation points. Therefore, they coincide at γ as well. The proof for the stencil points in the vertical direction is similar.

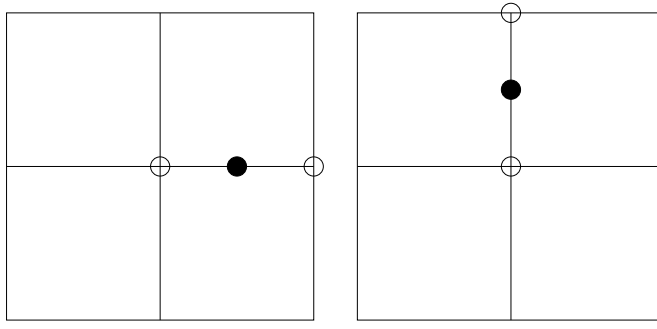


FIG. 3.1. Neighbors points (circle) of γ (bullet)

For $\gamma = (\gamma_x, \gamma_y) \in \Gamma^j$, we consider the following directional local scales $a_r^j(\gamma)$, $r \in \{x, y\}$:

$$(3.4) \quad a_x^j(\gamma) = \min \{ \Delta_x^q, |\gamma_x - \eta_x|, \eta = (\eta_x, \gamma_y) \in \Gamma^j, \eta \neq \gamma \}$$

$$(3.5) \quad a_y^j(\gamma) = \min \{ \Delta_y^q, |\gamma_y - \eta_y|, \eta = (\gamma_x, \eta_y) \in \Gamma^j, \eta \neq \gamma \},$$

where $q = 0$ for $\gamma \in X^0$, and $q = \ell + 1$ for $\gamma \in \Lambda^\ell$, $0 \leq \ell \leq j - 1$.

We can see that the statements (3.2) and (3.3) allways hold for the directional local scales (3.4) and (3.5). In fact, note that, if $\gamma \in \Lambda^\ell$, $0 \leq \ell \leq j - 1$, then, by definition, $a_r^j(\gamma) \leq \Delta_r^{\ell+1}$, and condition (3.3) holds. Condition (3.2) is allways true for $\gamma \in X^0 \subset \Gamma^q$, independently of q . For $\gamma \in \Lambda^\ell$, $0 \leq \ell \leq j - 1$, we know that $\Delta_r^q = a_r^j(\gamma) \leq \Delta_r^{\ell+1}$. This implies that $q \geq \ell + 1$, i.e., $\gamma \in \Gamma^{\ell+1} \subset \Gamma^q$, and the result is also true for this case. However, the verification of condition (3.1) is more subtle. For example, the refined grid $\hat{\Gamma}$ in Figure 2.2, which is formed by the points of Γ and the neighbors of γ in X^4 and X^5 , is an example of a grid for which the directional local scale (3.4) does not satisfies the condition (3.1). For instance, for the point $\nu \in X^0$, in the center of the grid, $a_x^5(\nu) = \frac{3}{4}\Delta_x^0$. The verification of condition (3.1) requiries more structured grids.

★ Tree condition

To facilitate the verification the the condition (3.1), let us consider grids having a tree sctucture as follows. A grid Γ^j satisfy the tree condition if for $0 \leq \ell \leq j - 1$:

- a) for a point $\gamma \in \Lambda^\ell$ of the form $\gamma = ((2k+1)\Delta_x^{\ell+1}, 2m\Delta_y^{\ell+1})$, then the two neighbors in X^ℓ $\gamma = (k\Delta_x^\ell, m\Delta_y^\ell)$ and $\gamma = ((k+1)\Delta_x^\ell, m\Delta_y^\ell)$ are in Γ^j (see Figure 3.1 left);
- b) for a point $\gamma \in \Lambda^\ell$ of the form $\gamma = (2k\Delta_x^{\ell+1}, (2m+1)\Delta_y^{\ell+1})$, then the two neighbors in X^ℓ $\gamma = (k\Delta_x^\ell, m\Delta_y^\ell)$ and $\gamma = (k\Delta_x^\ell, (m+1)\Delta_y^\ell)$ are in Γ^j (see Figure 3.1 right).

The refined grid $\hat{\Gamma}$ in Figure 2.2 is an example of a grid that does not have the tree structure. For instance, the four neighbors of the point $\gamma \in \Lambda^2$, which are in X^3 and located in the vertices, are new points in $X^3 \setminus X^2$, and they do not satisfy the item a), as supposed by the tree condition.

We observe that the tree structure is a more relaxed restriction than the cone condition, as defined in [5]. Cone like grids are supposed to have a smooth grid density variation. Tree like grids may occur with strong variation in the grid density, as the one indicated in Figure 3.2.

Lemma 3.2. *For a grid Γ^j satisfying the tree condition, the directional local scales defined in (3.4) and (3.5) satisfy the conditions (3.1-3.3).*

Proof: As observed before, it only remains to prove (3.1). Let us consider first the case $r = x$. We note that if the result is not true for some $\gamma \in \Gamma^j$, then there is some $\eta = (\eta_x, \gamma_y) \in \Gamma^j$ such that $a_x^j(\gamma) = |\gamma_x - \eta_x| < \Delta_x^q$, where $q = 0$ for

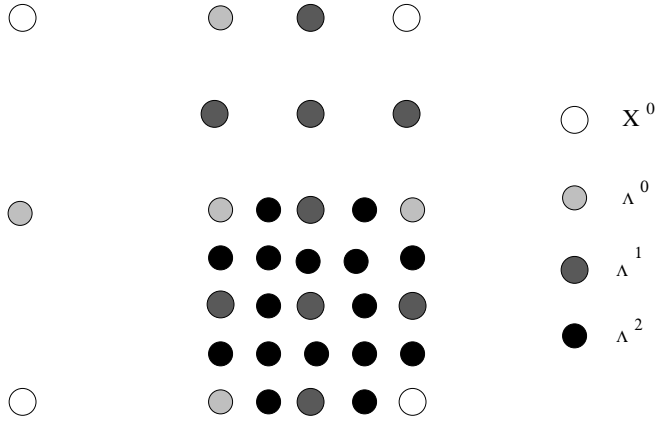


FIG. 3.2. Example of a tree like grid with strong variation in the grid density

$\gamma \in X^0$, and $q = \ell + 1$ for $\gamma \in \Lambda^\ell$, $0 \leq \ell \leq j - 1$. We first consider the case $\gamma \in X^0$. Therefore, η can not be a point in X^0 . Furthermore, since $\eta_y = \gamma_y$, then $\eta \in \Lambda^q$, $q \geq 0$ has the form $((2k + 1)\Delta_x^{q+1}, 2m\Delta_y^{p+1})$. By the tree condition a), the two neighbors (kh_x^p, mh_y^p) and $((k + 1)h_x^q, mh_y^q)$ are in Γ^j . Since one of them must coincide with γ , we obtain that $a_x^j(\gamma) = |\gamma_x - \eta_x| = \Delta_x^p$. Similarly, if $\gamma \in \Lambda^\ell$, $\ell \geq 0$, then η can not be a point in $\Gamma^{\ell+1}$. Recalling that $\eta_y = \gamma_y$, we see that η must have the form $((2k + 1)\Delta_x^{p+1}, 2m\Delta_y^{p+1})$, for some $p > \ell$. Again, by the tree condition a), the two neighbors (kh_x^p, mh_y^p) and $((k + 1)\Delta_x^q, m\Delta_y^q)$ are in Γ^j , and one of them must coincide with γ . Therefore, we obtain that $a_x^j(\gamma) = |\gamma_x - \eta_x| = \Delta_x^p$. The proof for the case $r = y$ is similar, where the tree condition b) is used.

4. Discretization of Derivatives . In this section, the focus is on discretizations of partial derivatives $L^r f = \frac{\partial}{\partial r} f$, $r \in \{x, y\}$, based on the approximation scheme $\mathcal{R}^j(\nu; f^j)$.

Collocation scheme: Let $M \geq 4$. Given the node values f_γ^j of a function f on the grid Γ^j , the collocation scheme $(L^{r,j} f^j)_\gamma$ is defined as the node values of the partial derivative of the reconstruction $\mathcal{R}^j(\cdot; f^j)$ computed at γ . That is,

$$(4.1) \quad (L^{r,j} f^j)_\gamma = (L^r \mathcal{R}^j(\cdot; f^j))(\gamma).$$

Recall from (2.1) that $\mathcal{R}^j(\nu; f^j)$ may be represented as

$$\mathcal{R}^j(\nu; f^j) = \sum_{\gamma \in X^j} \tilde{f}_\gamma^j \phi_\gamma^j(\nu),$$

where $\tilde{f}_\gamma = \mathcal{R}^j(\gamma; f^j)$. Using this representation, $(L^{x,j} f^j)_\gamma$ may be interpreted as the result of the action of a finite difference operator

$$(4.2) \quad \begin{aligned} (L^{x,j} f^j)_\gamma &= \left(L_{FD}^{x,j} \mathcal{R}^j(\cdot; f^j) \right) (\gamma) \\ &= \frac{1}{\Delta_x^j} \sum_k \tilde{f}(\gamma_x - k\Delta_x^j, \gamma_y) \beta(k), \end{aligned}$$

with coefficients $\beta(k) = \varphi'(k)$. Analogously,

$$(4.3) \quad \begin{aligned} (L^{y,j} f^j)_\gamma &= \left(L_{FD}^{y,j} \mathcal{R}^j(\cdot; f^j) \right) (\gamma) \\ &= \frac{1}{\Delta_y^j} \sum_m \tilde{f}(\gamma_x, \gamma_y - m\Delta_y^j) \beta(m). \end{aligned}$$

These finite difference operators are closely connected with the underlying approximation scheme, which is exact for polynomials of degree less or equal $M - 1$. Then the associated finite difference operator also have order M . Since $\varphi(x)$ is a symmetric function supported on the interval $[-M + 1, M - 1]$, the coefficients $\beta(k)$ end up being anti-symmetric, $\beta(k) = -\beta(-k)$, and $\beta(k) = 0$, $|k| \geq M - 1$. The applications of the present paper are for $M = 4$, for which the non vanishing values are $-\beta(-1) = \beta(1) = 2/3$ and $-\beta(-2) = \beta(2) = -1/12$, corresponding to the standard fourth order central difference operator. However, it should be noted that the formulations (4.2) and (4.3) also makes sense if $L_{FD}^{r,j}$ are replaced by any other finite difference operators.

4.1. Adaptive Formulation: $L_a^{r,j}$, $r \in \{x, y\}$. M. Holmström [1] introduced the idea of implementing the finite difference schemes (4.2) and (4.3) in an adaptive fashion. For each point $\gamma = (\gamma_x, \gamma_y) \in \Gamma^j$ with directional local scales $a_r^j(\gamma)$, define

$$(4.4) \quad (L_a^{x,j} f^j)_\gamma = \frac{1}{a_x^j(\gamma)} \sum_k \tilde{f}(\gamma_x - k a_x^j(\gamma), \gamma_y) \beta(k),$$

$$(4.5) \quad (L_a^{y,j} f^j)_\gamma = \frac{1}{a_y^j(\gamma)} \sum_m \tilde{f}(\gamma_x, \gamma_y - m a_y^j(\gamma)) \beta(m).$$

In view of equations (4.2) and (4.3) at level q , and assuming that the directional local scales $a_r^j(\gamma)$ satisfy the conditions (3.1-3.3), these adaptive finite difference formulae can be formulated as adaptive collocation schemes.

Lemma 4.1. *Suppose that for points in Γ^j we have defined directional local scales $a_r^j(\gamma)$ satisfying conditions (3.1-3.3). If $\gamma \in \Gamma^j$, and $a_r^j(\gamma) = h_r^q$ (that is, $\gamma \in \Gamma^q$), then $(L_a^{r,j} f^j)_\gamma$ can be formulated as*

$$(4.6) \quad (L_a^{r,j} f^j)_\gamma = (L_{FD}^{r,q} \mathcal{R}^q(., f^q))(\gamma) = (L^r \mathcal{R}^q(., f^q))(\gamma).$$

As we will see below, the adaptive schemes $L_a^{r,j} f^j$ may coincide with their non adaptive versions $L^{r,j} f^j$.

Theorem 4.1. *Assume that for point $\gamma \in \Gamma^j$ directional local scales are defined satisfying (3.1-3.3). Let $\gamma \in \Gamma^j$ with $a_r^j(\gamma) = \Delta_r^q$, $q \leq j$. Consider the fourth order FD formulae. Then, the adaptive formulation $(L_a^{r,j} f^j)_\gamma$ coincides with its non adaptive version $(L^{r,j} f^j)_\gamma$ at the points $\gamma \in \Gamma^j$.*

Proof: Lemma 3.1 assures that $\mathcal{R}^q(., f^q)$ and $\mathcal{R}^j(., f^j)$ coincide on the stencil points required for the evaluation of $L^{r,j}$ at γ . This means that

$$[L^{r,j} \mathcal{R}^q(., f^q)](\gamma) = [L^{r,j} \mathcal{R}^j(., f^j)](\gamma).$$

But, since $\gamma \in \Gamma^q$,

$$[L^{r,j} \mathcal{R}^q(., f^q)](\gamma) = [L^r \mathcal{R}^q(., f^q)](\gamma) = [L^{r,q} \mathcal{R}^q(., f^q)](\gamma).$$

Therefore, according to the equivalent adaptive collocation formulation of Lemma 4.1, we obtain

$$(L_a^{r,j} f^j)_\gamma = [L^r \mathcal{R}^q(., f^q)](\gamma) = [L^{r,j} \mathcal{R}^q(., f^q)](\gamma) = [L^{r,j} \mathcal{R}^j(., f^j)](\gamma) = (L^{r,j} f^j)_\gamma.$$

4.2. Consistency Analysis. Let $L_{FD}^{r,j}$ be uniform finite difference operators as in (4.2) and (4.3), corresponding to the collocation scheme (4.1). Given the samples f^j of a function in a grid Γ^j , consider

$$g^r(\nu) = [L_{FD}^{r,j} \mathcal{R}^j(., f^j)](\nu).$$

For an refined grid $\hat{\Gamma}^j$ containing Γ^j , let $\hat{g}_\gamma^r = [L_a^{r,j} \tilde{f}^j](\gamma)$, $\gamma \in \hat{\Gamma}^j$ be the result of applying the adaptive difference scheme based on $\hat{\Gamma}^j$. We are interested in the error TE^r in approximating g^r by the interpolation of $\hat{g}^r$. That is,

$$TE^r(\nu) = g^r(\nu) - \mathcal{R}^j(\nu; \hat{g}^r).$$

If we see the finite difference operator $L_{FD}^{r,j}$ as playing the role of the r -partial derivative, then TE^r can be interpreted as *the difference between the r -partial derivative of the reconstruction and the reconstruction of the adaptive discrete r -partial derivative*. In this sense, we call TE^r the *truncation error*. To be more specific, we should note that $g^r(\nu) \in \mathcal{V}^j$ is just an approximation of the r -partial derivative of $\mathcal{R}^j(x; f^j)$. But since we are assuming that the finite difference operator comes from the collocation scheme, $g^r(x)$ indeed coincides with $\frac{\partial}{\partial r} \mathcal{R}^j(\cdot; f^j)$ at X^j . That is why we call TE^r as a truncation error. It

The contributions to the truncation error come from two sides.

- For points $\gamma \in \hat{\Gamma}^j$, $TE^r(\gamma)$ represents the difference between $g^r(\gamma)$, which represents the r -derivative computed by the FD scheme using the spacing of finest scale level, and $\hat{g}_\gamma^r$, which is the result of using the FD with local directional scale instead. As stated in the previous section, this error can be avoided for fourth order schemes if for points in $\hat{\Gamma}^j$ the directional local scales satisfy the conditions (3.1-3.3). Precisely, as a consequence of Theorem 4.1, g^r coincides at $\gamma \in \hat{\Gamma}^j$ with the adaptive finite difference values $\hat{g}_\gamma^r = [L_a^j \tilde{f}^j](\gamma)$. This means that $TE^r(\gamma) = 0$ for all $\gamma \in \hat{\Gamma}^j$.
- Provided that $g_\gamma^r = \hat{g}_\gamma^r$, thus $TE^r(\gamma)$, as interpolation errors, are the wavelet coefficients of g^r at points γ outside $\hat{\Gamma}^j$. So, the question is what can be said about the wavelet content of g^r outside $\hat{\Gamma}^j$. Note that $\mathcal{R}^j(\nu; f^j)$ only contains contributions of wavelets ψ_ν^l , $\nu \in \Gamma^j$. However, differentiation is an action that activates some other wavelets. Thus, components $\psi_\gamma^l(\nu)$, with $\gamma \in X^j \setminus \Gamma^j$, may be present in $g^r(\nu)$. However, as analysed in [8], in the context of multiresolution for cell averages, one may expect that this is the kind of error that could be controlled by a special refinement strategy for the construction of $\hat{\Gamma}^j$. Thus, given a precision $\varepsilon > 0$, the goal is try to construct $\hat{\Gamma}^j$ so that $|TE^r(\gamma)| < \varepsilon$ for $\gamma \in X^j \setminus \hat{\Gamma}^j$.

For the applications of the present paper, before the application of the adaptive finite difference scheme, we use the refinement procedure described in Section

5. Numerical Example . As illustrating example, we consider the TE mode system of Maxwell's equations

$$\begin{aligned} \mu \frac{\partial H^z}{\partial t} &= \frac{\partial E^x}{\partial y} - \frac{\partial E^y}{\partial x}, \\ \epsilon \frac{\partial E^x}{\partial t} &= \frac{\partial H^z}{\partial y}, \\ \epsilon \frac{\partial E^y}{\partial t} &= -\frac{\partial H^z}{\partial x}, \end{aligned}$$

where $\mathbf{E} = (E^x, E^y, 0)$ and $\mathbf{H} = (0, 0, H^z)$ are the electrical and magnetic fields, ϵ is the electrical permittivity and μ is the magnetic permeability. We assume that the fields at $t = 0$ are $E^x(x, y) = E^y(x, y) = 0$ and $H^z(x, y) = e^{-3500(x-\frac{1}{2})^2} e^{-3500(y-\frac{1}{2})^2}$. The permittivity and permeability parameters are those for free space. We restrict the problem to the computational domain $R = [0, 1] \times [0, 1]$, and PEC boundary conditions are enforced, i.e., $E^y(0, y, t) = E^y(1, y, t) = 0$ and $E^x(x, 0, t) = E^x(x, 1, t) = 0$ for all $0 \leq x, y \leq 1$ and $t > 0$.

Our purpose here is to show the behavior of the SPR representation for the numerical solution of this 2D problem produced by a reference uniform FD scheme and to analyse the performance of the SPR discretization of spatial partial derivatives using either local directional scales or the grid spacing of the finest scale level.

In the reference scheme, both fields are sampled at the same uniform grid X^j , with spacing $\Delta_x^j = \Delta_y^j = 2^{-j}$. The fourth order FD scheme is applied for the discretization of spatial derivatives. Since some extra virtual finite difference stencils outside the boundary are required, we use the anti-symmetry properties for E^y at $x = 0$ and $x = 1$, and for E^x at $y = 0$

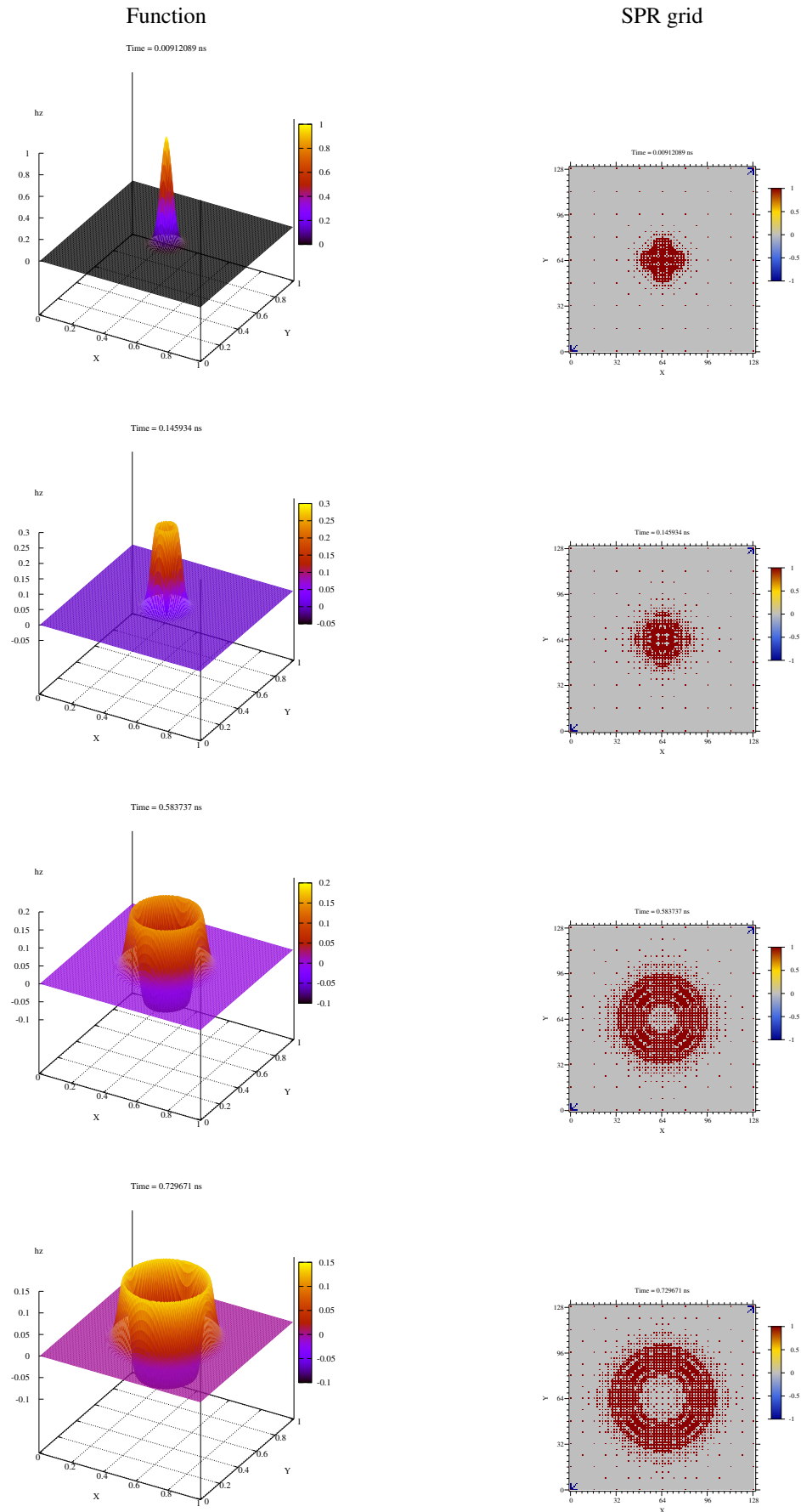


FIG. 5.1. Time evolution of the magnetic field (left) and the corresponding SPR grid (right) for $\varepsilon = 10^{-4}$.

time	error using (5.2)	error using (3.4)	% used points	% interpolations
t_1	2.07294×10^{-1}	15.6880×10^{-17}	5	13
t_2	1.36269×10^{-1}	335.240×10^{-17}	9	18
t_3	0.93914×10^{-1}	366.373×10^{-17}	17	25
t_4	1.85313×10^{-1}	277.572×10^{-17}	23	30

TABLE 5.1

Error in SPR derivatives using either the uniform grid spacing of the finest scale level or the local directional scales of each point: tests for SPR grids satisfying the tree condition.

and $y = 1$. Consequently, symmetry properties are assumed for the magnetic field at all boundary faces. During a time step $n\Delta t \leq t \leq (n+1)\Delta t$, the integration in time uses a first order Euler scheme for which the components of the electrical field are taken at the initial time level $n\Delta t$ to obtain the magnetic field at the next time level $(n+1)\Delta t$. Then, the magnetic field at $(n+1)\Delta t$ is used for the evolution of the electrical fields. To ensure stability and to get accuracy, the time step is constrained by the formula $\Delta t = 0.35 \times \sqrt{\mu\epsilon} \times 2^{-j}$. Such scheme is a generalization of the well know FDTD algorithm [9], in the sense that non-staggered grids are used in place of staggered ones and fourth order spatial FD scheme replaces the low order one of the Yee's scheme. We refer to [10] for detailed dispersion and stability analysis of such reference scheme as well as for 1D SPR simulations.

From top to down, the plots in Figure 5.1 (left) correspond to simulation results for the magnetic field at times (ns) $t_1 = 0.00912089$, $t_2 = 0.291869$, $t_3 = 0.583737$ and $t_4 = 0.875606$. The uniform grid corresponds to $j = 7$. On the right side of this figure we show the SPR representation of such fields, using cubic subdivision interpolation, by tresholding the wavelet coefficients with the parameter $\varepsilon_H = 10^{-4}$. It can be observed that the SPR grids are non uniformly sampled, with density varying according to the evolution of the field.

For these instances of the magnetic field, we compute the two contributions to the truncation error. Namely, at each point of the sparse grid, we first evaluate the information of the error coming from the difference of discretizations

$$(5.1) \quad \max\{|(L^{x,j}f)(\gamma) - (L_a^{x,j}f)(\gamma)|\},$$

corresponding to applying the FD operator either with the uniform grid spacing of the finest scale level or using the local directional scales. We test this error for the grid obtained after the enforcement of the tree structure to the SPR representation of the field and for the grid given by the refinement operation. Being sure that this difference of the SPR derivatives is negligible at all points in the sparse grid, next we evaluate the contribution to the truncation error at the points elsewhere.

In Table 5.1, the error is computed after the enforcement of the grid tree structure. For this case, using the directional local scale (3.4), the error indicated in the third column is zero within machine precision, as a consequence of Lemma 3.2, and in accordance with the statement of Theorem 4.1. For comparison, we also consider local directional scales for $\gamma = (\gamma_x, \gamma_y) \in \Gamma^j$ by just taking

$$(5.2) \quad a_x^j(\gamma) = \min \{|\gamma_x - \eta_x|, \eta = (\eta_x, \gamma_y) \in \Gamma^j, \eta \neq \gamma\},$$

which do not satisfy the hypotheses (3.1-3.3) for general tree like grids. Observe in the second column of Table 5.1 that the error for this form of directional local scale is significant.

Next we consider similar tests applied to the refined grids, as explained in the Section 2. It is interesting to observe in Table 5.2 that the differences (5.1) are also zero within machine precision, independently of the directional local scales used as step size. For this example, the sparse grids that come after the tresholding and refinement operators naturally hold the directional tree structure. Furthermore, both formulae for the directional local scale, (3.4) and (5.2), coincide.

In the fourth columns of Table 5.1 and Table 5.2 we indicate the percentage of points present in each of the sparse grids with respect to the total number of points of the uniform grid in the finest scale level. At the begining, since the features of the magnetic field are highly concentrated around the initial pulse, there is a large amount of data compression. As time

time	error using (5.2)	error using (3.4)	% used points	% interpolations
t_1	1.38777×10^{-17}	1.38777×10^{-17}	9	12
t_2	2.77555×10^{-17}	2.77555×10^{-17}	19	15
t_3	49.9600×10^{-17}	49.9600×10^{-17}	33	21
t_4	58.2867×10^{-17}	58.2867×10^{-17}	50	27

TABLE 5.2

Error in SPR derivatives using either the uniform grid spacing of the finest scale level or the local directional scales of each point: tests for refined SPR grids

evolves, the circular wave front spreads towards the boundary, requiring more points to be represented. As expected, the SPR refined grids are more numerous than the grids obtained by just enforcing the tree structure.

The last columns in Table 5.1 and Table 5.2 show the number of interpolations used in the computation of the SPR derivatives using the directional local scale (3.4) expressed as a percentage of the number of interpolations required to compute the derivatives using the step size of the finest scale level. In both studies, the savings in interpolations range from 87 to 70%, for the tree like SPR grids, and from 88 to 73%, for the refined SPR grids, the gain being more significant at the beginning of the simulation, when the grids have more sparsity.

Now, being sure that using the directional local scale (3.4) the resulting truncation error is practically zero at the points of the sparse grids holding the tree structure, either with or without refinement, we turn to the analysis at the points elsewhere. We recall from Section 4.2 that outside the grid, where the SPR derivative is evaluated, the truncation error represents the wavelet coefficients of the function $L_{FD}^{x,j} \mathcal{R}^j(., Hz^j)(\nu)$. Our purpose is to evaluate the efficiency of the refinement procedure in capturing a good SPR representation for this "derivative" of the magnetic field. The results of such consistency analysis are presented in Table 5.3, for tree like grids without refinement, and in Table 5.4, after the refinement operation. In both cases, we extend the analysis by considering $j = 7$, and $j = 8$, and letting the thresholding parameter to be $\varepsilon_H = 10^{-3}$, 10^{-4} and 10^{-5} . As expected, a comparison of the results in both tables reveal that the refinement operation helps to reduce the truncation error, at the cost of less sparsity. However, the ratio TE_{wr}/TE_r , of the truncation error without refinement and the truncation error with refinement, seems to be greater than the ratio NP_r/NP_{wr} , of the number of points in the refined grid and the number of points before refinement.

The plots in Figure 5.2 illustrate this behavior for the magnetic fields at $t = t_1$ (top) and $t = t_4$ (bottom), for the variation of the threshold parameter ε_H between 10^{-5} to 10^{-3} . For instance, we conclude from the curves in the top-left part, for SPR grids at $j = 7$ for the magnetic field at $t = t_1$, that the refinement doubles the number of points but the truncation error without refinement can be 15 to 30 times larger, specially for $\varepsilon_H < 7 \times 10^{-4}$. At higher resolution $j = 8$, the curves in the top-right part of Figure 5.2 show that the gain in the truncation error double-compensates the increase in the number of points in the refined grid, for $\varepsilon \geq 10^{-4}$. For smaller thresholding parameters $\varepsilon < 5 \times 10^{-5}$, the impact of refinement is again very important in reducing the truncation error with low cost in the additional number of points. A similar trend is verified for the field at $t = t_4$, as we can see in Figure 5.2 (bottom), both for $j = 7$ (left) and $j = 8$ (right). These plots show that for larger thresholding parameters there is a balance in the reduction of the truncation error with the increase of the number of points in the refined grids. For smaller values of ε_H , the gain in the truncation error with refinement can be very significant, up to 40 or 12 times larger than without it, for $j = 7$ and 8, respectively, with very low extra cost.

The plots in Figure 5.3 represent the normalized truncation error $TE/(100 \times \varepsilon_H)$ for the H_z field at $t = t_1$ (left) and $t = t_4$ (right). Since the truncation error represents the wavelet coefficients of the function $L_{FD}^{x,j} \mathcal{R}^j(., Hz^j)(\nu)$, the conclusion drawn from these plots is that, after the refinement of the SPR (ε_H) representation of the magnetic field, we obtain a SPR (ε_E) representation for its x-“derivative”, where $\varepsilon_E \approx 100 \times \varepsilon_H$.

6. Concluding Remarks . In this paper we consider adaptive finite difference schemes $L_a^{r,j} f^j$, for partial derivatives in the $r = x$ and $r = y$ directions, arising in the bidimensional SPR method. The information f^j is formed by point values in a sparse grid adapted to the function under consideration, and the main principle is to approximate directional derivatives by uniform finite differences, using step size proportional to each point local scale $a_r^j(\gamma)$ in the corresponding

time	tree structure			
	$j = 7$		$j = 8$	
	TE error	# points	TE error	# points
$\varepsilon_H = 10^{-3}$				
t_1	5.400440×10^{-1}	381	5.598303×10^{-1}	381
t_2	4.734112×10^{-1}	581	2.393129×10^{-1}	605
t_3	4.958390×10^{-1}	821	5.239063×10^{-1}	821
t_4	5.431991×10^{-1}	1109	5.425516×10^{-1}	1117
$\varepsilon_H = 10^{-4}$				
t_1	19.27483×10^{-2}	765	12.37215×10^{-2}	1037
t_2	10.00526×10^{-2}	1557	10.32946×10^{-2}	1557
t_3	6.279617×10^{-2}	2757	6.329703×10^{-2}	2749
t_4	5.862434×10^{-2}	3873	6.184630×10^{-2}	3913
$\varepsilon_H = 10^{-5}$				
t_1	10.29540×10^{-3}	1157	11.87374×10^{-3}	2509
t_2	92.42311×10^{-3}	2469	13.19974×10^{-3}	5377
t_3	8.507794×10^{-3}	4577	8.576567×10^{-3}	9501
t_4	31.28736×10^{-3}	6525	23.81233×10^{-3}	13325

TABLE 5.3

Truncation error in SPR derivatives: tests for tree structured grids without refinement

time	refined grid			
	$j = 7$		$j = 8$	
	TE error	#points	TE error	#points
$\varepsilon_H = 10^{-3}$				
t_1	0.9880369×10^{-1}	1018	$1.087488e \times 10^{-1}$	1597
t_2	1.223386×10^{-1}	1933	1.297945×10^{-1}	2622
t_3	1.510714×10^{-1}	3437	1.845292×10^{-1}	3625
t_4	1.224536×10^{-1}	5128	1.482000×10^{-1}	5154
$\varepsilon_H = 10^{-4}$				
t_1	0.6293835×10^{-3}	1485	2.812171×10^{-2}	3332
t_2	2.738631×10^{-2}	3118	2.305200×10^{-2}	6849
t_3	0.9277954×10^{-2}	5552	1.877493×10^{-2}	12585
t_4	1.380015×10^{-2}	8241	1.754093×10^{-2}	18203
$\varepsilon_H = 10^{-5}$				
t_1	0.7332167×10^{-3}	1990	1.116536×10^{-3}	5005
t_2	0.7992242×10^{-3}	3817	1.116536×10^{-3}	10769
t_3	0.7992242×10^{-3}	6516	2.181932×10^{-3}	20178
t_4	0.8040323×10^{-3}	9964	1.926749×10^{-3}	29610

TABLE 5.4

Truncation error in SPR derivatives: tests for grids after refinement

direction of differentiation. Required neighboring stencils that are not present in the adaptive grid are interpolated by means of a reconstruction operator. The main results of our study are:

- To preserve accuracy in such adaptive discretizations, the grid structure and the form of the directional local scales are important. Under conditions stated in (3.1-3.3), we prove an statement for the local behavior of the reconstruction operator. This crucial result is the basis for the proof of Theorem 4.1 that guaranties, for the fourth order case, that the adaptive SPR discretizations $L_a^{r,j} f^j$ coincide at the points of the adaptive SPR grid with their non adaptive formulations $L^{r,j} f^j$, where the step size corresponds to the spacing of the finest scale level. This result is favourable to the applications of the SPR method, in terms of operation count. In fact, the formulations $L_a^{r,j} f^j$ are more economic than $L^{r,j} f^j$, since they do not require the calculation of the stencils up to the most refined level, for those points having large local scales $a_r^j(\gamma) \gg \Delta_r^j$.
- A grid tree structure and directional local scales $a_r^j(\gamma)$ are defined satisfying conditions (3.1-3.3).

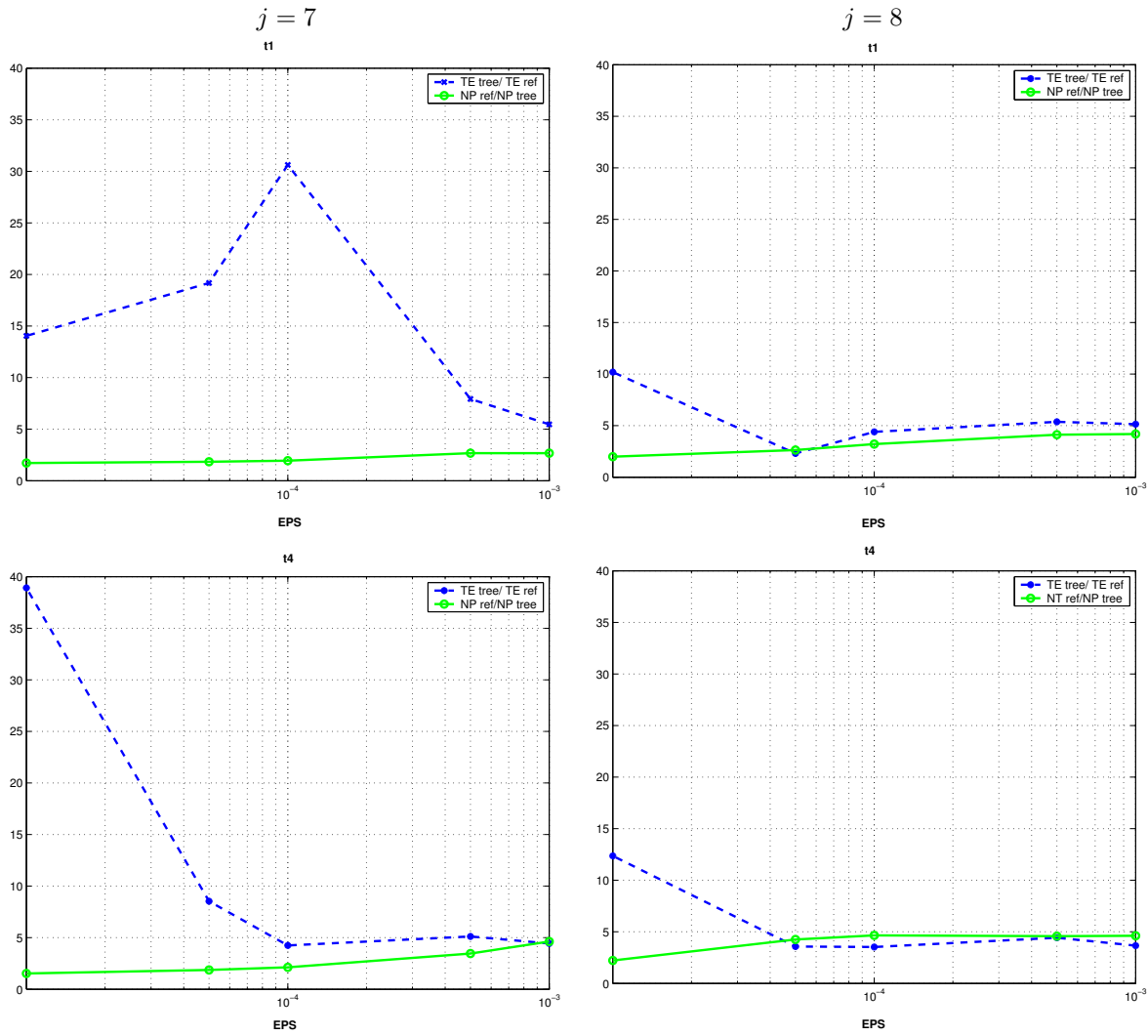


FIG. 5.2. Comparison of truncation error and grid densities, in terms of the threshold parameter ε_H , for tree like grids with and without refinement: $j = 7$ (left), $j = 8$ right; $t = t_1$ (top), $t = t_4$ (bottom).

- Furthermore, the consistency analysis indicates that, in order to keep the truncation error under a certain given precision, an adequate refinement strategy should be used.

Numerical tests are presented for functions representing magnetic fields occurring in the simulation of Maxwell equations. The numerical consistency analysis for such examples reveals that the structure of the grids that come from the application of the thresholding and refinement operators are sufficient to ensure a small truncation error. In fact, for the experiments of the present paper, the tree structure naturally holds for the SPR refined grids, which ensures that the adaptive SPR finite difference schemes $L_a^{r,j}$ can be safely applied. However, in general SPR representations, the grids obtained after thresholding and refinement operations may not have the tree structure. Therefore, before the application of adaptive SPR finite difference schemes $L_a^{r,j}$, the enforcement of the tree structure after refinement is recommended.

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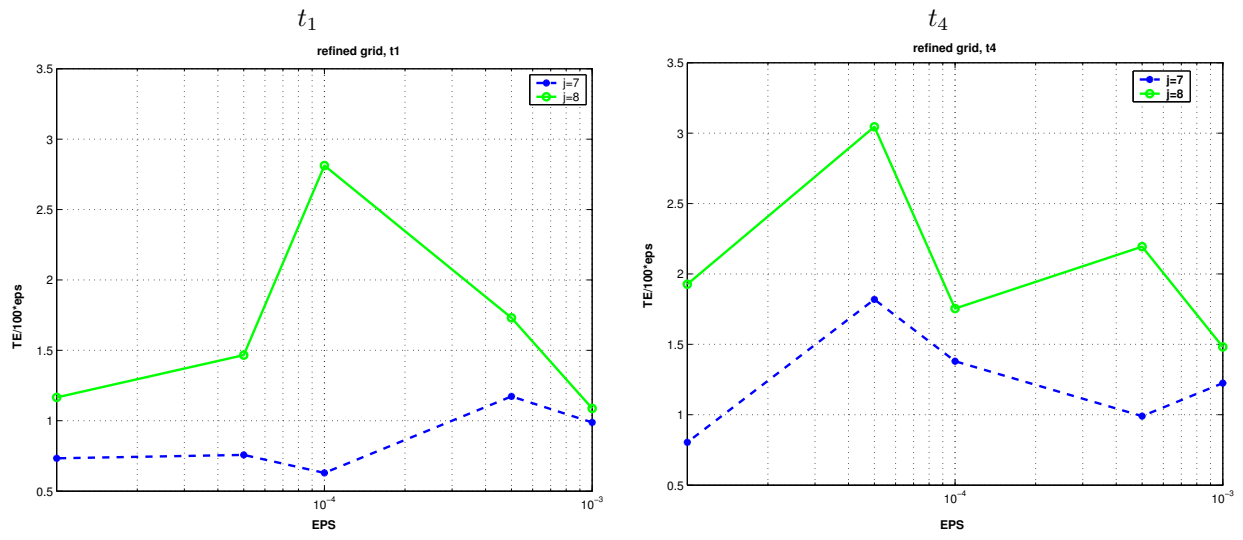


FIG. 5.3. Normalized truncation error $TE/(100 \times \epsilon_H)$ in terms of the threshold parameter ϵ_H

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