

© 1975 Premium Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, recorded in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilm, or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for \$15.00.

Moscow State University. Translated from Funktsional'nyi Analiz i Egzo Prilozheniya, Vol. 9, No. 1, pp. 59-60, January-March, 1975. Original article submitted October 29, 1973.

where $\phi_n: R \rightarrow R$ satisfies the following conditions: $\phi_n(t) = 0$ outside (a_n, c_n) , and on (a_n, c_n) it satisfies the inequality $0 < \phi_n(t) < 1/n$; the function $h: R \rightarrow R$ is defined as follows: $h(t) = 0$ for $t < 0$, $h(t) = t$ for $t \in [0, 1]$, and $h(t) = 1$ for $t > 1$. We can show that ϕ_n , P_n , and f are continuous. By a theorem of Dunford

$$\phi(t, x) = \sum_{n=1}^{\infty} \phi_n(t) h\left(\frac{x - p_n(x)}{e_n}\right)$$

We define the function $\phi: R \times L \rightarrow L$ by means of the formula

$$a_n = \frac{2a_n + 1}{4}, \quad b_n = \frac{2a_n}{4}, \quad e_n = \frac{a_n + b_n}{2}.$$

The function $P: R \times L \rightarrow L$ is given by the formula $P(t, x) = \sum_{n=1}^{\infty} \phi_n(t) e_n(x)$, where $\phi_n(t) = 0$ if $t \leq a_n$, $\phi_n(t) = 1$ if $t \geq b_n$, and ϕ_n is linear on the interval $[a_n, b_n]$, where

$$f(t, x) = \begin{cases} \phi(t, x), & \text{if } P(t, x) = 0, \\ \sqrt{\|P(t, x)\|} + \phi(t, x), & \text{if } P(t, x) \neq 0. \end{cases}$$

Proof. Suppose that Peano's theorem is true in some infinite-dimensional Banach space B . Let L be a closed subspace in B as described in Theorem 1. We introduce a function $f: R \times L \rightarrow L$ by means of the relation

THEOREM 2. Each Banach space in which Peano's theorem is true is finite-dimensional.

2) $\{e_i\}$ is a Schauder basis of the closed linear subspace L of B generated by the set $\{e_j\}$:
3) if $P_m(x) = \sum_{i=1}^m a_i(x)e_i$, then P_m is a projector in L with norm less than $1 + 1/m$ for an arbitrary natural number m .

THEOREM 1 (see Day [5]). Let B be an infinite-dimensional Banach space. There exist biorthogonal sequences $\{e_j\}$ and $\{a_i\}$ in B and B^* , respectively, such that

and the author [3] independently showed that Peano's theorem is not true in nonreflexive Banach spaces. We will now show that all of the showed that Peano's theorem is not true in the space C_0 of sequences which converge to zero. Yorke [2] showed that Peano's theorem is not true in the space C_0 of sequences which converge to zero. Yorke [2]

The first result in this direction was obtained by Dieudonne; in [1] he produced an example which

has a solution which is defined on some neighborhood of t_0 .

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0$$

We will say that in a Banach space B Peano's theorem is true if for each continuous function $f: R \times B \rightarrow B$ defined on some open set $V \subset R \times B$ and for each point $(t_0, x_0) \in V$ the Cauchy problem

A. N. Godunov

PEANO'S THEOREM IN BANACH SPACES

([6], p. 357) the function $f: R \times L \rightarrow L$ has a continuous extension $F: R \times B \rightarrow L$. We will show that the Cauchy problem

$$x'(t) = F(t, x(t)), \quad x(0) = 0 \quad (1)$$

does not have a solution in any neighborhood of zero. Suppose that some function $x: R \rightarrow B$, defined in a neighborhood of zero, is a solution of (1). Then $x(t) = \int_0^t F(t, x(t)) dt$, but $F(t, x(t)) \in L$, and therefore $x(t) \in L$ for each t . Consequently, $x(t)$ is a solution of the problem

$$x'(t) = f(t, x(t)), \quad x(0) = 0.$$

Two cases are possible: 1) there exist arbitrarily large n for which $x(b_n) \neq 0$; 2) for all sufficiently large n we have $x(b_n) = 0$.

We consider the first case. We choose m such that $x(b_m) \neq 0$. Clearly $x(t)$ is a solution of the Cauchy problem

$$u'(t) = f(t, u(t)), \quad u(b_m) = x(b_m). \quad (2)$$

The unique solution of (2) on $[b_m, 1]$ is the function

$$u(t) = \frac{x(b_m)}{\|x(b_m)\|} \cdot \frac{(t - \tilde{b}_m)^2}{4}, \quad (3)$$

where \tilde{b}_m satisfies the conditions $\tilde{b}_m < b_m$ and $(b_m - \tilde{b}_m)^2 = 4 \|x(b_m)\|^2$. The function $u(t)$ clearly satisfies the initial condition in (2). If $n < m$, since $x(b_m) = \int_0^{b_m} f(t, x(t)) dt$ and $\alpha_n(f(t, x(t))) = 0$ for $t < b_m$, we have $\alpha_n(x(b_m)) = 0$ and $\alpha_n(u(t)) = 0$. Consequently, for $t > b_m$ we have

$$P(t, u(t)) = \sum_{n=m}^{\infty} \alpha_n(u(t)) e_n = u(t) - \sum_{n=1}^{m-1} \alpha_n(u(t)) e_n = u(t).$$

We can show that $\varphi(t, u(t)) \equiv 0$ for $t \geq b_m$. Therefore, $u(t)$ is a solution of (2). Uniqueness of the solution of (2) follows from the fact that f is locally Lipschitzian in some neighborhood of the set $M = \{(t, u(t)) | t \in [b_m, 1]\} \subset R \times L$. In fact, $P(t, x)$ is Lipschitzian for $t > 0$ since for an arbitrary n we know that if $t \in [b_{n+1}, b_n]$, then

$$\|P(t, x) - P(t, y)\| = \left\| \sum_{k=1}^{\infty} \psi_k(t) \alpha_k(x - y) e_k \right\| \leq \|\psi_n(t) \alpha_n(x - y) e_n\| + \|(x - y) - P_n(x - y)\| \leq 4 \|x - y\|.$$

Since $P(t, x) \neq 0$ in some neighborhood of M , it follows that $P(t, x)/\sqrt{\|P(t, x)\|}$ is locally Lipschitzian in this neighborhood. Also, in some neighborhood of M we have $\varphi \equiv 0$. If $t > b_m$ and $n < m$, then $\alpha_n(x(t)) = \alpha_n(u(t)) = 0$. Since m can be chosen arbitrarily large and $\{e_n\}$ is a Schauder basis in L , we have $x(t) \equiv 0$. This contradiction shows that the first case is impossible.

We consider the second case. We choose m such that $x(b_m) = 0$. If $n > m$ and $t \in [b_{m+1}, b_m]$, then

$$\frac{d}{dt} \alpha_n(x(t)) = \frac{\alpha_n(x(t))}{\sqrt{\|P(t, x(t))\|}}. \quad (4)$$

Suppose that there exists $n > m$ such that $\alpha_n(x(t_0)) \neq 0$ for some $t_0 \in [b_{m+1}, b_m]$. From (4) it follows that $\alpha_n(x(b_m)) > \alpha_n(x(t_0)) > 0$, and $\alpha_n(x(b_m)) < \alpha_n(x(t_0))$ if $\alpha_n(x(t_0)) < 0$. We obtain $\alpha_n(x(t)) \equiv 0$ on $[b_{m+1}, b_m]$ for $n > m$. Therefore,

$$\frac{d}{dt} \alpha_m(x(t)) = \varphi_m(t) h\left(\frac{(t - b_{m+1})^2}{4}\right)$$

on $[0, c_m]$, that is, $(d/dt) \alpha_m(x(t)) = 0$ on $[0, a_m]$ and $(d/dt) \alpha_m(x(t)) > 0$ on (a_m, c_m) . Consequently, $\alpha_m(x(c_m)) > 0$. Now it is not hard to show that $\alpha_m(x(b_m)) > 0$, but this contradicts our assumptions. So the second case is impossible. Thus, problem (1) does not have a solution. The theorem is proved.

LITERATURE CITED

1. J. Dieudonné, Acta. Sci. Math. Szeged, 12, Pars B, 38-40 (1950).
2. J. A. Yorke, Funkcialaj Ekvacioj, 13, 19-21 (1970).

3. A. N. Godunov, "Counterexample of Peano's theorem in an infinite-dimensional Hilbert space," *Vestnik MGU, Seriya Matem. i Mekhan.*, No. 5, 31-34 (1972).
4. A. Cellina, Bull. Amer. Math. Soc., 78, No. 6, 1069-1072 (1972).
5. M. M. Day, Proc. Am. Math. Soc., 13, 655-658 (1962).
6. J. Duaguji, Pacific J. Math., 1, 353-367 (1951).