DIFFERENTIAL FORMS
AND
THEIR INTEGRALS

(preliminary, incomplete version)

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Contents

CHAPTER 0

A review of basic vector calculus

In this chapter we will review some basic facts of vector calculus which will be used extensively along these notes. We will assume the reader familiar with the differential and integral calculus for real valued functions of one real variable, as well as with the basic topology of Euclidean spaces: open and closed sets, continuity, compactness, Cauchy sequences etc. The material in sections 1 and 2 are quite standard in Calculus courses, while the one in section 3 is probably less "popular."

1. Differentiable functions

We will consider the Euclidean space $\mathbb{R}^n$ with its canonical inner product and associated norm.

For a point $x \in \mathbb{R}^n$ and $r \in \mathbb{R},\ r > 0$, we denote by $B^n(p, r) := \{x \in \mathbb{R}^n : \|x - p\| < r\}$ the *ball of radius r centred at p*. When $p = 0$ we simply write $B^n(r)$ for $B^n(0, r)$.

We will denote by $L(\mathbb{R}^n, \mathbb{R}^m)$ the space of linear maps of $\mathbb{R}^n$ into $\mathbb{R}^m$. There is a natural identification of $L(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{nm}$, associating to a linear transformation L, the entries (in a fixed order) of the matrix representing L in the canonical bases. This identification induces a scalar product in $L(\mathbb{R}^n, \mathbb{R}^m)$

$$\langle A, B\rangle \ = \ \text{trace } A^t B,$$

where A^t is the transpose of A. Often it is more convenient to consider the *operator norm*, defined by

$$\|L\| = \sup\{\|Lx\| : x \in \mathbb{R}^n,\ \|x\| = 1\}.$$

The two norms are equivalent (see Exercise 4.1), so for the basic topological concepts like convergence, continuity etc., it does not matter which one we use. In what follows we will consider the operator norm, unless otherwise stated. Observe that, for the operator norm, we have the inequality $\|L \circ T\| \leq \|L\|\|T\|$.

Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \longrightarrow \mathbb{R}^m$ a function.

1.1. Definition. f is *differentiable at $x \in U$* if there exist a linear map $\mathrm{d}f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that[1]

$$\lim_{\|h\|\to 0} \frac{\|f(x + h) - f(x) - \mathrm{d}f(x)(h)\|}{\|h\|} = 0.$$

The map $\mathrm{d}f(x)$ is called the *differential of f at x*.

f is *differentiable* in U if it is differentiable at every point of U.

1.2. Remark. For a function $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ the *derivative* of f at $x \in U$, $f'(x)$, is defined as

$$f'(x) := \frac{\mathrm{d}f}{\mathrm{d}t}|_{t=x} := \lim_{t\to 0}[f(x + t) - f(x)]t^{-1},$$

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[1] Observe that, if $\|h\|$ is sufficiently small, $x + h \in U$.

if the limit exists. The *differential* of f at x is the linear map

$$\mathrm{d}f(x) : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathrm{d}f(x)h = f'(x)h.$$

The following facts are easy to prove and and we leave the proofs to the reader (Exercise 4.2).

1.3. Proposition.

- *If f, g are differentiable at x and $a \in \mathbb{R}$, then $f+g$ and af are differentiable and* $\mathrm{d}(f+g)(x) = \mathrm{d}f(x) + \mathrm{d}g(x)$, $\mathrm{d}[af](x) = a[\mathrm{d}f(x)]$.
- *If f is differentiable at x, f is continuous at x.*
- *If f is differentiable at x, the differential is unique.*
- (The chain rule) *If $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $g : \Sigma \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^p$ are differentiable at x and $f(x)$ respectively, then $g \circ f$ is differentiable at x and* $\mathrm{d}[g \circ f](x) = \mathrm{d}g(f(x)) \circ \mathrm{d}f(x)$.

1.4. Example. Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map. Then L is differentiable and $\mathrm{d}L(x) = L, \ \forall\, x \in \mathbb{R}^n$, as follows directly from the definition.

1.5. Example. If $B : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a bi-linear map, B is differentiable and $\mathrm{d}B(x,y)(z,w) = B(x,w) + B(z,y)$. In particular, if $f, g : U \longrightarrow \mathbb{R}$ are differentiable functions, the map $F : U \longrightarrow \mathbb{R}^2$, $F(x) = (f(x), g(x))$ is differentiable with $\mathrm{d}F(x) = (\mathrm{d}f(x), \mathrm{d}g(x))$. Since the product $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is bilinear, by the chain rule the function fg is differentiable and the product rule $\mathrm{d}(fg)(x) = f(x)\mathrm{d}g(x) + g(x)\mathrm{d}f(x)$ holds.

In the theory of real valued functions of one real variable, an elementary but useful result is the Mean Value Theorem.

1.6. Theorem. [Mean Value Theorem] *If $f : [a, a+h] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function, then there exists $t_0 \in [0,1]$ such that*

$$f(a+h) - f(a) = f'(a + t_0 h)h.$$

The Theorem extends, with essentially the same proof, to the case of a differentiable function $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. For functions with values in $\mathbb{R}^m$, $m > 1$, such a Theorem does not hold any longer (see Exercise 4.4) but, at least, we have an inequality. The result will still be called the Mean Value Theorem.

1.7. Theorem. [Mean Value Theorem] *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a continuous function. Suppose that the segment with endpoints $a, a+h$ is contained in U and f is differentiable at the points of the segment. Then*

$$\|f(a+h) - f(a)\| \leq \|h\| \ \sup\{\|\mathrm{d}f(a+th)\| : t \in [0,1]\}.$$

Proof. Consider the function $\phi : [0,1] \longrightarrow \mathbb{R}^m$, $\phi(t) = f(a+th)$. ϕ is differentiable, by the chain rule, $\phi(0) = f(a)$, $\phi(1) = f(a+h)$ and $\mathrm{d}\phi(t)(1) = \mathrm{d}f(a+th)(h)$. Let $M = \sup\{\|\mathrm{d}\phi(t)\| : t \in [0,1]\}$. It is then sufficient to prove that $\|\phi(1) - \phi(0)\| \leq M$. For this purpose we will show that, given $\epsilon > 0$, $\|\phi(1) - \phi(0)\| \leq M + \epsilon$. Consider the set

$$A = \{t \in [0,1] : \|\phi(s) - \phi(0)\| \leq (M+\epsilon)s, \ \forall\, s \in [0,t]\}.$$

It is easy to see that $A = [0, a]$ for some $a \in (0, 1]$. We wish to prove that $a = 1$. Suppose $a < 1$. Then there exists a positive δ such that $a + \delta < 1$ and for $k \in [0, \delta)$

$$\phi(a + k) - \phi(a) = \mathrm{d}\phi(a)k + r(k) \quad \text{with } \|r(k)\| \leq \epsilon k$$

(by the definition of differentiability at a). Then $\|\phi(a + k) - \phi(a)\| \leq (M + \epsilon)k$. But $a \in A$, hence $\|\phi(a) - \phi(0)\| \leq (M + \epsilon)a$. Therefore $\|\phi(a + k) - \phi(0)\| \leq (M + \epsilon)(a + k)$. In particular $a + k \in A$, a contradiction. ☐

1.8. Definition. Let $f : U \longrightarrow \mathbb{R}^m$ be differentiable at $x \in U$ and $X \in \mathbb{R}^n$. The *directional derivative* of f at x in the X direction is defined as

$$\frac{\partial f}{\partial X}(x) := \mathrm{d}f(x)(X).$$

1.9. Remark. For reasons that will be clear later on will use often the notation $X_x(f)$ for $\mathrm{d}f(x)(X)$.

If $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$, $\dfrac{\partial f}{\partial e_i}(x)$ is the i^{th} *partial derivative* at x and will be denoted, as usual, by $\dfrac{\partial f}{\partial x_i}(x)$. If f is differentiable at x and $h = \sum_{i=1}^n \alpha_i e_i$, then

$$\mathrm{d}f(x)h = \sum_{i=1}^n \alpha_i \mathrm{d}f(x)(e_i) = \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial x_i}(x).$$

In particular, if $f(x) = (f_1(x), \ldots, f_m(x))$, where $f_i : U \longrightarrow \mathbb{R}$ are the coordinate functions of f, then the *Jacobian matrix* $[\dfrac{\partial f_j}{\partial x_i}(x)]$ is the matrix that represents $\mathrm{d}f(x)$ in the canonical bases. This is the multidimensional analogue of Remark 1.2.

Let $\gamma : (a, b) \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$ be a differentiable map. We will also say that γ is a *differentiable curve.* For such a function, the tangent vector at $t \in (a, b)$ (or, sometimes, at $\gamma(t)$) is the vector

$$\dot{\gamma}(t) := \mathrm{d}\gamma(t)(1) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=t}\gamma(s).$$

It is easy to see that if $\gamma : (-\epsilon, \epsilon) \longrightarrow R^n$ is a differentiable curve with $\dot{\gamma}(0) = X$,

$$\frac{\partial f}{\partial X}(x) = \mathrm{d}(f \circ \gamma)(0)(1) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(\gamma(t)).$$

1.10. Remark. The latter fact gives a geometric interpretation of the differential of f: *the image, via* $\mathrm{d}f$*, of the vector tangent to a given curve* γ *is the vector tangent to the image curve,* $f \circ \gamma$.

In particular the right hand side of the formula above does not depend on γ as long the curve passes trough x and its tangent vector, at x, is X. This observation allow us to define the directional derivative, hence partial derivatives, even for a class of not necessarily differentiable functions (in the sense of Definition 1.1). If $f : U \longrightarrow \mathbb{R}^m$ is a function and $X \in \mathbb{R}^n$, we define the directional derivative of f at $x \in U$, in the direction of X as

$$\frac{\partial f}{\partial X}(x) := \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(x + tX),$$

if it exists. The partial derivatives may exist even if the function is not differentiable (see Exercise 4.7). However we have

1.11. PROPOSITION. *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. If the partial derivatives of f exist and are continuous*[2]*, f is differentiable.*

PROOF. We will prove the Proposition for $n = 2$ to avoid notational complications. We want to show that the linear map $L(x, y)(h, k) = \dfrac{\partial f}{\partial x}h + \dfrac{\partial f}{\partial y}k$ is the differential of f at $(x, y) \in \mathbb{R}^2$. Hence we have to show that, given $\epsilon > 0$,

$$\|f(x + h, y + k) - f(x, y) - \frac{\partial f}{\partial x}h - \frac{\partial f}{\partial y}k\| \leq \epsilon\|(h, k)\|,$$

if $\|(h, k)\|$ is sufficiently small. Adding and subtracting $f(x, y + k)$ and using Exercise 4.5, we have that the quantity on the left of the inequality sign is less or equal to

$$\|h\| \sup\{\|\frac{\partial f}{\partial x}(x + th, y + k) - \frac{\partial f}{\partial x}(x, y)\| : t \in [0, 1]\} + \|k\| \sup\{\|\frac{\partial f}{\partial y}(x, y + tk) - \frac{\partial f}{\partial x}(x, y)\| : t \in [0, 1]\}.$$

The conclusion follows from the continuity of the partial derivatives. □

1.12. REMARK. Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. Let $\pi_i : \mathbb{R}^m \longrightarrow \mathbb{R}$, $\pi_i(x_1, \ldots, x_m) = x_i$ be the i^{th} projection. Then $f_i(x) = \pi_i \circ f(x)$ are the coordinates of $f(x)$. It is easy to see that f is differentiable at $x \in U$ if and only if the coordinate functions are differentiable at x and, in this case,

$$\mathrm{d}f(x)(X) = (\mathrm{d}f_1(x)(X), \ldots, \mathrm{d}f_m(x)(X)) = \sum[\mathrm{d}f_i(x)X]e_i.$$

Partial derivatives take care of the "opposite" situation. Given a splitting of $\mathbb{R}^n = \mathbb{E}_1 \oplus \mathbb{E}_2$ as a direct sum of complementary subspaces and a point $(x_0, y_0) \in \mathbb{E}_1 \oplus \mathbb{E}_2$, we can consider the inclusion $i_j : \mathbb{E}_j \longrightarrow \mathbb{R}^n$, $i_1(x) = (x, y_0)$, $i_2(y) = (x_0, y)$ and the functions $f^{(i)} = f \circ i_j : \mathbb{E}_j \cap U \longrightarrow \mathbb{R}^m$. If f is differentiable at (x_0, y_0), $f^{(1)}$ (resp. $f^{(2)}$) is differentiable at x_0 (resp. y_0) and $\mathrm{d}f(x_0, y_0)(X, Y) = \mathrm{d}f^{(1)}(x_0)(X) + \mathrm{d}f^{(2)}(y_0)(Y)$. So we can define the *partial differentials* relative to the given splitting, $\mathrm{d}_j f = \mathrm{d}f \circ i_j$. The existence of the partial differentials does not implies the existence of the differential of f. However, as in Proposition 1.11, if the partial differentials exist and are continuous, then f is differentiable. Obviously the same arguments work for a decomposition of $\mathbb{R}^n$ into the direct sum of k complementary subspaces. Partial derivatives are, essentially, partial differentials relative to the canonical splitting of $\mathbb{R}^n$ as the direct sum of the coordinate lines.

If $f : U \longrightarrow \mathbb{R}^m$ is a differentiable function, the differential can be seen as a map $\mathrm{d}f : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$, $x \rightsquigarrow \mathrm{d}f(x)$.

1.13. DEFINITION. We will say that f is *twice differentiable at* $x \in U$, if the function $\mathrm{d}f : U \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is differentiable at x. In this case, the differential of $\mathrm{d}f$ at x will be called the *second differential of f at x* and will be denoted by $\mathrm{d}^2 f(x)$.

Inductively, we define, if it exists, the k^{th} *differential of f at x*, $\mathrm{d}^k f(x)$, as the differential, at x, of $\mathrm{d}^{k-1} fU \longrightarrow L(\mathbb{R}^n, L(\mathbb{R}^n, \ldots)$

We will say that f is of *class C^k* in U if $\mathrm{d}^k(x)$ exists, $\forall\ x \in U$, and is continuous, as a function of x.

We will say that f is of *class C^∞*, if it is of class C^k, $\forall\ k$. If f is C^∞ we will also say that f is *smooth.*

1.14. REMARK. It is easy to produce examples of C^k functions that are not C^{k+1} (see Exercise 4.6). One of the important features of the class of smooth functions is that it is closed under differentiation, i.e. f is smooth if and only if $\mathrm{d}f$ is smooth.

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[2] As maps $\dfrac{\partial f}{\partial x_i} : U \longrightarrow \mathbb{R}^m$.

If f is twice differentiable at x, then $\mathrm{d}^2 f(x) \in L(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m))$, and so it can be seen as the bilinear map $\mathrm{d}^2 f(x)(X, Y) = \mathrm{d}(\mathrm{d}f)(x)(X)(Y)$. In a similar way, $\mathrm{d}^k f(x)$ can be viewed as a k-multilinear map.

1.15. THEOREM. [Schwarz's Theorem] *If* $\mathrm{d}^2 f(x)$ *exists, it is a symmetric bilinear form.*

We will sketch a proof in the case that f is C^2 in Exercise 4.22

1.16. REMARK. If f is k times differentiable at x, $\mathrm{d}^k f(x)$ is a k-multilinear *symmetric* map. Moreover we can define higher order partial derivatives. Schwarz Theorem 1.15 and a simple induction imply that if f is of class C^k, the result of successive partial derivatives, up to order k, does not depend on the order of derivations.

1.17. EXAMPLE. If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map $\mathrm{d}L(x) = L,\ \forall\ x \in \mathbb{R}^n$. In particular the differential $\mathrm{d}L : \mathbb{R}^n \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is the constant map. Hence $\mathrm{d}^k L = 0$, if $k \geq 2$, and L is C^∞. Similarly, if $B : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ is a bilinear map, $\mathrm{d}B(x, y)(z, w) = B(x, w) + B(z, y)$. In particular $\mathrm{d}B$ is a linear map, hence C^∞, and so is B.

1.18. EXAMPLE. Let $M(n, \mathbb{R})$ be the space of $n \times n$ matrices with real coefficients. The product map

$$m : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R}), \quad m(A, B) = AB,$$

is a bi-linear map, hence smooth. Also the map

$$g : M(n, \mathbb{R}) \times M(n, \mathbb{R}) \longrightarrow L(M(n, \mathbb{R}), M(n, \mathbb{R})), \quad g(A, B)H = AHB$$

is bilinear, hence smooth.

Since the determinant, $\det : M(n, \mathbb{R}) \longrightarrow \mathbb{R}$, is a continuous function, the set of invertible matrixes, $GL(n, \mathbb{R})$, is an open subset of $M(n, \mathbb{R})$. Consider the inversion map

$$\iota : GL(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R}), \quad \iota(A) = A^{-1}.$$

CLAIM The map ι is smooth.

PROOF. Differentiating the identity $(A + tB)(A + tB)^{-1} = \mathbb{1}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(A + tB)^{-1} = -A^{-1}BA^{-1}.$$

So, if ι is differentiable at $A \in GL(n, \mathbb{R})$, $\mathrm{d}\iota(A)(B) = -A^{-1}BA^{-1}$. It is easy to check that, in fact, the formula defines a linear map which is the differential of ι at A. In particular ι is continuous. The differential of ι is then given by the composition

$$GL(n, \mathbb{R}) \xrightarrow{\Delta} GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \xrightarrow{\iota \times \iota} GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \xrightarrow{-g} L(M(n, \mathbb{R}), M(n, \mathbb{R})),$$

where $\Delta : GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$ is the diagonal map, $\Delta(A) = (A, A)$, and g is as above. Hence $\mathrm{d}\iota$ is continuous and ι is of class C^1. At this point a simple induction proves the Claim. □

As we have seen, the differential of a function f at a point x, provides the best linear approximation of $f - f(x)$ in a neighborhood of x. The Taylor formula provides the best polynomial approximation, for functions with more differentiability.

1.19. Theorem. [Infinitesimal version of Taylor Theorem] *Let* $f : U \longrightarrow \mathbb{R}^m$ *be a function* s *times differentiable in an open neighborhood of* $a \in U$ *and such that* $\mathrm{d}^{s+1}(a)$ *exists. Then*

$$f(a+h) = f(a) + \sum_{k=1}^{s+1} \frac{1}{k!}\mathrm{d}^k f(a)(h, \dots, h) + r(h) \quad \textit{with} \quad \lim\nolimits_{h\to 0} r(h)\|h\|^{-(s+1)} = 0.$$

Proof. The proof is a simple consequence of the lemma below, which is of interest on its own

1.20. Lemma. *Let* $r : B = B^n(R) \longrightarrow \mathbb{R}^m$ *be a function* s *times differentiable in* B *and* $s+1$ *times differentiable in 0. Assume* $\mathrm{d}^j r(0) = 0,\ 0 \le j \le s+1$. *Then* $\lim_{x\to 0} r(x)\|x\|^{-(s+1)} = 0$.

Proof. We proceed by induction on s. If $s = 0$ the conclusion follow from the definition of differentiability. Suppose the conclusion true for s. By the mean value Theorem we have

$$\|r(x)\| \le M\|x\|, \qquad M = \sup\{\|\mathrm{d}r(tx)\|| : t \in [0,1]\}.$$

Applying the inductive hypothesis to $\mathrm{d}r$, given $\epsilon > 0$ there exist $\delta > 0$ such that, if $\|y\| < \delta,\ \|\mathrm{d}r(y)\| < \epsilon\|y\|^s$. Hence if $\|x\|\delta,\ M \le \epsilon\|x\|^s$ and $\|r(x)\| \le \epsilon\|x\|^{s+1}$. □

□

In the linear context, i.e. vector spaces and linear maps, we study properties that are invariant for linear isomorphisms, i.e. changes of bases. The analogue in the differential context are properties that are invariant for (local) diffeomorphisms , i.e. change of variables (or coordinates).

1.21. Definition. A map $\phi : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ between open sets is a C^k *diffeomorphism* if there exists a C^k map $\psi : V \longrightarrow U$, such that $\psi \circ \phi = \mathbb{1}_U,\ \phi \circ \psi = \mathbb{1}_V$. ϕ is a *local diffeomorphism* if $\forall\ x \in U$, there exists an open neighborhood $\tilde{U} \subseteq U$ of x, such that $\phi|_{\tilde{U}}$ is a diffeomorphism onto an open neighborhood $\tilde{V}$ of $\phi(x)$ in V. A local diffeomorphism will also be called a *change of variables* (or *coordinates*).

1.22. Remark. If ϕ is a diffeomorphism, then $\mathrm{d}\phi(x)$ is an isomorphism, by the chain rule. Hence $n = m$.

The following fact will be useful.

1.23. Lemma. *If* $\phi : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ *is a* C^k *map,* $k \ge 1$, *between open sets, and* ϕ *admits a differentiable inverse, then the inverse is of class* C^k.

Proof. From the chain rule $\mathrm{d}[\phi^{-1}](\phi(x)) \circ \mathrm{d}\phi(x) = \mathbb{1}$. In particular $\mathrm{d}[\phi^{-1}]$ is given by the composition

$$V \xrightarrow{\phi^{-1}} U \xrightarrow{\mathrm{d}\phi} GL(n, \mathbb{R}) \xrightarrow{\iota} GL(n, \mathbb{R}),$$

where ι is the matrix inversion map, that, by Example 1.18, is smooth. Hence $\mathrm{d}[\phi^{-1}]$ is continuous and ϕ^{-1} is of class C^1. In general the argument gives that, if ϕ^{-1} is $C^s,\ s < k$, $\mathrm{d}[\phi^{-1}]$ is also of class C^s so ϕ^{-1} is of class C^{s+1} and this concludes the proof.

□

Let $\mathbb{E}, \mathbb{F}$ be real, finite dimensional vector spaces and $L : \mathbb{E} \longrightarrow \mathbb{F}$ be a linear map. Then, in suitable bases, F has a very simple expression. In fact we can chose a basis $\{e_1, \dots, e_n\}$ of $\mathbb{E}$, such that $\{e_{k+1}, \dots, e_n\}$

is a basis of the kernel of F. Then $\{f_1 = F(e_1), \dots, f_k = F(e_k)\}$ is a basis of the image of F that we can complete with vectors $\{f_{k+1}, \dots, f_m\}$ to have a basis for $\mathbb{F}$. Then, in terms of coordinates in these bases,

$$F(x_1, \dots, x_n) = (x_1, \dots, x_k, 0, \dots, 0).$$

Since a differentiable function is (locally) approximated by a linear one, we can expect something similar to hold, locally, for differentiable maps, up to change of coordinates. In fact this is the case.

1.24. Theorem. [Rank Theorem] *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function of class C^k. Let $p \in U$ be such that rank $\mathrm{d}f(x) = k$ in an open neighborhood of p. Then there exist open neighborhoods U of p and V of $f(p)$, and diffeomorphisms $\phi : U \longrightarrow \tilde{U}$, $\psi : V \longrightarrow \tilde{V}$ such that:*

$$\psi \circ f \circ \phi^{-1}(x_1, \dots, x_n) = (x_1, \dots x_k, 0, \dots, 0), \qquad for \quad (x_1, \dots, x_n) \in \tilde{U}.$$

This Theorem will follow from the next three.

1.25. Theorem. [Inverse function Theorem] *Let $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^k map, $k \geq 1$, such that $\mathrm{d}f(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isomorphism. Then the exist an open neighborhood U' of p such that $f|U'$ is a C^k diffeomorphism onto an open neighborhood of $f(p)$.*

Proof. Without loss of generality we may assume $p = 0 = f(p)$. Moreover, by composing f with $\mathrm{d}f(0)^{-1}$, we may assume $\mathrm{d}f(p) = \mathbb{1}$. Consider the function $g(x) = f(x) - x$. Then $g(0) = 0$, $\mathrm{d}g(0) = 0$. Let r be a positive real number such that if $x \in B^n(r)$, $\mathrm{d}f(x)$ is invertible and $\|\mathrm{d}g(x)\| < \frac{1}{2}$.

Claim 1. *If $y \in B^n(\frac{r}{2})$ then there exists a unique $x \in B^n(r)$ such that $f(x) = y$.*

Proof. Define $x_0 = 0$, $x_1 = y$ and, inductively, $x_{n+1} = y + g(x_n)$. By the mean value Theorem we have:

$$\|x_{n+1} - x_n\| = \|g(x_n) - g(x_{n-1})\| \leq \frac{1}{2}\|x_n - x_{n-1}\|$$

$$\|x_{n+1}\| = \|g(x_n) + y\| \leq \|g(x_n)\| + \|y\| < \frac{1}{2}\|x_n\| + \|y\| \leq \frac{1}{2}\|x_n\| + \frac{r}{2} - \epsilon,$$

for some $\epsilon > 0$, independent of n. Hence:

1. $\|x_{n+1} - x_n\| \leq 2^{-n}\|y\|$ (from the first equation),
2. $\|x_n\| < r - \epsilon$ (from the second equation and induction).

By condition (1), $\{x_n\}$ is a Cauchy sequence, hence it converges to a point x, and by condition (2), $x \in B^n(r)$.

Let us show that x is unique. Suppose $f(z) = f(x) = y, z \in B^n(r)$. Then $\|x - z\| = \|g(z) - g(x)\| \leq \frac{1}{2}\|z - x\|$ which implies $z = x$. □

So we have a well defined surjective map $f^{-1} : B^n(\frac{r}{2}) \longrightarrow U' = B^n(r) \cap f^{-1}(B^n(\frac{r}{2}))$. Observe that U' is the intersection of two open sets, hence it is open.

Claim 2. *$f^{-1} : B^n(\frac{r}{2}) \longrightarrow U'$ is C^k.*

Proof. We start by observing that $\|f(x_1) - f(x_2)\| \geq \|x_1 - x_2\| - \|g(x_1) - g(x_2)\| \geq \frac{1}{2}\|x_1 - x_2\|$, hence f^{-1} is continuous. In order to show that f^{-1} is differentiable we observe that, since f is differentiable,

$$f(x) - f(x_1) = \mathrm{d}f(x_1)(x - x_1) + h(x, x_1) \text{ with } \lim_{x \to x_1} \frac{h(x, x_1)}{\|x - x_1\|} = 0.$$

Applying $A := \mathrm{d}f(x_1)^{-1}$ to the equality above we have

$$A(y-y_1) + Ah_1(y,y_1) = f^{-1}(y) - f^{-1}(y_1),$$

where $y = f(x), y_1 = f(x_1),\ h_1(y,y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Then

$$\frac{h_1(y,y_1)}{\|y-y_1\|} = -\frac{h(x,x_1)}{\|x-x_1\|}\frac{\|x-x_1\|}{\|y-y_1\|}.$$

Since $\dfrac{\|x-x_1\|}{\|y-y_1\|} \leq 2$, $\displaystyle\lim_{y\to y_1}\frac{h_1(y,y_1)}{\|y-y_1\|} = 0$ and $\mathrm{d}[f^{-1}](y_1) = [\mathrm{d}f(x_1)]^{-1}$. Hence f^{-1} is differentiable , the Claim follows from Lemma 1.23. ☐

☐

1.26. THEOREM. [Local form of immersions] *Let* $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{n+p}$ *be a* C^k *map,* $k \geq 1$*, such that* $0 \in U, f(0) = 0$. *If* $\mathrm{d}f(0) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+p} = \mathbb{R}^n \times \mathbb{R}^p$ *is injective, there exists an open neighborhood* U' *of* 0 *and a* C^k *diffeomorphism* ϕ *between neighborhoods of* $0 \in \mathbb{R}^{n+p}$ *such that if* $x \in U'$,

$$\phi \circ f(x) = (x, 0).$$

PROOF. Up to an isomorphism of $\mathbb{R}^{n+p}$ which sends $\mathrm{d}f(0)e_i$ to e_i, we can assume that $\mathrm{d}f(0)v = (v, 0)$. Consider the function

$$F : U \times \mathbb{R}^p \longrightarrow \mathbb{R}^{n+p}, \qquad F(x,y) = f(x) + y.$$

Observe that $F(x,0) = f(x)$ and $\mathrm{d}F(0) = \mathbb{1}$. By the Inverse Function Theorem there is a diffeomorphism ϕ between neighborhoods of $0 \in \mathbb{R}^{n+p}$ such that $\phi \circ F = \mathbb{1}$. Then

$$\phi \circ f(x) = \phi \circ F(x, 0) = (x, 0).$$

☐

1.27. THEOREM. [Local form of subimmersions] *Let* $f : U \subseteq \mathbb{R}^{n+p} \longrightarrow \mathbb{R}^n$ *be a* C^k *map,* $k \geq 1$*, such that* $0 \in U, f(0) = 0$. *If* $\mathrm{d}f(0) : \mathbb{R}^{n+p} \longrightarrow \mathbb{R}^n$ *is surjective, then there exists a* C^k *diffeomorphism* $\psi : U' \longrightarrow V$*, between open neighborhoods of* $0 \in \mathbb{R}^{n+p}$*, such that if* $(x, y) \in U$,

$$f \circ \psi(x, y) = x.$$

PROOF. Up to an isomorphism of $\mathbb{R}^{n+p} = \mathbb{R}^n \times \mathbb{R}^p$, we can assume that $\ker \mathrm{d}f(0) = \{0\} \times \mathbb{R}^p$ and $\mathrm{d}f(0)(v, 0) = v$. Consider the function

$$F : U \longrightarrow \mathbb{R}^n \times \mathbb{R}^p, \qquad F(x, y) = (f(x, y), y).$$

Then $\mathrm{d}F(0) = \mathbb{1}$ and $f = \pi \circ F$, where $\pi : R^{n+p} \longrightarrow R^n$ is the canonical projection. By the Inverse Function Theorem there exists a local inverse ψ of F. Then

$$f \circ \psi(x, y) = \pi \circ F \circ \psi(x, y) = \pi(x, y) = x.$$

☐

At this point we leave to the reader the task of proving Theorem 1.24.

2. Integration

We will recall now the basic fact of Riemann integration theory. We will take a limited approach which is enough for our purposes.

We will start with the case of functions of one real variable. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A *partition* of the interval is a set $P = \{t_0, \dots, t_k\} \subseteq \mathbb{R}$ such that $a = t_0 < \cdots < t_k = b$. We will set $|P| = \sup\{t_i - t_{i-1}\}$. Given a function $f : [a,b] \longrightarrow \mathbb{R}^m$ and a partition P of $[a,b]$, we define

$$\Sigma(f, P) = \sum_{i=0}^{k-1} (t_{i+1} - t_i) f(t_i).$$

2.1. Definition. A vector $X \in \mathbb{R}^m$ is said to be an *integral* of f on $[a,b]$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\Sigma(f,P) - X\| < \epsilon \quad \text{for all partitions } \ P \text{ with } \ |P| < \delta.$$

If an integral exists we will say that f is *integrable* on $[a,b]$ and we use the notation

$$X = \int_a^b f(t)\mathrm{d}t \quad \text{or, when clear from the contex, simply } \ X = \int_a^b f.$$

It is easy to see that if a function is integrable, the integral is unique.

2.2. Remark. Let $f : [a,b] \longrightarrow \mathbb{R}^m$ be an integrable function. Then we can compute the integral as limit of the sequence $\Sigma(f, P_n)$ where P_n is a sequence of partitions such that $\lim_{n\to\infty} |P_n| = 0$.

The proof of the following Proposition is simple and left to the reader (see Exercise 4.16).

2.3. Proposition. *Let $f, g : [a,b] \longrightarrow \mathbb{R}^m$ be integrable functions, $k \in \mathbb{R}$ and $T : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ be a linear map. Then $f+g$, kf and $T \circ f$ are integrable and*

1. $\int_a^b f + g = \int_a^b f + \int_a^b g, \qquad \int_a^b kf = k \int_a^b f,$
2. $\int_a^b T \circ f = T(\int_a^b f)$
3. $\| \int_a^b f \| \leq (b-a)\|f\|_0.$
4. *The function $F : [a,b] \longrightarrow \mathbb{R}^m$, defined by $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$ is well defined and continuous.*

We will denote by $\mathcal{B} := \mathcal{B}([a,b], \mathbb{R}^m)$ the set of bounded functions of $[a,b]$ into $\mathbb{R}^m$. $\mathcal{B}([a,b], \mathbb{R}^m)$ is a real vector space, with the obvious operations, and

$$\|f\|_0 = \sup\{f(t) : t \in [a,b]\}$$

is a norm in $\mathcal{B}$, called the *sup norm* or the norm of *uniform convergence.*

2.4. Definition. A sequence of functions $f_n : [a,b] \longrightarrow \mathbb{R}^m$ in $\mathcal{B}$ is *uniformly convergent to* $f \in \mathcal{B}$ if $\lim_{n\longrightarrow\infty} \|f_n - f\|_0 = 0$.

2.5. Remark. Proposition 2.3 tell us that the set of bounded integrable functions $\mathcal{I}$ is a linear subspace of $\mathcal{B}$, the integral maps $\mathcal{I}$ into $\mathbb{R}^m$ linearly (items (1)) and continuously (item (3)). The next Proposition tell us that $\mathcal{I}$ is closed in $\mathcal{B}$.

2.6. PROPOSITION. *Let $f_n : [a,b] \longrightarrow \mathbb{R}^m$ be a sequence of bounded integrable functions. If the sequence converges uniformly to a function f, the f is integrable and*

$$\int_a^b f = lim_{n\to\infty} \int_a^b f_n.$$

PROOF. Set $I_n = \displaystyle\int_a^b f_n$. By Proposition 2.3 (3), $\|I_m - I_k\| \leq (b-a)\|f_m - f_k\|_0$. Hence $\{I_n\}$ is a Cauchy sequence in $\mathbb{R}^m$ and therefore converges to a vector $I \in \mathbb{R}^m$. We claim that I is the integral of f. Fix $\epsilon > 0$. Then there exist n such that $\|f_m - f\|_0 < \epsilon/3(b-a)$, $\|I - I_m\| < \epsilon/3$ if $m > n$. Also there exist $\delta > 0$ such that $\|\Sigma(f_m, P) - I_m\| < \epsilon/3$ if $|P| < \delta$. Observe that $\|\Sigma(f,P) - \Sigma(f_m, P)\| \leq (b-a)\|f - f_m\|_0$. So, if $|P| < \delta$, $\|I - \Sigma(f,P)\| \leq \|I - I_m\| + \|I_m - \Sigma(F_m, P)\| + \|\Sigma(f_m, P) - \Sigma(f,P)\| < \epsilon$ and this prove the claim. ☐

We will describe classes of integrable functions.

2.7. DEFINITION. A function $f : [a,b] \longrightarrow \mathbb{R}^m$ is a *step function* if there exists a partition P of $[a,b]$, $P = \{t_0, \ldots, t_k\}$ and vectors $\{X_0, \ldots, X_{k-1}\}$ such that $f(t) = X_i$, $t \in (t_i, t_{i+1})$.

2.8. LEMMA. *Let $f : [a,b] \longrightarrow \mathbb{R}^m$ be a step function relative to a partition P. Then f is integrable and*

$$\int_a^b f = \sum (t_{i+1} - t_i) X_i.$$

PROOF. We can suppose $f(t_i) = X_i$ (see Exercise 4.19). Observe that $\Sigma(f, P') = \Sigma(f,P)$ if P' is obtained from P adding new points. Therefore $\Sigma(f,P) = \Sigma(f, P \cup Q)$ for all partitions Q and the conclusion follows. ☐

Since every continuous function is uniform limit of step functions (see Exercise 4.20), combining the last two Proposition we have

2.9. PROPOSITION. *If $f : [a,b] \longrightarrow \mathbb{R}^m$ is continuous, then it is integrable.*

We will recall now the basic relation between differentiation and integration.

2.10. LEMMA. *Let $f : [a,b] \longrightarrow \mathbb{R}^m$ be a continuous function and $x \in [a,b]$. Then*

$$\int_a^b f = \int_a^x f + \int_x^b f.$$

PROOF. Let P be a partition such that $x \in P$. then $P = P' \cup P''$ where P' is a partition of $[a,x]$ and P'' a partition of $[x,b]$. Since all three integrals exist, by Lemma 4.20, we can compute the integrals as limit of $\Sigma(f, P_n)$ where P_n is a partition as above and $\lim_{n\to\infty} |P_n| = 0$ (see Remark 2.2). Then the conclusion follows easily. ☐

2.11. REMARK. The Lemma still holds for functions that are just integrable. We just have to prove that an integrable function is integrable on any subinterval (see Exercise 4.18).

2.12. THEOREM. [Fundamental Theorem of Calculus] *Let $f : [a,b] \longrightarrow \mathbb{R}^m$ be a continuous function. Define*

$$F : (a,b) \longrightarrow \mathbb{R}^m, \qquad F(x) = \int_a^x f.$$

Then F is differentiable and $F'(x) = f(x)$.

Proof. F is continuous by Proposition 2.3 (item (4)). Fix $x \in (a,b)$ and let $h > 0$ be such that $x + h \in (a,b)$. Then

$$|\frac{F(x+h) - F(x)}{h} - f(x)| = |\frac{1}{h}\int_x^{x+h} f - f(x)| \le \sup\{|f(t) - f(x)| : t \in [x, x+h)\}.$$

Since f is continuous the expression on the right hand side goes to 0, when h goes to 0. The same argument works for $h < 0$ and the Claim follows. □

We will define now the integral of functions of several real variables. We will consider the case of two variables and the reader should not have any difficulty to extend these considerations for n variables.

Let $f : [a,b] \times [c,b] \longrightarrow \mathbb{R}^m$ be a function and let t, s be the first and second coordinate respectively. For $t \in [a,b]$ fixed, we set $f_t(s) = f(t,s)$. Suppose the function f_t integrable, $\forall\ t \in [a,b]$. Then we define the *iterated integral* (if it exists), as

$$\int_a^b \int_c^d f(t,s)\ \mathrm{d}s\ \mathrm{d}t := \int_a^b\ [\int_c^d f_t(s)\ \mathrm{d}s]\ \mathrm{d}t := \int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s.$$

The elementary properties of the iterated integrals follows from the corresponding ones for the integrals of functions of one real variable. For example

$$|\int_a^b \int_c^d f(t,s)\mathrm{d}s\mathrm{d}t| \le (b-a)(d-c)\|f\|_0, \quad \text{where } \ \|f\|_0 = \sup\{\|f(t,s)\| : (t,s) \in [a.b] \times [c,d]\}.$$

2.13. Example. Let $P = \{t_0, \dots, t_k\}$ be a partition of $[a,b]$, $Q = \{s_0, \dots, s_h\}$ a partition of $[c,d]$ and let $X_{ij} \in \mathbb{R}^n$ be fixed vectors. Let $g : [a,b] \times [c,d] \longrightarrow \mathbb{R}^n$ be a function such that $g(t,s) = X_{ij}, t \in (t_i, t_{i+1}),\ s \in (s_j, s_{j+1})$. For $s \in [c,d]$ the function $g_s(t) = g(t,s)$ is a step function and

$$\int_a^b g_s \mathrm{d}t = \sum_0^{k-1}(t_{j+1} - t_j)X_{ij} \quad \text{ if } \ s \in (s_i, s_{i+1}).$$

Therefore $h(s) = \displaystyle\int_a^b g_s\mathrm{d}t$ is also a step function, therefore integrable and

$$\int_c^d \mathrm{d}s \int_a^b g(t,s)\mathrm{d}t = \sum_{ij}(s_{i+1} - s_i)(t_{j+1} - t_j)X_{ij}.$$

Observe, in particular, that the iterated integral does not depend on the order of integration.

2.14. Proposition. *If f is continuous, the iterated integrals exist and*

$$\int_a^b \int_c^d f(t,s)\ \mathrm{d}s\ \mathrm{d}t = \int_c^d \int_a^b f(t,s)\ \mathrm{d}t\ \mathrm{d}s.$$

Proof. We will start with a general fact

Claim Let $U \subseteq \mathbb{R}^n$ and let $f : U \times [a,b] \longrightarrow \mathbb{R}^m$ be a continuous function. Then the function

$$F : U \longrightarrow \mathbb{R}^m, \qquad F(x) = \int_a^b f(x,t)$$

is a continuous function.

Proof. Fix $x_0 \in U$ and $\epsilon > 0$. The set $V = \{(x,t) \in X \times [a,b] : |f(x,t) - f(x_0,t)| < \epsilon(b-a)^{-1}$ is an open neighborhood of $x_0 \times [a,b]$. Since $[a,b]$ is compact, there exists a neighborhood W of x_0 such that $W \times [a,b] \subseteq V$. In particular, for all $x \in W,\ |f(x,t) - f(x_0,t)| < \epsilon(b-a)^{-1},\ \forall t \in [a,b]$. Hence, if $x \in W$

$$|F(x) - F(x_0)| \leq \int_a^b |f(x,t) - f(x_0,t)| \leq (b-a)\sup\{|f(x,t) - f(x_0,t)|\} < \epsilon.$$

☐

The Claim implies, in particular, that a continuous function admits iterated integrals. We will prove now the commutativity relation. More precisely, given $\epsilon > 0$, we will show that, if P, Q are partitions as in Example 2.13, there exists $\delta > 0$ such that, if $|P|, |Q| < \delta$,

$$|\int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s - \sum (s_{i+1} - s_i)(t_{j+1} - t_j) f(t_j, s_i)| < \epsilon.$$

The conclusion will follows, since the other integral is, by symmetry, approximated by a sum of the same type. By uniform continuity of f, it follows that there exists $\delta > 0$ such that $|f(t,s) - f(t',s')| < \epsilon/(b-a)(d-c)$ if $|s-s'|, |t-t'|$ are smaller than δ. Consider the function g as in Example 2.13, with $g(t,s) = f(t_j, s_i),\ t \in [t_j, t_{j+1}),\ s \in [s_i, s_{i+1})$. Then $\|f - g\| < \epsilon/(b-a)(d-c)$. Therefore

$$|\int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s - \sum (s_{i+1} - s_i)(t_{j+1} - t_j) f(t_j, s_i)| = |\int_a^b \mathrm{d}t \int_c^d f(t,s)\mathrm{d}s - \int_a^b \mathrm{d}t \int_c^d g(t,s)\mathrm{d}s| =$$

$$= |\int_a^b \mathrm{d}t \int_c^d [f(t,s) - g(t,s)]\mathrm{d}s \leq (b-a)(d-c)\|f-g\| < \epsilon.$$

☐

We will define now the *integral* of a function $f : C = [a,b] \times [c,d] \longrightarrow \mathbb{R}^n$. Let P, Q be partitions of the two intervals and set, in analogy with the 1-dimensional case,

$$\Sigma(f, P, Q) = \sum (t_{i+1} - t_i)(s_{j+1} - s_j) f(t_i, s_j).$$

2.15. Definition. We will say that $X = \lim_{|P|,|Q| \to 0} \Sigma(f,P,Q)$ if, given $\epsilon > 0$ there exist $\delta > 0$ such that $\|X - \Sigma(f,P,Q)\| < \epsilon$ if $|P|, |Q| < \delta$. If such a limit exists we will say that f is integrable and define the *duple integral* of f on C as

$$\int_C f(t,s)\mathrm{d}t\mathrm{d}s = X.$$

2.16. Lemma. *If f is integrable over C and one of the simple integral, let say $\int_a^p f(t,s)\mathrm{d}t$, exists, $\forall s \in [c,d]$, then the other simple integral exists and the iterated integrals are equal to the duple integral.*

Proof. The claim follows from the general relation between duple limits and iterated limits:

if $\lim\limits_{|P|,|Q| \to 0} \Sigma(f,P,Q)$ exists and $\lim\limits_{|P| \to 0} \Sigma(f,P,Q)$ exists $\forall Q$, $\lim\limits_{|Q| \to 0} [\lim\limits_{|P| \to 0} \Sigma(f,P,Q)] = \lim\limits_{|P|,|Q| \to 0} \Sigma(f,P,Q)$.

☐

2.17. Theorem. [Baby Fubini] *If $f : C \longrightarrow \mathbb{R}^n$ is continuous, then the duple integral exists and is equal to the iterated integrals.*

Proof. This is a corollary of the proof of Theorem 2.14. ☐

In particular we can define the integral of a continuous function with compact support.

2.18. Definition. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ be a continuous function with compact support and C a rectangle containing the support of F. We define

$$\int_{\mathbb{R}^2} f(t,s)\mathrm{d}t\mathrm{d}s = \int_C f(t,s)\mathrm{d}t\mathrm{d}s.$$

2.19. Remark. It is easy to see that the definition does not depends on the choice of the rectangle C.

Beside Fubini's Theorem, that allows us to reduce the calculus of multiple integrals to the case of simple integrals, the other basic fact on integration that we will need is the formula of *change of variables*.

2.20. Theorem. *Let $U, V \subseteq \mathbb{R}^2$ be open sets and let $F : U \longrightarrow V$ be a diffeomorphism. If $f : V \longrightarrow \mathbb{R}^n$ is an integrable function with compact support,*

$$\int_V f = \int_U |J[F]| f \circ F,$$

where $J[F] := \det[\mathrm{d}F]$ is the Jacobian determinant.

Proof. □

We invite the reader to extend the concepts and results above for the case of integration of function of several variables.

3. Vector fields, distributions and the local Frobenius Theorem

3.1. Definition. Let U be an open set of $\mathbb{R}^n$. A *(tangent) vector field* on U is a smooth map $X : U \longrightarrow \mathbb{R}^n$. We will denote by $\mathcal{H}(U)$ the space of vector fields on U.

3.2. Remark. Let X be a vector field. We want to think of $X(x)$ as a vector *based at* x. This is the reason why we use different names for the same thing[3]. We can make this point more precise as follows.

- The *tangent space of* U *at* $x \in U$ is the vector space

$$T_xU = \{(x,v) : v \in \mathbb{R}^n\}$$

  with the obvious operations on the second component.
- The *tangent bundle* of U is

$$TU = \cup_{x \in U} T_xU = U \times \mathbb{R}^n.$$

A vector field on U should be defined as a smooth map $\tilde{X} : U \longrightarrow TU$ of the form $\tilde{X}(x) = (x, X(x))$, $X : U \longrightarrow \mathbb{R}^n$. Of course, in our context, we are just complicating notations, but this point of view, that seems silly now, will prove to be useful when these concepts are extended to the case of differentiable manifolds.

We will review now some facts about solutions of differential equations.

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[3] B. Russel used to say that "Mathematics is the art of calling different things with the same name and the same thing with different names".

3.3. Definition. Let $X \in \mathcal{H}(U),\ x \in U$. An *integral curve of X with initial condition x* is a smooth map $\gamma_x : (a,b) \subseteq \mathbb{R} \longrightarrow U$ such that:

$$\mathrm{d}\gamma_x(t)(1) := \dot{\gamma}_x(t) = X(\gamma_x(t)), \qquad 0 \in (a,b) \ \text{ and } \ \gamma_x(0) = x.$$

When it is clear from the context, or irrelevant, we will ignore the subscript relative to the initial condition.

The basic result about integral curves is the following

3.4. Theorem. *If $X \in \mathcal{H}(U),\ x \in U$, there exists an integral curve with initial condition $x \in U$. This curve is unique in the sense that two such curves, with the same initial condition, coincide in the intersection of the domains. In particular there is a maximal interval of definition, $(\alpha(x), \beta(x)) \subseteq \mathbb{R}$. Moreover the curve is smooth and depends smoothly on the initial condition.*

3.5. Remark. Smooth dependence on the initial condition means that, for fixed x, the exists a neighborhood U of x and $\epsilon > 0$ such that the map

$$\Gamma : U \times (-\epsilon, \epsilon) \longrightarrow U, \qquad \Gamma(y,t) = \gamma_y(t),$$

is a smooth map.

3.6. Remark. Integral curves exist even if the vector field X is merely continuous. For unicity we need the field to be *locally lipschitzian.* If X is of class C^k, the curves are of class C^{k+1}.

3.7. Definition. The vector field is *complete* if its integral curves are defined on all of $\mathbb{R}$.

3.8. Proposition. *If X is complete, the map*

$$\gamma_t : U \longrightarrow U, \qquad \gamma_t(x) = \gamma_x(t),$$

is well defined and smooth. Moreover

(1) $\gamma_0 = \mathbb{1}$,
(2) $\gamma_{t+s} = \gamma_t \circ \gamma_s$.

Proof. The first property is obvious, by definition. As regards the second, we observe that, for fixed s, the curves $\gamma_{\gamma_x(s)}(t)$ and $\gamma_x(t+s)$ are integral curves of X with the same initial condition. The conclusion follows from the unicity of integral curves. ☐

In particular γ_t is a diffeomorphism of U with inverse γ_{-t}, and the map $t \in \mathbb{R} \rightsquigarrow \gamma_t$ is a homomorphism of the additive group $\mathbb{R}$ into the group of diffeomorphisms of U.

3.9. Remark. If X is not complete, the considerations above hold locally. We leave to the reader the task of making this claim precise.

3.10. Definition. A point $x \in U$ is a *singularity* (or a *singular point*) of $X \in \mathcal{H}(U)$, if $X(x) = 0$.

The behavior of X near a singularity can be quite complicated. On the contrary, the behavior near a non singular point is quite simple.

3.11. Theorem. *Let* $X \in \mathcal{H}(U)$, $x \in U$ *and* $X(x) \neq 0$. *Then there is a diffeomorphism* $\phi : U \subseteq U \longrightarrow V$ *of a neighborhood* U *of* $0 \in \mathbb{R}^n$ *onto a neighborhood* V *of* x, *such that* $\mathrm{d}\phi(y)(\dfrac{\partial}{\partial x_1}) = X(\phi(y))$.

Proof. We can assume $x = 0, X(0) = \dfrac{\partial}{\partial x_1}(0)$. For $p = (0, x_2, \dots, x_n) \in U$, consider the integral curve of X with initial condition p, $\gamma_p(t)$. Then the map $\phi(p,t) = \gamma_p(t)$ is well defined and smooth if $|t| < \epsilon$, with ϵ sufficiently small and p is in a sufficiently small neighborhood U' of $0 \in \mathbb{R}^{n-1} = \{(x_1, \dots x_n)\} \in \mathbb{R}^n : x_1 = 0\}$. It is clear that $\mathrm{d}\phi(p,t)(\frac{\partial}{\partial x_1}) = X(\phi(p,t))$ (see Remark 1.10). Also $\mathrm{d}\phi(0,0) = \mathbb{1}$, hence, by Theorem 1.25, ϕ is a diffeomorphism of a (possible smaller) neighborhood $U \subseteq U' \times (-\epsilon, \epsilon)$ of 0, onto its image. ☐

We can ask for a natural generalization of Theorem 3.11: given linearly independent vector fields $X_1, \dots, X_k \in \mathcal{H}(U)$, do there exist local coordinates $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ such that $X_i = \dfrac{\partial}{\partial x_i}$? In order to answer this question we will take a slight different approach to vector fields. First a few definitions.

3.12. Definition. An *algebra* over the reals is a real vector space $\mathbb{E}$ together with a bilinear map, the *product*, $b : \mathbb{E} \oplus \mathbb{E} \longrightarrow \mathbb{E}$. The algebra is said to be *associative* if $b(x, b(y,z)) = b(b(x,y), z)$ and *commutative* if $b(x,y) = b(y,x) \quad \forall\, x, y, z \in \mathbb{E}$.

When clear from the context we will write xy for $b(x,y)$.

Examples of such a structure are

- The real or complex numbers with the usual multiplication. They are associative and commutative algebras.
- The spaces $M(n, \mathbb{K})$ of $n \times n$ matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, with the usual product of matrices. They are associative but non commutative algebras (if $n > 1$!).
- The space $\mathcal{F}(U)$ of smooth real valued functions defined in $U \subseteq \mathbb{R}^n$.

3.13. Definition. An *algebra homomorphism* $h : \mathbb{E} \longrightarrow \mathbb{E}'$ between the algebras $\mathbb{E}$ and $\mathbb{E}'$ is a linear map such that the image of the product of two elements in $\mathbb{E}$ is the product of the images (in $\mathbb{E}'$).

3.14. Definition. An *ideal* $\mathcal{I}$ of an algebra $\mathbb{E}$ is a vector subspace of $\mathbb{E}$ such that if $x \in \mathcal{I}, y \in \mathbb{E}$, then $b(x,y),\ b(y,x) \in \mathcal{I}$

It is not difficult to see that if $\mathcal{I}$ is an ideal of $\mathbb{E}$, the quotient vector space $\mathbb{E}/\mathcal{I}$ has a natural product (and hence a structure of algebra) such that the quotient map is an algebra homomorphism. Moreover, given an algebra homomorphism $h : \mathbb{E} \longrightarrow \mathbb{E}'$, the kernel of h, $\ker h$, is an ideal and, in fact, every ideal $\mathcal{I}$ is the kernel of an algebra homomorphism, the projection $\pi : \mathbb{E} \longrightarrow \mathbb{E}/\mathcal{I}$.

Let $\mathcal{F}(U)$ be the algebra of smooth real valued functions defined in U.

3.15. Definition. A *derivation of* $\mathcal{F}(U)$ (resp. a *derivation at* $x \in U$) is a $\mathbb{R}$-linear map $Y : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ (resp. $Y(x) : \mathcal{F}(U) \longrightarrow \mathbb{R}$), such that:

$$Y(fg) = Y(f)g + fY(g) \qquad (\text{resp. } Y(x)(fg) = Y(x)(f)g(p) + f(p)Y(x)(g)) \ \ \forall\, f, g \in \mathcal{F}(U).$$

Both the set of derivations and the set of derivations at x have a natural structure of real vector space. We will denote by $\mathcal{D}er(U)$ and $\mathcal{D}er_x(U)$ these spaces. Observe that $\mathcal{D}er(U)$ is infinite dimensional (if $n > 0$!) while, as we will see soon, $\mathcal{D}er_x(U)$ is n-dimensional.

3.16. EXAMPLE. Let $X \in \mathbb{R}^n$, $x \in U$. Then the directional derivative of $f \in \mathcal{F}(U)$ at $x \in U$, in the direction X is a derivation at x. As we shall soon see, all derivations at x are directional derivatives.

3.17. EXAMPLE. If $X \in \mathcal{H}(U)$, we define a derivation $\mathcal{D}er(U)$, still denoted by X, $X(f)(x) := X(x)(f)$, where $X(x)(f)$ is the directional derivative at x as in the example above. It is easily seen that $X(f)(x) \in \mathcal{F}(U)$ so X is, in fact, a derivation in $\mathcal{D}er(U)$.

Some simple but basic facts are the following:

3.18. LEMMA. *Let $f \in \mathcal{F}(U)$ and $X_x \in \mathcal{D}er_x(U)$.*

- *If f vanishes on an open neighborhood $V \subseteq U$, then $X_x(f) = 0$. In particular, if two functions $f, g \in \mathcal{F}(U)$ coincide in a neighborhood of x, then $X_x f = X_x g$.*
- *If f is constant in a neighborhood of x, then $X_x f = 0$.*
- *If f is (locally) a product of functions vanishing at x, then $X_x f = 0$.*

PROOF. Let $\phi \in \mathcal{F}(U)$ be a function which vanishes in a neighborhood V_1 of x and is identically 1 outside V (see Exercise 4.13 for the existence of such a function). Then $f = \phi f$ and

$$X_x(f) = (X_x\phi)f(x) + \phi(x)X_x f = 0.$$

The second claim follows from $1 \cdot 1 = 1$ and the definition of a derivation. The third one is also immediate. □

Let $x \in \mathbb{R}^n$. Consider the set

$$\tilde{\mathcal{F}}_x := \{(f, V) : V \text{ is a neighborhood of } x,\ f \in \mathcal{F}(V)\}.$$

3.19. DEFINITION. The algebra of *germs of smooth functions at x*, $\mathcal{F}_x$, is the quotient of $\tilde{\mathcal{F}}_x$ by the equivalence relation $(f, U) \sim (g, V) \iff f = g$ in a neighborhood of x (contained in $U \cap V$). The operations are the usual sum and product of functions (which are defined in the intersections of the domains).

3.20. REMARK. The advantage of working with germs instead that with functions is that we do not have to worry about the domain of definition of the functions involved. Anyway, when clear from the context we will make no difference between a function and its germ.

We will denote by $\mathcal{D}_x$ the space of derivations of $\mathcal{F}_x$ at x (with the obvious definition). Lemma 3.18 implies, in particular, that an element of $\mathcal{D}er_x(U)$ induces a derivation of $\mathcal{F}_x$. We shall see next that all derivations in $\mathcal{D}_x$ are of this type.

3.21. THEOREM. *Given $x \in \mathbb{R}^n$ and a derivation $X_x \in \mathcal{D}_x$, there exist a unique vector $v \in \mathbb{R}^n$ such that $X_x = v(x)$. In particular $\mathcal{D}_x \cong T_x\mathbb{R}^n \cong \mathcal{D}er_x(U)$.*

PROOF. Let $f \in \mathcal{F}_x$. In a suitable neighborhood of x consider the Taylor formula

$$f(x_1, \ldots, x_n) = f(x) + \sum_1^n \frac{\partial f}{\partial x_i}(x)(x_i - x_i(x)) + \Phi(x_1 \ldots, x_n),$$

where $\Phi(x_1 \ldots, x_n)$ is a sum of products of functions vanishing at x (see Theorem 1.19 and Exercise 4.26).

Applying X_x to both sides and using Lemma 3.18 we have:

$$X_x(f) = \sum_1^n X_x(x_i)\frac{\partial f}{\partial x_i}(x).$$

Therefore:

$$X = \sum_1^n X(x_i)\frac{\partial}{\partial x_i}(x),$$

and the map that associates to e_i the derivation $\frac{\partial}{\partial x_i}(x)$ extends to an isomorphism of $\mathbb{R}^n$ (or, better T_xU) onto $\mathcal{D}_x$. ☐

In what follows we will identify T_xU with $\mathcal{D}_x$ and $\mathcal{H}(U)$ with $\mathcal{D}er(U)$.

The composition of two derivations is not a derivation, in general. However the commutator of two derivations is a derivation (Exercise 4.28). This fact suggests the following

3.22. Definition. Let $X, Y \in \mathcal{D}er(U)$. The *Lie product* (or *bracket*) of X and Y is the commutator $[X,Y] := X \circ Y - Y \circ X$.

The following properties are easy to prove and we leave the details to the reader (Exercise 4.29).

3.23. Proposition. *The Lie product* $[\,\cdot,\cdot\,] : \mathcal{H}(U) \times \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$ *is a* $\mathbb{R}$*-bilinear map. Moreover*

(1) $[X,Y] = -[Y,X]$,

(2) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ *(Jacoby identity).*

3.24. Remark. An algebra with a product which satisfies the properties above is called a *Lie algebra.*

3.25. Example. By Theorem 1.15 $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$.

We go back to the original question: given vector fields $X_1, \ldots, X_k \in \mathcal{H}(U)$, linearly independent at each point, there exist local coordinates $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that $X_i = \frac{\partial}{\partial x_i}$?

There is a natural necessary condition for a positive answer, the condition being $[X_i, X_j] = 0$. It turns out that the condition is also sufficient, at least locally. We will take a slightly more general approach.

3.26. Definition. Let $U \subseteq \mathbb{R}^n$ be an open set. A *k-dimensional distribution D* on U is a law that associates to a point $x \in U$ a k-dimensional subspace $D_x \subseteq \mathbb{R}^n$. Moreover:

- The distribution D is *smooth* if there exist, locally, k smooth vector fields $X_1, \ldots, X_k$ such that $D_x = span\{X_1(x), \ldots, X_k(x)\}$.
- A smooth distribution D is *involutive* (or *integrable*) if for all vector fields $X, Y \in \mathcal{H}(U)$ such that $X(x),\ Y(x) \in D_x,\ \forall\ x \in U$, then $[X,Y](x) \in D_x$.

3.27. Theorem. [Frobenius Theorem, local version] *Let D be a k-dimensional involutive smooth distribution on $U \subseteq \mathbb{R}^n$. Then there exist (local) coordinates $x_1, \ldots, x_n$ such that* $D_x = span\{\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_k}(x)\}$.

3.28. Remark. The word "local" means that the claim of the Theorem holds in a sufficiently small open neighborhood of a fixed point, that we can assume to be $0 \in \mathbb{R}^n$.

PROOF. We will proceed by induction on k. If $k=1$, the Theorem follows directly from Theorem 3.11. So we assume that the Theorem is true for $(k-1)$-dimensional involutive distributions. Let us suppose that D is a k-dimensional distribution spanned, locally, by smooth vector fields $X_1,\dots,X_k$. By Theorem 3.11 we can assume that there are coordinates $y_1,\dots,y_n$ such that $X_1=\dfrac{\partial}{\partial y_1}$. Consider the set

$$\overline{D}=\{X\in\mathcal{H}(U):X(x)\in D_x,\ X(y_1)=0\}.$$

CLAIM 1. $\overline{D}$ is a smooth $(k-1)$-dimensional involutive distribution spanned by the vector fields

$$Y_i=X_i-X_i(y_1)X_1,\quad i=2,\dots,k.$$

PROOF. It is easy to see that the vector fields $X_1,Y_2,\dots,Y_k$ are linearly independent at every point. Moreover $Y_i\in\overline{D}$ and $X_1\notin\overline{D}$, since $X_1(y_1)=1$. So $\overline{D}$ is a smooth $(k-1)$-dimensional distribution. Let us show that $\overline{D}$ is involutive. If $Y,Z\in\overline{D}$, $[Y,Z]\in D$ since D is involutive. Moreover $[Y,Z](y_1)=Y(Z(y_1))-Z(Y(y_1))=0$, hence $[Y,Z]\in\overline{D}$. □

Observe that $\overline{D}$ is tangent to the slides $\mathbb{R}^{n-1}_c:=\{(y_1,\dots,y_n)\in\mathbb{R}^n:y_1=c\}$, since the first coordinate of Y_i is $Y_i(y_1)=0$. By the inductive hypothesis the are (local) coordinates $z_2,\dots z_n$, in $\mathbb{R}^{n-1}_0$ such that $\overline{D}_{(0,z)}=span\{\dfrac{\partial}{\partial z_i},\ i=2,\dots,k\}$. Consider the coordinates $x_1=y_1,x_i=z_i,i=2,\dots,n$. The proof of the Theorem follow from the following

CLAIM 2. D is spanned by $\dfrac{\partial}{\partial x_i},\ i=1,\dots,k$.

PROOF. We want to show that $Y_1:=X_1=\dfrac{\partial}{\partial x_1},\ Y_2,\dots,Y_k$ are linear combinations of $\dfrac{\partial}{\partial x_i},\ i=1,\dots,k$. For this is sufficient to show that $Y_i(x_j)=0$ for $i\le k,\ j>k$ (this is obviously true for $i=1$).

Since the distribution is involutive, there are real valued smooth functions g_{irs} such that $[Y_i,Y_r]=\sum_{s=1}^k g_{irs}Y_s$. Now

$$Y_1(Y_i)(x_j)=[Y_1,Y_i](x_j)=\sum_1^k g_{irs}Y_s(x_j).$$

Hence the functions $Y_i(x_j)$ are solutions of the system of differential equations

$$\frac{\partial}{\partial x_1}Y_i(x_j)=\sum_1^k g_{irs}Y_s(x_j).$$

This is a *linear homogeneous system of ordinary differential equations*, along the x_1 curves, hence it admits the zero functions as solutions. Now the initial condition, for $x_1=0$, is $Y_i(x_j)(0,x_2,\dots,x_n)$ which vanishes (for $j>k$) since there $x_j=z_j$. Hence, by unicity of the solutions of the initial value problem, the solutions vanish identically. □

□

3.29. REMARK. The Frobenius Theorem is really a result on existence and unicity of solutions of first order partial differential equations. We will sketch the proof of a simple fact that will explain this claim.

Let $a,b:\mathbb{R}^2\longrightarrow\mathbb{R}$ be smooth functions and consider the problem of finding a function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ such that

$$\frac{\partial f}{\partial x}=a,\qquad\frac{\partial f}{\partial y}=b.$$

The Theorem of Schwarz (Theorem 1.15) gives an obvious necessary condition for the existence of such a function, that is $\frac{\partial b}{\partial x} = \frac{\partial a}{\partial y}$. We will use the Theorem of Frobenius to shows that, at least locally, such condition is also sufficient[4]. Consider the vector fields

$$X = \frac{\partial}{\partial x} + a\frac{\partial}{\partial z} \qquad Y = \frac{\partial}{\partial y} + b\frac{\partial}{\partial z}.$$

A simple calculation gives $[X, Y] = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})\frac{\partial}{\partial z}$. Hence the distribution spanned by X, Y is involutive if and only if $[X, Y] = 0$. In this case, by the Frobenius Theorem, there is a local diffeomorphism $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\mathrm{d}\Phi(\frac{\partial}{\partial x}) = X$, $\mathrm{d}\Phi(\frac{\partial}{\partial y}) = Y$. The "surface" $\Phi(x, y, c)$ has the distribution spanned by X and Y as "tangent space" and, since the normal vector is not horizontal, it project (locally) on the plane $e_3^{\perp}$, injectively. Hence it is the graph of a function f that is, as it is easily seen, a solution of our problem.

The differential equation above is the simplest case of a class of differential equation, called *total differential equations*, for which necessary and sufficient conditions for existence and unicity of solutions may be given in terms of the Theorem of Frobenius.

There is an interpretation of the Lie product of vector fields worth mentioning.

3.30. PROPOSITION. *Let $X, Y \in \mathcal{H}(U)$ and ϕ_t be the (local) flow of X. Then, for $x \in U$,*

$$[X, Y](x) = \lim_{t \to 0} t^{-1}[\mathrm{d}\phi_{-t}(\phi_t(x))Y(\phi_t(x)) - Y(x)].$$

PROOF. By Theorem 3.11 we can assume $X = \frac{\partial}{\partial x_1}$. Let $Y = \sum y_i \frac{\partial}{\partial x_i}$. By linearity we can assume $Y = y_i \frac{\partial}{\partial x_i}$. Observe that the flow of X is just translations, i.e. $\phi_t(x_1, \ldots, x_n) = (x_1 + t, \ldots, x_n)$. Then the right hand side is just $\frac{\partial Y}{\partial x_1}(x) = \frac{\partial y_i}{\partial x_1}(x)\frac{\partial}{\partial y_i}(x)$. On the other hand, the left hand side is also $\frac{\partial y_i}{\partial x_1}(x)\frac{\partial}{\partial y_i}(x)$ (see Exercise 4.30). ☐

An important fact about Lie product is that it "behaves well with respect to smooth maps". First a definition to make the statement precise. Let $F : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^m$ be a smooth map between open sets.

3.31. DEFINITION. We say that $\tilde{X} \in \mathcal{H}(V)$ is F-related to $X \in \mathcal{H}(U)$ if $\mathrm{d}F(x)(X) = \tilde{X}(F(x))$, $\forall\, x \in U$.

3.32. PROPOSITION. *If $\tilde{X}$, $\tilde{Y} \in \mathcal{H}(V)$ are F-related to $X, Y \in \mathcal{H}(U)$, then $[\tilde{X}, \tilde{Y}]$ is F-related to $[X, Y]$.*

PROOF. Let $f \in \mathcal{F}(V)$. We must show that, fixed $x \in U$, $\mathrm{d}F([X, Y](x))(f) = [\tilde{X}, \tilde{Y}](F(x))(f)$.

$$\mathrm{d}F([X,Y](x))(f) = [X,Y](x)(f \circ F) = X(x)(Y(f \circ F)) - Y(x)(X(f \circ F)) = X(x)(\tilde{Y}(f) \circ F) - Y(x)(\tilde{X}(f) \circ F) =$$

$$= \mathrm{d}F(X(x))(\tilde{Y}(f)) - \mathrm{d}F(Y(x))(\tilde{X}(f)) = \tilde{X}(F(x))(\tilde{Y}(f)) - \tilde{Y}(F(x))(\tilde{X}(f)) = [\tilde{X}, \tilde{Y}](x)(f).$$

☐

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[4] A different proof will be given in Chapter 1.

4. Exercises

4.1. For $L, T \in L(\mathbb{R}^n, \mathbb{R}^m)$ consider the norms $\|L\|_2 = \mathrm{trace} L^t L$, $\|L\| = \sup\{\|L(x)\| : \|x\| = 1\}$. Prove that $\|L\| \leq \|L\|_2 \leq n\|L\|$ and $\|L \circ T\| \leq \|L\|\|T\|$.

4.2. Prove Proposition 1.3.

4.3. Let $f, g : U \longrightarrow \mathbb{R}^n$ be differentiable functions. Define $F : U \longrightarrow \mathbb{R}$, $F(x) = \langle f(x), g(x)\rangle$. Prove that F is differentiable and compute $\mathrm{d}F(x)$ (see Example 1.5).

4.4. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$, $f(t) = (\cos t, \sin t)$. Compute $\mathrm{d}f(t)$ and show that there is no $t_0 \in [0, 2\pi]$ such that $f(2\pi) - f(0) = \mathrm{d}f(t_0)(1)2\pi$ (So the mean value Theorem, in the form 1.6, is not true if the dimension of the target space is greater than 1).

4.5. Let $f : U \longrightarrow \mathbb{R}^m$ be a differentiable function. Use Theorem 1.7 to prove

(1) if $\mathrm{d}f(x) = 0$, $\forall\, x \in U$, then f is locally constant. In particular, if U is connected, f is constant,
(2) if $T \in L(\mathbb{R}^n, \mathbb{R}^m)$. Then

$$\|f(a+h) - f(a) - T(h)\| \leq \|h\| \sup\{\|\mathrm{d}f(a+th) - T\| : t \in [0,1]\}.$$

4.6. Prove that the function

$$f_k(t) = \begin{cases} t^k & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

is of class C^{k-1} but is not of class C^k.

4.7. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}, \quad (x,y) \neq (0,0), \qquad f(0,0) = 0.$$

Prove that the partial derivatives at $(0,0)$ exist, but f is not differentiable at $(0,0)$.

4.8. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function.

(1) Prove that if $f(tx) = tf(x)$, $\forall\, t \in \mathbb{R}$, and f is differentiable at 0, then f is linear.
(2) Prove that if $f(tx) = |t|f(x)$, $\forall\, t \in \mathbb{R}$, and f is differentiable at 0, then f vanishes identically.
(3) Prove that if $f(tx) = t^2 f(x)$, $\forall\, t \in \mathbb{R}$, and f is twice differentiable at 0, then f is bilinear.
(4) Prove that if $f(tx) = t^k f(x)$, $\forall\, t \in \mathbb{R}$ and f is of class C^k, then

$$\mathrm{d}^i f(x)(h_1, \ldots, h_i) = \frac{1}{(k-i)!}\mathrm{d}^k f(0)(x, \ldots, x, h_1, \ldots, h_i).$$

4.9. Let $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a norm.

(1) Prove that the function $f(x) = \|x\|$ is not differentiable at 0.
(2) Prove that the function $f^2(x) = \|x\|^2$ is differentiable at 0 and $\mathrm{d}f^2(0) = 0$.
(3) Prove that the norm is induced by a scalar product if and only if f^2 is twice differentiable at 0.

4.10. Prove that, if $\{x_n\} \subseteq \mathbb{R}^n$ is a Cauchy sequence admitting a subsequence converging to $x \in \mathbb{R}^n$, then the whole sequence converges to x.

4.11. Let $\{x_n\} \subseteq \mathbb{R}^n$ be a sequence. Define convergence for the series $\sum x_n$ and prove the Cauchy convergence criterion for series.

4.12. Let $M(n, \mathbb{R})$ the space of $n \times n$ matrices with real entries. Consider the natural identification with $\mathbb{R}^{n^2}$ and define $\exp : M(n, \mathbb{R}) \longrightarrow M(n, \mathbb{R})$, by:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

(1) Prove that exp is well defined (i.e. the series converges).

(2) Prove that, if $AB = BA$, $(A+B)^k = \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} A^i B^{k-i}$. Conclude that, if $AB = BA$, $\exp(A+B) = \exp(A)\exp(B)$.

(3) Prove that $\exp(PAP^{-1}) = P\exp(A)P^{-1}$.

(4) Let A be an upper triangular matrix. Compute the diagonal entries of $\exp(A)$.

(5) Show that $\det(\exp(A)) = e^{\mathrm{trace}(A)}$, $\forall A \in M(n, \mathbb{R})$. Conclude that $\exp(A)$ is invertible $\forall A \in M(n, \mathbb{R})$. Hint: put A in upper diagonal form.

(6) Show that exp is differentiable and compute $\mathrm{d}\exp(A)(B)$. Hint: compute $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} \exp(A + tB)$.

(7) Show that $\mathrm{d}\exp(0) = \mathbb{1}$. In particular exp maps diffeomorphically a neighborhood of 0 onto a neighborhood of $\exp(0) = \mathbb{1}$. The (local) inverse is the *logarithm*.

4.13. Consider the function

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

(1) Prove that f is smooth.

(2) Let $0 < \delta_1 < \delta_2$. Prove that the function

$$\phi(x) = \frac{f(\|x\|^2 - \delta_1^2)}{f(\|x\|^2 - \delta_1^2) + f(\delta_2^2 - \|x\|^2)}$$

is a well defined smooth function with values in $[0, 1]$, that vanishes for $\|x\| \leq \delta_1$ and is identically 1 for $\|x\| \geq \delta_2$.

4.14. Consider the map $\phi : B^n(1) \longrightarrow \mathbb{R}^n$, $\phi(x) = x(1 - \|x\|^2)^{-1}$. Prove that ϕ is a diffeomorphism.

4.15. Use the local form of subimmersions (Theorem 1.27), to prove the following

<u>THEOREM</u> [Implicit function Theorem] *Let* $U \subseteq \mathbb{R}^n \times \mathbb{R}^m$ *be an open set.* $f : U \longrightarrow \mathbb{R}^m$ *a smooth function such that, for* $z_0 = (x_0, y_0) \in U$, $f(z_0) = 0$, *and* $\mathrm{d}_2 f(z_0) : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ *is an isomorphism. Then there exists a neighborhood* $V \subseteq \mathbb{R}^n$ *of* x_0 *and a unique smooth function* $g : V \longrightarrow \mathbb{R}^m$ *such that* $f(x, g(x)) = 0 \ \forall x \in V$. *Moreover*

$$\mathrm{d}g((x) = -[\mathrm{d}_2 f(x, g(x))]^{-1} \circ \mathrm{d}_1 f(x, g(x))$$

($\mathrm{d}_i f$ is defined in Remark 1.12).

4.16. Prove Proposition 2.3

4.17. Let $T : [a, b] \subseteq \mathbb{R} \longrightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ be an integrable function and $Y \in \mathbb{R}^m$. Prove that the function $f(t) = T(t)Y$ is integrable and

$$\int_a^b f = [\int_a^b T]Y.$$

4.18. Prove that, if $f : [a, b] \longrightarrow \mathbb{R}^m$ is integrable and $x \in [a, b]$ then $f|_{[a,x]}$ and $f|_{[x,b]}$ are integrable and

$$\int_a^b f = \int_a^x f + \int_x^b f.$$

4.19. Let $f : [a, b] \longrightarrow \mathbb{R}^m$ be such that $f(t) = 0$ for t outside a finite set. Prove that f is integrable and $\int_a^b f = 0$. Conclude that if two functions $f, g : [a, b] \longrightarrow \mathbb{R}^m$ differ only on a finite set, then one is integrable if and only if the other one is integrable and, in this case, the two integral coincide.

4.20. Prove that any continuous function $f : [a, b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m$ is uniform limit of step functions.

4.21. A curve $\gamma : [a, b] \longrightarrow \mathbb{R}^m$ is said to be *rectificable* if there exists $l = l(\gamma) \in \mathbb{R}$ (called the *length of* γ) such that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $P = \{t_0, \ldots, t_k\}$ is a partition with $|P| < \delta$, we have

$$|l - \sum_0^{k-1} \|\gamma(t_{i+1}) - \gamma(t_i)\| \; | < \epsilon.$$

Prove that if γ is of class C^1, γ is rectificable and $l(f) = \int_a^b \dot{\gamma}(t)\mathrm{d}t := \int_a^b \mathrm{d}\gamma(t)(1)\mathrm{d}t$.

4.22. Use Fubini's Theorem (Theorem 2.17) to prove Theorem 1.15. Hint: it is nor restrictive to assume $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (why?). If $\dfrac{\partial f}{\partial x \partial y} - \dfrac{\partial f}{\partial y \partial x} > 0$ at $z_0 = (x_0, y_0)$ so it is in a small rectangle $C = [a, b] \times [c, d]$ containing z_0. Show that the integral over C of the difference is zero.

4.23. Let $X : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth vector field. Prove that if $\mathrm{supp}(X) := \overline{\{x \in U : X(x) \neq 0\}}$ is compact, then X is complete.

4.24. Let $X : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth vector field. Prove that, if there is a constant M with $\|X(x)\| \leq M$, $\forall\ x \in \mathbb{R}^n$, then X is complete (hint: show that an integral curve $\gamma : [0, a) \longrightarrow \mathbb{R}^n$ has finite length, if $a < \infty$, so its image has compact closure).

4.25. Give an example of a *non complete* vector field in $\mathbb{R}$.

4.26. Let $f \in \mathcal{F}(U),\ 0 \in U \subseteq \mathbb{R}^n,\ f(0) = 0$. Prove that there exist functions $g_i \in \mathcal{F}(U_i)$ where $U' \subseteq U$ is an open neighborhood of 0, such that $f(x_i, \ldots x_n) = \sum_{i=1}^n x_i g_i(x_i, \ldots x_n)$ and $g_i(0) = \dfrac{\partial f}{\partial x_i}(0)$. Hint: write $f(x) = \displaystyle\int_0^1 \frac{\mathrm{d}f(tx)}{\mathrm{d}t}\mathrm{d}t$.

4.27. Consider $\mathcal{F}_0$, the algebra of germs of smooth functions at $0 \in \mathbb{R}^n$ and $\mathcal{I}_0 = \{[f] \in \mathcal{F}_0 : f(0) = 0\}$.

(1) Prove that $\mathcal{I}_0$ is the *unique* maximal (non trivial) ideal of $\mathcal{F}_0$.

(2) Let $\mathcal{I}_0^2$ be the ideal generated by products of two elements in $\mathcal{I}_0$. Prove that $\mathcal{I}_0/\mathcal{I}_0^2$ is a n-dimensional real vector space spanned by the (equivalence classes of) the germs of the coordinate functions. Conclude that $\mathcal{I}_0/\mathcal{I}_0^2$ is canonically isomorphic to $[\mathbb{R}^n]^*$.

4.28. Prove that if $X, Y \in \mathcal{D}er(U)$ then $[X, Y] := X \circ Y - Y \circ X \in \mathcal{D}er(U)$.

4.29. Prove Proposition 3.23.

4.30. Let

$$X = \sum_k a_k(x) \frac{\partial}{\partial x_k}, \qquad Y = \sum_k b_k(x) \frac{\partial}{\partial x_k}$$

be smooth vector fields in $\mathbb{R}^n$.

(1) Compute $[X, Y]$ in the basis $\dfrac{\partial}{\partial x_k}$.

(2) Let $X_1, \ldots, X_p$ be linear independent vectors in $\mathbb{R}^n$. Show that there exist smooth vector fields $\tilde{X}_1, \ldots, \tilde{X}_p$ in $\mathbb{R}^n$ such that, for a fixed $x \in U$, $\tilde{X}_i(x) = X_i$ and $[\tilde{X}_i, \tilde{X}_j] = 0$ in $\mathbb{R}^n$.

CHAPTER 1

The de Rham cohomology for open sets of $\mathbb{R}^n$

1. Exterior forms

Let $\mathbb{E}$ be a finite dimensional real vector space and $\mathbb{E}^*$ its dual. We will identify, as usual, $\mathbb{E}$ with the double dual $(\mathbb{E}^*)^* := \mathbb{E}^{**}$.

1.1. Definition. A *tensor of type* (p,q) in $\mathbb{E}$ is a multilinear[1] map:

$$t\colon \underbrace{\mathbb{E}^* \times \cdots \times \mathbb{E}^*}_{p \text{ times}} \times \underbrace{\mathbb{E} \times \cdots \times \mathbb{E}}_{q \text{ times}} \longrightarrow \mathbb{R}$$

We will denote by $\mathbb{E}_{(p,q)}$ the space of these tensors. This is a real vector space with the operations of sum of multilinear maps (summing the values) and product by a scalar (multiplying the values by the scalar).

1.2. Examples.

- $\mathbb{E}_{(0,1)} = \mathbb{E}^*$, $\mathbb{E}_{(1,0)} = \mathbb{E}^{**} = \mathbb{E}$.
- A scalar product in $\mathbb{E}$ is an element of $\mathbb{E}_{(0,2)}$.
- It is convenient to define $\mathbb{E}_{(0,0)} := \mathbb{R}$.

We will be interested mainly in tensors of type $(0,q)$. To simplify the notations we will set $\mathbb{E}_q := \mathbb{E}_{(0,q)}$. Beside adding tensors, we can multiply them.

1.3. Definition. Given $\omega \in \mathbb{E}_p$, $\tau \in \mathbb{E}_q$, we define the *tensor product* $\omega \otimes \tau \in \mathbb{E}_{p+q}$ as

$$\omega \otimes \tau(x_1, \ldots, x_{p+q}) := \omega(x_1, \ldots, x_p)\tau(x_{p+1}, \ldots x_{p+q}).$$

1.4. Remark. It is easy to see that the tensor product is associative and distributive (Exercise 7.1) and therefore, suitable extended, define an associative algebra structure in $\mathbb{E}_* := \oplus \mathbb{E}_p$. Whit this structure $\mathbb{E}_*$ is called the *tensor algebra*.

1.5. Proposition. *Let* $\{\omega_1, \ldots, \omega_n\}$ *be a basis of* $\mathbb{E}_1 = \mathbb{E}^*$. *Then the set* $\{\omega_{i_1} \otimes \cdots \otimes \omega_{i_q} : i_1, \ldots, i_q \in \{1, \ldots, n\}\}$ *is a basis of* $\mathbb{E}_q$.

Proof. Let $\{e_1, \cdots, e_n\}$ be the dual basis, i.e., $\omega_i(e_j) = \delta_{ij}$. Then:

$$\sum a_{i_1 \cdots i_q} \omega_{i_1} \otimes \cdots \otimes \omega_{i_q}(e_{j_1}, \ldots, e_{j_q}) = a_{j_1 \cdots j_q}.$$

---

[1] i.e. linear in each variable.

It follows, by a standard argument, that the the elements of the set in question are linearly independent. Now, given $\omega \in \mathbb{E}_q$ we define $a_{i_1 \cdots i_q} = \omega(e_{i_1}, \ldots, e_{i_q})$. It is easy to check that $\omega = \sum a_{i_1 \cdots i_q} \omega_{i_1} \otimes \cdots \otimes \omega_{i_q}$, and this concludes the proof.

□

We will be interested in special elements of $\mathbb{E}_q$. Let $\Sigma(p)$ be the group of permutations of $\{1, \ldots, p\} \subseteq \mathbb{N}$. If $\pi \in \Sigma(p)$, we will denote by $|\pi|$ the sign of π, i.e. $|\pi| = 1$ if π is the product of an even number of transpositions and $|\pi| = -1$ otherwise.

1.6. Definition. Let $\omega \in \mathbb{E}_p$. We will say that

- ω is *a symmetric form* if $\omega(x_1, \ldots, x_p) = \omega(x_{\pi(1)}, \ldots, x_{\pi(p)}), \quad \forall\ \pi \in \Sigma(p)$.
- ω is an *exterior form* [2] if $\omega(x_1, \ldots, x_p) = |\pi|\omega(x_{\pi(1)}, \ldots, x_{\pi(p)}), \quad \forall\ \pi \in \Sigma(p)$.

We will denote by $\Sigma^p(\mathbb{E})$ the space of symmetric tensors in $\mathbb{E}_p$ and with $\Lambda^p(\mathbb{E})$ the space of exterior p-forms. These are subspaces of $\mathbb{E}_p$. Clearly $\Lambda^0(\mathbb{E}) = \mathbb{R} = \Sigma^0(\mathbb{E}), \quad \Lambda^1(\mathbb{E}) = \mathbb{E}_1 = \mathbb{E}^* = \Sigma^1(\mathbb{E})$.

We will be mostly interested in exterior forms and we will describe now the basic example.

1.7. Example. Let $\{e_1, \ldots, e_n\}$ be a fixed basis of $\mathbb{E}$ and $\{\phi_1, \ldots, \phi_n\}$ be the dual basis. Let us fix indexes $1 \leq i_1 < \cdots < i_p \leq n$ and define:

$$\omega_{(i_1, \ldots, i_p)}(x_1, \ldots, x_p) := \det(\phi_{i_j}(x_k)).$$

In other words we consider the matrix whose k^{th} column is given by the coordinates of x_k in the fixed basis, and compute the determinant of the sub matrix obtained considering only the lines $(i_1, \ldots, i_p)$ of the original matrix. The $\omega_{(i_1, \ldots, i_p)}$'s are exterior p-forms since the determinant is multilinear in the columns and, permuting the columns the sign changes according to the parity of the permutation. As we shall see (Proposition 1.20 and Remark 1.18), these forms are a basis of $\Lambda^p(\mathbb{E})$.

1.8. Remark. By Example 1.7 p-forms are, essentially, determinants of $p \times p$ matrices and, therefore, "p-dimensional (oriented) volume elements". So they appear as the *natural integrands of the multiple (oriented) integrals.* This statement will be made precise in the next chapter.

The tensor product of exterior forms is not, in general, an exterior form. But we can "alternate" the tensor product in order to obtain an exterior form. Define the linear operator

$$A : \mathbb{E}_p \longrightarrow \mathbb{E}_p, \qquad A(\tau)(x_1, \ldots, x_p) = \frac{1}{p!} \sum_{\pi \in \Sigma(p)} |\pi| \tau(x_{\pi(1)}, \ldots, x_{\pi(p)}).$$

1.9. Proposition.

1. *If* $\tau \in \mathbb{E}_p, \quad A(\tau) \in \Lambda^p(\mathbb{E})$.
2. *If* $\tau \in \Lambda^p(\mathbb{E}), \quad A(\tau) = \tau$.

In particular $A^2 = A$.

---

[2] The terms alternating tensor or skew symmetric tensor are also used in the literature.

PROOF. If $p = 1$ there is nothing to prove, so we assume $p > 1$. For $i, j \in \{1, \dots, p\}$, we will denote by (ij) the element of $\Sigma(p)$ that interchanges i and j and leaves the other integers fixed. If $\pi \in \Sigma(p)$, we set $\pi' = \pi \circ (ij)$. Then $|\pi'| = -|\pi|$ and

$$A(\tau)(x_1, \dots, x_j, \dots, x_i, \dots, x_p) = \frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi(1)}, \dots, x_{\pi(j)}, \dots, x_{\pi(i)}, \dots, x_{\pi(p)}) =$$

$$\frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi'(1)}, \dots, x_{\pi'(i)}, \dots, x_{\pi'(j)}, \dots, x_{\pi'(p)}) =$$

$$\frac{1}{p!} \sum_{\pi'} -|\pi'| \tau(x_{\pi'(1)}, \dots, x_{\pi'(i)}, \dots, x_{\pi'(j)}, \dots, x_{\pi'(p)}) = -A(\tau)(x_1, \dots, x_i, \dots, x_j, \dots, x_p)$$

It is easy to see that the equation above implies that $A(\tau) \in \Lambda^p(\mathbb{E})$ (see Exercise 7.2). Moreover, if $\tau \in \Lambda^p(\mathbb{E})$,

$$A(\tau)(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\pi} |\pi| \tau(x_{\pi(1)}, \dots x_{\pi(p)}) = \frac{1}{p!} \sum_{\pi} |\pi|^2 \tau(x_1, \dots x_p) = \tau(x_1, \dots, x_p)$$

and this proves the second claim.

□

Observe that, in general, $A(\phi \otimes \psi) \neq A(\phi) \otimes A(\psi)$. However we have

1.10. LEMMA. *If $\phi_1, \dots, \phi_p \in \mathbb{E}^*$, then:*

$$A(\phi_1 \otimes \dots \otimes \phi_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_{\sigma(1)} \otimes \dots \otimes \phi_{\sigma(p)}.$$

PROOF.

$$A(\phi_1 \otimes \dots \otimes \phi_p)(x_1, \dots, x_p) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_1 \otimes \dots \otimes \phi_p(x_{\sigma(1)}, \dots, x_{\sigma(p)}) =$$

$$\frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_1(x_{\sigma(1)}) \cdots \phi_p(x_{\sigma(p)}) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \phi_{\sigma(1)}(x_1) \cdots \phi_{\sigma(p)}(x_p).$$

□

Using the operator A we can define *product of exterior forms.*

1.11. DEFINITION. The *exterior (or wedge) product* is defined as the map

$$\wedge : \Lambda^p(\mathbb{E}) \times \Lambda^q(\mathbb{E}) \longrightarrow \Lambda^{p+q}(\mathbb{E}), \quad \wedge(\omega, \tau) := \omega \wedge \tau = \frac{(p+q)!}{p!q!} A(\omega \otimes \tau).$$

(The reason for the coefficient $\frac{(p+q)!}{p!q!}$ will be discuss in Remark 1.19.)

It is easy to prove that the exterior product is distributive (see Exercise 7.3). In particular, suitable extended, it defines an algebra structure on $\Lambda^*(\mathbb{E}) := \oplus \Lambda^p(\mathbb{E})$. $\Lambda^*(\mathbb{E})$ is called the *exterior algebra.*

It is also true that the exterior product is associative, but this fact is a little bit tricky. The proof involves a characterization of the kernel of A. The problem is that A is not an algebra homomorphism, hence we can not conclude, directly, that $\ker A$ is an ideal. We will prove that, in fact, $\ker A$ is an ideal.

Consider the ideal $\mathcal{I} \subseteq \mathbb{E}_*$ generated by $\phi \otimes \phi,\ \phi \in \mathbb{E}^*$. This is the vector subspace of $\mathbb{E}_*$ generated by elements of the form $\tau \otimes \phi \otimes \phi, \psi \otimes \psi \otimes \eta,\ \phi, \psi \in \mathbb{E}^*, \tau, \eta \in \mathbb{E}_*$ or, alternatively, it is the intersection of all ideals containing the elements of the form $\phi \otimes \phi,\ \phi \in \mathbb{E}^*$.

1.12. Theorem. $\ker A = \mathcal{I}$.

Proof. It is easily seen that $\mathcal{I} \subseteq \ker A$. We will prove that $\ker A \subseteq \mathcal{I}$. Consider the quotient algebra $\mathbb{E}_*/\mathcal{I}$. Denote by $\cdot$ the product in this quotient and by $\pi : \mathbb{E}_* \longrightarrow \mathbb{E}_*/\mathcal{I}$ the projection map, which is an algebra homomorphism. First observe that, if $\phi, \psi \in \mathbb{E}^*$:

$$0 = \pi((\phi+\psi) \otimes (\phi+\psi)) = \pi(\phi \otimes \phi + \phi \otimes \psi + \psi \otimes \phi + \psi \otimes \psi) = \pi(\phi \otimes \psi) + \pi(\psi \otimes \phi),$$

i.e. $\pi(\phi \otimes \psi) = -\pi(\psi \otimes \phi)$. Therefore, for $\phi_1, \dots, \phi_p \in \mathbb{E}^*$ and $\sigma \in \Sigma(p)$, we have

$$\pi(\phi_{\sigma(1)}, \otimes \dots, \otimes \phi_{\sigma(p)}) = \pi\left(\phi_{\sigma(1)}\right) \cdots \pi(\phi_{\sigma(p)}) = |\sigma| \pi(\phi_1) \cdots \pi(\phi_p) = |\sigma| \pi(\phi_1 \otimes \cdots \otimes \phi_p).$$

Hence

$$\pi(A(\phi_1 \otimes \cdots \otimes \phi_p)) = \pi(\frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma| \pi(\phi_{\sigma(1)} \otimes \cdots \otimes \phi_{\sigma(p)})) = \frac{1}{p!} \sum_{\sigma \in \Sigma(p)} |\sigma|^2 \pi(\phi_1 \otimes \cdots \otimes \phi_p) = \pi(\phi_1 \otimes \cdots \otimes \phi_p).$$

So any element in $\ker A$ is in $\mathcal{I} := \ker\ \pi$. ☐

1.13. Corollary. *Let* $\omega \in \mathbb{E}_p,\ \tau \in \mathbb{E}_q$. *If* $A(\omega) = 0,\quad A(\omega \otimes \tau) = 0 = A(\tau \otimes \omega)$.

Proof. This follows from the fact that $\ker A$ is an ideal. ☐

At this point we can prove the announced result

1.14. Proposition. *The wedge product is associative.*

Proof. First we observe that:

$$A(A(\omega \otimes \eta) \otimes \theta)) = A(\omega \otimes \eta \otimes \theta) = A(\omega \otimes A(\eta \otimes \theta)).$$

In fact, by 1.9, $A(A(\eta \otimes \theta) - \eta \otimes \theta) = 0$ and, by 1.13, we have that:

$$0 = A(\omega \otimes [A(\eta \otimes \theta) - \eta \otimes \theta]) = A(\omega \otimes A(\eta \otimes \eta) - \omega \otimes \eta \otimes \theta) = A(\omega \otimes A(\eta \otimes \theta)) - A(\omega \otimes \eta \otimes \theta),$$

which proves the second equality. The first one is proved in a similar way.

Therefore, if $\omega \in \Lambda^k(\mathbb{E}), \eta \in \Lambda^l(\mathbb{E}), \theta \in \Lambda^m(\mathbb{E})$, we have:

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!m!} A((\omega \wedge \eta) \otimes \theta) = \frac{(k+l+m)!}{(k+l)!m!} \frac{(k+l)!}{k!l!} A(\omega \otimes \eta \otimes \theta),$$

and the associativity follows from the associativity of the tensor product. ☐

1.15. Example. Let $\phi_1, \phi_2 \in \mathbb{E}^* = \mathbb{E}_1,\ x_1, x_2 \in \mathbb{E}$. Then:

$$\phi_1 \wedge \phi_2(x_1, x_2) = 2\frac{1}{2}(\phi_1(x_1)\phi_2(x_2) - \phi_1(x_2)\phi_2(x_1)) = \det[\phi_i(x_j)].$$

More generally, an induction on p gives:

1.16. Proposition. *Let* $\phi_i \in \mathbb{E}^*, x_j \in \mathbb{E} \;\; i,j = 1, \ldots, p$. *Then:*

$$\phi_1 \wedge \cdots \wedge \phi_p(x_1, \ldots, x_p) = \det[\phi_i(x_j)].$$

In particular if $\sigma \in \Sigma(p)$, $\phi_1 \wedge \cdots \wedge \phi_p = |\sigma| \phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(p)}$.

1.17. Remark. Observe that, by 1.14, the form $\phi_1 \wedge \cdots \wedge \phi_p$ is well defined.

1.18. Remark. In the Example 1.7 the form $\omega_{(i_1, \ldots, i_p)}$ is just $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$.

1.19. Remark. The coefficient $\dfrac{(p+q)!}{p!q!}$ in 1.11 is convenient in order to avoid unpleasant coefficients in 1.16 and also for a geometric reason: let $\mathbb{E}$ be an inner product space, $\{e_1, \ldots, e_n\}$ an orthonormal basis and $\{\phi_1, \ldots, \phi_n\}$ the dual basis (so $\phi_i(e_j) = \langle e_i, e_j \rangle = \delta_{ij}$). Given vectors $x_1, \ldots, x_n \in \mathbb{E}$, $\phi_1 \wedge \cdots \wedge \phi_n(x_1, \ldots, x_n)$ is the "volume" of the parallelepiped of edges the $x_i' s$. The coefficient above is such that the "unit cube", i.e. the parallelepiped spanned by the e_i's, has volume 1 (see Definition 1.28).

1.20. Proposition. *Let* $\{\phi_1, \ldots, \phi_n\}$ *be a basis for* $\mathbb{E}^*$. *Then*

$$\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$$

is a basis of $\Lambda^p(\mathbb{E})$. *In particular* $\Lambda^p(\mathbb{E})$ *has dimension* $\binom{n}{p}$ *and* $\Lambda^p(\mathbb{E}) = \{0\}$, *if* $p > n$.

Proof. Let $\{e_1, \ldots, e_n\}$ be the dual basis. First observe that $\phi_1 \wedge \cdots \wedge \phi_n(e_1, \ldots, e_n) = \det[\phi_i(e_j)] = 1$. Also observe that $\phi_i \wedge \phi_j = -\phi_j \wedge \phi_i$ and, in particular, if we interchange two elements in the product $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$ the form changes sign. We will prove now that the forms $\{\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} : i_1 < \cdots < i_p\}$ are linearly independent. Suppose

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \cdots i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p} = 0.$$

We want to show that $a_{i_1 \cdots i_p} = 0$. We will do it for $a_{1 \cdots p}$, the other cases being analogous. Observe that

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \cdots i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p} \wedge \phi_{p+1} \wedge \cdots \wedge \phi_n(e_1, \ldots, e_n) = a_{1 \cdots p} = 0,$$

since the terms with $\{i_1, \ldots, i_p\} \neq \{1, \ldots, p\}$ vanish (they contain two equal indexes), and the conclusion follows. We leave to the reader the task of showing that they span $\Lambda^p(\mathbb{E})$ (Exercise 7.4).

□

1.21. Corollary. *The algebra* $\Lambda^*(\mathbb{E})$ *is graded commutative*[3] *, i.e. if* $\omega \in \Lambda^p(\mathbb{E})$, $\tau \in \Lambda^q(\mathbb{E})$

$$\omega \wedge \tau = (-1)^{pq} \tau \wedge \omega.$$

In particular the square of a form of odd degree is zero.

Proof. As we have seen this is true for products of decomposable elements (i.e. elements of the form $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$). The general case follows from the fact that such forms span the exterior algebra. □

1.22. Remark. There is a restriction, in Proposition 1.20, on the set of indexes with respect to Proposition 1.5 and this is due to the graded commutativity of the exterior algebra.

---

[3] An algebra $\mathbb{E}$, with product $b : \mathbb{E} \oplus \mathbb{E} \longrightarrow \mathbb{E}$ is a *graded algebra* if there is a sequence of vector subspaces $\mathbb{E}_i$ such that $\mathbb{E} = \oplus \mathbb{E}_i$ and $b(\mathbb{E}_i \oplus \mathbb{E}_j) \subseteq E_{i+j}$. Such an algebra is said to be *graded commutative* if for $\omega \in \mathbb{E}_p$, $\tau \in \mathbb{E}_q$, $b(\omega, \tau) = (-1)^{pq} b(\tau, \omega)$.

Let $L : \mathbb{E} \longrightarrow \mathbb{F}$ be a linear map. Recall that the *transpose* of L is the map

$$L^* : \mathbb{F}^*(=\mathbb{F}_1) \longrightarrow \mathbb{E}^*(=\mathbb{E}_1), \qquad L^*(\phi)(x) := \phi(Lx).$$

This map extends to a linear map

$$\mathbb{E}_p(L) : \mathbb{F}_p \longrightarrow \mathbb{E}_p, \qquad \mathbb{E}_p(L)(\omega)(x_1, \ldots, x_p) = \omega(L(x_1), \ldots, L(x_p)).$$

It is simple to see that if $\omega \in \Lambda^p(\mathbb{F})$ then $\mathbb{E}_p(L)(\omega) \in \Lambda^p(\mathbb{E})$. So we get, by restriction, a linear map

$$\Lambda^p(L) := \mathbb{E}_p(L)|_{\Lambda^p(\mathbb{F})} : \Lambda^p(\mathbb{F}) \longrightarrow \Lambda^p(\mathbb{E}),$$

and, by additivity, a linear map $\Lambda^*(L) : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$.

When clear from the context we will write L^*_p, or just L^*, for $\Lambda^p(L)$ and $\Lambda^*(L)$.

1.23. Proposition. *$L^*(\omega \wedge \tau) = L^*(\omega) \wedge L^*(\tau)$. This means that L induces a graded algebra homomorphism $L^* : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$. Moreover we have the following properties, called the* funtorial properties[4]

(1) $(\mathbb{1}_{\mathbb{E}})^* = \mathbb{1}_{\Lambda^*(\mathbb{E})}$.
(2) *If $L : \mathbb{E} \longrightarrow \mathbb{F}$ and $T : \mathbb{F} \longrightarrow \mathbb{G}$ are linear maps, then $(T \circ L)^* = L^* \circ T^*$.*

Proof. To prove the first assertion, we just observe that, if $\phi_i \in \mathbb{E}^*, x_j \in \mathbb{E},\ i, j = 1, \ldots, p$, we have:

$$L^*_p(\phi_1 \wedge \cdots \wedge \phi_p)(x_1, \ldots, x_p) = \det[\phi_i(Lx_j)] = \det[L^*(\phi_i)(x_j)] = L^*(\phi_1) \wedge \cdots \wedge L^*(\phi_p)(x_1, \ldots, x_p).$$

Since $\Lambda^p(\mathbb{E})$ is spanned by elements of the form $\phi_1 \wedge \cdots \wedge \phi_p$, by Proposition 1.20, the conclusion follows by linearity. The functorial properties are obvious. □

1.24. Remark. We will meet often, along these notes, "functorial properties". These properties are usually trivial to prove, but important. For example, in the context of Proposition 1.23, they imply that, if L is an isomorphism, then L^* is also an isomorphism (see Exercise 7.15).

Let $\mathbb{E}$ be a finite dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle : \mathbb{E} \times \mathbb{E} \longrightarrow \mathbb{R}$.

1.25. Definition. The isomorphisms

$$\flat : \mathbb{E} \longrightarrow \mathbb{E}^*, \qquad \flat(x)(y) = \langle x, y \rangle, \qquad \sharp : \mathbb{E}^* \longrightarrow \mathbb{E}, \qquad \sharp := \flat^{-1},$$

are called the *musical isomorphisms*.

We define an inner product in $\mathbb{E}^*$ by requiring $\flat$ to be an isometry. We can also define an inner product in $\Lambda^p(\mathbb{E})$ extending, by bi-linearity, the formula

$$\langle \phi_1 \wedge \cdots \wedge \phi_p, \psi_1 \wedge \cdots \wedge \psi_p \rangle = \det(\langle \phi_i, \psi_j \rangle).$$

Observe that, if $\{\omega_i\}$ is an orthonormal basis for $\mathbb{E}^*$, the basis $\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} : i_1 < \cdots < i_p\}$ is orthonormal.

We recall that two bases of a n-dimensional real vector space $\mathbb{E}$ are *equioriented* if the matrix that gives the change of bases has positive determinant. This relation is an equivalence relation and the set of bases of $\mathbb{E}$ is divided into two equivalence classes.

---

[4] In the language of category theory this means that the low that associate to a finite dimensional real vector space $\mathbb{E}$ the graded algebra $\Lambda^*(\mathbb{E})$ and to a linear maps $L : \mathbb{E} \longrightarrow \mathbb{F}$ the map L^* is a contravariant functor from the category of finite dimensional real vector spaces and linear maps, to the category of algebras and their homomorphisms.

1.26. Definition. An orientation on $\mathbb{E}$ is the choice of one of two equivalence classes of equioriented bases. $\mathbb{E}$ is *oriented* if such a choice has been made and the bases in the chosen class will be called *positive.*

1.27. Remark. Naturally an orientation in $\mathbb{E}$ induces an orientation on $\mathbb{E}^*$, by declaring positive the bases that are dual of positive bases of $\mathbb{E}$.

1.28. Definition. Let $\mathbb{E}$ be a n-dimensional *oriented* inner product space and let $\{\omega_1, \dots, \omega_n\}$ be a positive orthonormal basis of $\mathbb{E}^*$. The *volume form* of $\mathbb{E}$ is the n-form $v = \omega_1 \wedge \cdots \wedge \omega_n$.

1.29. Lemma. *The volume form is well defined, i.e. it does not depend on the choice of the basis.*

Proof. Let $\{\omega_i\}$, $\{\phi_j\}$ be bases of $\mathbb{E}^*$ and $A = (a_{ij})$ such that $\phi_k = \sum a_{kj}\omega_j$. Then

$$\phi_1 \wedge \cdots \wedge \phi_n = \sum_{\sigma \in \Sigma(n)} |\sigma| a_{1\sigma(1)} \cdots a_{n\sigma(n)} \omega_1 \wedge \cdots \wedge \omega_n = \det(A) \omega_1 \wedge \cdots \wedge \omega_n.$$

If the bases are orthonormal and positive, then $A \in SO(n)$. In particular $\det(A) = 1$. ☐

1.30. Definition. Let $\mathbb{E}$ be a n-dimensional oriented inner product space. The *Hodge (star) operator* is the operator

$$*_p : \Lambda^p(\mathbb{E}) \longrightarrow \Lambda^{(n-p)}(\mathbb{E}), \qquad *_p(\eta)(x_1, \dots, x_{(n-p)}) := \langle \eta \wedge \flat(x_1) \wedge \cdots \wedge \flat(x_{(n-p)}), v \rangle,$$

where v is the volume form. When clear from the context, we will write simply $*$ instead of $*_p$.

1.31. Remark. Let $\{\omega_i\}$ be a positive orthonormal basis for $\mathbb{E}^*$. Then the Hodge operator may be defined by extending, linearly, the map

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-p}},$$

where $\{i_1, \dots, i_p, j_1, \dots j_{n-p}\}$ is an even permutation of $\{1, \dots, n\}$.

The following properties are easily established

1.32. Proposition. $*$ *is a linear isometry and* $*_{n-p} \circ *_p = (-1)^{p(n-p)} \mathbb{1}_{\Lambda^p(\mathbb{E})}$.

2. Differential forms and the de Rham cohomology

2.1. Definition. A *differential p-form* on an open set $U \subseteq \mathbb{R}^n$ is a smooth map $\omega : U \longrightarrow \Lambda^p(\mathbb{R}^n) \cong \mathbb{R}^{\binom{n}{p}}$. When clear from the context we will just say that ω is a differential form or simply a form.

2.2. Remark. According to Remark 3.2 of Chapter 0, we can complicate the definition in order to have one that make sense in the context of smooth manifold. Consider the *bundle of exterior p-forms*

$$\Lambda^p(U) := \cup_{x \in U} \Lambda^p(T_x U)$$

that can be identified with $U \times \Lambda^p(\mathbb{R}^n)$. Then a differential p-form is a smooth map $\tilde{\omega} : U \longrightarrow \Lambda^p(U)$ such that $\tilde{\omega}(x) \in \Lambda^p(T_x U)$, i.e, $\tilde{\omega}(x) = (x, \omega(x))$, $\omega(x) \in \Lambda^p(\mathbb{R}^n)$.

We will denote by $\Omega^p(U)$ the set of differential p-forms on U. $\Omega^p(U)$ has an obvious structure of real vector space. Moreover we can multiply a differential form by a function and this operation is associative and distributive, in the appropriate sense, i.e. $\Omega^p(U)$ is a *module over* $\mathcal{F}(U)$.

A differential form $\omega \in \Omega^p(U)$ induces a $\mathcal{F}(U)$-multilinear map, denoted by the same symbol,

$$\omega : \mathcal{H}(U) \times \cdots \times \mathcal{H}(U) \longrightarrow \mathcal{F}(U), \quad \omega(X_1, \dots, X_p)(x) = \omega(x)(X_1(x), \dots, X_p(x)).$$

Conversely, we have

2.3. Theorem. [Tensoriality Criterion] *A* $\mathbb{R}$*-multilinear map*

$$\omega : \mathcal{H}(U) \times \cdots \times \mathcal{H}(U) \longrightarrow \mathcal{F}(U),$$

is induced by a differential form if and only if it is $\mathcal{F}(U)$*-multilinear.*

Proof. Clearly, if ω is induced by a form, it is $\mathcal{F}(U)$-multilinear. Suppose that ω is $\mathcal{F}(U)$-multilinear. Let $x \in U, X_i \in T_xU$. Extend the X_i's to vector fields $\tilde{X}_i \in \mathcal{H}(U)$, $\tilde{X}_i(y) = \sum_j a_{ij}(y)e_j$, and define:

$$\omega(x)(X_1, \dots, X_p) := \omega(\tilde{X}_1, \dots, \tilde{X}_p)(x).$$

In order to show that the above equality defines a form it is sufficient to show that it does not depend on the extensions. In fact, by $\mathcal{F}(U)$-multilinearity,

$$\omega(\tilde{X}_1, \dots, \tilde{X}_p)(x) = \sum_{i_1, \dots, i_p = 1}^{n} a_{1i_1}(x) \cdots a_{pi_p}(x) \omega(e_{i_1}, \dots, e_{i_p}).$$

□

2.4. Example. Since $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$, $\Omega^0(U) = \mathcal{F}(U)$.

The basic example of a differential form is the following. Let $f \in \mathcal{F}(U)$. Then the differential of f is the the 1-form

$$(\mathrm{d}f)(x)(X) := X(x)(f), \qquad X \in \mathcal{D}er(U).$$

In particular, we can consider the coordinate functions $x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$. At each point $x \in U$, the differentials at x, $\mathrm{d}x_i(x)$ [5] are a basis of $\Lambda^1(\mathbb{R}^n)$. Therefore $\{\mathrm{d}x_{i_1}(x) \wedge \cdots \wedge \mathrm{d}x_{i_p}(x) : \ 1 \leq i_i < \cdots < i_p \leq n\}$ is a basis of $\Lambda^p(\mathbb{R}^n)$. So we have

2.5. Proposition. *Let* $\omega \in \Omega^p(U)$*. Then* ω *can be written in a unique way as:*

$$\omega = \sum_{i_1 < \cdots < i_p} \omega_{i_1, \dots, i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p},$$

where $\omega_{i_1, \dots, i_p} \in \mathcal{F}(U)$*.*

2.6. Example. If $f \in \mathcal{F}(U)$, $\mathrm{d}f = \sum_1^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i$.

2.7. Remark. As a real vector space, $\Omega^p(U)$ is infinite dimensional (if $n > 0$!), but as a $\mathcal{F}(U)$-module, it is a free module of dimension $\binom{n}{p}$.

---

[5] Since x_i is linear, $\mathrm{d}x_i = x_i$, and $\mathrm{d}x_i$ is the form that associates to a vector its i^{th} coordinate in the canonical basis.

Let $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth function, $F(x) = (F_1(x), \ldots, F_m(x))$. Then $\mathrm{d}F(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear map and we have an induced map $F^* : \Lambda^p(\mathbb{R}^m) \longrightarrow \Lambda^p(\mathbb{R}^n)$. This map induces a linear map:

$$F^* : \Omega^p(V) \longrightarrow \Omega^p(U), \qquad F^*(\omega)(X_1, \ldots, X_p)(x) := \omega(\mathrm{d}F(x)(X_1), \ldots, \mathrm{d}F(x)(X_p)).$$

If $x_1, \ldots, x_n,\ y_1, \ldots y_m$ are the canonical coordinates in $\mathbb{R}^n,\ \mathbb{R}^m$ respectively, we have

$$F^*(\mathrm{d}y_i) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_j} \mathrm{d}x_j, \tag{1}$$

and therefore, if $\omega = \sum\limits_{i_1, \ldots, i_p} \omega_{i_1, \ldots, i_p} \mathrm{d}y_{i_1} \wedge \cdots \wedge \mathrm{d}y_{i_p}$,

$$F^*(\omega)(x) = \sum_{i_1, \ldots, i_p} \omega_{i_1, \ldots, i_p}(F(x)) F^*(\mathrm{d}y_{i_1}) \wedge \ldots \wedge F^*(\mathrm{d}y_{i_1}).$$

We have the *functorial properties*:

- $\mathbb{1}_U^* = \mathbb{1}_{\Omega^p(U)}$,
- If $F_1 : U_1 \longrightarrow U_2$ e $F_2 : U_2 \longrightarrow U_3$ are smooth maps, $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

In particular, if F is a diffeomorphism, F^* is an isomorphism.

2.8. Example. Let $U \subseteq \mathbb{R}^n$ and $j : U \longrightarrow U \times \mathbb{R}^m, j(x_1 \ldots, x_n) = (x_1 \ldots, x_n, 0 \ldots, 0)$, be the inclusion. If $\omega = f(x_1, \ldots, x_{n+m})\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}, i_1 < \cdots < i_p$, $j^*\omega = 0$, if $i_p > n$, and $j^*\omega = f(x_1, \ldots x_n, 0, \ldots, 0)\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$ is $i_p \leq n$.

Differentiating a function can be viewed as a $\mathbb{R}$-linear map:

$$\mathrm{d} : \Omega^0(U) = \mathcal{F}(U) \longrightarrow \Omega^1(U).$$

Now we extend extend now this operation to higher dimensional forms.

2.9. Theorem. *There exists a unique family of* $\mathbb{R}$ *linear operators* $\mathrm{d}^p : \Omega^p(U) \longrightarrow \Omega^{p+1}(U),\ p = 0, \ldots, n$, *such that:*

(1) $\mathrm{d}^0 = \mathrm{d}$ *(the usual differential).*
(2) $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$.
(3) *If* $\omega \in \Omega^p(U),\ \tau \in \Omega^q(U), \mathrm{d}^{p+q}\omega \wedge \tau = \mathrm{d}^p\omega \wedge \tau + (-1)^p \omega \wedge \mathrm{d}^q\tau$.

Moreover, if $F : U \longrightarrow V$ *is a smooth map and* $\omega \in \Omega^p(V)$, *then* $\mathrm{d}^p F^*\omega = F^*\mathrm{d}^p\omega$.

When clear from the context we will write simply d *for* d^p.

Proof. Let us suppose that such a family exists. If $\omega = f(x)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$, we have:

$$\mathrm{d}\omega = (\mathrm{d}f) \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} + f\, \mathrm{d}(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}).$$

Now, from (1), $\mathrm{d}f = \sum\limits_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i$, and, from (2) and (3)

$$\mathrm{d}(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) = \sum_j \pm \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}\mathrm{d}x_{i_j} \wedge \cdots \wedge \mathrm{d}x_{i_p} = 0.$$

Therefore, if $\omega = \sum_{i_1<\cdots<i_p} \omega_{i_1\ldots i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$,

$$\mathrm{d}\omega = \sum_k \sum_{i_1<\cdots<i_p} \frac{\partial \omega_{i_1\ldots i_p}}{\partial x_k} \mathrm{d}x_k \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}.$$

This shows that if such a family exists, it is unique. Conversely, if we define d^p by the formula above we obtain a family of operators that, as it is easily seen, has the desired properties.

The last claim follows from

$$F^*(\mathrm{d}y_i) = \sum_j \frac{\partial F_i}{\partial x_j} \mathrm{d}x_j = \mathrm{d}(y_i \circ F) = \mathrm{d}(F^*(y_i))$$

and the fact that F^* is an algebra homomorphism. □

The operator d is called the *de Rham differential* or *exterior differential* or simply the *differential.* A simple but useful consequence of the properties above is the following

2.10. Corollary. d *is a local operator, i.e. if* $\omega \equiv \tau$ *in an open set* U*, then* $\mathrm{d}\omega = \mathrm{d}\tau$ *in* U.

Proof. The proof is essentially the same as the proof of the first claim in Lemma 3.18 of Chapter 0. □

We can also give an alternative definition of the exterior differential that does not depend on coordinates.

2.11. Proposition. *Let* $\omega \in \Omega^p(U)$, $X_0, \ldots, X_p \in \mathcal{H}(U)$. *Then*

$$\mathrm{d}\omega(X_0,\ldots,X_p) = \sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0,\ldots,\hat{X}_i,\ldots X_p) + \sum_{i<j} (-i)^{i+j} \omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p).$$

Proof. We sketch the proof leaving the details to the reader (Exercise 7.25). First observe that the right hand side of the equality above is $\mathcal{F}(U)$-multilinear, and so, by the tensoriality criterium, it is a differential form. In particular, to compute $\mathrm{d}\omega(X_0,\ldots,X_p)$ at a given point $x_0 \in U$, we can take arbitrarily extensions of the $X_i(x_0)$. So will be enough to prove the equality for the case the X_i's are coordinate vector fields. In this case $[X_i, X_j] = 0$ so the second term on the right hand side vanishes while the firs term is just the expression of $\mathrm{d}\omega$ given in Theorem 2.9. □

We have a sequence of vector spaces and $\mathbb{R}$-linear maps:

$$0 \longrightarrow \Omega^0(U) \xrightarrow{\mathrm{d}^0} \Omega^1(U) \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{n-1}} \Omega^n(U) \longrightarrow 0$$

which is a *cochain complex*, i.e. $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$, or, equivalently, $\mathrm{Im}\,\mathrm{d}^{p-1} \subseteq \ker\ \mathrm{d}^p$ (see next section for the definition and basic properties of cochain complexes). This sequence is called the *de Rham complex of* U. We define

- $Z^p(U) := \ker\ \mathrm{d}^p$, *the space of* p*-cocycles* or *closed* p*-forms.*
- $B^p(U) := \mathrm{Im}\ \mathrm{d}^{p-1}$, *the space* p*-coboundaries* or *exact* p*-forms.*
- $H^p(U) := Z^p(U)/B^p(U)$, *the* p*-dimensional (de Rham) cohomology of* U.

2.12. Remark. Let $\omega,\ \tau$ be closed forms in U. Then $\mathrm{d}(\omega \wedge \tau) = 0$, i.e. $\omega \wedge \tau$ is closed. Moreover if $\tau = \mathrm{d}\beta,\ \ \omega \wedge \tau = \pm\mathrm{d}(\omega \wedge \beta)$, i.e. $\omega \wedge \tau$ is exact. In particular the wedge product induces a well defined bilinear map $\cup : H^p(U) \oplus H^q(U) \longrightarrow H^{p+q}(U),\ \ [\omega] \cup [\tau] = [\omega \wedge \tau]$. This product, suitably extended, define an algebra structure on $H^*(U) := \oplus H^p(U)$. With this structure $H^*(U)$ is called the *cohomology algebra* of U[6].

Let $U \subseteq \mathbb{R}^n,\ V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth function. As we already observed, F induces a map $F^* : \Omega^p(V) \longrightarrow \Omega^p(U)$. Since, by Theorem 2.9, $F^* \circ \mathrm{d} = \mathrm{d} \circ F^*,\ \ F^*$ maps closed forms to closed forms and exact forms to exact forms. Therefore it induces a $\mathbb{R}$-linear map, that we still denote by F^*,

$$F^* : H^p(V) \longrightarrow H^p(U).$$

It is also simple to see that F^* induces an algebra homomorphism $F^* : H^*(V) \longrightarrow H^*(U)$.

The basic functorial properties are easily verified

- $\mathbb{1}_U^* = \mathbb{1}_{H^p(U)}$,
- If $F_1 : U_1 \longrightarrow U_2$ and $F_2 : U_2 \longrightarrow U_3$ are smooth maps, then $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

In particular, if F is a diffeomorphism, F^* is an isomorphism. So the de Rham cohomology is a (differential) topological invariant of U.

3. Algebraic aspects of cohomology

The construction of the de Rham cohomology fits into a general algebraic setting called *homological algebra.* In this section we will discuss some elementary facts that will be used in these notes. For simplicity we will restrict to the case of real vector spaces (not necessarily finite dimensional) although most of the matter could be extended to the case of modules over commutative rings (see Remarks 3.10 and 3.22).

The objects we study are sequences of (real) vector spaces and linear maps of the type

$$\mathcal{E} := \{(\mathbb{E}^p, \mathrm{d}^p) : \mathrm{d}^p : \mathbb{E}^p \longrightarrow \mathbb{E}^{p+1}\}.$$

When we introduce “objects” it is a good strategy to introduce “morphisms” between such objects, i.e. maps that preserves the structure of the objects.

3.1. Definition. A *morphism* $\phi : \mathcal{E} \longrightarrow \mathcal{F}$, between two sequences is a sequence of linear maps $\phi^p : \mathbb{E}^p \longrightarrow \mathbb{F}^p$ such that the diagrams

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{E}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\phi^p"] & \mathbb{E}^{p+1} \arrow[r] \arrow[d, "\phi^{p+1}"] & \cdots \\ \cdots \arrow[r] & \mathbb{F}^p \arrow[r, "\mathrm{d}^p"] & \mathbb{F}^{p+1} \arrow[r] & \cdots \end{tikzcd}$$

commute, i.e. $\mathrm{d}^p \circ \phi^p = \phi^{p+1} \circ \mathrm{d}^p$ (we are using the same symbols d^p for the linear maps in the two sequences).

The morphism is an *isomorphism* if all ϕ^p are vector spaces isomorphisms.

We have some special sequences.

---

[6] This product is usually called the *cup product.* The use of this terminology, instead of the more natural wedge product, is due to the fact that the cup product can be defined for different cohomology theories, where the wedge product is not defined.

3.2. Definition. A sequence $\mathcal{E} = \{\mathbb{E}^p, \mathrm{d}^p\}$ is *exact at* $\mathbb{E}^p$ if $\operatorname{Im} \mathrm{d}^{p-1} = \ker \mathrm{d}^p$. The sequence is an *exact sequence* if it is exact at each $\mathbb{E}^p$.

3.3. Examples.

(1) A sequence of the type $\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F}$ is exact at $\mathbb{E}$ if and only if ϕ is injective.
(2) A sequence of the type $\mathbb{E} \xrightarrow{\phi} \mathbb{F} \longrightarrow \{0\}$ is exact at $\mathbb{F}$ if and only if ϕ is surjective.
(3) A sequence of the type $\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \longrightarrow \{0\}$ is exact if and only if ϕ is an isomorphism.

3.4. Definition. A sequence of the type:

$$\{0\} \longrightarrow \mathbb{E} \longrightarrow \mathbb{F} \longrightarrow \mathbb{G} \longrightarrow \{0\}$$

is called a short sequence.

3.5. Remark. Short (exact) sequences are important since they are the "building blocks" of (long exact) sequences. Let

$$\cdots \longrightarrow \mathbb{E}_{i-1} \xrightarrow{\phi_{i-1}} \mathbb{E}_i \xrightarrow{\phi_i} \mathbb{E}_{i+1} \longrightarrow \cdots$$

be a sequence. Consider the short sequence

$$\{0\} \longrightarrow \mathbb{E}_{i-1}/\ker \phi_{i-1} \xrightarrow{\tilde{\phi}_{i-1}} \mathbb{E}_i \xrightarrow{\tilde{\phi}_i} \operatorname{Im}(\phi_i) \longrightarrow \{0\}$$

where $\tilde{\phi}_{i-1}$, $\tilde{\phi}_i$ are the induced maps. Since $\operatorname{Im}(\tilde{\phi}_{i-1}) = \operatorname{Im}(\phi_{i-1})$, $\ker(\tilde{\phi}_i) = \ker(\phi_i)$, the (long) sequence is exact at $\mathbb{E}_i$ if and only if the short sequence is exact.

3.6. Proposition. *A short* exact *sequence*

$$\{0\} \longrightarrow \mathbb{E} \xrightarrow{\phi} \mathbb{F} \xrightarrow{\psi} \mathbb{G} \longrightarrow \{0\}$$

is isomorphic to the sequence

$$\{0\} \longrightarrow \mathbb{E} \xrightarrow{i} \mathbb{E} \oplus \mathbb{G} \xrightarrow{\pi} \mathbb{G} \longrightarrow \{0\},$$

where $i(v) = (v, 0)$ *and* $\pi(v, w) = w$.

Proof. Let $\tilde{\mathbb{G}}$ be a complement[7] of $\operatorname{Im} \phi = \ker \psi$, i.e $\mathbb{F} = \phi(\mathbb{E}) \oplus \tilde{\mathbb{G}}$. The map $\psi \mid_{\tilde{\mathbb{G}}} : \tilde{\mathbb{G}} \longrightarrow \mathbb{G}$ is an isomorphism. Therefore the map $k \colon \mathbb{F} \longrightarrow \mathbb{E} \oplus \mathbb{G}$, $k(v + w) = (\phi^{-1}(v), \psi(w))$ $(v \in \phi(\mathbb{E}), w \in \tilde{\mathbb{G}})$ is the required isomorphism. ☐

The following result appears often in the applications

3.7. Lemma. [The five Lemma] *Consider the diagram:*

$$\begin{tikzcd}
\mathbb{E}_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \mathbb{E}_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & \mathbb{E}_3 \arrow[r, "f_3"] \arrow[d, "\phi_3"] & \mathbb{E}_4 \arrow[r, "f_4"] \arrow[d, "\phi_4"] & \mathbb{E}_5 \arrow[d, "\phi_5"] \\
\mathbb{F}_1 \arrow[r, "g_1"] & \mathbb{F}_2 \arrow[r, "g_2"] & \mathbb{F}_3 \arrow[r, "g_3"] & \mathbb{F}_4 \arrow[r, "g_4"] & \mathbb{F}_5
\end{tikzcd}$$

If the squares commute, the lines are exact and the ϕ_i*'s are isomorphisms for* $i = 1, 2, 4, 5$ *then* ϕ_3 *is an isomorphism.*

---

[7] Recall that a complement of a subspace is obtained by starting from a basis $\{e_\alpha\}$ of the subspace and completing it to a basis of the ambient space with elements $\{f_\beta\}$ and then considering the subspace spanned by the $\{f_\beta\}$.

Proof. Suppose $\phi_3(e_3) = 0$. Then $\phi_4(f_3(e_3)) = g_3(\phi_3(e_3)) = 0$. Therefore $f_3(e_3) = 0$ and, by the exactness of the first line, $e_3 = f_2(e_2)$. Now $g_2(\phi_2(e(2)) = \phi_3(e_3) = 0$, and therefore $\phi_2(e_2) = g_1(\mu_1)$, for some $\mu_1 \in \mathbb{F}_1$, by the exactness of the second line. Since ϕ_1 is surjective, there exists $e_1 \in \mathbb{E}_1$ such that $\phi_1(e_1) = \mu_1$. Finally

$$0 = f_2(f_1(e_1)) = f_2(\phi_2^{-1} g_1 \phi_1(e_1)) = f_2(e_2) = e_3$$

and therefore ϕ_3 is injective. We will show now that ϕ_3 is surjective. Let $\mu_3 \in \mathbb{F}_3$, $\mu_4 = g_3(\mu_3)$ and $e_4 \in \phi_4^{-1}(\mu_4)$. Now $\phi_5(f_4(e_4)) = g_4(\mu_4) = 0$ and therefore $f_4(e_4) = 0$, since ϕ_5 is injective. In particular there exists $e_3 \in \mathbb{E}_3$ such that $f_3(e_3) = e_4$. Let $\overline{\mu}_3 = \phi_3(e_3)$ and $\omega = \mu_3 - \overline{\mu}_3$. Now $g_3(\omega) = 0$ and therefore $\omega = g_2(\mu_2)$. Let $e_2 = \phi_2^{-1}(\mu_2)$. We have $\phi_3(f_2(e_2)) = g_2(\phi_2(e_2)) = \omega = \phi(e_3) - \mu_3$ and therefore $\mu_3 = \phi_3(e_3 - f_2(e_2)) \in \operatorname{Im} \phi_3$.

□

3.8. Remark. We observe that in the proof of Theorem 3.7 we use only that ϕ_2, ϕ_4 are isomorphisms, ϕ_1 is surjective and ϕ_5 is injective. However the lemma is used, generally, as it is stated.

A more general and very important class of sequences is the class of cochain complexes.

3.9. Definition. A sequence $\mathcal{E} = \{\mathbb{E}^p, \mathrm{d}^p\}$ is *semiexact* or a *cochain complex* if $\operatorname{Im} \mathrm{d}^{p-1} \subseteq \ker \mathrm{d}^p, \quad \forall p$. Equivalently, it is a cochain complex if $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$.

If $\mathcal{E}$ is a cochain complex we define:

- $Z^p(\mathcal{E}) := \ker \mathrm{d}^p$, the *group of p-dimensional cocycles*,
- $B^p(\mathcal{E}) := \operatorname{Im} d^{p-1}$, the *group of p-dimensional coboundaries*,
- $H^p(\mathcal{E}) := Z^p(\mathcal{E})/B^p(\mathcal{E})$, the *p-dimensional cohomology group.*

3.10. Remark. Naturally $Z^p(\mathcal{E}),\ B^p(\mathcal{E}),\ H^p(\mathcal{E})$ are vector spaces. The use of the term “group” is due to the fact that they can be defined in the more general context of complexes of Abelian groups, or modules over a commutative ring.

The cohomology gives a measure of how much the complex is not an exact sequence.

3.11. Example. The de Rham complex $\cdots \longrightarrow \Omega^p(U) \xrightarrow{\mathrm{d}^p} \Omega^{(p+1)}(U) \longrightarrow \cdots$ is a cochains complex whose cohomology is the de Rham cohomology $H^p(U)$.

Consider now a morphism between two cochain complexes, $\phi : \mathcal{E} \longrightarrow \mathcal{F}$. The commutativity condition implies that cocycles are sent to cocycles and coboundaries to coboudaries. In particular ϕ induces linear maps

$$\phi^{*,p} : H^p(\mathcal{E}) \longrightarrow H^p(\mathcal{F}).$$

When clear from the context we will write simply ϕ^* or ϕ^p.

The following “functorial” properties are easily verified:

- $\mathbb{1}^* = \mathbb{1}$,
- $(\phi \circ \psi)^* = \phi^* \circ \psi^*$.

In particular if ϕ is an isomorphism, ϕ^* is also an isomorphism.

It is convenient to consider also sequences with "decreasing indexes", i.e. a sequence of the type

$$\mathcal{E} := \{(\mathbb{E}_p, \partial_p) : \partial_p : \mathbb{E}_p \longrightarrow \mathbb{E}_{p-1}\}.$$

If such a sequence is semiexact, we will call it a *chain complex.* For such a chain complex we define:

- $Z_p(\mathcal{E}) := \ker \partial_p$, the *group of p-dimensional cycles.*
- $B_p(\mathcal{E}) := \operatorname{Im} \partial_{p+1}$, the *group of p-dimensional boundaries.*
- $H_p(\mathcal{E}) := Z_p(\mathcal{E})/B_p(\mathcal{E})$, the *p-dimensional homology group.*

As in the case of cochains, a morphism $\phi : \mathcal{E} \longrightarrow \mathcal{F}$, between two chain complexes, sends cycles to cycles and boundaries to boundaries, so it induces a sequence of maps $\phi_{*,p} : H_p(\mathcal{E}) \longrightarrow H_p(\mathcal{F})$ and the functorial properties are easily verified. When clear from the context we will write simply ϕ_* or ϕ_p.

3.12. Remark. Naturally chain and cochain complexes are, essentially, the same objects. For example, changing the index p in $-p$ we pass from a chain complex to a cochain complex. But a more interesting approach is *duality* and we will discuss this now.

Let $\mathcal{E} := \{(\mathbb{E}_p, \partial_p) : \partial_p : \mathbb{E}_p \longrightarrow \mathbb{E}_{p-1}\}$ be a chain complex. We define the *dual complex* $\mathcal{E}^* = \{(\mathbb{E}^p, \mathrm{d}^p)\}$ where $\mathbb{E}^p := (\mathbb{E}_p)^*$ is the dual space, and $\mathrm{d}^p = (\partial_p)^*$ is the transpose of ∂_p. It is simple to show that $\mathrm{d}^p \circ \mathrm{d}^{p-1} = 0$ so $\mathcal{E}^*$ is, in fact, a cochain complex. We will denote with H_p (resp. H^p) the homology of $\mathcal{E}$ (resp. the cohomology of $\mathcal{E}^*$). Consider the bi-linear map

$$b : \mathbb{E}^p \times \mathbb{E}_p \longrightarrow \mathbb{R}, \qquad b(\phi, c) = \phi(c).$$

It is easily seen that this map induces a bi-linear map

$$\tilde{b} : H^p \times H_p \longrightarrow \mathbb{R}, \qquad \tilde{b}([\phi], [c]) = \phi(c),$$

and therefore a linear map

$$K : H^p \longrightarrow [H_p]^*, \qquad K([\phi])([c]) = \phi(c).$$

3.13. Theorem. [Universal coefficients Theorem] *The map K is an isomorphism.*

Proof. We start observing that we have two short exact sequences

$$\{0\} \longrightarrow Z_p \longrightarrow \mathbb{E}_p \xrightarrow{\partial_p} B_{p-1} \longrightarrow \{0\}, \qquad \{0\} \longrightarrow B_{p-1} \longrightarrow Z_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\} \tag{2}$$

where the maps are the obvious ones. By Proposition 3.6, we have the decompositions

$$\mathbb{E}_p \cong Z_p \oplus B_{p-1}, \qquad Z_{p-1} \cong B_{p-1} \oplus H_{p-1} \tag{3}$$

Claim 1.: *K is surjective.* Let $[\phi] \in [H_p]^*$. Consider the map $\phi \circ \pi : Z_p \longrightarrow \mathbb{R}$, where $\pi : Z_p \longrightarrow H_p$ is the quotient map. Using the first decomposition in (3), we can extend this map to a map $\tilde{\phi} : \mathbb{E}_p \longrightarrow \mathbb{R}$ with $\tilde{\phi} = 0$ on B_{p-1}. Let $e \in \mathbb{E}_p$. Then $\mathrm{d}\tilde{\phi}(e) = \tilde{\phi}(\partial(e)) = 0$, hence $\tilde{\phi}$ is a cocycle and $K([\tilde{\phi}]) = [\phi]$.

CLAIM 2.: *K is injective.* Let $\psi \in Z^p$ be such that $\psi(c) = 0 \,\forall\, c \in Z_p$. The map $\phi = \psi \circ \partial^{-1} : B_{p-1} \longrightarrow \mathbb{R}$ is well defined since, by the first sequence in (2), the difference of two elements in $\partial^{-1}(B_{p-1})$ is a cycle. Using the decompositions in (3), we can extend ϕ to a map $\tilde{\phi} : E_{p-1} \longrightarrow \mathbb{R}$. Now, $\forall e \in E_p$, we have:

$$\mathrm{d}\tilde{\phi}(e) = \tilde{\phi}(\partial e) = \psi \circ \partial^{-1}(\partial e) = \psi(e).$$

Hence $[\psi] = [\mathrm{d}\tilde{\phi}] = 0$. □

A useful consequence is the following

3.14. COROLLARY. *If a sequence is exact, the dual sequence is also exact.*

PROOF. An exact sequence is a chain complex with vanishing homology. Therefore the dual sequence is a cochain complex with vanishing cohomology, by Theorem 3.13, hence an exact sequence [8] □

We will study now when two morphism between cochain (resp. chain) complexes induces the same map in cohomology (resp. homology).

3.15. DEFINITION. An *algebraic homotopy* between two morphisms $\phi, \psi : \mathcal{E} \longrightarrow \mathcal{F}$ of cochain (resp. chain) complexes is a family of maps $K^p : \mathbb{E}^p \longrightarrow \mathbb{F}^{p-1}$ (resp. $K_p : \mathbb{E}_p \longrightarrow \mathbb{F}_{p+1}$), such that:

$$\phi - \psi = \mathrm{d} \circ K + K \circ \mathrm{d} \quad (\text{resp. } \phi - \psi = \partial \circ K + K \circ \partial).$$

If there exists such an algebraic homotopy, we will say the the two morphisms are (algebraically) *homotopic.*

From the very definition of induced morphisms we have:

3.16. PROPOSITION. *Two algebraically homotopic maps induce the same morphism in cohomology (resp. in homology).*

Consider now a short exact sequence of cochain complexes:

$$\{0\} \longrightarrow \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow \{0\}.$$

In particular ϕ_i is injective and ψ_i is surjective. In general, at the cohomology level, ϕ^* is not injective and ψ^* is not surjective. In any case, we still have a good relation between the cohomology groups of the three complexes.

3.17. THEOREM. [Algebraic Mayer-Vietoris Theorem] *In the situation above there exists a family of linear maps $\Delta^{*,p} : H^p(\mathcal{G}) \longrightarrow H^{p+1}(\mathcal{E})$ such that the sequence:*

$$\cdots \longrightarrow H^p(\mathcal{E}) \xrightarrow{\phi^*} H^p(\mathcal{F}) \xrightarrow{\psi^*} H^p(\mathcal{G}) \xrightarrow{\Delta^{*,p}} H^{p+1}(\mathcal{E}) \longrightarrow \cdots$$

is a (long) exact sequence. When clear from the context we will write dimply Δ^p or Δ^*.

---

[8] We could also give a direct proof, and the reader is invited to do so (Exercise 7.17).

Proof. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{E}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\phi_p"] & \mathbb{E}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d, "\phi_{p+1}"] & \mathbb{E}^{p+2} \arrow[r] \arrow[d, "\phi_{p+2}"] & \cdots \\
\cdots \arrow[r] & \mathbb{F}^p \arrow[r, "\mathrm{d}^p"] \arrow[d, "\psi_p"] & \mathbb{F}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d, "\psi_{p+1}"] & \mathbb{F}^{p+2} \arrow[r] \arrow[d, "\psi_{p+2}"] & \cdots \\
\cdots \arrow[r] & \mathbb{G}^p \arrow[r, "\mathrm{d}^p"] \arrow[d] & \mathbb{G}^{p+1} \arrow[r, "\mathrm{d}^{p+1}"] \arrow[d] & \mathbb{G}^{p+2} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the columns are exact and the rows are the cochain complexes under consideration. The idea is to construct a map from $\mathbb{G}^p$ to $\mathbb{E}^{p+1}$. A natural choice would be $(\phi_{p+1})^{-1} \circ \mathrm{d}^p \circ \psi_p^{-1}$. The point is that this map is not well defined. Let us see how we can overcome this problem. Consider a cocycle $c \in \mathbb{G}^p$. Since ψ_p is surjective, there exists $b \in \mathbb{F}^i$ such that $c = \psi_p(b)$. The element $\mathrm{d}^p(b) \in \mathbb{F}^{p+1}$ is in $\ker \psi_{p+1}$ since the diagrams commute and c is a cocycle. Since $\ker \psi_{p+1} = \operatorname{Im} \phi_{p+1}$ we have $\mathrm{d}^p(b) = \phi_{p+1}(a)$ for some $a \in \mathbb{E}^{p+1}$ and this a is unique since ϕ_{p+1} is injective. Observe that $\mathrm{d}^{p+1}(a) = 0$, since $\phi_{p+2}(d^{p+1}(a)) = \mathrm{d}^{p+1}(\phi_{p+1}(a)) = \mathrm{d}^{p+1} \circ \mathrm{d}^p(b) = 0$ and ϕ_{i+2} is injective. Therefore a is a cocycle. We define: $\Delta^* \colon H^p(\mathcal{G}) \longrightarrow H^{p+1}(\mathcal{E})$, $\Delta^*([c]) = [a]$. We have to show that $[a]$ is well defined. The first choice we made was $b \in \mathbb{F}^p$. If b' is an other choice, i.e. $\psi^p(b') = \psi^p(b)$, then $b - b' \in \ker \psi_p = \operatorname{Im} \phi_p$. Therefore $b' - b = \phi_p(a')$, for some $a' \in \mathbb{E}^p$, and $b' = b + \phi_p(a')$. So, changing b by $b + \phi_p(a')$, we change a by $a + \mathrm{d}^p(a')$ and this does not change $[a]$. Next we shall show that $[a]$ does not depend on the choice of $c \in [c]$. Consider $c + \mathrm{d}^p(c')$. Since $c' = \psi_{p-1}(\tilde{b})$, for some $\tilde{b} \in \mathbb{F}^{p-1}$, we have $c + \mathrm{d}^{p-1}(c') = c + \mathrm{d}^{p-1}(\psi_{p-1}(\tilde{b})) = c + \psi_p(\mathrm{d}^{p-1}(\tilde{b})) = \psi_p(b + \mathrm{d}^{p-1}(\tilde{b}))$. Therefore b is substituted by $b + \mathrm{d}^{p-1}(\tilde{b})$, and this does not change $\mathrm{d}^p(b)$ and, therefore, $[a]$.

It is easy to see that Δ^* is linear. We leave to the reader the task of proving the exactness of the sequence (Exercise 7.22). □

3.18. Remark. The map Δ^* is well defined in cohomology but *not* at the cocycles level.

3.19. Definition. The sequence in Theorem 3.17 is called the (algebraic) *Mayer-Vietoris sequence.* The maps Δ^*, are the *Mayer-Vietoris coboundaries.*

3.20. Remark. Naturally we have a similar sequence in homology, associated to a short exact sequence of chain complexes. The similar maps $\Delta_{*,p}$ or simply Δ_*, are called the *Mayer-Vietoris boundaries.* We leave the details to the reader.

An important aspect of the Mayer-Vietoris (co)boundaries is that they are "natural" in the sense of the following Proposition, whose proof we leave to the reader (Exercise 7.22).

3.21. Proposition. *A morphism between short exact sequences of (co)chain complexes induces a morphism between the associated Mayer-Vietoris exact sequences, i.e. the Mayer-Vietoris (co)boundaries commutes with the induced maps.*

3.22. Remark. As suggested in Remark 3.10, instead of chain and cochain complexes of vector spaces we could consider chain and cochain complexes of Abelian groups (or modules over a commutative ring). Almost all we have done in this section extends to the case of complexes of Abelian groups. The "almost" refers to two exceptions:

- Proposition 3.6 does not hold in this more general setting. For example the sequence of abelian groups

$$\{0\} \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}, \quad \cdot 2(a) := 2a,$$

is a short exact sequence, but it is *not* isomorphic to the sequence

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow \{0\}.$$

A short exact sequence of Abelian groups that verify Proposition 3.6 is called a *split short exact sequence.* A sufficient condition for splitting is given by the following simple fact

3.23. Proposition. *A short exact sequence of Abelian groups*

$$\{0\} \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow \{0\}$$

splits if and only if there is a map $r : C \longrightarrow B$ such that $\psi \circ r = 1\!\!1_C$. This always happens if C is free [9].

- We can consider "duality" in the context Abelian groups. If G is such a group, $G^* := Hom(G, \mathbb{Z})$ is the group of homomorphisms from G to $\mathbb{Z}$. Therefore we can define the dual of a chain complex of Abelian groups. However Theorem 3.13 does not hold in this context. In fact, one of the points in the proof was that the sequence of vector spaces

$$\{0\} \longrightarrow B_{p-1} \longrightarrow \mathbb{Z}_{p-1} \longrightarrow H_{p-1} \longrightarrow \{0\}$$

splits. As observed above, this is not the case, in general, for short exact sequences of Abelian groups. However, if H_{p-1} is a free Abelian group, then the sequence splits, by Proposition 3.23, and the Theorem holds true. In the general case there is still a relation between the homology of a chain complex of Abelian groups and the cohomology of the dual complex, still known as *the Universal Coefficients Theorem.*

4. Basic properties of the de Rham cohomology

The natural problem that the de Rham cohomology attacks is the problem of (indefinite) integration, i.e. the problem of solving the equation $\mathrm{d}\omega = \beta$, for a given $\beta \in \Omega^{p+1}(U)$. A necessary condition for the existence of a solution ω is $\mathrm{d}\beta = 0$. In general the problem has two aspects:

- *The local problem*: given $x \in U$, $\beta \in \Omega^{p+1}(U)$ do there exist a neighborhood $V \subseteq U$ of x and a solution $\omega \in \Omega^p(V)$ of the equation $\mathrm{d}\omega = \beta|V$? In this case, as we shall see, the condition $\mathrm{d}\beta = 0$ is also sufficient.
- *The global problem*: given $\beta \in \Omega^{p+1}(U)$, does there exist a solution $\omega \in \Omega^p(U)$ of the equation $\mathrm{d}\omega = \beta$? In this case, the condition $\mathrm{d}\beta = 0$ is no longer sufficient, in general, and the answer will depend on the particular β and/or the *topology* of U.

We will start computing the de Rham cohomology in some simple cases.

---

[9] A *free Abelian group* G is an Abelian group that admits a basis, i.e. a subset $\mathcal{B} \subseteq G$ such that for any Abelian group H and map $\phi : \mathcal{B} \longrightarrow H$, there exists a homomorphism $\tilde{\phi} : G \longrightarrow H$, extending ϕ.

4.1. Example. For $U = \mathbb{R}^0$ we have:

$$H^p(\mathbb{R}^0) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

4.2. Example. Let $U = \coprod_\alpha U_\alpha$ be the union of disjoint open sets U_α. Then $\Omega^p(U) = \prod_\alpha \Omega^p(U_\alpha)$ (direct product) and the differential preserves the decomposition, i.e. if $\omega = \{\omega_\alpha\}$, $\mathrm{d}\omega = \{\mathrm{d}\omega_\alpha\}$. It follows that

$$H^p(U) \cong \prod_\alpha H^p(U_\alpha).$$

4.3. Example. Let us analyze the 0-dimensional cohomology. In this case, the only exact 0-form is the zero form so $H^0(U)$ is the space of closed 0-forms, i.e. functions in $\mathcal{F}(U)$ with zero differential. Such a function is locally constant, in particular it is constant on the connected components of U. It follows that $H^0(U)$ is the direct product of copies of $\mathbb{R}$, as many as the connected components of U.

Let us give a further look at the 0-dimensional cohomology. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open *connected* sets, and $F : U \longrightarrow V$ a smooth map. As we observe in 4.3, the zero dimensional cohomology of U is the space of constant functions, and the same for V. Given a 0-form $f \in \Omega^0(V) = \mathcal{F}(V)$, $F^*(f) = f \circ F$ and therefore $F^* : H^0(V) \longrightarrow H^0(U)$ is an isomorphism. Modulo the identification of the zero dimensional cohomology groups with $\mathbb{R}$, we have $F^* = \mathbb{1} : \mathbb{R} \longrightarrow \mathbb{R}$.

We want to look now at the induced maps in higher dimensional cohomology groups. The question is the following: when do two smooth maps $F_i : U \longrightarrow V$, $i = 0, 1$ induce the same morphism in cohomology? We will give a sufficient condition in terms of *homotopy*.

4.4. Definition. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F_i : U \longrightarrow V$, $i = 0, 1$ be smooth functions.

- A *homotopy* between the two functions is a smooth map[10]

  $$H : U \times [0,1] \subseteq \mathbb{R}^{n+1} \longrightarrow V,$$

  such that $H(x, i) = F_i(x), i = 0, 1$.
- We will say that the two functions are *homotopic* if there exist a homotopy between them. In this case we write $F_0 \sim F_1$.
- We will say that U and V are *homotopy equivalent* if there exist functions $F : U \longrightarrow V$, $G : V \longrightarrow U$, such that $G \circ F \sim \mathbb{1}_U$, $F \circ G \sim \mathbb{1}_V$. F (resp. G) is called a homotopy inverse of G (resp. of F)[11].
- We will say that U is *contractible* if U is homotopy equivalent to $\mathbb{R}^0$.

4.5. Example. A subset $U \subseteq \mathbb{R}^n$ is star shaped if there exists $p \in U$ such that, for all $q \in U$, the segment joining p and q is contained in U. For example convex sets are star shaped. Star shaped subsets are contractible since the map $H(q, t) := tp + (1 - t)q$ is a homotopy between $\mathbb{1}_U$ and the constant map $F(q) = p$. It follows that $\mathbb{1}$ and F are homotopy inverses.

4.6. Remark. Given a homotopy $H : U \times [0,1] \longrightarrow V$, there is a smooth function $\overline{H} : U \times \mathbb{R} \longrightarrow V$, such that $\overline{H}(x, i) = F_i(x), i = 0, 1$. In fact, if $\lambda : \mathbb{R} \longrightarrow [0,1]$ is a smooth function such that $\lambda(t) = 0$ if $t \leq 0$, $\lambda(t) = 1$ if $t \geq 1$, just take $\overline{H}(x, t) = H(x, \lambda(t))$.

---

[10] A map $f : V \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$, defined in a *non necessarily open subset* $V \subseteq \mathbb{R}^N$ is smooth, if for all $p \in V$, f extends to a smooth map defined in an open neighborhood of p.

[11] Observe that a homotopy inverse *is not*, in general, unique.

A homotopy between two functions may be viewed as a curve in the space of smooth maps joining the two functions. Also it may be viewed as a "smooth deformation" of one function to the other.

4.7. Theorem. [Homotopy invariance for cohomology] *If $F_i : U \longrightarrow V, i = 0, 1$ are two homotopic smooth functions, then $F_0^* = F_1^* : H^p(V) \longrightarrow H^p(U)$, for all p.*

Proof. By Remark 4.6 we can suppose that there is a homotopy $H : U \times \mathbb{R} \longrightarrow V$. Let $j_i : U \longrightarrow U \times \mathbb{R}, \ \ i = 0, 1, \ j_i(x) = (x, i)$, be the canonical inclusions. We claim that it is sufficient to prove that $j_0^* = j_1^*$. In fact, if so, we have:

$$F_0^* = (H \circ j_0)^* = j_0^* \circ H^* = j_1^* \circ H^* = (H \circ j_1)^* = F_1^*.$$

To prove that $j_0^* = j_1^*$ we will construct an algebraic homotopy between j_0^* and j_1^* (at the cochain level, see Definition 3.15 and Proposition 3.16), i.e. an $\mathbb{R}$-linear map $\tilde{H} : \Omega^p(U \times \mathbb{R}) \longrightarrow \Omega^{p-1}(U)$ such that

$$\tilde{H} \mathrm{d}\omega + \mathrm{d}\tilde{H}\omega = j_1^*\omega - j_0^*\omega.$$

Let us construct such a map. If $\omega \in \Omega^p(U \times \mathbb{R})$, $\omega = \mathrm{d}t \wedge \alpha + \beta$, with

$$\alpha = \sum_{i_1 < \ldots < i_{p-1}} \alpha_{i_1, \ldots, i_{p-1}}(x, t) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}}, \quad \beta = \sum_{j_1 < \cdots < j_p} \beta_{j_1, \ldots, j_p}(x, t) \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p}.$$

We define

$$\tilde{H}(\omega) = \sum_{i_1 < \ldots < i_{p-1}} \left(\int_0^1 \alpha_{i_1, \ldots, i_{p-1}}(x, t) \mathrm{d}t \right) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}}.$$

Then

$$\mathrm{d}\omega = -\mathrm{d}t \wedge \mathrm{d}\alpha + \mathrm{d}\beta = -\mathrm{d}t \wedge \sum_{j, i_1 < \cdots < i_p} \frac{\partial \alpha_{i_1 \ldots i_{p-1}}}{\partial x_j} \, \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}} +$$
$$+ \mathrm{d}t \wedge \sum_{j_1 < \cdots < j_p} \frac{\partial \beta_{j_1, \ldots, j_p}}{\partial t} \, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} + \gamma$$

where γ does not contain terms with $\mathrm{d}t$. Therefore

$$\tilde{H}\mathrm{d}\omega = \sum_{j_1 < \cdots < j_p} \left(\int_0^1 \frac{\partial \beta_{j_1, \ldots, j_p}}{\partial t} \, \mathrm{d}t \right) \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} \quad -$$
$$\sum_{j, i_1 < \cdots < i_p} \left(\int_0^1 \frac{\partial \alpha_{i_1 \ldots i_{p-1}}}{\partial x_j} \quad \mathrm{d}t \quad \right) \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}},$$

$$\mathrm{d}\tilde{H}\omega = \sum_{j, i_1 < \cdots < i_p} \left(\int_0^1 \frac{\partial \alpha_{i_1 \ldots i_{p-1}}}{\partial x_j} \, \mathrm{d}t \right) \mathrm{d}x_j \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{p-1}},$$

and (see Example 2.8)

$$\tilde{H}\mathrm{d}\omega + \mathrm{d}\tilde{H}\omega = \sum_{j_1 < \cdots < j_p} \left(\int_0^1 \frac{\partial \beta_{j_1, \ldots, j_p}}{\partial t} \, \mathrm{d}t \right) \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} =$$
$$= \sum_{j_1 < \cdots < j_p} [\beta_{j_1, \ldots, j_p}(x, 1) - \beta_{j_1, \ldots, j_p}(x, 0)] \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p} = j_1^*\omega - j_0^*\omega.$$

□

From 4.7, and the funtorial properties, we have

4.8. Corollary. *If $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are homotopically equivalent open sets, then they have isomorphic cohomology.*

In particular we have the so called *Poincaré Lemma*

4.9. Corollary. [Poincaré Lemma] *If U is a star shaped open set in $\mathbb{R}^n$, $H^p(U) = \{0\}$ if $p \geq 1$.*

4.10. Remark. Theorem 4.7 allows to define the map induced in cohomology by a *continuous map.* In fact, as we shall see in the Appendix, a continuous map $F : U \longrightarrow V$ is homotopic, via a continuous homotopy $H : U \times [0,1] \longrightarrow V$, to a smooth map $\tilde{F} : U \longrightarrow V$ and if there is a continuous homotopy between two smooth maps, there is a smooth one. So $F^* := \tilde{F}^*$ is well defined and invariant by continuous homotopies.

A basic method to compute the cohomology of an open set $U \subseteq \mathbb{R}^n$ is to write U as union of two, possibly simpler open sets U_1, U_2, and look for relations between the cohomology of U, U_i and $V := U_1 \cap U_2$.

4.11. Lemma. *Consider the sequence :*

$$\{0\} \longrightarrow \Omega^p(U) \overset{(j_1^*, j_2^*)}{\longrightarrow} \Omega^p(U_1) \oplus \Omega^p(U_2) \overset{(k_1^* - k_2^*)}{\longrightarrow} \Omega^p(V) \longrightarrow \{0\},$$

where $j_i : U_i \longrightarrow U$ and $k_i : V \longrightarrow U_i$ are the inclusions. Then the sequence is a short exact sequence of cochain complexes.

Proof. Observe that $j_i^* \omega = \omega|_{U_i}$ and, if $(\omega_1, \omega_2) \in \Omega^p(U_1) \oplus \Omega^p(U_2)$, $(k_1^* - k_2^*)(\omega_1, \omega_2) = \omega_1|_V - \omega_2|_V$ (see Example 2.8). So the exactness of the sequence is obvious, except for the surjectivity of $(k_1^* - k_2^*)$. To prove that $(k_1^* - k_2^*)$ is surjective we consider a partition of unity dominated by the covering $\{U_1, U_2\}$, i.e. smooth functions $\phi_i : U \longrightarrow [0,1]$, $i = 1, 2$ such that:

$$\phi_1(x) + \phi_2(x) = 1 \quad \forall x \in U, \qquad \text{supp}(\phi_i) := \overline{\{x \in U : \phi_i(x) > 0\}} \subseteq U_i$$

(see Theorem 6.2 for a proof of the existence of partitions of unity).

Given $\omega \in \Omega^p(V)$, we define:

$$\omega_i(x) = \begin{cases} \phi_j(x)\omega(x) & \text{if } x \in V \\ 0 & \text{if } x \in U_i \setminus V \end{cases}$$

where $i \neq j$. Then ω_i is well defined since ϕ_j vanishes outside $\overline{U_j}, j \neq i$. Moreover,

$$(k_1^* - k_2^*)(\omega_1, -\omega_2) = \omega_1|_V + \omega_2|_V = \phi_1\omega + \phi_2\omega = \omega.$$

Therefore $(k_1^* - k_2^*)$ is surjective. ☐

At this point Theorem 3.17 gives:

4.12. Theorem. [Mayer Vietoris sequence for de Rham cohomology] *There exists a sequence of linear maps $\Delta^* : H^p(V) \longrightarrow H^{p+1}(U)$, such that the sequence below is exact:*

$$\cdots \longrightarrow H^p(U) \overset{(j_1^*, j_2^*)}{\longrightarrow} H^p(U_1) \oplus H^p(U_2) \overset{(k_1^* - k_2^*)}{\longrightarrow} H^p(V) \xrightarrow{\Delta^*} H^{i+1}(U) \longrightarrow \cdots$$

4.13. Definition. The sequence above is called the *Mayer-Vietoris sequence for the de Rham cohomology* and the maps Δ^* are called the *Mayer-Vietories coboundaries.*

4.14. Remark. The Mayer-Vietoris coboundaries can be decribed explicitly. If $[\omega] \in H^p(V), \Delta^*[\omega]$ is the class of the form

$$\tau(x) = \begin{cases} -\mathrm{d}(\phi_2\omega)(x) & \text{if } x \in U_1 \\ \mathrm{d}(\phi_1\omega)(x) & \text{if } x \in U_2 \end{cases}$$

Since d commutes with induced maps, so does Δ^*. We invite the reader to check the details.

4.15. Example. Let us apply the Mayer-Vietoris sequence to compute the cohomology of $\Sigma_n := \mathbb{R}^n \backslash \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq 1\}$.

Consider the open sets:

$$U_1 = \{(x_1, \ldots, x_n) \in \Sigma_n : x_n > -1/2\}, \quad U_2 = \{(x_1, \ldots, x_n) \in \Sigma_n : x_n < 1/2\}.$$

The following facts are easy to prove:

- $\Sigma_n = U_1 \cup U_2$.
- U_i is contractible, $i = 1, 2$. In fact the projection $(x_1, \ldots, x_n) \rightsquigarrow (x_1, \ldots, x_{n-1}, 2)$ is a homotopy equivalence between U_1 and the hyperplane $x_n = 2$. Similarly for U_2.
- $U_1 \cap U_2$ is homotopy equivalent to $\Sigma_{(n-1)}$ (the projection of $U_1 \cap U_2$ into the hyperplane $x_n = 0$ is a homotopy equivalence).

If $n = 1$, Σ_1 is the disjoint union of two contractible sets, hence by Corollary 4.8 and Example 4.3

$$H^p(\Sigma_1) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

For the case $n \geq 2$ we will prove that

$$H^p(\Sigma_n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ \{0\} & \text{if } p \neq 0, n-1 \end{cases}$$

We proceed by induction. Let $n = 2$. Since Σ_2 and the U_i's are connected, $H^0(\Sigma_2) \cong H^0(U_i) \cong \mathbb{R}$. Consider the Mayer-Vietoris sequence:

$$\begin{aligned} &\{0\} \longrightarrow H^0(\Sigma_2) \longrightarrow H^0(U_1) \oplus H^0(U_2) \longrightarrow H^0(\Sigma_1) \longrightarrow H^1(\Sigma_2) \longrightarrow H^1(U_1) \oplus H^1(U_2) \longrightarrow \\ &\longrightarrow \cdots \longrightarrow H^{p-1}(\Sigma_1) \longrightarrow H^p(\Sigma_2) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow \cdots. \end{aligned}$$

The first row reduces to:

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(\Sigma_2) \longrightarrow \{0\}.$$

The first arrow is injective hence the kernel of the second one, as well as its image, are 1-dimensional. Hence the the third one is surjective with 1-dimensional kernel and $H^1(\Sigma_2) \cong \mathbb{R}$.

From the second row we get $H^p(\Sigma_2) = \{0\}$ if $p > 1$. Hence the formula holds true.

Suppose now $n \geq 3$ and that the formula holds true for $n - 1$. Consider again the Mayer-Vietoris sequence:

$$H^{p-1}(\Sigma_n) \longrightarrow H^{p-1}(U_1) \oplus H^{p-1}(U_2) \longrightarrow H^{p-1}(\Sigma_{n-1}) \longrightarrow H^p(\Sigma_n) \longrightarrow H^p(U_1) \oplus H^p(U_2) \longrightarrow$$

If $p > 1$ we have $H^p(\Sigma_n) \cong H^{p-1}(\Sigma_{n-1})$, and, for $p = 1$ we get

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(\Sigma_n) \longrightarrow \{0\}.$$

Hence $H^1(\Sigma_n) = \{0\}$ and the formula holds true for n.

4.16. Remark. Observe that the inclusion $\Sigma_n \longrightarrow \mathbb{R}^n \setminus \{0\}$ is a homotopy equivalence (Exercise 7.28).

5. An application: the Jordan-Alexander duality Theorem

It is convenient, as we shall see, in order to avoid special arguments for the 0-dimensional case and to have more clean statements, to introduce *reduced cohomology*. Define

$$\Omega^{-1}(U) := \mathbb{R} \qquad \mathrm{d}^{(-1)} : \Omega^{-1}(U) \longrightarrow \Omega^0(U), \qquad \mathrm{d}^{(-1)}(a) := a \in \Omega^0(U).$$

Then the sequence

$$\{0\} \longrightarrow \Omega^{-1}(U) \xrightarrow{d^{(-1)}} \Omega^0(U) \xrightarrow{d} \Omega^1(U) \longrightarrow \cdots$$

is a cochain compex called the *augmented de Rham complex*.

5.1. Definition. The *reduced de Rham cohomology* of U, $\tilde{H}^p(U)$, is the cohomology of the augmented de Rham complex.

5.2. Remark. It is clear that $\tilde{H}^{-1}(U) = \{0\}$, $H^0(U) \cong \tilde{H}^0(U) \oplus \mathbb{R}$ and $\tilde{H}^p(U) = H^p(U)$, if $p > 0$. In particular $\tilde{H}^p(U) = \{0\}$, $\forall\ p \geq 0$, if U is contractible.

The basic properties, such as homotopy invariance and the Mayer-Vietoris exact sequence, continue to hold true for the reduced cohomology and we will leave the proofs to the reader (see Exercise 7.24).

We will discuss now a nice application of the Mayer-Vietoris argument, the so called *Jordan-Alexander duality principle*, that has, as a simple consequence, the celebrated Jordan closed curve Theorem. We will follow closely [**4**].

Let F_i, $i = 1, 2$ be closed subsets of $\mathbb{R}^n$. Suppose that there exists a homeomorphism $\phi : F_1 \longrightarrow F_2$. It is natural to ask if there exists some relation between the complementary sets $\mathbb{R}^n \setminus F_i$. The illusion that they are homeomorphic or, at least, homotopy equivalent is soon frustrated. For example consider $F_1 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \cup \{x \in \mathbb{R}^2 : \|x\| = 2\}$ and $F_2 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \cup \{x \in \mathbb{R}^2 : \|x - (3,0)\| = 1\}$. The complement of F_1 is homotopy equivalent to the disjoint union of a point and two circles, while the complement of F_2 is homotopy equivalent to the disjoint union of two points and the wedge [12] of two circles. It is easily seen that these spaces *are not* homotopy equivalent.

5.3. Remark. (For the reader familiar with the concept of foundamental group,) The fact that the complements of two homeomorphic closed set are not homotopy equivalent is important in several contexts, for example in Knot Theory. Recall that a knot in $\mathbb{R}^3$ is a function $\gamma : S^1 \longrightarrow \mathbb{R}^3$ which is a homeomorphism onto its image. Two knots are equivalent if there exists an isotopy, i.e. a homotopy through homeomorphisms, which takes one into the other. One of the most important invariants for equivalence classes of knots is the fundamental group of the complement of the image. Now, the images of two knots are homeomorphic and

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[12] Recall that the wedge of two topological spaces is the space obtained from the disjoint union identifying a fixed point in the first space with one in the second one.

if the complements were homotopy equivalent, they would have isomorphic fundamental group and so the invariant would be trivial.

There is, however, an interesting relation between the complements of homeomorphic closed sets.

5.4. THEOREM. [Jordan Alexander duality Theorem] *Let* $F_i, i = 1, 2,$ *be closed sets in* $\mathbb{R}^n$ *and* $\phi : F_1 \longrightarrow F_2$ *an homeomorphism. Then:*

$$\tilde{H}^k(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^k(\mathbb{R}^n \setminus F_2).$$

PROOF. We will consider $\mathbb{R}^n$ as the subspace of vectors in $\mathbb{R}^{n+k}$ with the last k coordinates zero. The proof of the Theorem will be an easy consequence of the following two Lemmas.

5.5. LEMMA. *Let* $F \subsetneq \mathbb{R}^n$ *be a closed subset. Then* $\tilde{H}^{i+1}(\mathbb{R}^{n+1} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \quad i \geq -1.$

PROOF. Consider the subsets of $\mathbb{R}^{n+1}$:

- $Z_+ := \mathbb{R}^{n+1} \setminus F \times \{t \in \mathbb{R} : t \leq 0\}.$
- $Z_- := \mathbb{R}^{n+1} \setminus F \times \{t \in \mathbb{R} : t \geq 0\}.$
- $Z := Z_+ \cup Z_- = \mathbb{R}^{n+1} \setminus F.$
- $Z_+ \cap Z_- \sim \mathbb{R}^n \setminus F.$

The orthogonal projection of Z_+ onto the hyperplane $x_{n+1} = 1$ is a homotopy equivalence. Hence the reduced cohomology of Z_+ vanishes in all dimensions. The same is true for Z_- and the Lemma follows from the Mayer-Vietoris sequence for the reduced cohomology:

$$\tilde{H}^i(Z_+) \oplus \tilde{H}^i(Z_-) = \{0\} \longrightarrow \tilde{H}^i(Z_+ \cap Z_-) \longrightarrow \tilde{H}^{i+1}(Z) \longrightarrow \tilde{H}^{i+1}(Z_+) \oplus \tilde{H}^{i+1}(Z_-) = \{0\}.$$

□

5.6. COROLLARY. *If* $F \subseteq \mathbb{R}^n$ *is a closed set, then* $\tilde{H}^{i+k}(\mathbb{R}^{n+k} \setminus F) \cong \tilde{H}^i(\mathbb{R}^n \setminus F), \quad \forall\, i \geq -k.$

5.7. LEMMA. *Let* $F_i \subseteq \mathbb{R}^n, i = 1, 2$ *be closed subsets and* $\phi : F_1 \longrightarrow F_2$ *an homeomorphism. Then* $\mathbb{R}^{2n} \setminus F_1 \times \{0\}$ *is homeomorphic to* $\mathbb{R}^{2n} \setminus \{0\} \times F_2$.

PROOF. Let $\psi = \phi^{-1}$. The homeomorphisms ϕ, ψ extend, by Tietze's Theorem[13], to continuous maps $\Phi, \Psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Define:

- $L : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \ L(x, y) = (x, y - \Phi(x)).$
- $R : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \ R(x, y) = (x - \Psi(y), y).$

The maps L, R are homeomorphisms. In fact $L^{-1}(x, y) = (x, y + \Phi(x)), R^{-1}(x, y) = (x + \Psi(y), y)$. Consider $\Gamma := \{(x, y) \in \mathbb{R}^{2n} : x \in F_1, \ y = \phi(x)\} = \{(x, y) \in \mathbb{R}^{2n} : y \in F_2, \ x = \psi(y)\}$. We have $L(F_1 \times \{0\}) = \Gamma = R(\{0\} \times F_2)$ and therefore a homeomorphism:

$$\mathbb{R}^{2n} \setminus F_1 \times \{0\} \xrightarrow{L} \mathbb{R}^{2n} \setminus \Gamma \xrightarrow{R^{-1}} \mathbb{R}^{2n} \setminus \{0\} \times F_2.$$

□

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[13] Tietze's Theorem states that a continuous real valued function defined in a closed subset of $\mathbb{R}^n$ extends to a continuous function defined in the all of $\mathbb{R}^n$ (this fact is true, more generally, for normal topological spaces).

The proof of the Theorem is, at this point, immediate:

$$\tilde{H}^i(\mathbb{R}^n \setminus F_1) \cong \tilde{H}^{i+n}(\mathbb{R}^{2n} \setminus F_1) \cong \tilde{H}^{i+n}(\mathbb{R}^{2n} \setminus F_2) \cong \tilde{H}^i(\mathbb{R}^n \setminus F_2).$$

☐

As an immediate consequence of the Jordan-Alexander duality we have have the celebrated Jordan curve Theorem.

5.8. THEOREM. [Jordan curve Theorem] *Let* $\gamma : S^1 \longrightarrow \mathbb{R}^2$ *be a homeomorphism onto its image*[14]. *Then* $\mathbb{R}^2 \setminus \gamma(S^1)$ *has exactly two connected components.*

PROOF. Consider the unit circle $S^1 \subseteq \mathbb{R}^2$. It is clear that the complement of S^1 in $\mathbb{R}^2$ has exactly two connected components and therefore $\tilde{H}^0(\mathbb{R}^2 \setminus S^1) \cong \mathbb{R}$. By the duality Theorem 5.4, $\tilde{H}^0(\mathbb{R}^2 \setminus \gamma(S^1)) \cong \mathbb{R}$ and therefore the complement of $\gamma(S^1)$ in $\mathbb{R}^2$ has also exactly two connected components. ☐

5.9. REMARK. It is clear that the argument in the proof of Theorem 5.8 may be extended to the case of a closed hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ (see Chapter 3 for definitions) any time we have a “model”, i.e. a close hypersurface homeomorphic to M^n and information on the complement of the model. For example this happens in the case of closed oriented surfaces in $\mathbb{R}^3$ or in the case of closed hypersurfaces of $\mathbb{R}^{n+1}$, homeomorphic to a sphere. A different approach will be discussed in Chapter 4 (Theorem 3.1).

6. Appendix: partitions of unity and smooth approximations

In order not to interrupt the flow of the arguments, we left, in the previous sections, a couple of “gaps”, namely the proof of existence of partitions of unity (in the proof of Theorem 4.12) and the approximation of continuous maps by smooth ones (see Remark 4.10). In this Appendix we will fill up these gaps.

Partitions of unity is a basic tool that allows you to glue together locally defined objects (such as functions, forms etc.) in order to obtain a globally defined object. We start with the basic definition.

6.1. DEFINITION. Let $U \subset \mathbb{R}^n$ be an open set and let $\{V_\alpha\}$ be an open covering of U. A *partition of unity dominated by the covering* $\{V_\alpha\}$ is a family of smooth functions $\lambda_i : \mathbb{R}^n \longrightarrow [0,1]$ such that:

1. For all i there exist α such that $\operatorname{supp}(\lambda_i) := \overline{\{x \in \mathbb{R}^n : \lambda_i(x) \neq 0\}} \subseteq V_\alpha$.
2. For all $x \in U$ there exist a neighborhood U_x of x such that $U_x \cap \operatorname{supp}(\lambda_i) = \emptyset$ for all but finitely many of the λ_i's.
3. For $x \in U$, $\sum_i \lambda_i(x) = 1$ (observe that, by (2), the sum is finite).

Our aim is to prove the following result:

6.2. THEOREM. *Let* $U \subset \mathbb{R}^n$ *be an open set and let* $\{V_\alpha\}$ *be an open covering of* U*. Then there exist a partition of unity dominated by* $\{V_\alpha\}$.

PROOF. We will use the following notations:

$$B(p,r) = \{x \in \mathbb{R}^n : \|x-p\| < r\}, \qquad D(p,r) = \{x \in \mathbb{R}^n : \|x-p\| \leq r\} = \overline{B(p,r)}\ .$$

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[14] Such a map is usually called a *Jordan curve.*

We recall (Exercise 4.13 of Chapter 0) that given $\delta_1, \delta_2 \in \mathbb{R}$, $0 < \delta_1 < \delta_2$,and $p \in \mathbb{R}^n$, there exists a smooth function $\phi : \mathbb{R}^n \longrightarrow [0,1]$ such that $\phi(x) = 0$ in $B(p,\delta_1)$ and $\phi(x) = 1$ in $\mathbb{R}^n \setminus B(p,\delta_2)$.

Claim 1. Let $K \subseteq \mathbb{R}^n$ be a compact set and $V \subseteq \mathbb{R}^n$ an open set with $K \subseteq V$. Then there exist a smooth function $\psi : \mathbb{R}^n \longrightarrow [0,1]$ such that $\psi(x) = 1$, if $x \in K$ and $\psi(x) = 0$ if $x \notin V$.

Proof. For any $p \in K$ consider $\delta(p)$ such that $D(p, 2\delta(p)) \subseteq V$. Then there is a finite number of points, $p_1, \dots, p_r \in K$, such that $K \subseteq \bigcup D(p_i, \delta(p_i))$. For each i we have a function $\phi_i : \mathbb{R}^n \longrightarrow [0,1]$ such that $\phi_i(x) = 0, \quad x \in D(p_i, \delta(p_i))$ and $\phi(y) = 1, \quad y \notin D(p_i, 2\delta(p_i))$. Then the function

$$\psi(x) = 1 - \phi_1(x) \cdots \phi_r(x)$$

has the required properties. □

Claim 2. There exist a continuous proper function[15] $\phi : U \longrightarrow [0,\infty)$.

Proof. Since $\mathbb{R}^n$ is homeomorphic to the open ball $B(0,1)$ (Exercise 4.14, Chapter 1) and the composition of a proper continuous function with a homeomorphism is still proper, we can assume that $U \subseteq B(0,1)$. For $x \in U$, define $d(x)$ to be the distance of x to the boundary of U. Then $d : U \longrightarrow \mathbb{R}$ is a *positive* continuous function. Consider $\phi : U \longrightarrow [0,\infty)$, $\phi(x) = d(x)^{-1}$. Then ϕ is continuous and for all $n \in \mathbb{N}, \phi^{-1}[0,n]$ is a closed bounded set in U, hence compact. So ϕ is proper. □

We will now prove the Theorem. Consider a proper function $\phi : U \longrightarrow [0,\infty)$ and set

$$A_n = \phi^{-1}[n, n+1], \qquad W_n = \phi^{-1}(n - \frac{1}{2}, n + \frac{3}{2}).$$

Then A_n is compact and therefore may be covered with a finite number of balls $B_{k,n}$ such that each disk $D_{k.n} := \overline{B_{k,n}}$ is contained in some $V_\alpha \cap W_n$. For each such disk we have a smooth function $\phi_{k,n} : U \longrightarrow [0,1]$ vanishing outside $V_\alpha \cap W_n$ and identically 1 in $D_{k,n}$. It is clear from the construction that the A_n's cover U and so, for all $x \in U$, there is at least one of the $\phi_{n.k}$'s not vanishing at x. Also $W_n \cap W_{n+2} = \emptyset$ so the supports of the $\phi_{n,k}$ are a locally finite covering and $\sum_{k,n} \phi_{k,n}(x) < \infty, \ \forall x \in U$. So the family of functions

$$\lambda_{n,k} = \frac{\phi_{n,k}}{\sum_{i,j} \phi_{i,j}}$$

is a well defined partition of unity dominated by the covering V_α. □

6.3. Remark. Observe that the partition of unity we constructed is a *countable* set of smooth functions.

We shall prove now that a continuous function may be approximate by a smooth function, homotopic to it. The proof is a good example of how to use partition of unity.

6.4. Theorem. *Let $U \subseteq \mathbb{R}^n$ be an open set and let $F : U \longrightarrow W \subseteq \mathbb{R}^m$ be a continuous function which is smooth on a closed subset $N \subseteq U$. Then, given a real valued positive continuous function $\delta : U \longrightarrow \mathbb{R}$ there exists a smooth function $G : U \longrightarrow W$ such that $\|F(x) - G(x)\| < \delta(x), \ \forall x \in U$ and $F(x) = G(x)$ if $x \in N$. Moreover $G \sim F$.*

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[15] A function is *proper* if the inverse image of a compact set is compact.

Proof. We recall that F smooth on N means that for all $x \in N$ there exists a neighborhood V_x of x and a smooth extension h_x of $F|[V_x \cap N]$. For $x \in U$ we consider a neighborhood V_x of x and a function $h_x : V_x \longrightarrow \mathbb{R}$ with the following conditions:

1. $F(V_x)$ is contained in a subset of an open ball contained in W.
2. If $x \in N,\ \ h_x$ is a smooth extension of $F|[V_x \cap N]$ and $\|h_x(y) - F(x)\| < \dfrac{\delta(x)}{2}$.
3. If $x \notin N,\ V_x \cap N = \emptyset$ and $h_x(y) = F(x),\ \forall y \in V_x$.
4. $\forall y \in V_x,\ \ \|F(y) - F(x)\| < \dfrac{\delta(x)}{2} < \delta(y)$.

Consider a smooth partition of unity, λ_i, dominated by the covering V_x. Then $\forall\ i$ there exists $x = x(i)$ with $\mathrm{supp}(\lambda_i) \subseteq V_{x(i)}$. For every i fix such a $x(i)$ and set

$$G(z) = \sum_i \lambda_i(z) h_{x(i)}(z).$$

Then G is a smooth function since in a neighborhood of a point G is a finite sum of smooth functions.

Let $z \in N$ and $\lambda_{i_1}, \ldots \lambda_{i_k}$ be the functions of the partition which do non vanish at z. Then $h_{x(i_j)}$ is an extension of F, hence equal, in z, to $F(z)$. Hence $G(z) = F(z)$ and G is an extension of $F|N$.

Let $y \in M \setminus N$. If $\lambda_i(y) \neq 0,\ \ y \in \mathrm{supp}(\lambda_i) \subseteq V_{x(i)}$. Hence $\|F(y) - h_{x(i)}\| < \delta(x(i))/2$. Hence

$$\|F(y) - G(y)\| = \|\sum_i \lambda_i(y)F(y) - \sum_i \lambda_i h_{x(i)}(y)\| \le \sum_i \lambda_i(y)\|F(y) - h_{x(i)}(y)\| < \frac{\delta(x(i))}{2} < \delta(y).$$

Finally $H(x,t) = tF(x) + (1-t)G(x)$ is the required homotopy.

□

6.5. Corollary. *If two smooth maps $F, G : U \longrightarrow W$ are homotopic via a continuous homotopy, then they are homotopic via a smooth one.*

7. Exercises

7.1. Prove that the tensor product of tensors is associative and distributive.

7.2. Prove that $\omega \in \mathbb{E}_p$ is an exterior form if and only if

$$\omega(x_1, \ldots, x_i, \ldots, x_j, \ldots x_p) = -\omega(x_1, \ldots, x_j, \ldots, x_i, \ldots x_p).$$

7.3. Prove that the exterior product is distributive with respect to the sum.

7.4. Complete the proof of Proposition 1.20.

7.5. Prove that $\phi_1, \ldots, \phi_p \in \mathbb{E}^*$ are linearly independent if and only if $\phi_1 \wedge \cdots \wedge \phi_p \neq 0$.

7.6. Prove that two sets of linearly independent elements of $\mathbb{E}^*$, $\{\phi_1, \ldots, \phi_p\}$ and $\{\psi_1, \ldots, \psi_p\}$ span the same subspace of $\mathbb{E}^*$, if and only if $\phi_1 \wedge \cdots \wedge \phi_p = d\ \psi_1 \wedge \cdots \wedge \psi_p,\ d \in \mathbb{R}$. In this case, d is the determinant of the matrix that gives the change of basis for the subspace.

7.7. Let $\omega \in \Lambda^*(\mathbb{E}),\ \omega = \sum_0^n \omega_i,\ \omega_i \in \Lambda^i(\mathbb{E})$. Prove that ω is invertible in $\Lambda^*(\mathbb{E})$[16] if and only if $\omega_0 \neq 0$.

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[16] i.e. there exists $\omega^{-1} \in \Lambda^*(\mathbb{E})$ such that $\omega \wedge \omega^{-1} = 1$.

7.8. Let $\mathbb{E}$ be a n-dimensional vector space. Let $\pi : \mathbb{E}^* \times \cdots \times \mathbb{E}^* \longrightarrow \Lambda^p(\mathbb{E})$ be the p-linear extension of $(\phi_1, \dots, \phi_p) \longrightarrow \phi_1 \wedge \cdots \wedge \phi_p$. Prove that the following universal property of the exterior algebra holds:

- (UP$\wedge$) If $\mathbb{K}$ is a vector space and $b : \mathbb{E}^* \times \cdots \times \mathbb{E}^* \longrightarrow \mathbb{K}$ is an alternated p-linear map, then there exists a unique linear map $l : \Lambda^p(\mathbb{E}) \longrightarrow \mathbb{K}$ such that $l \circ \pi = b$.

7.9. Prove that the universal property (UP$\wedge$) characterizes $\Lambda^p(\mathbb{E})$ i.e., given a vector space $\mathbb{L}$ and a p-linear map $\tilde{\pi} : \mathbb{E}^* \times \cdots \times \mathbb{E}^* \longrightarrow \mathbb{L}$ such that $(\tilde{\pi}, \mathbb{L})$ verifies UP$\wedge$, then $\mathbb{L} \cong \Lambda^p(\mathbb{E})$.

7.10. Prove that $\Lambda^p(\mathbb{E}^*) \cong [\Lambda^p(\mathbb{E})]^*$.

7.11. Let $v \in \Lambda^n(\mathbb{E}) \setminus \{0\}$. Define a map:

$$b_v : \Lambda^p(\mathbb{E}) \times \Lambda^{(n-p)}(\mathbb{E}) \longrightarrow \mathbb{R}, \qquad b_v(\omega, \tau)v := \omega \wedge \tau.$$

Prove that b_v is non degenerate and hence defines an isomorphism $\tilde{b}_v : \Lambda^p(\mathbb{E}) \longrightarrow [\Lambda^{(n-p)}(\mathbb{E})]^*$.

7.12. Let $\phi_1, \dots, \phi_r \in \mathbb{E}^*$ be linearly independent. Let $\psi_1, \dots, \psi_r \in \mathbb{E}^*$ be such that $\sum_i \phi_i \wedge \psi_i = 0$. Prove that $\psi_i = \sum_j a_{ij}\phi_j$ with $a_{ij} = a_{ji}$.

7.13. A form $\omega \in \Lambda^p(\mathbb{E})$ is *decomposable* if $\omega = \phi_1 \wedge \cdots \wedge \phi_p,\ \phi_i \in \mathbb{E}^*$. By Proposition 1.20, any p-form is a sum of decomposable forms.

(1) Show that, if $\dim(\mathbb{E}) = n$, any $(n-1)$-form is decomposable.
(2) Show that, if $\dim(\mathbb{E}) = 4$ and $\{\phi_1, \dots, \phi_4\}$ is a basis of $\mathbb{E}^*$, then $\phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$ is *not* decomposable.

7.14. Let $\mathbb{E}$ be a n-dimensional vector space. A vector space $G(\mathbb{E})$, with an associative product denoted by $\wedge$, is called a *Grassman algebra for* $\mathbb{E}$ if

(1) $G(\mathbb{E})$ contains a subspace isomorphic to $\mathbb{R} \oplus \mathbb{E}$ and is generated, as an algebra, by this subspace,
(2) $1 \wedge x = x, x \wedge x = 0,\ \forall x \in \mathbb{E}$,
(3) $\dim(G(\mathbb{E})) = 2^n$.

Prove that $G(\mathbb{E})$ is isomorphic, as an algebra, to $\Lambda^*(\mathbb{E}^*)$.

7.15. Prove, using the functorial properties, that if $L : \mathbb{E} \longrightarrow \mathbb{F}$ is an isomorphism, $L^* : \Lambda^*(\mathbb{F}) \longrightarrow \Lambda^*(\mathbb{E})$ is an isomorphism (see Remark 1.24).

7.16. Let $\phi \in \mathbb{E}^* \setminus \{0\}$ and $\omega \in \Lambda^p(\mathbb{E})$. Show that, if $\phi \wedge \omega = 0$, then there exists $\tau \in \Lambda^{p-1}$ such that $\omega = \phi \wedge \tau$. Conclude that the sequence:

$$\cdots \longrightarrow \Lambda^{p-1}(\mathbb{E}) \xrightarrow{\phi\wedge} \Lambda^p(\mathbb{E}) \xrightarrow{\phi\wedge} \Lambda^{p+1}(\mathbb{E}) \longrightarrow \cdots$$

is exact. Hint: choose a basis containing ϕ.

7.17. Prove directly, i.e. without using Theorem 3.13, Proposition 3.14.

7.18. Let $\mathbb{L}$ be a finite dimensional real Lie algebra, i.e. a finite dimensional real vector space with a bi-linear map $[\ ,\ ] : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L},\ \ (X, Y) \longrightarrow [X, Y]$ such that, $\forall\ X, Y, Z \in \mathbb{L}$ we have:

(1) $[X, Y] = -[Y, X]$,
(2) $[[X, Y]Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity).

Define a map $\mathrm{d}^p : \Lambda^p(\mathbb{L}) \longrightarrow \Lambda^{p+1}(\mathbb{L})$,

$$\mathrm{d}^p(\omega)(X_1,\dots,X_{p+1}) = \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1}).$$

Show, at least for $p=1$, that $\mathrm{d}^{p+1}\circ\mathrm{d}^p = 0$.

In particular the sequence above is a cochain complex and its cohomology is called the *cohomology of the Lie algebra* $\mathbb{L}$.

7.19. Let $\mathbb{L}$ be a Lie algebra. $\omega \in \Lambda^p(\mathbb{L})$ is said to be Ad-invariant if, $\forall\ Y, X_1,\dots,X_p \in \mathbb{L}$, we have

$$\sum_i (-1)^{i-1}\omega([Y,X_i],X_1,\dots,\hat{X}_i,\dots,X_p) = 0.$$

(The terminology will be clear in the section on Lie groups).

(1) Show that if $\omega \in \Lambda^p(\mathbb{L})$ is Ad-invariant, then $\mathrm{d}^p\omega = 0$.
(2) Show that *span* $\{[X,Y] : X,Y \in \mathbb{L}\} = \mathbb{L}$ if and only if the only Ad-invariant 1-form is zero.
(3) Show that if the only Ad-invariant 1-form is the zero form, the only Ad-invariant 2-form is zero.

REMARK: Under suitable hypothesis the cohomology of the Lie algebra is isomorphic to the space of Ad-invariant forms.

7.20. Let $\mathcal{E} = \{0\} \longrightarrow \mathbb{E}_n \longrightarrow \cdots \longrightarrow \mathbb{E}_0 \longrightarrow \{0\}$ be a chain complex. Assume that the $\mathbb{E}_i$'s are finite dimensional and let H_i be the homology groups of the complex. Prove that

$$\chi(\mathcal{E}) := \sum_0^n (-1)^i \dim(\mathbb{E}_i) = \sum_0^n (-1)^i \dim(H_i).$$

$\chi(\mathcal{E})$ is called the *Euler characteristic of the complex.*

7.21. Let $\mathcal{E},\mathcal{F}$ be chain complexes as in Exercise 7.20, and let $\phi : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism. Prove that:

$$\lambda(\phi) := \sum(-1)^i \mathrm{trace}(\phi_i) = \sum(-1)^i \mathrm{trace}(\phi_{*,i}).$$

$\lambda(\phi)$ is called the *Leftchetz number of* ϕ (this number is of great importance in fixed point theory).

7.22. Show that the (algebraic) Mayer-Vietoris sequence (Theorem 3.17) is exact and the (co)boundaries are natural (Proposition 3.21).

7.23. Prove that the Mayer-Vietoris cobundary, for the de Rham cohomology, are given by

7.24. Show that the Mayer-Vietoris sequence for the reduced cohomology (see Definition 5.2) is exact.

7.25. Give details of the proof of Proposition 2.11.

7.26. Use Example 4.15 and Remark 4.16 to prove the *Theorem of invariance of dimension*:

THEOREM: If $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a homeomorphism, then $n = m$.

7.27. Redo the computations in Example 4.15, using reduced cohomology.

7.28. Prove the Claim in Remark 4.16

7.29. Let $U \subseteq \mathbb{R}^n$ be an open set and $v = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ be the volume form. We will identify vectors fields and 1-forms via the "musical isomorphisms" $\flat : \mathcal{H}(U) \longrightarrow \Omega^1(U)$ and its inverse $\sharp : \Omega^1(U) \longrightarrow \mathcal{H}(U)$. Also $*$ will denote the Hodge operator. We define the classical differential operators of calculus:

- The *gradient* $\nabla : \mathcal{F}(U) \longrightarrow \mathcal{H}(U)$, $\nabla f := \sharp \mathrm{d}f = \sum \dfrac{\partial f}{\partial x_i} \dfrac{\partial}{\partial x_i}$.
- The *divergence* div $: \mathcal{H}(U) \longrightarrow \mathcal{F}(U)$, $\operatorname{div} \left(\sum X_i \dfrac{\partial}{\partial x_i} \right) = \sum \dfrac{\partial X_i}{\partial x_i}$.
- The (geometers) *Laplacian* $\Delta : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$, $\Delta f = -\operatorname{div} \nabla f$.
- The *rotational* rot $: \Omega^1(U) \longrightarrow \Omega^{n-2}(U)$ $\operatorname{rot} \omega = *\mathrm{d}\omega$.

.

Prove that:

1. $\Delta f = -\mathrm{d} * (\mathrm{d}f) = -\sum_1^n \dfrac{\partial^2 f}{\partial x_i^2}$.
2. $\Delta(fg) = g\Delta f + f\Delta g - 2\langle \nabla f, \nabla g \rangle$.
3. ω is closed if and only if rot $\omega = 0$.
4. rot $\nabla f = 0$.
5. If $n = 3$ compute rot $\sum X_i \dfrac{\partial}{\partial x_i}$ and show that div rot $\omega = 0$.

7.30. Let $U \subseteq \mathbb{R}^n$ be an open set. Show that $H^n(U) = \{0\}$ if and only if $\forall\ f \in \mathcal{F}(U)$ there exists a vector field $X \in \mathcal{H}(U)$ such that $\operatorname{div} X = f$.

REMARK: It can be shown that the Laplacian $\Delta : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is surjective (this is a non trivial fact). In particular the equation $\operatorname{div} X = f$ has a solution $\forall\ f \in \mathcal{F}(U)$. Hence $H^n(U) = \{0\}$.

7.31. Identify $\mathbb{R}^2$ with the complex line $\mathbb{C}$, $(x, y) \longrightarrow x + iy, i = \sqrt{-1}$. If $U \subseteq \mathbb{R}^2$ is an open set and $f : U \longrightarrow \mathbb{C}$, we will write $f(z) := f(x, y) = u(x, y) + iv(x, y), u, v \in \mathcal{F}(U)$. f is said to be *holomorphic* if it is C^1 and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(Cauchy-Riemann equations)}.$$

It can be shown that a holomorphic function is smooth, and, even more than that, complex analytic, i.e. it is locally the sum of its (complex) Taylor series.

1. Show that the Cauchy-Riemann equations just say that the differential $\mathrm{d}f(z) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is $\mathbb{C}$-linear (i.e. commutes with multiplication by $i = \sqrt{-1}$).
2. Define *complex* 1-forms:

   $$\mathrm{d}z := \mathrm{d}x + i\mathrm{d}y, \quad f\mathrm{d}z := (u + iv)\mathrm{d}z := (u\mathrm{d}x - v\mathrm{d}y) + i(u\mathrm{d}y + v\mathrm{d}x).$$

   and the complex derivative $f'(z)$ by the identity $f'(z)\mathrm{d}z = \mathrm{d}f$. Prove that f is holomorphic if and only if the real and imaginary parts of $f\mathrm{d}z$ are closed. In this case $f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$.
3. Prove that if $f = u + iv$ is holomorphic, then $u, v : U \longrightarrow \mathbb{R}$ are harmonic functions (i.e. $\Delta u = \Delta v = 0$).
4. Show that, if U is star shaped, given a harmonic function $u : U \longrightarrow \mathbb{R}$, there exists a harmonic function $v : U \longrightarrow \mathbb{R}$ such that $f(x, y) = u(x, y) + iv(x, y)$ is holomorphic. The function v is unique, up to an additive constant (if U is connected), and is called the *harmonic conjugate* of u.

7.32. Let $\mathbb{E}$ be a real vector space and $J : \mathbb{E} \longrightarrow \mathbb{E}$ a linear map such that $J^2 = -\mathbb{1}$. Prove that the dimension of $\mathbb{E}$ is even an J induces a structure of complex vector space on $\mathbb{E}$.

7.33. Let $\mathbb{E}$ be a real vector space with an inner product, $\dim \mathbb{E} = 2n$ and let $J : \mathbb{E} \longrightarrow \mathbb{E}$ be a linear isometry such that $J^2 = -\mathbb{1}$.

(1) Prove that there exist orthonormal vectors $\{e_1, \dots, e_n\} \subseteq \mathbb{E}$ such that the set $\{e_i, J(e_i), i = 1, \dots n\}$ is an orthonormal basis for $\mathbb{E}$.
(2) Prove that $\omega(x, y) := \langle x, J(y) \rangle$ is an exterior form.
(3) Let $\phi_i = \flat e_i, \psi_i = \flat J(e_i)$. Prove that $\omega = -\sum \phi_i \wedge \psi_i$.
(4) Prove that $\omega^n = (-1)^n n! * 1$

CHAPTER 2

Integration and the singular homology of open sets of $\mathbb{R}^n$

In Remark 1.8 of Chapter 1, we observed that p-forms are "p-dimensional (oriented) volume elements" and hence the natural integrands for the (oriented) multiple integrals. In this Chapter we will make this statement precise, we will introduce the singular homology of open sets in $\mathbb{R}^n$ and see how integration gives a duality between singular homology and the de Rham cohomology.

1. Integration on singular chains and Stokes Theorem

1.1. Definition. Let $U \subseteq \mathbb{R}^n$ be an open set and $\omega = f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(U)$. Let $D \subseteq U$ be the closure of an open bounded set. We define

$$\int_D \omega = \int_D f(x_1, \ldots, x_n)\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where the integral on the right hand side is the usual Riemann integral.

1.2. Remark. The integral defined above is "oriented" in the sense that if $\omega_\sigma = f(x)\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)}$, $\sigma \in \Sigma(n)$, then

$$\int_D \omega = |\sigma| \int_D \omega_\sigma.$$

In particular the integral depends on the ordering of the coordinates, i.e., it depends on the choice of an orientation in R^n, while the usual Riemann integral of a function does not depend on such a choice (see also Exercise 6.5).

In order to define the integral of a p-form, we first define the "domain of integration".

1.3. Definition.

- A *p-simplex in* $\mathbb{R}^n$ is the *convex hull*[1] of $(p+1)$ points $\{v_0, \ldots, v_p\} \subset \mathbb{R}^n$ in general position[2]. The points v_i are called the *vertices* of the simplex. Any subset of $q+1$ (distinct) vertices determines a q-simplex called a *face* of the original one.
- Let $\{e_1, \ldots, e_p\}$ be the canonical basis of $\mathbb{R}^p$ and $e_0 = 0$. The *standard p-simplex*, $\Delta^p \subset \mathbb{R}^p$ is the simplex with vertices $\{e_0, e_1, \ldots, e_p\}$.
- A *differentiable singular p-simplex in U* is a smooth map $\sigma : \Delta^p \longrightarrow U$ (i.e. σ extends to a smooth map of an open neighborhood of Δ^p) . If it is clear from the context we shall omit the term differentiable.

---

[1] We recall that the convex hull of a subset of $\mathbb{R}^n$ is the smallest convex set that contains the given set. More precisely, it is the intersection of all convex sets that contain the given set.

[2] The points $\{v_0, \ldots, v_p\}$ are in general position if they are not contained in any affine subspace of dimension less than p. This is equivalent to the fact that the vectors $\{v_i - v_0 : i = 1, \ldots p\}$ are linearly independent.

1.4. Remark. Given a p-simplex with vertices $\{v_o, \dots, v_p\}$, a point in the simplex can be written in a unique way in the form $v = \sum_{i=0}^{p} \lambda_i v_i$ with $\lambda_i \in [0,1] \subset \mathbb{R}$ and $\sum_{i=0}^{p} \lambda_i = 1$. The numbers λ_i are the *barycentric coordinates of* v.

1.5. Example. An important example of a singular simplex is the following: Let $\{v_0, \dots, v_p\}$ be points of $\mathbb{R}^n$, not necessarily in general position. Define $L(v_0, \dots, v_p)$ to be the singular simplex of $\mathbb{R}^n$ that maps the point of Δ^p with barycentric coordinates $\{\lambda_0, \dots, \lambda_p\}$ to the point $\sum_{i=0}^{p} \lambda_i v_i \in \mathbb{R}^n$. This simplex will be called the *linear simplex with vertices* $\{v_0, \dots, v_p\}$.

1.6. Definition. Let $\omega \in \Omega^p(U)$ be a differential p-form and $\sigma : \Delta^p \to U$ a singular p-simplex. We define

$$\int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega,$$

where the integral on the right hand side is in the sense of Definition 1.1.

1.7. Example. If $f \in \mathcal{F}(U)$ is a smooth function, i.e. a 0-form, and $p \in U$ a fixed point, i.e. a 0-simplex, then the integral of the form on the simplex is just $f(p)$.

1.8. Example. If $\omega = \sum \omega_i \mathrm{d}x_i \in \Omega^1(U)$ is a 1-form and $\sigma : \Delta^1 \longrightarrow U$ a smooth 1-simplex, then

$$\sigma^* \omega = \tilde{\omega}(t) \mathrm{d}t, \quad \text{with } \ \tilde{\omega}(t) = \sigma^*\omega(t)(1) = \omega(\sigma(t))(\mathrm{d}\sigma(t)(1)) = \omega(\sigma(t))(\dot{\sigma}(t)) = \sum_{i=1}^{n} \omega_i(\sigma(t)) \dot{\sigma}_i(t),$$

where $\sigma_i(t) = \langle \sigma(t), e_i \rangle$ is the i^{th} coordinate of σ. Hence

$$\int_\sigma \omega = \int_0^1 \left[\sum_{i=1}^{n} \omega_i(\sigma(t)) \dot{\sigma}_i(t) \right] \mathrm{d}t.$$

The fundamental result in the elementary integration theory is Stokes Theorem. It relates the integral of a p-form on a domain to the integral of a primitive on the boundary. For $p = 1$ Stokes Theorem is just the *foundamental Theorem of calculus*

$$\int_a^b \mathrm{d}f(t) \mathrm{d}t = \int_{\partial[a,b]} f = f(b) - f(a) \quad \text{(see Example 1.7)}.$$

We will define now the ingredients necessary to state this Theorem in higher dimensions. We start by introducing more general domains of integration for a p-form.

1.9. Definition. A *singular p-chain* is a (formal) finite linear combination of singular p-simplices, with real coefficients. The set $C_p(U)$ of all such p-chains is a real vector space, with the obvious operations.

If $\omega \in \Omega^p(U), c \in C_p(U),\ c = \sum a_i \sigma_i$, we define the integral of ω on c by:

$$I(c, \omega) := \int_c \omega := \sum a_i \int_{\sigma_i} \omega.$$

Next we have to define the boundary of a p chain. Intuitively, the boundary of a singular simplex will be the restriction of the simplex to the boundary of the standard p-simplex Δ^p (which is a chain and not a simplex). More precisely

1.10. Definition. The *boundary operator* $\partial_p : C_p(U) \longrightarrow C_{p-1}(U)$ is defined as the linear extension of

$$\partial_p\ \sigma := \sum_0^p (-1)^i \sigma \circ F_i,$$

where σ is a singular p-simplex and $F_i : \Delta^{p-1} \longrightarrow \Delta^p$ is the linear simplex $F_i = L(e_0, \ldots, \hat{e_i}, \ldots, e_p)$.

1.11. Remark. The signs in the definition above guarantee that the $(p-1)$ faces of Δ^p are taken with the *induced orientations.*

1.12. Example. For a linear simplex, we have the formula:

$$\partial_p L(v_0, \ldots, v_p) = \sum_{i=0}^p (-1)^i L(v_0, \ldots, \hat{v_i}, \ldots, v_p).$$

In our context we have the following version of the classical Stokes Theorem:

1.13. Theorem. [Stokes Theorem] *If* $c \in C_{p+1}(U),\ \omega \in \Omega^p(U)$, *then*

$$I(\partial c, \omega) := \int_{\partial c} \omega = \int_c \mathrm{d}\omega := I(c, \mathrm{d}\omega).$$

Proof. By linearity, it is sufficient to prove the Theorem when c is a singular simplex $\sigma : \Delta^{p+1} \longrightarrow U$. In this case

$$\int_\sigma \mathrm{d}\omega = \int_{\Delta^{p+1}} \sigma^* \mathrm{d}\omega = \int_{\Delta^{p+1}} \mathrm{d}\sigma^*\omega$$

(see Theorem 2.9 of Chapter 1 for the last equality). Also

$$\int_{\partial\sigma} \omega = \int_{\partial\Delta^{p+1}} \sigma^*\omega,$$

where $\partial\Delta^{p+1}$ is the linear chain $\sum_{i=0}^{p+1}(-1)^i L(e_0, \ldots, \hat{e_i}, \ldots e_{p+1}) \in C_p(\Delta^{p+1})$.

Now $\eta := \sigma^*\omega = \sum_i f_i(x_1, \ldots, x_{p+1})\mathrm{d}x_1 \wedge \cdots \hat{\mathrm{d}x_i} \cdots \wedge \mathrm{d}x_{p+1}$. Again by linearity, it is sufficient to prove the Theorem for each monomial. Since we can permute coordinates, up to sign, it is not restrictive to assume

$$\eta = f(x_1, \ldots, x_{p+1})\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p.$$

Then:

$$\mathrm{d}\eta = (-1)^p \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{p+1}.$$

Hence, by Fubini's Theorem

$$\int_{\Delta^{p+1}} \mathrm{d}\eta = (-1)^p \int_{\Delta^{p+1}} \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_1 \cdots \mathrm{d}x_{p+1} = (-1)^p \int_{\Delta^p} \left[\int_0^{1-\sum_i^p x_i} \frac{\partial f}{\partial x_{p+1}} \mathrm{d}x_{p+1}\right] \mathrm{d}x_1 \cdots \mathrm{d}x_p =$$

$$= (-1)^p \int_{\Delta^p} \left[f(x_1, \ldots, x_p, 1 - \sum_{i=1}^p x_i) - f(x_1, \ldots, x_p, 0)\right] \mathrm{d}x_1 \cdots \mathrm{d}x_p,$$

where Δ^p is the standard simplex $\{e_0, \ldots e_p\} \subseteq \mathbb{R}^p \subseteq \mathbb{R}^{p+1}$.

Now $\partial\Delta^{p+1} = L(e_1, \ldots e_{p+1}) + (-1)^{p+1} L(e_0, \ldots, e_p) + \gamma$ where γ is a chain of linear simplices that are faces of Δ^{p+1} containing both e_0 and e_{p+1}. Since on each of such faces at least one of the first p coordinates vanishes, $\eta = 0$ on γ. Hence:

$$\int_{\partial\Delta^{p+1}} \eta = \int_{L(e_1, \ldots e_{p+1})} \eta + (-1)^{p+1} \int_{L(e_0, \ldots e_p)} \eta =$$

$$= (-1)^p \int_{\Delta^p} f(x_1, \dots, x_p, 1 - \sum_{i=1}^{p} x_i) \mathrm{d}x_1 \cdots \mathrm{d}x_p + (-1)^{p+1} \int_{\Delta^p} f(x_1, \dots, x_p, 0) \mathrm{d}x_1 \cdots \mathrm{d}x_p = \int_{\Delta^{p+1}} \mathrm{d}\eta.$$

□

2. Singular homology

We will now look a little more deeply at the boundary operator.

2.1. Lemma. *$\partial_{(p-1)} \circ \partial_p = 0$.*

Proof. Let σ be a singular simplex. From Example 1.12 we have

$$\partial_p(\sigma) = \sum_i (-1)^i \sigma \circ L(e_0, \dots, \hat{e_i}, \dots, e_p).$$

Therefore:

$$\partial_{(p-1)} \partial_p(\sigma) = \sum_{i=0}^{p} (-1)^i \sum_{j<i} (-1)^j \sigma \circ L(e_0, \dots, \hat{e_j}, \dots, \hat{e_i}, \dots, e_p) +$$

$$+ \sum_{i=0}^{p} (-1)^i \sum_{j>i} (-1)^{(j-1)} \sigma \circ L(e_0, \dots, \hat{e_i}, \dots, \hat{e_j}, \dots, e_p).$$

Note that the term $\sigma \circ L(e_0, \dots, \hat{e_i}, \dots, \hat{e_j}, \dots, e_p)$, i, j fixed, appears twice in the above sum with opposite signs, and therefore $\partial_{(p-1)} \partial_p(\sigma) = 0$.

□

In particular the sequence:

$$\cdots \longrightarrow C_{(p+1)}(U) \xrightarrow{\partial_{(p+1)}} C_p(U) \xrightarrow{\partial_p} C_{(p-1)}(U) \xrightarrow{\partial_{(p-1)}} \cdots,$$

is a chain complex and we define:

- $Z_p(U) := \ker \partial_p$ the group of *p-dimensional cycles.*
- $B_p(U) := \operatorname{Im} \partial_{(p+1)}$ the group of *p-dimensional boundaries.*
- $H_p(U) := Z_p(U)/B_p(U)$ the p^{th} *dimensional (singular smooth) homology group.*

From Stokes Theorem 1.13 we get:

2.2. Theorem. *If $a \in Z_p(U)$, $I(a, \mathrm{d}\omega) = 0$. If $\sigma \in Z^p(U)$, $I(\partial b, \sigma) = 0$. Therefore the operator $I : C_p(U) \times \Omega^p(U) \to \mathbb{R}$ induces a $\mathbb{R}$-bilinear operator:*

$$\tilde{I} : H_p(U) \times H^p(U) \longrightarrow \mathbb{R}, \qquad \tilde{I}([c], [\omega]) := I(c, \omega).$$

2.3. Remark. The classical Theorem of de Rham states that the linear map induced by $\tilde{I}$,

$$dR_U : H^p(U) \longrightarrow [H_p(U)]^*, \quad dR_U([\omega])([c]) = \int_c \omega,$$

is an isomorphism, called de *de Rham isomorphism.* We will prove this Theorem in the next section.

Let $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ be a smooth map. Then F induces a linear map $F_* : C_p(U) \longrightarrow C_p(V)$, obtained by extending by linearity the map which sends a singular simplex $\sigma : \Delta^p \longrightarrow U$ to the singular simplex $F \circ \sigma : \Delta^p \to V$. It is easy to check that F_* commutes with the boundary operator and hence it is a morphism between chain complexes. Therefore it induces a morphism in homology, that we will denote with the same symbol,

$$F_* : H_p(U) \longrightarrow H_p(V).$$

The following *functorial properties* are easily established[3]

- $(\mathbb{1}_U)_* = \mathbb{1}_{H_p(U)}$,
- $(G \circ F) = G_* \circ F_*$.

An important feature of the de Rham map is that it is *natural* with respect to smooth maps.

2.4. PROPOSITION. *Let $F : U \longrightarrow V$ be a smooth map. Then*

$$[F_*]^*(dR_U(\omega)) = dR_V(F^*\omega).$$

PROOF. Let $\sigma \in C_p(U), \omega \in \Omega^p(V)$. Then

$$\int_{F\circ\sigma} \omega = \int_\sigma F^*\omega$$

(essentially by definition), and the conclusion follows. ☐

Now we will look at some examples that are the analogs, for homology, of Examples 4.1 4.2, and 4.3 of Chapter 1.

2.5. EXAMPLE. Let $U = \mathbb{R}^0$. Then there is a unique singular p-simplex, the constant one. His boundary is the alternated sum of $(p + 1)$ elements, all equal to the (unique) $(p - 1)$-simplex. Therefore the boundary operator is null if p is odd and it is the identity if p is even. The complex of singular chains is given by:

$$\longrightarrow C_{(2p+1)}(U) = \mathbb{R} \xrightarrow{0} C_{2p}(U) = \mathbb{R} \xrightarrow{\mathbb{1}} C_{(2p-1)}(U) = \mathbb{R} \xrightarrow{0} \cdots \xrightarrow{0} C_0(U) = \mathbb{R} \longrightarrow \{0\}.$$

Therefore:

$$H_p(\mathbb{R}^0) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0 \\ \{0\} & \text{if } p > 0 \end{cases}$$

2.6. REMARK. It might appear more natural and, in fact, some times more convenient, to define chains and homology using *singular cubes*, i.e., smooth maps of the unit cube $[0, 1]^p \subseteq \mathbb{R}^p$ into U. Since a p-cube has always an even number of $(p - 1)$-faces, this construction gives, for $U = \mathbb{R}^0$, a chain complex with p-dimensional chain group $\mathbb{R}$ and null boundary operators. So the homology would be isomorphic to $\mathbb{R}$ in all dimensions, which is not what we would like to have. However if we take the quotient of the complex of singular cubes by a suitable subcomplex, we obtain a new complex whose homology is the same as the homology of the complex of singular simplices.

---

[3] This means that the homology is a *covariant* functor from the category of open sets of $\mathbb{R}^n$ and smooth maps into the category of (graded) vector spaces and linear maps.

2.7. EXAMPLE. Let $U = \coprod_\alpha U_\alpha$ be the disjoint union of the open sets U_α. Since Δ^p is connected, the image of a singular simplex is contained in some U_α. Therefore $C_p(U) = \bigoplus_\alpha C_p(U_\alpha)$ (direct sum) and the boundary maps preserve the decomposition, i.e. if $c = \{c_\alpha\},\ \partial c = \{\partial c_\alpha\}$. It follows that

$$H_p(U) \cong \bigoplus_\alpha H_p(U_\alpha).$$

2.8. REMARK. We observe explicitly that we are dealing with *finite* linear combinations of simplices, hence we have a direct sum instead of a direct product, as in the case of cohomology. Furthermore, this is in agreement with the de Rham Theorem 2.3, since the dual of the direct sum of vector spaces is the direct product of the duals.

2.9. EXAMPLE. Let us analyze the 0-dimensional homology. Let us suppose first that U is connected. A 0-simplex is a constant map, i.e. a point in U. Such a simplex is a cycle, by definition. On the other hand, given two points in U they can be joined by a smooth curve, i.e. a 1-simplex. The boundary of such simplex is the difference of the two points, so the two points are in the same homology class. It follows that $H_0(U) \cong \mathbb{R}$. Also, as in the case of cohomology, if $U \subseteq \mathbb{R}^n,\quad V \subseteq \mathbb{R}^m$ are connected open sets and $F : U \longrightarrow V$ is a smooth map, the induced map $F_* : H_0(U) \longrightarrow H_0(V)$ is an isomorphism.

If U is not connected, let us say with connected components U_α, it follows from Example 2.7 that

$$H_0(U) \cong \bigoplus_\alpha \mathbb{R}.$$

Next we will prove the *homotopy invariance for homology*:

2.10. THEOREM. *Let $F, G : U \to V$ be homotopic smooth maps. Then $F_* = G_*$.*

PROOF. Let $H : U \times [0,1] \to V$ be a homotopy between F and G. We will construct an *algebraic homotopy* between the induced maps, i.e. a map $\tilde{H}_p : C_p(U) \longrightarrow C_{(p+1)}(V)$ such that

$$\partial \circ \tilde{H}(\sigma) = G_*(\sigma) - F_*(\sigma) - \tilde{H} \circ \partial\sigma.$$

The Theorem then follows since if $c \in Z_p(U), G_*(c) - F_*(c) \in B_p(V)$, i.e. $[G_*(c)] = [F_*(c)]$ in $H_p(V)$.

Consider the product $\Delta^p \times [0,1] \subset \mathbb{R}^{p+1}$. If σ is a singular p-simplex of U, we consider the map $H \circ (\sigma \times \mathbb{1}) : \Delta^p \times [0,1] \longrightarrow V$. The problem is that $\Delta^p \times [0,1]$ is not a simplex[4]. The strategy will be to subdivide $\Delta^p \times [0,1]$ into simplices and to take a suitable alternated sums of the restrictions of $H \circ (\sigma \times \mathbb{1})$ to such simplices.

Consider $v_i = (e_i, 0), w_i = (e_i, 1)$, and the linear $(p+1)$-simplices $L(v_0, \dots, v_i, w_i, \dots w_p)$. If $\sigma : \Delta^p \longrightarrow U$ is a singular p-simplex, we define

$$\tilde{H}(\sigma) = \sum_{i=0}^{p} (-1)^i H \circ (\sigma \times \mathbb{1}) \circ L(v_0, \dots, v_i, w_i, \dots, w_p) \in C_{p+1}(V).$$

Extending the formula by linearity we get a morphism $\tilde{H} : C_p(U) \longrightarrow C_{p+1}(V)$.

We observe that, geometrically, the left hand side of the first equation is the restriction of $\sigma \times \mathbb{1}$ to the boundary of the prism $\Delta^p \times [0,1]$ while the right hand side is, with appropriate signs, the restriction of $\sigma \times \mathbb{1}$

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[4] This is a case in which would be more convenient to work with singular cubes instead that simplices since the product of two cubes is a cube (Remark 2.6).

to the bases of the prism, $\Delta \times \{0,1\}$, essentially $G_*(\sigma) - F_*(\sigma)$, plus the restriction of $\sigma \times 1\!\!1$ to the "lateral faces" $\partial\Delta^p \times [0,1]$. We will make this precise.

Using 1.12 and the functorial properties, we get:

$$\partial\tilde{H}(\sigma) = \sum_i \sum_{j\leq i} (-1)^i(-1)^j H \circ (\sigma \times 1\!\!1) \circ L(v_0, \dots, \hat{v_j}, \dots, v_i, w_i, \dots w_p) +$$
$$+ \sum_i \sum_{j\geq i} (-1)^i(-1)^{j+1} H \circ (\sigma \times 1\!\!1) \circ L(v_0, \dots, v_i, w_i, \dots, \hat{w_j}, \dots, w_p).$$

For $i = j$ the terms on the right hand side cancel except for

$$H \circ (\sigma \times 1\!\!1) \circ L(\hat{v_0}, w_0, \dots, w_p) = G \circ \sigma \quad \text{and} \quad - H \circ (\sigma \times 1\!\!1) \circ L(v_0, \dots, v_p, \hat{w_p}) = -F \circ \sigma.$$

The rest of the sum is the opposite of

$$\sum_i \sum_{j<i} (-1)^{i-1}(-1)^j H \circ (\sigma \times 1\!\!1) \circ L(v_0, \dots, \hat{v_j}, \dots, v_i, w_i, \dots w_p) +$$
$$+ \sum_i \sum_{j>i} (-1)^i(-1)^j H \circ (\sigma \times 1\!\!1) \circ L(v_0, \dots, v_i, w_i, \dots, \hat{w_j}, \dots, w_p) = \tilde{H}\partial(\sigma).$$

$\tilde{H}$ is then the required homotopy. ☐

From Theorem 2.10 and the funtorial properties we have

2.11. Corollary. *If $F : U \longrightarrow V$ is a homotopy equivalence, then $F_* : H_p(U) \to H_p(V)$ is an isomorphism. In particular, a contractible space has the same homology as $\mathbb{R}^0$.*

2.12. Remark. As in the case of cohomology, the homotopy invariance allows us to define the map induced in homology by a continuous map (see Remark 4.10 in Chapter 1).

We also have a Mayer-Vietoris exact sequence for homology. Let $U_i \subseteq \mathbb{R}^n, i = 1, 2$ be open sets and define $U = U_1 \cup U_2, V = U_1 \cap U_2$. Consider the sequence of chain complexes

$$\{0\} \longrightarrow C_p(V) \overset{((j_1)_*,(j_2)_*)}{\longrightarrow} C_p(U_1) \oplus C_p(U_2) \overset{((k_1)_*-(k_2)_*)}{\longrightarrow} C_p(U) \longrightarrow \{0\},$$

where $j_i : V \to U_i$, $k_i : U_i \to U$ are the inclusions and the boundary maps are the obvious ones.

We would like to proceed like in the case of cohomology. The problem we have here is that the sequence above is not exact. More precisely, $((k_1)_* - (k_2)_*)$ is not surjective, since a chain in U might not be the sum of chains in U_i. To overcome this problem, we consider the chain complex $C_p(U_1 + U_2) \subseteq C_p(U)$ spanned by the singular simplices of U_1 and U_2. Substituting $C_p(U)$ with this complex, we have a short exact sequence of chain complexes. The point that makes this idea work is the following result

2.13. Theorem. [Small simplicies Theorem] *The inclusion $C_p(U_1 + U_2) \longrightarrow C_p(U)$ induces an isomorphism in homology.*

The proof requires some new constructions and we will give it in the Appendix in order not to interrupt the flow of our discussion.

Using Theorem 2.13 and Theorem 3.17 of Chapter 1, we deduce, as for cohomology

2.14. THEOREM. *There are linear maps* $\Delta_{*,p} : H_p(U) \longrightarrow H_{(p-1)}(V)$ *such that the sequence*

$$\cdots \longrightarrow H_p(V) \xrightarrow{((j_1)_*,(j_2)_*)} H_p(U_1) \oplus H_p(U_2) \xrightarrow{((k_1)_*-(k_2)_*)} H_p(U) \xrightarrow{\Delta_{*,p}} H_{(p-1)}(V) \longrightarrow \cdots$$

is a (long) exact sequence. Again, we often write Δ_* *or* Δ_p *for* $\Delta_{*,p}$.

2.15. DEFINITION. The exact sequence above is called the *Mayer-Vietoris sequence for singular homology* and the maps Δ_*, the *Mayer-Vietoris boundary operators.*

3. The de Rham Theorem for open sets of $\mathbb{R}^n$

Let $U \subseteq \mathbb{R}^n$ be an open set. As we have seen, integration induces a linear map:

$$dR : H^p(U) \longrightarrow [H_p(U)]^*, \qquad dR([\omega])([c]) = \int_c \omega.$$

We have already announced that this map is an isomorphism and the aim of this section is to prove this fact. We will start with a Lemma, known as the *Mayer Vietoris argument*, useful in several situations.

3.1. LEMMA. [Mayer Vietoris argument][5] *Let* $U \subseteq \mathbb{R}^n$ *be an open set and* $\mathcal{P}$ *a statement about the open subsets* $V \subseteq U$. *Suppose that:*

(1) $\mathcal{P}$ *is true for open convex sets,*
(2) *If* $\mathcal{P}$ *is true for disjoint sets, then it is true for their union,*
(3) *If* $\mathcal{P}$ *is true for two sets and for their intersection, then it is true for their union.*

Then $\mathcal{P}$ *is true for* U.

PROOF. First we observe that $\mathcal{P}$ is true for the union of n convex sets. In fact, for $n = 2$ this follows from (3) observing that the intersection of two convex sets is convex. Suppose that $\mathcal{P}$ is true for the union of $(n-1)$ convex sets. Let $V_1, \dots, V_n$ be convex sets and $V = V_1 \cup \ldots \cup V_{(n-1)}$. Then $\mathcal{P}$ is true for V_n and, by the inductive hypothesis, for V. But it is also true for $V \cap V_n$ since

$$V \cap V_n = (V_1 \cap V_n) \cup \ldots \cup (V_{(n-1)} \cap V_n)$$

is the union of $(n-1)$ convex sets. From (3), $\mathcal{P}$ is true for the union of all the V_i's.

Let $\phi : U \longrightarrow [0, \infty)$ be a proper function (see Claim 2. in the proof of Theorem 6.2, Chapter 1). Define:

$$A_n = \phi^{-1}([n, n+1]).$$

Since ϕ is proper, A_n is compact and we can cover it with a finite number of open convex sets, $U_{k,n}$, contained in $\phi^{-1}\left((n - \frac{1}{2}, n + \frac{3}{2})\right)$. Let $U_n = \cup_k U_{k,n}$. Now $\mathcal{P}$ is true for U_n, since it is a finite union of convex sets. Let us consider $U_{even} = \cup_n U_{2n}$ and $U_{odd} = \cup_n U_{2n+1}$. Then, by (2), $\mathcal{P}$ is true for U_{even} and U_{odd} since each one is a disjoint union of sets for which $\mathcal{P}$ is true. Finally $U_{even} \cap U_{odd} = \cup_{n,k,h} U_{k,2n} \cap U_{h,2n+1}$ and therefore it is a disjoint union of sets that are finite unions of convex sets. Therefore, by (3), $\mathcal{P}$ is true for $U = U_{even} \cup U_{odd}$. ☐

We can now prove the de Rham Theorem.

3.2. THEOREM. *The map* $dR : H^p(U) \longrightarrow [H_p(U)]^*$ *is an isomorphism.*

---

[5] The lemma is also called the *onion lemma* and the reason for this will be clear from the proof (see [**2**]).

Proof. Since we will work with several open sets, it is convenient to denote with dR_V the de Rham map relative to the open set $V \subseteq U \subseteq \mathbb{R}^n$. We are going to use Lemma 3.1. Let us consider the statement

$$\mathcal{P}(V) := dR_V : H^p(V) \longrightarrow [H_p(V)]^* \text{ is an isomorphism.}$$

Clearly the statement is true for convex sets. In fact they are contractible and we have to check the statement in dimension 0, which is trivial. Also, if it is true for a family of disjoint open sets, it is also true for their union (recall that the dual of the direct sum is the direct product).

Let us suppose that $\mathcal{P}$ is true for the open sets V, W and for $V \cap W$. Consider the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(V \cap W) \arrow[r] \arrow[d, "dR_{V\cap W}"] & H^{p+1}(V \cup W) \arrow[r] \arrow[d, "dR_{V\cup W}"] & H^{p+1}(V) \oplus H^{p+1}(W) \arrow[r] \arrow[d, "dR_V \oplus dR_W"] & \cdots \\
\cdots \arrow[r] & {[H_p(V \cap W))]^*} \arrow[r] & {[H_{p+1}(V \cup W)]^*} \arrow[r] & {[H_{p+1}(W)]^* \oplus [H_{p+1}(W)]^*} \arrow[r] & \cdots
\end{tikzcd}$$

where the upper row is the Mayer-Vietoris sequence for cohomology and the lower row is the *dual* of the Mayer-Vietoris sequence in homology. The latter is exact by Proposition 3.14 of Chapter 1. Since integration commutes with induced maps (Proposition 2.4), the diagram above are commutative. Since $dR_{V\cap W}$ and $dR_V \oplus dR_W$ are isomorphisms by hypothesis, it follows from the five Lemma (Lemma 3.7 of Chapter 1) that $dR_{V\cup W}$ is an isomorphism. So $\mathcal{P}$ verifies the hypothesis of Lemma 3.1 and hence $dR = dR_U$ is an isomorphism. ☐

3.3. Remark. Starting with the singular complex $\mathcal{C}(U) = \{C_p(U), \partial_p\}$, we can consider the *dual complex* $\mathcal{C}^*(U) = \{C_p(U)^*, \partial_p^*\}$ (see Remark 3.12 of Chapter 1). The cohomology of $\mathcal{C}^*(U)$ is called the *singular cohomology* of U and it is isomorphic, by Theorem 3.13 of Chapter 1, to the dual of the singular homology of U. So the de Rham Theorem states that the singular cohomology and the de Rham cohomology are isomorphic. The de Rham cohomology $H^*(U) = \oplus_{p\geq 0} H^p(U)$ has a natural product, induced by the exterior product of forms, which is distributive, associative and graded commutative, (see Remark 2.12 of Chapter 1). In the singular cohomology it is possible to introduce, by geometric arguments, a product, also called the *cup product*, which is distributive, associative and graded commutative. The de Rham Theorem actually says that dR, extended by linearity, is an isomorphism of *algebras*.

3.4. Remark. Singular homology is usually defined by starting with *continuous simplices* i.e., continuous maps $\sigma : \Delta^p \longrightarrow U$ [6]. The singular (continuous) chain complex $\mathcal{C}^0(U) = \{C^0_p(U), \partial_p\}$ is defined in the obvious way, i.e. the spaces $C^0_p(U)$ are the vector spaces with basis the singular continuous simplices and the boundary operator is defined just as in the smooth case. The basic properties, such as homotopy invariance and the Mayer-Vietoris exact sequence, are also proved just as in the smooth case. The inclusion $\mathcal{C}(U) \longrightarrow \mathcal{C}^0(U)$ is a morphism of chain complexes, so it induces a map between the homology groups. Using the same arguments as in the proof of the de Rham Theorem, it is easy to prove that the inclusion induces an isomorphism in homology.

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[6] Here U can be any topological space.

4. Tensor product of vector spaces and the Künnet's Theorem

A natural question to ask is the following: given open sets $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ find a relation between the cohomology groups of $U_1,\ U_2$ and $U_1 \times U_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$.

To answer this question we need some preliminary algebraic facts. To start with we need a slightly different approach to tensors.

4.1. Definition. Let $\mathbb{E}, \mathbb{F}$ be two real vector spaces (not necessarily finite dimensional). Consider the vector space freely generated by $\{(x, y) : x \in \mathbb{E}, y \in \mathbb{F}\}$ and the subspace generated by the elements

- $(x_1 + x_2, y) - (x_1, y) - (x_2, y), \quad (x, y_1 + y_2) - (x, y_1) - (x, y_2), \quad x_i \in \mathbb{E},\ y_i \in \mathbb{F}.$
- $r(x, y) - (rx, y), \quad r(x, y) - (x, ry), \quad x \in \mathbb{E},\ y \in \mathbb{F},\ r \in \mathbb{R}.$

The quotient space is called the *tensor product* of $\mathbb{E}$ and $\mathbb{F}$ and will be denoted by $\mathbb{E} \otimes \mathbb{F}$. The class of (x, y) in $\mathbb{E} \otimes \mathbb{F}$ will be denoted by $x \otimes y$.

In other words we may think of $\mathbb{E} \otimes \mathbb{F}$ as the space of finite (formal) linear combinations of elements of the type $x \otimes y$ with the "calculus rules"

- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y, \quad x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2,$
- $r(x \otimes y) = rx \otimes y = x \otimes ry.$

The following facts are easily verified

4.2. Proposition.

(1) $\mathbb{E} \otimes \mathbb{F} \cong \mathbb{F} \otimes \mathbb{E}, \quad \mathbb{E} \otimes \mathbb{R} \cong \mathbb{E}.$
(2) $(\mathbb{E} \otimes \mathbb{F}) \otimes \mathbb{P} \cong \mathbb{E} \otimes (\mathbb{F} \otimes \mathbb{P}).$
(3) $\mathbb{E} \otimes (\mathbb{F} \oplus \mathbb{P}) \cong \mathbb{E} \otimes \mathbb{F} \oplus \mathbb{E} \otimes \mathbb{P}.$
(4) *If* $\{e_i\}, \{f_j\}$ *are bases for* $\mathbb{E}, \mathbb{F}$ *respectively, then* $\{e_i \otimes f_j\}$ *is a basis for* $\mathbb{E} \otimes \mathbb{F}$. *In particular, if* $\mathbb{E}$, $\mathbb{F}$ *are finite dimensional,* $\dim(\mathbb{E} \otimes \mathbb{F}) = \dim(\mathbb{E}) \dim(\mathbb{F})$.
(5) *If* $\mathbb{E}$ *is finite dimensional,* $\mathbb{E}^* \otimes \mathbb{E}^* \cong \mathbb{E}_2$.

Let $\pi : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ be the bi-linear extension of $\pi(x, y) = x \otimes y$.

4.3. Proposition. *The following* universal property of the tensor product *holds*

• (UP$\otimes$) *If* $\mathbb{K}$ *is a vector space and* $b : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{K}$, *is a bilinear map, then there exists a unique linear map* $l : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{K}$ *such that* $l \circ \pi = b$.

Proof. Set $l(x \otimes y) = b(x, y)$. By the "calculus rules", l extend to a linear map of $\mathbb{E} \otimes \mathbb{F}$ into $\mathbb{K}$ such that $l \circ \pi = b$. If $l' : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{K}$ is a linear map with $l' \circ \pi = b$, then $l'(x \otimes y) = b(x, y) = l(x \otimes y)$. Since the elements of the type $x \otimes y$ span $\mathbb{E} \otimes \mathbb{F}$, we have $l = l'$. □

The general philosophy is that objects defined by *universal properties* are *unique*.

4.4. Proposition. *If* $\mathbb{H}$ *is a vector space and* $\tilde{\pi} : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{H}$ *is a bi-linear map such that* UP$\otimes$ *is verified for* $(\tilde{\pi}, \mathbb{H})$, *then* $\mathbb{H} \cong \mathbb{E} \otimes \mathbb{F}$.

Proof. From the universal property for $\pi : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ follows that there is a unique linear map $l : \mathbb{E} \otimes \mathbb{F} \longrightarrow \mathbb{H}$ such that $l \circ \pi = \tilde{\pi}$. From the universal property of $\tilde{\pi} : \mathbb{E} \times \mathbb{F} \longrightarrow \mathbb{H}$ follows that there is a unique map $l' : \mathbb{H} \longrightarrow \mathbb{E} \otimes \mathbb{F}$ such that $l' \circ \tilde{\pi} = \pi$. Now, $l \circ l' : \mathbb{H} \longrightarrow \mathbb{H}$ is such that $\tilde{\pi} \circ (l \circ l') = \tilde{\pi}$. But also $\tilde{\pi} \circ \mathbb{1} = \tilde{\pi}$. Hence, by uniqueness, $(l \circ l') = \mathbb{1}$. Analogously $l' \circ l = \mathbb{1}$, hence l and l' are inverse isomorphisms. $\square$

The important feature of the tensor product is that it allows us *to transform a bi-linear problem into a linear problem*, which is, generally, easier to solve.

Let $\mathbb{E}_i$, $\mathbb{F}_i$, $i = 1, 2$ be vector spaces and let $L_i : \mathbb{E}_i \longrightarrow \mathbb{F}_i$ be linear map. We define

$$L_1 \otimes L_2 : \mathbb{E}_1 \otimes \mathbb{E}_2 \longrightarrow \mathbb{F}_1 \otimes \mathbb{F}_2, \qquad L_1 \otimes L_2(v \otimes w) := L_1(v) \otimes L_2(w).$$

We will need also the following result , whose proof we will leave to the reader (Exercise 7.23)

4.5. PROPOSITION. *Let* $\cdots \longrightarrow \mathbb{E}_1 \xrightarrow{\phi} \mathbb{E}_2 \xrightarrow{\psi} \mathbb{E}_3 \longrightarrow \cdots$ *be an exact sequence and let* $\mathbb{F}$ *be a vector space. Then the sequence* $\cdots \longrightarrow \mathbb{E}_1 \otimes \mathbb{F} \xrightarrow{\phi \otimes \mathbb{1}} \mathbb{E}_2 \otimes \mathbb{F} \xrightarrow{\psi \otimes \mathbb{1}} \mathbb{E}_3 \otimes \mathbb{F} \longrightarrow \cdots$ *is exact.*

The result we have promised, called the Künnet Theorem, or also the *Künnet formula*, is the following

4.6. THEOREM. [Künnet's Theorem] *Let* $U_1 \subseteq \mathbb{R}^n$ *and* $U_2 \subseteq \mathbb{R}^m$ *be open set. Then*

$$H^k(U_1 \times U_2) \cong \oplus_{p+q=k} H^p(U_1) \otimes H^q(U_2).$$

PROOF. Let us denote by $\pi_i : U_1 \times U_2 \longrightarrow U_i$ the projection maps. Let $W \subseteq U_1$ be an open set and consider the map

$$\kappa_W : \Omega^p(W) \otimes \Omega^q(U_2) \longrightarrow \Omega^k(W \times U_2),\ k = p + q, \qquad \kappa_W(\omega \otimes \tau) = \pi_1^* \omega \wedge \pi_2^* \tau.$$

Since π_i^* commutes with d, κ_W induces a morphism in cohomology. Summing up these morphism, for $p + q = k$, we get a map, still denoted by κ_W,

$$\kappa_W : \oplus_{p+q=k} H^p(W) \otimes H^q(U_2) \longrightarrow H^k(W \otimes U_2).$$

We want to prove that κ_{U_1} is an isomorphism. For this we will use the Mayer Vietoris argument (Lemma 3.1) as in the proof of the de Rham Theorem. Let us consider the statement

$$\mathcal{P}(W) = \kappa_W : \oplus_{p+q=k} H^p(W) \otimes H^q(U_2) \longrightarrow H^k(W \otimes U_2) \quad \text{is an isomorphism.}$$

We have to show that the conditions of the Lemma 3.1 are verified. Clearly $\mathcal{P}(W)$ is true if W is convex. Also if W_α are disjoint open sets such that $\mathcal{P}(W_\alpha)$ is true, the same holds for $W = \cup_\alpha W_\alpha$. It remains to show that if $V, W \subseteq U_1$ are open sets such that $\mathcal{P}(V)$, $\mathcal{P}(W)$ and $\mathcal{P}(V \cap W)$ are true, them $\mathcal{P}(V \cup W)$ is true. Consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^p(V \cup W) \longrightarrow H^p(V) \oplus H^p(W) \longrightarrow H^p(V \cap W) \xrightarrow{\Delta^*} H^{p+1}(V \cup W) \longrightarrow \cdots$$

Tensoring with $H^q(U_2)$ and summing for $p + q = k$ we obtain the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \oplus_{p+q=k} H^p(V \cap W) \otimes H^q(U_2) \arrow[r, "\oplus \Delta^* \otimes \mathbb{1}"] \arrow[d, "\kappa_{V \cap W}"] & \oplus_{p+q=k} H^{p+1}(V \cup W) \otimes H^q(U_2) \arrow[r] \arrow[d, "\kappa_{U \cup V}"] & \cdots \\
\cdots \arrow[r] & H^k((V \cap W) \times U_2) \arrow[r, "\Delta^*"] & H^{k+1}((V \cup W) \times U_2) \arrow[r] & \cdots
\end{tikzcd}$$

The upper line is exact by Corollary 4.5, the lower one is exact being the Mayer-Vietoris sequence of $V \times U_2, W \times U_2 \subseteq (V \cup W) \times U_2$. Moreover $\kappa_{V \cap W}$ and $\kappa_V \oplus \kappa_W$ are isomorphisms, by hypothesis. So we can use the five Lemma to conclude that $\kappa_{V\cup W}$ is an isomorphism, once we show that the squares commute. This is obvious for all squares but the one that appear in the picture above. For this square we have

- $\kappa_{V\cup W}\circ(\Delta^*\otimes 1\!\!1)(\omega\otimes\phi)=\pi_1^*\Delta^*\omega\wedge\pi_2^*\phi$,
- $\Delta^*\circ\kappa_{V\cap W}(\omega\otimes\phi)=\Delta^*\pi_1^*\omega\wedge\pi_2^*\phi$.

The conclusion follows from the fact that Δ^* commutes with induced maps (Remark 4.14 of Chapter 1). $\square$

5. Integration of 1-forms and some applications

Let $U\subseteq\mathbb{R}^n$ be an open set. A smooth curve $\gamma:[a,b]\longrightarrow U$ can be seen as the smooth 1-simplex $\tilde\gamma=\gamma\circ L(a,b)$ where $L(a,b)=(1-t)a+tb$. If $\omega\in\Omega^1(U)$ is a 1-form, we define

$$\int_\gamma\omega:=\int_{\tilde\gamma}\omega=\int_0^1\left[\sum\omega_i(\tilde\gamma(t))\dot{\tilde\gamma}_i(t)\right]\mathrm{d}t=\int_a^b\left[\sum\omega_i(\gamma(t))\dot\gamma_i(t)\right]\mathrm{d}t,$$

where the second integral is the integral of ω on the 1-simplex $\tilde\gamma$ and the last equality comes from the formula of change of variable in 1-dimensional integrals (see also Example 1.8). For the rest of this section, when clear from the context, *we will make no difference between the curve γ and the 1-simplex $\tilde\gamma$.*

Let $\gamma:[a,b]\subseteq\mathbb{R}\longrightarrow U$ be a piecewise smooth curve, i.e. a continuous curve such that there exists a partition $t_0=a<t_1<\cdots<t_k=b$ of $[a,b]$ such that $\gamma_i:=\gamma|[t_i,t_{i+1}]$ is smooth. Then γ can be viewed either as the (smooth) 1-chain $\gamma=\sum\gamma_i$ or as a continuous 1-simplex. Clearly, in both cases, $\partial\gamma=\gamma(b)-\gamma(a)$.

Let $\gamma:[a,b]\subseteq\mathbb{R}\longrightarrow U$ be a continuous *closed* curve, i.e. $\gamma(a)=\gamma(b)$. Consider the map $\pi:[a,b]\longrightarrow S^1:=\{x\in\mathbb{R}^2:\|x\|=1\}$, $\pi((1-t)a+tb))=(\cos 2\pi t,\sin 2\pi t)$. Since γ is closed, $\overline{\gamma}=\gamma\circ\pi^{-1}$ is a well defined continuous map of S^1 into U. Conversely, any such map defines a continuous closed curve. From this point of view, continuous closed curves and continuous maps of the circle into U look like the same thing. However, there are some differences:

- If γ is a smooth curve $\overline{\gamma}$ will be just piecewise smooth. It will be smooth if and only if the derivatives of all orders of γ at a, coincide with the derivatives of the corresponding order of γ at b.
- Any curve $\gamma:[a,b]\longrightarrow U$ is homotopic to a constant (see Exercise 7.3). This is not the case for maps of S^1 into U. The following result,whose proof is quite obvious, relates the two situations:

5.1. Lemma. *Let $\overline{\gamma}_i:S^1\longrightarrow U,\ i=0,1$ be continuous maps and γ_i be the corresponding closed curves. Then $\overline{\gamma}_0\sim\overline{\gamma}_1$ if and only if there is a homotopy $H:[a,b]\times[0,1]\longrightarrow U$ between γ_0 and γ_1 such that $H(a,s)=H(b,s)\ \ \forall s\in[0,1]$.*

5.2. Remark. A homotopy like the one in Lemma 5.1 is called a *free homotopy* and the maps $\overline{\gamma}_i$ are said to be *freely homotopic.* The word "free" is to distinguish this concept from the one of *based homotopy,* frequently used in homotopy theory, for example in the definition of the fundamental group.

When clear from the context we will make no distinction between γ and $\overline{\gamma}$.

Let $\gamma:[a,b]\longrightarrow U$ be a closed piecewise smooth curve. Then, if we think of γ as a smooth 1-chain, $\partial\gamma=0$ and therefore it determines an element $[\gamma]\in H_1(U)$.

5.3. Lemma. *If γ_0 and γ_1 are freely homotopic piecewise smooth closed curves, then $[\gamma_0]=[\gamma_1]$ in $H_1(U)$.*

Proof. Let $H : [a, b] \times [0, 1] \longrightarrow U$ be a free homotopy between the two curves. Subdividing $[a, b] \times [0, 1]$ into triangles and using linear simplices as in the proof of homotopy invariance for singular homology (see Theorem 2.10), we get a chain $\tilde{H}$ with $\partial \tilde{H} = \gamma_1 - \gamma_0$. □

An other important variant of the concept of homotopy of curves is the following:

5.4. Definition. Let $\gamma_i : [a, b] \longrightarrow U,\ i = 0, 1$ be curves such that $\gamma_0(a) = \gamma_1(a),\ \gamma_0(b) = \gamma_1(b)$. An *endpoints fixing homotopy* between the two curves is a homotopy $H : [a, b] \times [0, 1] \longrightarrow U$ such that $H(a, s) = \gamma_0(a),\ H(b, s) = \gamma_0(b),\ \forall\ s \in [0, 1]$.

If such homotopy exists, we will say that the curves are *homotopic relative to the endpoints.*

The following (well known) facts follow easily from the Theorem of de Rham (we invite the reader to give a more elementary proof, see Exercise 7.11).

5.5. Proposition. *Let $\omega \in \Omega^1(U)$ be a closed 1-form.*

(1) *If $\gamma_i,\ i = 0, 1$ are freely homotopic piecewise smooth closed curves (resp. curves homotopic relative to the endpoints) then:*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

(2) *ω is exact if and only if for all closed curves γ*

$$\int_{\gamma} \omega = 0.$$

5.6. Definition. A connected open set $U \subseteq \mathbb{R}^n$ is *simply connected* if every closed curve is freely homotopic to a constant curve [7].

From Proposition 5.5 we have:

5.7. Corollary. *If U is simply connected, then $H^1(U) = \{0\}$.*

5.8. Remark. A natural question is whether $H^1(U) = \{0\}$ implies that U is simply connected. The answer to this question is affirmative for $n = 2$ (see Exercise 7.25) and negative if $n \geq 3$. For example there are, in $\mathbb{R}^3$, (complicated) closed sets, homeomorphic to the 3-dimensional closed disk, whose complements are not simply connected (for example the so called "horned sphere"). The complement of such a disk has, by the Jordan-Alexander duality (see Theorem 5.4 of Chapter 1), the same cohomology as the complement of the standard 3-dimensional disk, hence vanishing first cohomology group (see Example 4.15 of Chapter 1). We do not know of any simpler example in dimension 3. For $n \geq 4$ there are simpler examples that we will discuss in Chapter 4.

We will focus now on closed curves in $U = \mathbb{R}^2 \setminus \{0\}$. In U there is an important 1-form, the *angle form*

$$\omega = \frac{-y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y.$$

---

[7] The concept of simply connectedness is usually defined in terms of the vanishing of the fundamental group. In this group, two freely homotopic closed curves are in the same conjugacy class (and conversely), but they may not be the same element of the group. However, the *vanishing* of the fundamental group is equivalent to the fact that every two closed curves are freely homotopic.

It is easily seen that $\mathrm{d}\omega = 0$, in fact, locally, $\omega = \mathrm{d}\arctan(y/x)$. But ω is not exact since, if $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$,

$$\int_\gamma \omega = \int_0^1 2\pi[\sin^2(2\pi t) + \cos^2(2\pi t)]\mathrm{d}t = 2\pi \neq 0.$$

In particular, $dR([\omega])([\gamma]) = 2\pi$. Since $H^1(U) \cong \mathbb{R}$, by Examples 4.15 of Chapter 1, $[\omega]$ spans $H^1(U)$ and $[\gamma]$ spans $H_1(U) \cong \mathbb{R}$.

5.9. Definition. Let $\gamma : [0,1] \longrightarrow U$ be a piecewise smooth curve. An *angular function* for γ is a piecewise smooth function $\theta : [0,1] \longrightarrow \mathbb{R}$ such that $\theta(t)$ is one of the determinations, in radians, of the (oriented) angle between e_1 and $\gamma(t)$.

5.10. Lemma. *Any piecewise smooth curve $\gamma : [0,1] \longrightarrow U$ admits angular functions and two angular functions for γ differ by an entire multiple of 2π.*

Proof. Let $\theta_0 \in [0, 2\pi)$ be the angle between e_1 and $\gamma(0)$, and ω the angle form. Define

$$\theta(t) = \int_{\gamma|[0,t]} \omega \ + \ \theta_0.$$

Since, locally, $\omega = \mathrm{d}\arctan(y/x)$, θ is an angular function for γ. Finally we observe that two angular functions, at a given time, are determinations of the same angle, so they differ, at that time, by an entire multiple of 2π. This multiple does not depend on the time since the difference of the two angular functions is, divided by 2π, an integers valued continuous function defined on a connected set, hence constant. ☐

5.11. Remark. The advantage of having angular functions is that we can write γ in *polar coordinates*

$$\gamma(t) = \|\gamma(t)\| \ e^{i\theta(t)} = \|\gamma(t)\| \ (\cos\theta(t), \sin\theta(t)).$$

Let $\gamma : [0,1] \longrightarrow U$ be a closed curve and θ an angular function. Since $\gamma(0) = \gamma(1)$, $\theta(1) - \theta(0)$ is an entire multiple of 2π.

5.12. Definition. The *winding number* of γ is the integer

$$w(\gamma) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

5.13. Remark. Since two angular functions differ by a multiple of 2π, the winding number does not depend on the particular angular function. Moreover

$$w(\gamma) = \frac{1}{2\pi}\int_\gamma \omega.$$

5.14. Example. Consider the curve $\xi_n(t) = (\cos 2\pi n t, \sin 2\pi n t)$, $t \in [0,1]$, n a given integer. Then $\theta(t) = 2\pi n t$ is an angular function and $w(\xi_n) = n$.

The main fact about winding numbers is the following

5.15. Theorem. [Homotopy classification] *Two piecewise smooth closed curves $\gamma_i : [0,1] \longrightarrow U$, $i = 0,1$, are freely homotopic if and only if they have the same winding number.*

Proof. If the two curves are freely homotopic, by Proposition 5.5 and Remark 5.13, they have the same winding number. Let γ be a piecewise smooth closed curve with angular function θ and winding number $w(\gamma) = n \in \mathbb{Z}$. Let ξ_n be as in Example 5.14. Define

$$H : [0,1] \times [0,1] \longrightarrow U, \quad H(t,s) = [s\|\gamma(t)\| + (1-s)](\cos(s\theta(t) + (1-s)2\pi nt), \sin(s\theta(t) + (1-s)2\pi nt)).$$

Then $H(t,0) = \xi_n(t),\ H(t,1) = \gamma(t)$ and the condition $w(\gamma) = n$ implies $H(0,s) = H(1,s), \quad \forall\ s \in [0,1]$. Hence H is a free homotopy between ξ_n and γ. This concludes the proof since the relation of being freely homotopic is an equivalence relation. ☐

5.16. Remark. Any *continuous* curve in U admits *continuous* angular functions. Once we have angular functions, we can define the winding number for a continuous closed curve. Theorem 5.15 holds true in this more general situation (see Exercise 7.15).

We will see now some applications of the homotopy invariance of the winding number.

Let $D^2(r) := \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ be the disk of radius r and $S^1(r) := \{x \in \mathbb{R}^2 : \|x\| = r\}$ be its boundary. Consider a smooth function[8] $f : D^2(r) \longrightarrow \mathbb{R}^2$. A basic question is to find solutions of the equation $f(x) = 0$. In the case of a function $f : [-r,r] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, the celebrated Theorem of Bolzano states that if $f(r)f(-r) < 0$ the equation has a solution. We will prove a similar result for our case, similar in the sense that we shall give a condition on f, at the boundary of the disk, that is be sufficient (but not necessary, in general) for the existence of solutions of our equation.

5.17. Definition. Let $f : D^2(r) \longrightarrow \mathbb{R}^2$ be a smooth function. Suppose $f(x) \neq 0$ if $\|x\| = r$. The degree of f, $dg(f)$, is defined as the winding number of the closed curve:

$$\gamma_f : [0,1] \longrightarrow U := \mathbb{R}^2 \setminus \{0\}, \qquad \gamma_f(t) = f(r(\cos 2\pi t, \sin 2\pi t))$$

.

5.18. Example. Consider the complex plane $\mathbb{C} \cong \mathbb{R}^2$ with complex variable $z = x + iy$, and the map $g(z) = z^n$. Then $\gamma_g(t) = r(\cos 2\pi nt, \sin 2\pi nt)$. Hence $dg(g) = n$.

The announced result is the following:

5.19. Theorem. *If $dg(f) \neq 0$ then the equation $f(x) = 0$ has a solution.*

Proof. Suppose $dg(f) \neq 0,\ f(x) \neq 0\ \forall\ x \in D^2(r)$. Consider the map

$$H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}, \qquad H(t,s) = f(sr(\cos 2\pi t, \sin 2\pi t)).$$

Since $f(x) \neq 0$, for $\|x\| \leq r$, H is a free homotopy, in $\mathbb{R}^2 \setminus \{0\}$, between γ_f and the constant curve $\alpha(t) = f(0)$. Therefore, by Theorem 5.15, $dg(f) := w(\gamma_f) = w(\alpha) = 0$, a contradiction. ☐

In order to compute degrees, the following fact is often useful

5.20. Lemma. [Lemma of Poincaré-Bohl] *Let $\gamma_i : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\},\ i = 0,1$ be two closed curves. If $\|\gamma_0(t) - \gamma_1(t)\| < \|\gamma_0(t)\| \quad \forall\ t \in [0,1]$, the two curves are freely homotopic.*

---

[8] By Remark 5.16 we only need continuity of the function.

Proof. Consider the map:

$$H : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2, \qquad H(t,s) = s\gamma_1(t) + (1-s)\gamma_0(t).$$

The condition $\|\gamma_0(t) - \gamma_1(t)\| < \|\gamma_0(t)\|$ implies that the segment joining $\gamma_0(t)$ and $\gamma_1(t)$ does not contain the origin. Hence $H([0,1] \times [0,1]) \subseteq \mathbb{R}^2 \setminus \{0\}$ and H is a free homotopy between the two curves. ☐

As an application of Theorem 5.19, we prove now the *Fundamental Theorem of Algebra*:

5.21. Theorem. *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ be a polynomial in the complex variable z. If $n \geq 1$, f has a complex root.*

Proof. Let $r > 1 + \sum_1^n |a_i|$. If $f(z) = 0$, for some $z \in S^1(r)$, there is nothing to prove. Suppose $f(z) \neq 0$ for $\|z\| = r$ and consider the function $g(z) = z^n$. For $\|z\| = r$ we have:

$$\|f(z) - g(z)\| \leq \sum_1^n |a_i| \|z\|^{n-i} < r^n = \|g(z)\|.$$

Hence, by Lemma 5.20, f and g have the same degree and $dg(g) = n \neq 0$, by Example 5.18. Hence, by Theorem 5.19, the polynomial f has a root in $D^2(r)$. ☐

5.22. Remark. We can take a slightly different approach to the winding number. Let $\gamma : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$ be a closed smooth curve. Then we can extend γ to a map

$$\Gamma : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^2 \setminus \{0\}, \qquad \Gamma(tx) = t\gamma(x), \quad x \in S^1.$$

Hence we have an induced map

$$\Gamma_* : H_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R} \longrightarrow H_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$$

which is multiplication by a real number, which is, as it is easily seen, the winding number of γ. In this context the winding number is also called the *degree* of γ and is denoted by $dg(\gamma)$. This point of view is useful in extending the concept to higher dimensions (see Exercise 7.21).

6. Appendix: baricentric subdivision and the proof of Theorem 2.13

Let $U = U_1 \cup U_2, U, U_i$ open sets in $\mathbb{R}^n$. We want to show that the inclusion $i : C_p(U_1 + U_2) \longrightarrow C_p(U)$ induces an isomorphism in homology. The idea of the proof goes as follow: consider a singular simplex $\sigma : \Delta^p \longrightarrow U$ and the covering of Δ^p given by $\sigma^{-1}U_i, i = 1, 2$. We "subdivide" Δ^p into sub simplices of very small diameter, smaller than the Lebesgue number of the covering, so that σ sends each one of the small simplecis into U_1 or U_2. With this operation we pass from a chain $c \in C_p(U)$ to a chain $\tilde{c} \in C_P(U_1 + U_2)$. Finally we must show that if c is a cycle, $\tilde{c}$ is also a cycle that represent, in $H_p(U)$, the same class of c and if $\tilde{c}$ is the boundary of a chain in $C_{p+1}(U)$ it is also the boundary of a chain in $C_{p+1}(U_1 + U_2)$. To formalize this idea we start introducing the concept of *baricentric subdivision*.

Let $\Gamma = [v_0, \ldots, v_p]$ be an (affine) p-simplex in $\mathbb{R}^n$.

7. Exercises

7.1. Let $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_p \in \Omega^p(\mathbb{R}^n)$ and Δ^p be the standard p-simplex. Show that

$$\int_{\Delta^p} \omega = \frac{1}{p!} \quad (= \text{volume of } \ \Delta^p).$$

7.2. Let $U, V \subseteq \mathbb{R}^n$ be connected open sets and $F : U \longrightarrow V$ be a diffeomorphism. Let $D \subseteq U$ be the closure of a bounded open set and $f : V \longrightarrow \mathbb{R}$ a smooth function. The change of variables Theorem for multiple integrals states that:

$$\int_{F(D)} f(y_1, \ldots, y_n)\mathrm{d}y_1 \cdots \mathrm{d}y_n = \int_D f(F(x_1, \ldots, x_n))|\det(\mathrm{d}F)|\mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

Let $\omega \in \Omega^n(V)$. Prove that:

$$\int_{F(D)} \omega = \pm \int_D F^*\omega,$$

with the sign $+$ (resp. $-$) if F preserves (resp. reverse) the orientation, i.e. $\det(\mathrm{d}F) > 0$ (resp. $\det(\mathrm{d}F) < 0$).

7.3. Let $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a continuous map. Prove that if U (resp. V) is contractible, then F is homotopic to a constant map.

7.4. Let $D^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$, $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} = \partial D^{n+1}$ and $V \subseteq \mathbb{R}^m$. Show that a continuous map $F : S^n \longrightarrow V$ is continuously homotopic to a constant map if and only if it extends to a continuous map $\tilde{F} : D^{n+1} \longrightarrow V$.

7.5. Prove that an open set $U \subseteq \mathbb{R}^n$ is connected if and only if $H_0(U) \cong \mathbb{R}$ (see Example 2.9).

7.6. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open sets and $F : U \longrightarrow V$ a smooth map. Prove that if U is connected, $F_* : H_0(U) \longrightarrow H_0(V)$ is injective. Study the case in which U is not connected (see Example 2.9).

7.7. For an open set $U \subseteq \mathbb{R}^n$ define the reduced homology, $\tilde{H}_p(U)$, as the homology of the *augmented chain complex*

$$\cdots \longrightarrow C_p(U) \longrightarrow C_{p-1}(U) \longrightarrow \cdots \longrightarrow C_0(U) \longrightarrow \mathbb{R} \longrightarrow \{0\},$$

where the last map sends any singular 0-simplex to $1 \in \mathbb{R}$ and is extended by linearity (the other maps are the usual boundaries). Find the relation between $H_p(U)$ and $\tilde{H}_p(U)$ and prove the homotopy invariance and the exactness of Mayer-Vietoris sequence for reduced homology.

7.8. Compute the homology of $\Sigma_n \subseteq \mathbb{R}^n$ using the Mayer Vietoris sequence for reduced homology (see Example 4.15 of Chapter 1 for the definition of Σ_n).

7.9. Let $U \subseteq \mathbb{R}^n$ be an open set and $p \in U$. Assume known that $H_n(U) = \{0\}$ (see Remark in Exercise 7.30 of Chapter 1). Find the relation between $H_k(U \setminus \{p\})$ and of $H_k(U)$.

7.10. Prove the claim made in Remak 3.4 that the homology of the complex of continuous singular simplices is isomorphic to the homology of the complex of the smooth singular simplices.

7.11. Give a proof, without using the Theorem of de Rham, of Proposition 5.5.

7.12. Let $U \subseteq \mathbb{R}^m$ be an open set and let $\alpha, \beta : [0,1] \longrightarrow V$ be continuous curves with $\alpha(1) = \beta(0)$. Define $\alpha * \beta$ as

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{se } 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \text{se } \frac{1}{2} \leq t \leq 1 \end{cases}$$

and $\alpha^{-1}(t) = \alpha(1-t)$. Assuming that the products below are well defined. Prove that

1. If $\alpha_1 \sim \beta_1, \quad \alpha_2 \sim \beta_2$, then $\alpha_1 * \alpha_2 \sim \beta_1 * \beta_2$.
2. $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$.
3. $\alpha * \epsilon_p \sim \alpha \sim \epsilon_p * \alpha$.
4. $\alpha * \alpha^{-1} \sim \epsilon_p \sim \alpha^{-1} * \alpha$.

where the homotopies are relative to the endpoints and ϵ_p is the constant path $\epsilon_p(t) = p$.
Hint: consider the homotopies

(1) If H_i are homotopies between α_i e β_i,

$$H(t,s) = \begin{cases} H_1(2t,s) & \text{se } 0 \leq t \leq \frac{1}{2} \\ H_2(2t-1,s) & \text{se } \frac{1}{2} \leq t \leq 1 \end{cases}$$

(2)

$$H(t,s) = \begin{cases} \alpha(\frac{4t}{s+1}) & \text{se } 0 \leq 4t \leq \ s+1 \\ \beta(4t-s-1) & \text{se } s+1 \leq 4t \leq s+2 \\ \gamma(\frac{4t-s-2}{2-s} & \text{se } s+2 \leq 4t \leq 4 \end{cases}$$

(3)

$$H(t,s) = \begin{cases} \epsilon_p(t) & \text{se } 0 \leq 2t \leq 1-s \\ \alpha(\frac{2t-1+s}{1+s}) & \text{se } 1-s \leq 2t \leq 2 \end{cases}$$

(4)

$$H(t,s) = \begin{cases} \alpha(2t) & \text{se } 0 \leq t \leq \frac{1-s}{2} \\ \alpha(1-s) & \text{se } \frac{1-s}{2} \leq t \leq \frac{1+s}{2} \\ \alpha^{-1}(2t-1) & \text{se } \frac{1+s}{2} \leq t \leq 1 \end{cases}$$

Give a “geometric” interpretation of the homotopies above.

7.13. Let $U \subseteq \mathbb{R}^n$ be an open set and $p \in U$. Consider the set $\Omega(U,p) = \{\gamma : [0,1] \longrightarrow U : \gamma \text{ is continuous and } \gamma(0) = \gamma(1) = p\}$. Prove that $*$ induces a group structure on the quotient set $\pi_1(U,p) := \Omega(U,p)$ modulo the equivalence relation $\alpha \sim \beta$ if and only if α, β are homotopic relative to the endpoints.

Remark. With this structure, $\pi_1(U)$ is called the *fundamental group* of U with respect to p.

7.14. Prove that an open set $U \subseteq \mathbb{R}^n$ is simply connected if and only if any two curves $\gamma_i : [0,1] \longrightarrow U, \ i = 0,1$ with with the same endpoints are homotopic relative to the endpoints.

7.15. Prove that any *continuous* curve $\gamma : [a,b] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ admits angular functions (hint: use polar coordinates to prove the claim when the image of γ is contained in a half plane. Then...). Extend Theorem 5.15, Definition 5.17 and Theorem 5.19 to the case of continuous functions.

7.16. Let $\gamma : S^1 \longrightarrow R^2 \setminus \{0\}$ be an *odd* closed curve, i.e. $\gamma(-t) = -\gamma(t), \quad t \in S^1$. Prove that $w(\gamma)$ is odd.

7.17. Prove the following Theorem of Borsuk: if $f, g : S^2 \longrightarrow \mathbb{R}$ are *odd* continuous functions, there exists $p \in S^2$ such that $f(p) = 0 = g(p)$ (hint: use the projection of the closed upper hemisphere onto the unit disk to define a function of the disk in $\mathbb{R}^2$).

7.18. Let $f, g : S^2 \longrightarrow \mathbb{R}$ be continuous functions. Prove that there exists $p \in S^2$ such that $f(p) = f(-p), \ g(p) = g(-p)$.

7.19. Prove that there are no injective continuous functions $F : S^2 \longrightarrow \mathbb{R}^2$.

7.20. Let $\omega = a(x,y)\mathrm{d}x + b(x,y)\mathrm{d}y$ be a smooth closed 1-form in $\mathbb{R}^2 \setminus \{0\}$. Suppose that, for $0 < x^2 + y^2 \leq K$, the functions a, b are bounded. Prove that ω is exact (hint: use homotopy invariance to show that for all closed curves $\gamma : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $\int_\gamma \omega = 0$).

7.21. Let $F : S^n \longrightarrow S^n$ be a smooth function and $\tilde{F} : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}, \ \tilde{F}(tx) = tF(x)$. Then we have an induced linear map $\tilde{F}_* : H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong \mathbb{R} \longrightarrow H_n(\mathbb{R}^{n+1} \setminus \{0\}) \cong \mathbb{R}$. This map is multiplication by a real number $dg(F)$, called the *degree* of F. It is known that $dg(F) \in \mathbb{Z}$ [9] . Let D^{n+1} be the unit disk and $G : D^{n+1} \longrightarrow \mathbb{R}^{n+1}$ a smooth function not vanishing on the unit sphere $S^n = \partial D^{n+1}$. Then the degree of G, $dg(G)$, is defined as the degree of the map $\tilde{G}(x) = \frac{G(x)}{\|G(x)\|}$. Prove that, if $dg(G) \neq 0$, then the equation $G(x) = 0$ has a solution.

7.22. Prove that there are no smooth maps $F : D^{n+1} \longrightarrow S^n = \partial D^{n+1}$ such that $F(x) = x \ \ \forall x \in S^n$. Use this fact to prove the celebrated *Brouwer fixed point Theorem*: any continuous map $G : D^{n+1} \longrightarrow D^{n+1}$ has a fixed point, i.e. a point $x \in D^{n+1}$ such that $G(x) = x$ (hint for the Brouwer fixed point Theorem: suppose $G(x) \neq x \ \ \forall \, x \in D^{n+1}$. For $x \ D^{n+1}$ consider the ray starting at $G(x)$ containing x and define $F(x)$ to be the intersection of this half line with S^n. Then ...).

7.23. Let $L : E_1 \longrightarrow \mathbb{E}_2$ be a linear map and let $\mathbb{F}$ be a given vector space. Prove that $\ker(L \otimes \mathbb{1}) = \ker(L) \otimes \mathbb{F}$ and $\mathrm{Im}(L \otimes \mathbb{1}) = \mathrm{Im}(L) \otimes \mathbb{F}$. Prove Proposition 4.5.

7.24. Compute the cohomology of $\mathbb{R}^n \setminus \{0\} \times \mathbb{R}^m \setminus \{0\}$.

7.25. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ be a Jordan curve. Then, by Theorem 5.8 of Chapter 1, $\mathbb{R}^2 \setminus \gamma([0,1])$ has two connected components. It is easily seen that one component is bounded and the other is unbounded. Assume the following (non trivial) Theorem

THEOREM. *The bounded component of* $\mathbb{R}^2 \setminus \gamma([0,1])$ *is homeomorphic to a disk.*

Let $U \subseteq \mathbb{R}^2$ be an open set such that $H^1(U) = \{0\}$. Prove that any smooth Jordan curve $\gamma : S^1 \longrightarrow U$ is homotopic, in U, to a constant curve (hint: by the Theorem above, $\gamma(S^1)$ is the boundary of a disk in $\mathbb{R}^2$. If the disk is contained in U, the curve is contractible by Exercise 7.4. If not, use the angle form to get a contradiction).

REMARK: This fact implies that U is simply connected (see Remark 5.8), and, by the Riemann mapping theorem, U is diffeomorphic to $\mathbb{R}^2$.

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[9] It follows, from homotopy invariance, that homotopic maps have the same degree. A basic fact in homotopy theory is the Theorem of Hopf: if two maps from S^n to S^n have the same degree, then they are homotopic.

7.26. Let $U \subseteq \mathbb{R}^2$ be an open set and $X : U \longrightarrow \mathbb{R}^2$ a smooth vector field. Let $D_\epsilon \subseteq U$ be a disk of radius ϵ, with center $p \in U$, and assume that $X(q) \neq 0$, $\forall q \in D_\epsilon \setminus \{p\}$. The point p is called an (*isolated*) *singularity* of X. The *index* of X at p, $i(X, p)$, is defined as the degree of $X_{|D_\epsilon}$, i.e. the winding number of the curve $X(p + \epsilon \cos 2\pi t, p + \epsilon \sin 2\pi t)$, $t \in [0, 1]$.

(1) Let $\gamma : [0, 1] \longrightarrow U$ be a piecewise smooth, positively oriented closed Jordan curve bounding a disk in U containing p in its interior. Prove that $i(X, p)$ is the winding number of $X \circ \gamma$.

(2) If $X(x, y) = (f(x, y), g(x, y))$, prove that

$$i(x, p) = \frac{1}{2\pi} \int_\gamma \theta,$$

where γ is as in the preceding item and

$$\theta = \frac{-g\mathrm{d}x}{f^2 + g^2} + \frac{f\mathrm{d}y}{f^2 + g^2} = X^*\omega,$$

where ω is the angle form.

(3) Prove that if $X(p) \neq 0$, then $i(X, p) = 0$.

(4) Let $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear isomorphism. Prove that $i(x, 0) = 1$ if $\det X > 0$ and $i(x, 0) = -1$ if $\det X < 0$.

(5) Assume that $X(p) = 0$ and $\mathrm{d}X(p)$ is invertible. In this case we say that p is a *simple* singularity of X, positive, if $\det \mathrm{d}X(p) > 0$, negative otherwise. Prove that a simple singularity is isolated and $i(X, p) = \pm 1$, depending on whether p is a positive or negative simple singularity (hint: by Taylor's formula $X(q) = \mathrm{d}X(0)(q) + R(q)\|q\|$, with $\lim_{q \to 0} R(q) = 0$. Prove that $H(q, s) = \mathrm{d}X(0)(q) + (1 - t)R(q)\|q\| \neq 0$, if $\|q\|$ is sufficiently small. Hence...).

(6) Prove the following formula, called the *Kronecker formula.*
Let $D \subseteq \mathbb{R}^2$ be a closed disk, with center q and radius r, and $X : D \longrightarrow \mathbb{R}^2$ be a vector field with only simple singularities, none of which is in ∂D. Then

$$\frac{1}{2\pi} \int_\gamma \theta = P - N,$$

where $\gamma(t) = p + r(\cos 2\pi t, \sin 2\pi t)$, P is the number of the positive singularities and N the number of the negative ones.

Remark: The condition $i(X, p) = 0$ does not imply $X(p) \neq 0$ (find an example!). However, if $i(X, p) = 0$, given $\epsilon > 0$, we can find a vector field $\tilde{X}$ which coincides with X outside a disk of radius ϵ and center p and has no zeros in that disk.

7.27. Let $f : U \subseteq \mathbb{C} = \mathbb{R}^2 \longrightarrow \mathbb{C}$ be a holomorphic function (see Exercise 7.31 of Chapter 1), $f = u + iv$.

(1) Prove the following Theorem:
Theorem: [Cauchy] If U is simply connected and $\gamma : S^1 \longrightarrow U$ is a closed piecewise smooth curve then

$$\int_\gamma f(z)\mathrm{d}z := \int_\gamma (u\mathrm{d}x - v\mathrm{d}y) + i \int_\gamma (u\mathrm{d}y + v\mathrm{d}x) = 0.$$

(2) Suppose that $f'(z) \neq 0$ for z in a disk $D \subseteq U$ and $f(z) \neq 0$ for $z \in \partial U$. Prove that the number of zeros in D is given by

$$\frac{1}{2\pi i} \int_{\partial D} \frac{\mathrm{d}f}{f}$$

(hint: prove that the singularities of the vector field $X(x, y) = (u(x, y), (v(x, y))$ are all simple and positive. Then....).

7.28. Use Exercise 7.21 to define the index of a vector field $X : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ at a point $p \in U$ and try to extend, as much as you can, the facts claimed in Exercise 7.26 for this situation.

CHAPTER 3

A short course on differentiable manifolds

The concept of a differentiable manifold is a natural extension of the concept of a regular surface in $\mathbb{R}^3$ or, more generally, of a submanifold of $\mathbb{R}^N$. In this Chapter we will recall, briefly, some basic facts about submanifolds of $\mathbb{R}^N$, introduce (abstract) differentiable manifolds, as a natural extension of the concept of submanifold, and discuss the basic facts of the theory of differentiable manifolds.

We will use the words *smooth* and *differentiable* as synonymous of C^∞.

1. Submanifolds of Euclidean spaces

1.1. Definition. An *n-dimensional submanifold of* $\mathbb{R}^N$ is a subset $M^n \subseteq \mathbb{R}^N$, such that $\forall p \in M$ there exist an open set $U \subseteq \mathbb{R}^N$, an open set $A \subseteq \mathbb{R}^n$ and a smooth map $\phi : U \longrightarrow \mathbb{R}^N,\ p \in \phi(U)$, such that:

(1) ϕ is a homeomorphism of U onto $A \cap M$, i.e. ϕ has a continuous inverse $\phi^{-1} : A \cap M \longrightarrow U$,

(2) $\forall\ x \in U,\ \mathrm{d}\phi(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is injective.

- The integer n is called the *dimension* of M^n.
- The map ϕ, or better the pair (U, ϕ), is called a (local regular) *parametrization* or a (local) *chart* and, if $p \in \phi(U),\ \ (x_1, \dots, x_n) = \phi^{-1}(p)$ are called the *local coordinates of* p in the chart (U, ϕ).
- A set of regular parameterizations whose images cover M will be called an *atlas*.

1.2. Remark. It may appear, at first, that the first condition in the definition is, at least locally, a consequence of the second one. In fact it follows from Theorem 1.26 of Chapter 0 that there is a neighborhood U' of x and a local diffeomorphism Φ of a neighborhood of $\phi(x)$ such that $\Phi \circ \phi(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0)\ \ \forall x \in U'$. The point is that the image of U' may not be the intersection of M with an open set of $\mathbb{R}^N$. For example the set $M = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ may be covered with images of maps with injective differentials but it *is not* a submanifold of $\mathbb{R}^2$, since the intersection of an open set containing $(0, 0)$ with M is not homeomorphic to an open interval. We will came back to this question later on, when we will talk about immersions and embeddings of manifolds (see Remark 4.2).

1.3. Definition. If $\phi_i : U_i \subseteq \mathbb{R}^n \longrightarrow M,\ i = 1, 2$ are regular parameterizations of M, the map $\phi_2^{-1} \circ \phi_1$, defined in $\phi_1^{-1}(\phi_1(U_1) \cap \phi_2(U_2))$, is called a *change of coordinates*.

It follows easily from Theorem 1.26 of Chapter 0 that, for a local regular parametrization ϕ, ϕ^{-1} is, locally, the restriction of a smooth map defined on an open set of $\mathbb{R}^N$. Therefore

1.4. Lemma. *If* $\phi_i : U_i \subseteq \mathbb{R}^n \longrightarrow M,\ i = 1, 2$ *are regular parameterizations of* M*, the change of coordinates* $\phi_2^{-1} \circ \phi_1$ *is a diffeomorphism from its domain onto its image.*

1.5. Definition. Let M be a submanifold of $\mathbb{R}^N$, $p \in M$ and $\phi : U \longrightarrow M$ be a regular parametrization with image in a neighborhood of p. Set $x = \phi^{-1}(p)$. The n-dimensional vector space

$$T_pM := (\mathrm{d}\phi)(x)(\mathbb{R}^n) \subseteq \mathbb{R}^N,$$

is called the *tangent space of* M *at* p.

1.6. Remark. It is easy to see that the definition does not depend on the choice of the local chart.

Observe that, in general, $p \notin T_pM$. The "*geometric tangent space*" is the affine subspace $p + \mathrm{d}\phi(x)(\mathbb{R}^n)$.

1.7. Definition. Let M be a submanifold of $\mathbb{R}^N$ and $F : M \longrightarrow \mathbb{R}^K$ a function. We will say that F is *smooth* if, $\forall\ p \in M$, there exists a regular parametrization $\phi : U \subset \mathbb{R}^n \longrightarrow M$ of a neighborhood of p in M, such that the function $\phi \circ F : U \longrightarrow \mathbb{R}^K$ is smooth[1]. Moreover

- A map $F : M \subseteq \mathbb{R}^n \longrightarrow N \subseteq \mathbb{R}^K$ between submanifolds is *smooth* if it is smooth as a map into $\mathbb{R}^K$.
- A smooth map $F : M \longrightarrow N$ is a *diffeomorphism* if it has a smooth inverse.

1.8. Remark. It follows from Lemma 1.4 that the definition of smooth maps does not depend on the parametrization.

1.9. Remark. We can give an "intrinsic" definition of smooth maps between submanifolds that will be useful later on. In fact, as it is easily seen, the definition above is equivalent to the following one

1.10. Definition. F is smooth if and only if, given regular parameterizations $\phi : U \longrightarrow M,\ \ \psi : \Sigma \longrightarrow N$ of neighborhoods of p and $f(p)$, the map $\psi^{-1} \circ F \circ \phi$ is smooth, where defined.

1.11. Remark. If $U \subseteq M$ is an open set, $p \in U$, we can consider the algebra of smooth real valued functions defined on U, $\mathcal{F}(U)$, the algebra of germs at $p \in U$, $\mathcal{F}_p$, and their derivations, $\mathcal{D}er(U)$, $\mathcal{D}_p$ (see Chapter 0, Definitions 3.15 and 3.19). At this point we can look at the tangent space in three ways

- As in the definition: $T_pM = \mathrm{d}\phi(x)(\mathbb{R}^n) \subseteq \mathbb{R}^N,\ p = \phi(x)$. A basis for this space is given by the vectors

$$(d\phi)(x)(e_i) = \frac{\partial \phi}{\partial x_i}(x) := \phi_{x_i}(x), \quad i = 1, \ldots, n.$$

We will call those vectors the *coordinate tangent vectors.*

- $T_pM = \{\dot\gamma(0) : \gamma : (-\epsilon, \epsilon) \longrightarrow M \subseteq \mathbb{R}^N$ is a smooth curve with $\gamma(0) = p\}$.
- As the space $\mathcal{D}_p$ of derivations of the algebra of germs at p. Given a regular parametrization $\phi : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$ of a neighborhood of p and a smooth function $f : U \longrightarrow \mathbb{R}$, we define the derivations

$$\frac{\partial}{\partial x_i}(p)(f) = \frac{\partial f \circ \phi}{\partial x_i}(\phi^{-1}(p)), \quad i = 1, \ldots n.$$

These derivations are a basis of $\mathcal{D}_p$.

We leave to the reader the task of proving the equivalence of these definitions (Exercise 10.2).

We shall define the *differential* of a smooth map.

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[1] This is equivalent, again by Theorem 1.26, to the fact that $\forall\ p \in M$ there exist a smooth extension of F to an open neighborhood of $p \in \mathbb{R}^N$.

1.12. Definition. Let $M,\ N$ be submanifolds of $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively and $F : M \longrightarrow N$ a smooth map. We define the *differential* of F at p

$$\mathrm{d}F(p) : T_pM \longrightarrow T_{F(p)}N, \qquad \mathrm{d}F(p)(\mathrm{d}\phi(\phi^{-1}(p))(v) = \mathrm{d}\psi(\psi^{-1}(F(p))\mathrm{d}(\psi^{-1} \circ F \circ \phi)(\phi^{-1}(p))(v)$$

where $\phi : U \longrightarrow M,\ \ \psi : \Sigma \longrightarrow N$ are regular parameterizations of neighborhoods of p and $F(p)$ respectively and $v \in \mathbb{R}^n$. It is easily seen that the definition does not depend on the choice of the charts.

1.13. Remark. Since we have alternative definitions of T_pM (see Remark 1.11) it is convenient to have alternative definitions of $\mathrm{d}F(p)$.

- If we look at T_pM as the space of tangent vectors to smooth curves $\gamma : (-\epsilon, \epsilon) \longrightarrow M, \gamma(0) = p$,

$$\mathrm{d}F(p)(\dot\gamma(0)) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F \circ \gamma(t).$$

- If we look at T_pM as derivations, we have

$$\mathrm{d}F(p)(X_p)(g) = X_p(F \circ g).$$

We leave to the reader the task of proving the equivalences.

We will discuss now a few examples.

1.14. Example. If M is a submanifold of $\mathbb{R}^N$, any open subset U of M is a submanifold of $\mathbb{R}^N$ (of the same dimension) and $T_pU = T_pM$.

1.15. Example. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ a smooth function. Then

$$\phi : U \longrightarrow \mathbb{R}^{n+1}, \qquad \phi(x) = (x_1, \dots, x_n, f(x)), \qquad x = (x_1, \dots, x_n) \in U,$$

is a regular parametrization of the graph of the function f, $\Gamma(f) = \{(x, f(x)) : x \in U\} \subseteq \mathbb{R}^{n+1}$. The vectors $\{(e_i, \dfrac{\partial f}{\partial x_i}(x)) \in \mathbb{R}^n \times \mathbb{R}\}$, where $\{e_i\}$ is the canonical bases of $\mathbb{R}^n$, span the tangent space at $(x, f(x))$.

1.16. Example. If $M_i^{n_i} \subseteq \mathbb{R}^{N_i},\ i = 1, 2$ are two submanifolds, then $M_1^{n_1} \times M_2^{n_1} \subseteq \mathbb{R}^{N_1+N_2}$ is a submanifold. Local charts for $M_1^{n_1} \times M_2^{n_1}$ are given by products of local charts for the M_i's. The tangent space at (x, y) is isomorphic to $T_xM_1 \times T_yM_2$.

1.17. Example. Consider $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. We shall to show that S^n is a submanifold. Consider a point $x \in S^n$. Up to an orthogonal transformation we can suppose $x = e_{n+1}$, the last vector of the canonical bases. Then the neighborhood $U = S^n \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ is parameterized by

$$\phi : B(1) = \{y \in \mathbb{R}^n : \|y\| < 1\} \longrightarrow U, \qquad \phi(y_1, \dots, y_n) = (y_1, \dots y_n, (1 - \sum y_i^2)^{\frac{1}{2}}).$$

For any other point $x \in S^n$ we can define a parametrization $\phi_x = A_x \circ \phi$ where A_x is an orthogonal operator such that $A_x(e_{n+1}) = x$. Observe that, in order to cover S^n, we need at least $2(n+1)$ of such parameterizations.

A more economic way of parameterize S^n is via the stereographic projections. The stereographic projection from the point $e_{n+1}, \pi_{e_{n+1}} : S^n \setminus \{e_{n+1}\} \longrightarrow \mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : \langle x, e_{n+1}\rangle = 0\}$ is the map that associate to $x \in S^n \setminus \{e_{n+1}\}$ the point of intersection of the straight line $tx + (1-t)e_{n+1}$ with $\mathbb{R}^n$. The condition $\langle tx + (1-t)e_{n+1}, e_{n+1}\rangle = 0$ gives

$$\pi_{e_{n+1}}(x) = [1 - \langle x, e_{n+1}\rangle]^{-1}x - \langle x, e_{n+1}\rangle[1 - \langle x, e_{n+1}\rangle]^{-1}e_{n+1},$$

whose inverse is given by

$$\pi^{-1}_{e_{n+1}}(y) = 2[1 + \|y\|^2]^{-1}y + [\|y\|^2 - 1][1 + \|y\|^2]^{-1}e_{n+1}.$$

Clearly $\pi^{-1}_{e_{n+1}} : \mathbb{R}^n \longrightarrow S^n \setminus \{e_{n+1}\}$ is a regular parametrization. Taking the inverse of the stereographic projection from $-e_{n+1}$ we get an other regular parametrization, and the two parametrization are an atlas for S^n.

Let $x \in S^n$ and $\gamma : (-\epsilon, \epsilon) \longrightarrow S^n$ be a smooth curve with $\gamma(0) = x$. Differentiating the identity $\langle\gamma(t), \gamma(t)\rangle = 1$ at $t = 0$, we get $\langle x, \dot\gamma(0)\rangle = 0$. It follows that

$$T_xS^n = x^\perp := \{v \in \mathbb{R}^{n+1} : \langle x, v\rangle = 0\}.$$

1.18. Example. Let $M \subseteq \mathbb{R}^N$ be an n-dimensional submanifold. Consider the *tangent bundle*

$$TM := \{(p, X) \in M \times \mathbb{R}^N : X \in T_pM\} \subseteq \mathbb{R}^N \times \mathbb{R}^N.$$

This is a $2n$-dimensional submanifold of $\mathbb{R}^{2N}$. Il fact, if $\phi : U \subseteq \mathbb{R}^n \longrightarrow M$ is a regular parametrization,

$$\Phi : U \times \mathbb{R}^n \longrightarrow TM, \qquad \Phi(x_1, \dots, x_n, t_1, \dots, t_n) = (\phi(x), \sum_1^n t_i\phi_{x_i}) \in \mathbb{R}^N \times \mathbb{R}^N,$$

is a regular parametrization for TM. If $\{(U_i, \phi_i)\}$ is an atlas for M, the corresponding parameterizations $\{(U_i \times \mathbb{R}^n, \Phi_i)\}$ form an atlas for TM. We also observe that the tangent bundle comes with a natural projection $\pi : TM \longrightarrow M$, $\pi(p, X) = p$, which is a smooth map. TM is a prototype of what is called a *(smooth) vector bundle.* Although locally TM is diffeomorphic to a product, it is not so globally, in general, as we will see along those notes.

1.19. Example. A similar example is the *normal bundle* of a submanifold $M \subseteq \mathbb{R}^N$. It is defined as

$$\nu M = \{(p, \xi) \in M \times \mathbb{R}^N : \xi \in [T_pM]^\perp\}.$$

We want to produce regular parameterizations for νM. Let $\phi : U \longrightarrow M$, $\phi(x) = p$, be a parameterizations of a neighborhood of $p \in M$.

Claim. There is a neighborhood $U' \subseteq U$ of x and smooth functions $\xi_1, \dots, \xi_{N-n} : U' \longrightarrow \mathbb{R}^N$, such that the $\xi_i(y)$' span $[T_{\phi(y)}M]^\perp$, $\forall\, y \in U'$.

Proof. The maps ϕ_{x_i} are smooth maps that span $T_{\phi(y)}M$, $\forall\, y \in U$. Using the orthonormalization process we can find smooth maps $X_i : U \longrightarrow \mathbb{R}^n$ such that $\langle X_i(y), X_j(y)\rangle = \delta_{ij}$ and they span $T_{\phi(y)}M$, $\forall\, y \in U$. Fix now a basis $\{\xi_i\}$ of $[T_{\phi(x)}M]^\perp$ and define $\xi_i(y) = \xi_i - \sum_j\langle\xi_i, X_j(y)\rangle X_j(y)$. Clearly $\xi_i(y) \in [T_{\phi(y)}M]^\perp$. Since $\xi_i(x) = \xi_i$, they are linearly independent at x, hence in a neighborhood $U' \subseteq U$ of x. $\square$

At this point we define local parameterizations (with the notations of the Claim) by

$$\Phi : U' \times \mathbb{R}^{N-n} \longrightarrow \nu M, \qquad \Phi(y, t_1, \dots, t_{N-n}) = (\phi(y), \sum t_j\xi_j(y)).$$

Also the normal bundle cames with a natural projection $\pi : \nu M \longrightarrow M$ which is a smooth map.

1.20. EXAMPLE. We will discuss now an example that will be useful later on. Consider the space $M(p,n,\mathbb{R})$ of $p\times n$ matrices with real entries, and its natural identification with $\mathbb{R}^{pn}$. Consider the subspace $M(p,n;k)$ of the matrices of rank k. We want to show that $M(p,n;k)$ is a submanifold of $M(p,n,\mathbb{R})$. Let

$$E_0 = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$$

be in $M(p,n;k)$. We will suppose initially that $A_0 \in GL(k,\mathbb{R})$, i.e. $\det A_0 \neq 0$. In particular there is $\epsilon > 0$ such that, if $\|A - A_0\| < \epsilon,\ A \in Gl(k,\mathbb{R})$. Set

$$U = \{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(p,n,\mathbb{R}) : \|A - A_0\| < \epsilon \}.$$

U is an open neighborhood of E_0 in $M(p,n,\mathbb{R})$.

<u>CLAIM.</u> If $A \in GL(k)$, then $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(p,n;k)$ if and only if $D = CA^{-1}B$.

PROOF. Consider

$$\begin{bmatrix} 1\!\!1 & 0 \\ X & 1\!\!1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA + C & XB + D \end{bmatrix}$$

Since the first matrix is invertible,

$$\mathrm{rank} E = \mathrm{rank} \begin{bmatrix} A & B \\ XA + C & XB + D \end{bmatrix}.$$

Take now $X = -CA^{-1}$. The matrix at the right hand side become

$$\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Hence E has rank k if and only if $CA^{-1}B - D = 0$. □

We exhibit now a (local) regular parametrization of a neighborhood of E_0. Identify $\mathbb{R}^{k(p+n-k)}$ with the space of matrices of the type $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, and consider the open subset $U = \{ \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} : \|A - A_0\| < \epsilon \}$. Consider the map

$$\phi : U \longrightarrow M(p,n,\mathbb{R}), \quad \phi\left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) = \begin{bmatrix} A & B \\ C & CA^{-1}B \end{bmatrix}.$$

It is easily seen that ϕ is a homeomorphism onto its image. Moreover

$$\mathrm{d}\phi \left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) \left(\begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix} \right) = \begin{bmatrix} X & Y \\ Z & * \end{bmatrix}.$$

Hence $\mathrm{d}\phi$ is injective ϕ is a regular parametrization with image $U \cap M(p,n;k)$.

In the general case we can proceed in the following way. Let $E \in M(p,n;k)$. Then there exists a submatrix $A \in GL(k)$. By suitably permutations of rows and columns, which do not affect the rank, we can construct an isomorphism L of $M(p,n,\mathbb{R})$ in itself that takes E onto a matrix $L(E) \in M(p,n;k)$ such that the upper left block is in $GL(k)$. Then, by the argument above, we have a regular parametrization of a neighborhood of $L(E)$. Composing this parametrization with L^{-1} we obtain a regular parametrization of a neighborhood of E.

Hence $M(p,n;k)$ is a submanifold of $M(p,n,\mathbb{R})$ and its dimension is $k(p+n-k)$.

A useful tool for constructing examples of submanifolds is given by the following

1.21. Proposition. *Let $U \subseteq \mathbb{R}^N$ be an open set and $F : U \to \mathbb{R}^p$ a smooth map. Let $c \in \mathbb{R}^p$ be a regular value of F, i.e. $\mathrm{d}F(x)$ is surjective $\forall\, x \in F^{-1}(c)$. Then $M := F^{-1}(c)$ is a $(N-p)$-dimensional submanifold of $\mathbb{R}^N$ and $T_pM = \ker \mathrm{d}F(p)$.*

Proof. Let $p \in M$. We can suppose, without loss of generality, $p = 0,\ F(0) = 0$. By Theorem 1.27 there is a neighborhood of U of 0 and a local diffeomorphism $\Psi : U \longrightarrow \Psi(U) \subseteq \mathbb{R}^N$ such that $F \circ \Psi(x_i, \dots, x_N) = (x_1, \dots, x_{N-p})$. Then $\phi(x_1, \dots, x_{N-p}) = \Phi^{-1}(x_1, \dots, x_{N-p}, 0, \dots 0)$ is a regular parametrization of $U \cap f^{-1}(c)$. We also claim that $T_pM = \ker \mathrm{d}f(p)$. In fact, if $\gamma(-\epsilon, \epsilon) \longrightarrow M$ is a smooth curve with $\gamma(0) = p,\ f(\gamma(t)) = c$. Therefore $\mathrm{d}f(p)(\dot\gamma(0)) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(\gamma(t)) = 0$. Hence $T_pM \subseteq \ker \mathrm{d}f(p)^2$. Since both subspaces are $(N-p)$-dimensional, they agree. ☐

1.22. Example. Consider again the sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. Then $S^n = F^{-1}(1)$ where $F(x) = \|x\|^2$. Now 1 is a regular value of F. In fact, since $\mathrm{d}F(x)(v) = 2\langle x, v\rangle$, $\mathrm{d}F(x) : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is non zero, if $x \neq 0$, hence surjective. Finally $T_xS^n = \ker \mathrm{d}F(x) = x^\perp := \{v \in \mathbb{R}^{n+1} : \langle x, v\rangle = 0\}$.

1.23. Example. Let $M(n,\mathbb{R})$ be the space of $n \times n$ matrices with real entries, identified with $\mathbb{R}^{n^2}$. Consider the group of linear isometries of $\mathbb{R}^n$, $O(n) = \{A \in M(n,\mathbb{R}) : AA^t = \mathbb{1}\}$ (A^t being the transpose of A). We want to show that $O(n)$ is a submanifold of $M(n,\mathbb{R})$. It is natural to consider the function

$$F : M(n,\mathbb{R}) \longrightarrow M(n,\mathbb{R}), \qquad F(A) = AA^t.$$

Then $O(n) = F^{-1}(\mathbb{1})$. The first idea would be to show that $\mathbb{1}$ is a regular value of F. This is not the case since, if it were, $O(n)$ would be 0-dimensional. More formally, a simple computation gives

$$\mathrm{d}F(A)(B) = BA^t + AB^t.$$

Observe that $BA^t + AB^t$ is a symmetric matrix, so the image of $\mathrm{d}F(A)$ is not the whole $M(n,\mathbb{R})$. But this observation suggests a way out. First observe that $F(A)$ belongs to the subspace $Sym(n,\mathbb{R}) \subseteq M(n,\mathbb{R})$ of symmetric matrices. So we may consider F as a map with values in $Sym(n,\mathbb{R})$. As such, $\mathbb{1} \in Sym(n,\mathbb{R})$ is a regular value, since, given $A \in O(n)$ and $C \in Sym(n,\mathbb{R})$, $\mathrm{d}F(A)(B) = C$, for $B = \frac{1}{2}CA$, hence $\mathrm{d}F(A) : M(n,\mathbb{R}) \longrightarrow Sym(n,\mathbb{R})$ is surjective. Since $Sym(n,\mathbb{R})$ has dimension $n(n+1)/2$, $O(n)$ has dimension $n(n-1)/2$.

Finally since $\ker \mathrm{d}F(\mathbb{1})$ is the space $Skw(n,\mathbb{R})$ of antisymmetric matrices, $T_{\mathbb{1}}O(n) = Skw(n,\mathbb{R})$.

1.24. Example. Consider the set $V_{2,n} = \{(x,y) \in \mathbb{R}^n \oplus \mathbb{R}^n : \|x\| = 1 = \|y\|,\ \langle x, y\rangle = 0\}$. This set is called the *Stiefel manifold of 2-frame (orthonormal) in* $\mathbb{R}^n$. We will show that $V_{2,n}$ is a submanifold of $\mathbb{R}^{2n}$ of dimension $2n-3$. Observe that $V_{2,2} = O(2)$, which is two (disjoint) copies of S^1. Consider the function

$$F : \mathbb{R}^n \oplus \mathbb{R}^n \longrightarrow \mathbb{R}^3, \qquad F(x,y) = (\|x\|^2, \|y\|^2, \langle x, y\rangle).$$

Clearly $V_{2,n} = F^{-1}(1,1,0)$. We shall show that $(1,1,0)$ is a regular value of F. A simple computation gives

$$\mathrm{d}F(x,y)(X,Y) = (2\langle x, X\rangle,\ 2\langle y, Y\rangle,\ \langle X, y\rangle + \langle x, Y\rangle).$$

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[2] We are using the second definition of T_pM in Remark 1.11

Now, given $(x, y) \in V_{2,n}$, $(a, b, c) \in \mathbb{R}^3$, we have $\mathrm{d}F(x, y)(X, Y) = (a, b, c)$ for $X = ax/2 + cy/2$, $Y = by/2 + cx/2$. Then $(1, 1, 0)$ is a regular value for F and $V_{2,n}$ is a submanifold of dimension 2n - 3.

More in general, we can consider $V_{k,n} = \{(x_1, \ldots, x_k) \in \mathbb{R}^{kn} : \langle x_i, x_j\rangle = \delta_{ij}$. $V_{k,n}$ is a submanifold of $\mathbb{R}^{kn}$.

1.25. Definition. Let M be a submanifold of $\mathbb{R}^N$. A *smooth (tangent) vector field on* M is a smooth map $X : M \longrightarrow \mathbb{R}^N$ such that $X(p) \in T_pM$, $\forall\ p \in M$. We will denote by $\mathcal{H}(M)$ the real vector space of such vector fields on M.

1.26. Remark. If $W \subseteq M$ is an open set (in M), then W is a submanifold of $\mathbb{R}^n$ and we can talk about vector fields on W and of the restrictions of vector fields on M to W.

1.27. Remark. We can think of a tangent vector field as a smooth map $X : M \longrightarrow TM$, such that $\pi \circ X = \mathbb{1}_M$ (see Example 1.18). This point of view will be useful in the theory of (abstract) differentiable manifolds.

1.28. Example. Let M be a submanifold and $W \subseteq M$ an open subset, image of a regular parametrization $\phi : U \subseteq \mathbb{R}^n \longrightarrow W$. Then ϕ_{x_i}, $i = 1, \ldots, n$, are tangent vector fields on W. Since they are, at each point $p \in W$, a basis for $T_pM(= T_pW)$, any other vector field in $\mathcal{H}(W)$ may be written as $X = \sum_1^n X_i\phi_{x_i}$, where the $X_i : W \longrightarrow \mathbb{R}$ are smooth functions. If we think of tangent vectors as derivations we can write

$$X = \sum_1^n X_i \frac{\partial}{\partial x_i}.$$

The vector fields $\dfrac{\partial}{\partial x_i}$ are called *coordinate vector fields.*

As we have observed previously, the tangent bundle to a n-dimensional submanifold $M^n \subseteq \mathbb{R}^N$, may not be, globally, diffeomorphic to the product $M^n \times \mathbb{R}^n$.

1.29. Definition. A submanifold $M^n \subseteq \mathbb{R}^N$ is *parallelizable* if there exists a diffeomorphism $t : M^n \times \mathbb{R}^n \longrightarrow TM^n$ such that $t|\{p\} \times \mathbb{R}^n$ is a (linear) isomorphism onto T_pM^n. Such a diffeomorphism is called a *trivialization* of TM^n.

1.30. Proposition. *Let $M^n \subseteq \mathbb{R}^N$ be a submanifold. Them M^n is parallelizable if and only if there exist vector fields $X_1, \ldots, X_n \in \mathcal{H}(M^n)$ such that $\forall\ p \in M^n$, $X_1(p), \ldots, X_n(p)$ are linearly independent.*

Proof. Let $t : M^n \times \mathbb{R}^n \longrightarrow TM^n$ be a trivialization. Then $X_i(x) = t(x, e_i)$ are smooth vector fields linearly independent at each point. Conversely, given n vector fields, $X_1, \ldots, X_n$, linearly independent at each point, we define $t(x, t_1, \ldots, t_n) = \sum t_iX_i(x)$. t is clearly smooth, a bijection and $t|\{p\} \times \mathbb{R}^n$ is a linear isomorphism onto T_pM. We have just to show that t^{-1} is smooth. Let $\phi : U \subseteq \mathbb{R}^n \longrightarrow M$ be a local chart and $\Phi : U \times \mathbb{R}^n \longrightarrow TM$ be the associated local chart for TM as in Example 1.18. Then $\phi_{x_i} = \sum a_{ij}X_j$ for some smooth functions $a_{ij} : U \longrightarrow \mathbb{R}$. Therefore

$$t^{-1}\Phi(x, t_1, \ldots, t_n) = t^{-1}(\phi(x), \sum_i t_i\phi_{x_i}) = t^{-1}(\phi(x), \sum_{ij} t_ia_{ij}X_j) = (x, \sum_i t_ia_{i1}, \ldots \sum_i t_ia_{in}).$$

It follows that t^{-1} is smooth and this conclude the proof. □

1.31. Example. If M can be cover by only one regular parametrization, in particular if it is an open set of $\mathbb{R}^n$, then M is parallelizable.

1.32. Example. Consider the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$. As we have seen in Example 1.22, $T_xS^n = x^\perp = \{v \in \mathbb{R}^{n+1} : \langle x, v\rangle = 0\}$. Hence a vector field on S^n can be viewed as a smooth map $X : S^n \longrightarrow \mathbb{R}^{n+1}$ such that $\langle X(x), x\rangle = 0$. If $n = 2k - 1$ a nowhere zero vector field is given by

$$X(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}).$$

We will see later that, if $n = 2k$, every vector field has must have a zero. Hence, by Proposition 1.30, *even dimensional spheres are not parallelizable.* It turns out that the only parallelizable spheres of positive dimension are S^1, S^3, S^7. This is a highly non trivial result, connected with the existence of a division algebra structure[3] on $\mathbb{R}^{n+1}$ (see also the comments in Remark 2.13).

1.33. Example. Consider the submanifold $O(n) \subseteq \mathbb{R}^{n^2}$ (Example 1.23). We have seen that $T_{\mathbb{1}}O(n) \cong Skw(n, \mathbb{R})$. For $A \in O(n)$, consider the map $L_A : O(n) \longrightarrow O(n)$, $L_A(B) = AB$. Then L_A is a smooth map, being the restriction of a linear map, and, in fact, it is a diffeomorphism, with $[L_A]^{-1} = L_{A^{-1}}$. Let $\{E_1, \dots, E_k\}$, $k = n(n-1)/2$, be a basis of $T_{\mathbb{1}}O(n)$. The maps $\tilde{E}_i : O(n) \longrightarrow TO(n), \tilde{E}_i(A) = \mathrm{d}L_A(E_i)$ are smooth vector fields on $O(n)$ [4] and, since $\mathrm{d}L_A(\mathbb{1}) : T_{\mathbb{1}}O(n) \longrightarrow T_AO(n)$ is an isomorphism, they are linearly independent at each point. Therefore $O(n)$ is parallelizable [5].

2. Differentiable manifolds

The careful reader has probably noted that all we have done up to now depends, essentially, only on the fact that the change of coordinates are smooth functions. This observation leads to the concept of an abstract differentiable manifold, that we will introduce in this section.

In what follows we will always assume that the topological spaces involved are *Hausdorff and second countable.* We recall that a topological space is second countable if there is a countable bases[6] for the topology. Such spaces, as it easily seen, have the Lindeloff property: *any open covering contains a countable subcovering.* The reader more familiar with metric spaces may think of *separable* metric spaces (which are Hausdorff and second countable). This assumption is not restrictive (see Remark 3.3).

We start with a few definitions.

2.1. Definition. Let M be a topological space. A *topological atlas*, or simply an atlas, for M is a collection of open sets $U_\alpha \subseteq \mathbb{R}^n$ and continuous maps $\phi_\alpha : U_\alpha \longrightarrow M$ such that:

(1) $\phi_\alpha(U_\alpha)$ is open in M and ϕ_α is a homeomorphism onto its image,
(2) $\bigcup_\alpha \phi_\alpha(U_\alpha) = M$.

---

[3] An algebra A is a *division algebra* if the maps $L_a(x) = ax$ and $R_a(x) = xa$ are surjective, if $a \neq 0$. If the algebra has a multiplicative unit, this is equivalent to the fact that any non zero element is invertible.

[4] Observe that, since multiplication of matrices is bilinear, hence smooth, L_A is smooth also with respect to A.

[5] This is a particular case of the fact that a *Lie group* is parallelizable (see section 5).

[6] If X is a topological space, a bases for the topology of X is a family of open sets such that every open set of X is union of elements of the bases.

The maps $\phi_\alpha : U_\alpha \longrightarrow M$ are called *charts* or *local coordinates* and the maps $\phi_\beta^{-1} \circ \phi_\alpha$ (where defined) are the *change of coordinates.*

A topological space which admits an atlas is called a *topological manifold.*

2.2. DEFINITION. An atlas is a *differentiable* (or a *smooth*) atlas if the change of coordinates $\phi_\beta^{-1} \circ \phi_\alpha$ are smooth functions (where defined).

Two smooth atlases on M are said to be *equivalent* if their union is a smooth atlas.

An equivalence class of smooth atlases is called a *differentiable* or *smooth structure.*

2.3. REMARK. It is clear that each equivalence class of differentiable atlases contains a maximal element, the union of all atlases in the class. We can think that a differential structure is such a maximal atlas.

At this point we can give the desired definition.

2.4. DEFINITION. A *differentiable* or *smooth manifold* is a topological manifold together with a differentiable structure.

2.5. REMARK. A word on the topological conditions in the definition of smooth manifold. The Hausdorff condition is a natural one to avoid pathologies. It is clear that the existence of an atlas on M implies that M is a *locally* Hausdorff spaces, i.e. $\forall\ p \in M$ there exists an open neighborhood U of p which is an Hausdorff space (for example the image of a chart). However there are spaces that admit atlases which are not Hausdorff. A classical example is the *real line with two origins*, i.e. the quotient of $\mathbb{R} \times \{-1, 1\}$ modulo the equivalence relation $(t, \alpha) \sim (s, \beta) \iff t = s \neq 0$ or $(t, \alpha) = (s, \beta)$. In fact $M = \mathbb{R} \times \{-1, 1\}/\sim$, with the quotient topology[7], can be covered by the charts $\phi_\pm : \mathbb{R} \longrightarrow M$, $\phi_\pm(t) = [t, \pm 1]$[8]. But $[0, 1]$ and $[0, -1]$ can not be separated by disjoint open sets.

The second countability condition will allow us to construct partitions of unity, a basic tool in order to glue together locally defined objects to obtain a globally defined one.

Clearly submanifolds of $\mathbb{R}^N$ are example of smooth manifolds. We will carry on the program of extending to this context what we have done for submanifolds.

For differentiable manifolds we can copy Definition 1.10 in order to define smooth maps.

2.6. DEFINITION. Let M, N be a differentiable manifolds. A map $F : M \longrightarrow N$ is smooth if for every pair of chart $\phi_\alpha : U_\alpha \longrightarrow M$ and $\psi_\beta : \Sigma_\beta \longrightarrow N$, $\psi_\beta^{-1} \circ F \circ \phi_\alpha$ is smooth, where defined.

F is a *diffeomorphism* if it is smooth and has a smooth inverse.

2.7. REMARK. Since change of coordinates are smooth, it is sufficient to check differentiability for given local charts (see also Remark 1.8).

---

[7] Let X be a topolgical space and $\sim$ an equivalence relation.Consider the quotient set $X/\sim$ and the quotient map $\pi : X \longrightarrow X/\sim$. The quotient topology on $X/\sim$ is the topology such that $U \subseteq X/\sim$ is open if and only if $\pi^{-1}(U)$ is open in X. This topology is characterized by the property that given a topological space Y and a map $f : X/\sim \longrightarrow Y$, f is continuous if and only if $f \circ \pi : X \longrightarrow Y$ is continuous.

[8] $[t, \epsilon]$ is the class of (t, ϵ) in M.

2.8. Remark. The natural equivalence relation in the theory of smooth manifolds is the relation of being diffeomorphic, i.e. we consider two smooth manifolds M, N to be the same if there exists a diffeomorphism $F : M \longrightarrow N$. This relation is not to be confused with equivalence of atlases. We make this point clear. Suppose that M is a topological manifold with two smooth atlases, $\mathcal{A}$, $\mathcal{B}$. Then the two atlases are equivalent if and only if $\mathbb{1} : (M, \mathcal{A}) \longrightarrow (M, \mathcal{B})$ is a diffeomorphism. This is more restrictive that requiring the existence of a diffeomorphism $F : (M, \mathcal{A}) \longrightarrow (M, \mathcal{B})$. For example consider in $\mathbb{R}$ the two differentiable structures containing $\phi(t) = t$ and $\psi(t) = t^3$ respectively. Those structure are not the same since the changes of coordinates $\psi^{-1}\phi(t) = \sqrt[3]{t}$ is not differentiable at $0 \in \mathbb{R}$. However the two structure are diffeomorphic, since the map $F(t) = \sqrt[3]{t}$ is a diffeomorphism. We invite the reader to check the details.

The considerations above lead us to the following questions

1. Given a topological manifold M there exists a smooth structure on M?
2. Given a topological manifold M there are differentiable structures on M that are *not* diffeomorphic?

Both questions have a definitive answer in low dimension.

2.9. Theorem. *A n-dimensional topological manifold admits a unique (up to diffeomorphism) differentiable structure if $n \leq 3$.*

But the situation in higher dimensions is quite complicated and it is still object of active research. To give a very approximative idea we will state some known facts and an important open question.

- There are very many n-dimensional topological manifolds, $n \geq 4$, which do not admit differentiable structures.
- $\mathbb{R}^n$ ha a unique differentiable structure (up to diffeomorphisms), if $n \neq 4$. $\mathbb{R}^4$ has infinitely many non diffeomorphic differentiable structure, in fact an uncountable number.
- The number Γ_n of (non diffeomorphic) differentiable structures on S^n is finite, if $n \neq 4$. $\Gamma_n = 1$ if $n = 5, 6$, $\Gamma_7 = 28$. It is still an open question if $\Gamma_4 = 1$ (differentiable Poincarè conjecture).

2.10. Example. The *real projective space* $\mathbb{R}P^n$ is defined as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ modulo the equivalence relation $x \sim \lambda x, \lambda \in \mathbb{R} \setminus \{0\}$. The topology is the quotient topology. It can be seen also as the quotient of $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ modulo the equivalence relation $x \sim \pm x$. In fact the inclusion $S^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeomorphism of the second model onto the first one. $\mathbb{R}P^n$ is compact, being the image, by the quotient map, of S^n, which is compact. It is easily seen that $\mathbb{R}P^n$ is Hausdorff and second countable. We shall define a smooth atlas on $\mathbb{R}P^n$. We will denote by $[x_0, \ldots, x_n]$ the equivalence class determined by $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

Let $A_i \subseteq \mathbb{R}P^n = \{[x_0, \ldots, x_n] \in \mathbb{R}P^n : x_i \neq 0\}$. Observe that A_i is well defined sice if the i^{th} coordinate of a (non zero) vector is non zero, it is non zero for all equivalent vectors. Moreover A_i is open in $\mathbb{R}P^n$ since the inverse image via the quotient map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. Define the maps

$$\phi_i : \mathbb{R}^n \longrightarrow A_i, \quad \phi_i(y_1, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n], \quad i = 0, \ldots, n.$$

Then ϕ_i is a bijection and its inverse is given by

$$\phi^{-1}([x_0, \ldots, x_n]) = x_i^{-1}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

The changes of coordinates are given by

$$\phi_j^{-1} \circ \phi_i(y_1, \dots, y_n) = y_j^{-1}(y_1, \dots, y_{i-1}, 1, y_i, \dots, y_{j-1}, y_{j+1}, \dots, y_n) \quad (i < j),$$

hence they are smooth. Observe that $\mathbb{R}P^n = \cup_i A_i$, so $\{(\mathbb{R}^n, \phi_i), i = 0, \dots n\}$ is a smooth atlas and determines a smooth structure on $\mathbb{R}P^n$.

2.11. Remark. The numbers $(x_0, \dots, x_n)$ are called the *homogeneous coordinates* of $[x]$.

If we think at $\mathbb{R}P^n$ as the quotient of S^n, as above, we can use the parameterizations of S^n as in Example 1.17, to produce an atlas for $\mathbb{R}P^n$. In fact we can define maps

$$\psi_i : B^n(1) \longrightarrow \mathbb{R}P^n, \qquad \psi_i(y_1, \dots, y_n) = [y_1, \dots, y_{i-1}, (1 - \sum_{j \neq i} y_j^2)^{\frac{1}{2}}, y_i, \dots, y_n].$$

the inverse maps are

$$\psi_i^{-1}([x_0, \dots, x_n]) = \frac{x_i}{|x_i|}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

The change of coordinates are the same as the ones in Example 1.17, hence smooth.

2.12. Example. We can consider the *complex projective space*, $\mathbb{C}P^n$, as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation $z \sim \lambda z$, $\lambda \in \mathbb{C} \setminus \{0\}$. This is equivalent to consider the unit sphere $S^{2n+1} = \{z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1} : \sum_0^n |z_i|^2 = 1\}$ and the equivalence relation $z \sim \lambda z, \lambda \in S^1 = \{w \in \mathbb{C} : |w| = 1\}$. In particular $\mathbb{C}P^n$ is compact, as a quotient of a compact space, and, as it is easily seen, Hausdorff and second countable. Proceeding, formally, as in the Example above, we can construct smooth atlases on $\mathbb{C}P^n$. Thus $\mathbb{C}P^n$ has, at least, one differentiable structure and with this structure it is a differentiable manifold of dimension $2n$.

2.13. Remark. The the construction of real and complex projective spaces relays on the fact that we have an associative division algebra structure[9] on $\mathbb{R}$ and in $\mathbb{C} \cong \mathbb{R}^2$. The existence of such a structure is a stringent constrain on the dimension n. In fact it can be proved that such a product exist if and only if $n = 1, 2, 4, 8$ (a non trivial result).

For $n = 1$ (resp. $n = 2$), a product is given by the usual product of real (resp. complex) numbers.

For $n = 4$ such a structure can be defined in the following way: let us denote by $\{1, i, j, k\}$ the canonical bases of $\mathbb{R}^4$. Define 1 to be the neutral element for the product and

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

We can extend this product, by bi-linearity, to a product in $\mathbb{R}^4$. With this product $\mathbb{R}^4$ is a division algebra, associative but *not commutative*, called the *algebra of quaternions*, that we will denote by $\mathbb{H}$. In analogy to the case of complex numbers we can define conjugation: if $q = x_0 + x_1 i + x_2 j + x_3 k$, $\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$. Then $q\bar{q} = \|q\|^2$. For $q \in \mathbb{H} \setminus \{0\}$ we have $q^{-1} = \bar{q}\|q\|^{-2}$.

For $n = 8$ the situation is a bit more complicated. We can define the *algebra of octonions* or the *Caley algebra* as $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$, as a vector space, equipped with the product

$$(a, b) * (c, d) = (ac - d\bar{b}, da - b\bar{c}).$$

---

[9] An algebra A is a *division algebra* if for $a \in A \setminus \{0\}$, the maps $L_a(x) = ax$ and $R_a(x) = xa$ are surjective. If A has a (multiplicative) neutral element, this is equivalent to require that any nonzero element is invertible.

With this product $\mathbb{O}$ becomes a *non commutative, non associative division algebra.*

We can define the *quaternionic projective space* $\mathbb{H}P^n$ as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim \lambda x,\ \lambda \in \mathbb{H} \setminus \{0\}$. Proceeding formally as above we can construct a smooth structure on $\mathbb{H}P^n$ that, with this structure, is a differentiable manifold of dimension $4n$.

We can *try* to define the Caley projective space $\mathbb{O}P^n$ as the quotient of $\mathbb{O}^{n+1} \setminus \{0\}$ by the relation $z \sim \lambda z,\ \lambda \in \mathbb{O} \setminus \{0\}$. It turns out that, for the lack of associativity, this relation *is not* an equivalence relation. We can still define the Caley projective line ($n = 1$) as the 1-point compactification $\mathbb{O}$, which is S^8, in analogy to the cases of the real, complex and quaternionic projective lines. Also it is possible to define a 16-dimensional manifold which deserves the name of Caley projective plane, and it is denoted by $\mathbb{O}P^2$, but not higher dimensional Caley projective spaces.

Once we have the concept of real valued differentiable functions, we have the concept of algebra of germs of smooth functions at $p \in M,\ \mathcal{F}_p$ and we can use it to define the tangent space.

2.14. Definition. The *tangent space* of M at p is the space of derivations of $\mathcal{F}_p$.

Also, given a smooth map between differentiable manifolds, $F : M \longrightarrow N$, we can define the *differential* of F at $p \in M$ as the linear map:

$$\mathrm{d}F(p)(X)(f) = X(f \circ F), \qquad \forall\ X \in T_pM,\ \ f \in \mathcal{F}_{F(p)}.$$

Also we can define the tangent vector to a curve $\gamma : (a, b) \subseteq \mathbb{R} \longrightarrow M$, at $t \in (a, b)$, as the tangent vector $\dot\gamma(t)$ that, on a function $f \in \mathcal{F}(M)$, takes the value:

$$\dot\gamma(t)(f) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=t} f(\gamma(s)).$$

Naturally, if M is a submanifold of $\mathbb{R}^N$, $\dot\gamma(t)$ is the usual tangent vector to γ.

2.15. Remark. Starting with the observation above we will suggest, in Exercise 10.4, a more geometric definition of T_pM.

2.16. Definition. The *tangent bundle* of a smooth manifold M is the the set

$$TM = \{(p, V) : p \in M, V \in T_pM\}.$$

So the tangent bundle is, roughly speaking, the disjoint union of all tangent spaces. We want to put a topology and a differentiable structure on TM. Consider a maximal smooth atlas (U_α, ϕ_α) for M. Define

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow TM, \qquad \Phi_\alpha(x, X) = (\phi_\alpha(x), \mathrm{d}\phi_\alpha(x)(X)).$$

It is easily seen that Φ_α is a bijection onto $\{(p, V) \in TM : p \in \phi_\alpha(U_\alpha)\} \subseteq TM$. We define a topology on TM declaring open the sets that are union of images of open sets in $U_\alpha \times \mathbb{R}^n$. With this topology the $\Phi'_\alpha s$ are homeomorphisms onto their images (which are open). Therefore we have a topological atlas. The change of coordinates are given by

$$\Phi_\beta^{-1} \circ \Phi_\alpha(x, X) = (\phi_\beta^{-1} \circ \phi_\alpha(x), \mathrm{d}[\phi_\beta^{-1} \circ \phi_\alpha](x)(X)).$$

Hence the change of coordinates are smooth and the atlas define a differentiable structure on TM.

Also we define the *projection map* $\pi : TM \longrightarrow M,\ \ \pi(p, V) = p$. It is obvious that π is smooth (with the smooth structure on TM defined above).

3. Special atlas and partitions of unity

In this section we will prove the existence of partition of unity for smooth manifolds. For the case $M = U \subseteq \mathbb{R}^n$ (see Appendix of Chapter 1) the essential fact in proving the existence of partitions of unity was that we could write U as the union of an incrising sequence of compact sets. We will extend this fact to the case of smooth manifolds and proceed as in the case of open sets of $\mathbb{R}^n$. We start recalling a few concept and facts in General Topology.

3.1. Definition. Lat X be a topological space and let $\mathcal{U} = \{U_\alpha, \alpha \in \mathcal{A}\}$ be a covering of X.

(1) The covering $\mathcal{U}$ is *locally finite* if $\forall\, x \in X$ there exists a neighborhood U of x such that $U \cap U_\alpha = \emptyset$ for all but a finite number of the $\alpha' s$.
(2) A covering $\mathcal{V} = \{V_i, i \in \mathcal{I}\}$ is a *refinement* of $\mathcal{U}$ if for each $i \in \mathcal{I}$ there exists $\alpha \in \mathcal{A}$ such that $V_i \subset U_\alpha$.
(3) An Hausdorff space is *paracompact* if every open covering has a locally finite refinement.

3.2. Theorem. *An Hausdorff space which is second countable and locally compact*[10] *is paracompact.*

Proof. We start by proving the following
Claim. There exists a sequence of compact sets K_i such that $\cup_i K_i = X$ and $K_i \subseteq Int(K_{i+1})$.[11]

Proof. Let $\{U_i,\ i \in \mathbb{N}\}$ be a countable bases for the open sets. Since X is locally compact, every point $x \in X$ has a relatively compact neighborhood U. Then there is U_i such that $x \in U_i \subseteq U$. Hence X is covered by a sequence of relatively compact open sets and we can suppose, to start with, that $\overline{U_i}$ is compact. Define $K_1 = \overline{U_1}$. We define K_k inductively. Given K_i compact, let k be the smaller integer such that $K_i \subseteq U_1 \cup \cdots \cup U_k$. Define $K_{i+1} = \overline{U_1 \cup \cdots \cup U_k} \cup \overline{U_{i+1}}$. Then the sequence K_i has the required property. ☐

We conclude now the proof of the Theorem. Let $\mathcal{U} = \{U_\alpha, \alpha \in \mathcal{K}\}$ be an open covering of X. Cover the compact set $K_{i+1} \setminus Int(K_i)$ by open sets $V_1, \dots, V_k$ such that $V_j \subseteq U_\alpha$, for some $\alpha \in \mathcal{K}$ and $V_j \subseteq K_{i+2} \setminus Int(K_{i-1})$. Let P_i be the collection of such sets. Then $\{P_i\}$ is a locally finite refinement of $\mathcal{U}$. ☐

3.3. Remark. We should mention also a classical result due to Uryson.

3.4. Theorem. [Uryson's Theorem] *An Hausdorff, paracompact and second countable topological space X is metrizable, i.e. there is a metric d on X such that the identity map $\mathbb{1}_X : (X, d) \longrightarrow X$ is a homeomorphisms.*

In particular *differentiable manifolds are metrizable spaces.*

We will need special atlases.

3.5. Definition. A countable smooth atlas $\{(\mathbb{R}^n, \phi_i)\}$ on a smooth manifold M is a *special atlas* if

(1) $\mathcal{V} = \{\phi_i(B^n(3))\}$ is a *locally finite* open covering of M,
(2) $\mathcal{W} = \{\phi_i(B^n(1))\}$ is a (necessarily locally finite) covering of M.

---

[10] A topological space is *locally compact* if every point has an open neighborhood with compact closure.

[11] The *interior* of a subset $Y \subseteq X$, $Int(Y)$ is the union of all open sets contained in Y.

For such an atlas we will use, consistently, the notations

$$V_i = \phi_i(B^n(3)), \quad U_i = \phi(B(2)), \quad W_i = \phi_i(B^n(1)).$$

If $\mathcal{U}$ is a covering such that $\mathcal{V}$ refines $\mathcal{U}$ we will say that the special atlas is *dominated* by $\mathcal{U}$.
Also, since the domain of the charts is fixed, we will denote a special atlas simply by $\{\phi_i\}$.

3.6. PROPOSITION. *Let M be an n-dimensional smooth manifold and let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M. Then there exists a special atlas such that $\mathcal{V} = \{V_i\}$ refines $\mathcal{U}$.*

PROOF. We start with the following
CLAIM 1. Given $p \in M$ there is a chart $\phi : \mathbb{R}^n \longrightarrow M$ with $\phi(0) = p$.

PROOF. Let $\psi : U \longrightarrow M$ be a chart such that $\psi(x) = p$. Let $r \in \mathbb{R}$ be positive and such that $B(x, r) \subseteq U$. Consider a diffeomorphism $F : \mathbb{R}^n \longrightarrow B(x, r)$, $F(0) = x$ (see Exercise 4.14 in Chapter 0). Then $\phi = \phi \circ F : \mathbb{R}^n \longrightarrow M$ is a chart with the desired property. ☐

Then we can proceed as in the proof of Theorem 3.2 and construct a special atlas such that $\mathcal{V}$ refines $\mathcal{U}$. We really need a little extra attention in order to guarantee that $\mathcal{W}$ is a covering of M. For this we need the following fact
CLAIM 2. Given a chart $\phi : \mathbb{R}^n \longrightarrow M$ and an opens set $\tilde{W} \supseteq \overline{W}$, the exist $\epsilon > 0$ and a chart $\tilde{\phi} : \mathbb{R}^n \longrightarrow M$ such that $\tilde{\phi}|B(1+\epsilon) = \phi|B(1+\epsilon)$ and $\tilde{\phi}(B(3)) \subseteq \tilde{W}$.
We leave to the reader the tasks of proving the Claim and to conclude the proof of the Proposition. ☐

3.7. DEFINITION. Let M be an n-dimensional smooth manifold and $\mathcal{U} = \{U_\alpha\}$ an open covering of M. A *partition of unity dominated by $\mathcal{U}$* is a countable family of smooth functions $\lambda_i : M \longrightarrow [0, 1]$ such that

(1) $\text{supp}(\lambda_i) := \overline{\{x \in M : \lambda_i(x) \neq 0\}} \subseteq U_\alpha$ for some $\alpha = \alpha(i)$,
(2) $\forall\ x \in M$ there exists an open neighborhood $U \ni x$ such that $\text{supp}(\lambda_i) \cap U = \emptyset$ for all but a finite number of i's.
(3) For all $x \in M$, $\sum_i \lambda_i(x) = 1$ ($\forall$ x the sum is finite by (2)).

Our aim is to prove the following result

3.8. THEOREM. *Given an open covering $\mathcal{U}$ of M there exists a partition of unity dominated by $\mathcal{U}$.*

PROOF. Let $\{\phi_i\}$ be a special atlas such that $\mathcal{V}$ refines $\mathcal{U}$ and $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ a smooth function such that

$$\lambda(x) = 1 \text{ if } \|x\| \leq 1, \quad \lambda(x) = 0 \text{ if } \|x\| \geq 2, \quad \lambda(x) \in (0, 1) \text{ if } \|x\| \in (1, 2)$$

(see Exercise 4.13, Chapter 1, for the existence of such a function). Define $\tilde{\lambda}_i : M \longrightarrow \mathbb{R}$,

$$\tilde{\lambda}_i(x) = \begin{cases} \lambda \circ \phi_i^{-1}(x) & \text{if } x \in V_i \\ 0 & \text{if } x \in M \setminus \overline{U_i} \end{cases}$$

Clearly $\tilde{\lambda}_i$ is a well defined smooth function. Set $\tilde{\lambda}(x) = \sum_i \tilde{\lambda}_i(x)$ (observe that there are only finitely many non zero summands at every point). Then $\tilde{\lambda}(x)$ is a well defined *positive* smooth function, since the W_i's cover M, and $\{\lambda_i = \tilde{\lambda}_i/\tilde{\lambda}\}$ is a partition of unity dominated by $\mathcal{U}$. ☐

The following approximation Theorem is proved just like in the case of open sets of $\mathbb{R}^n$ and we will leave the details to the reader.

3.9. Theorem. *Let M be a smooth manifold, $F : M \longrightarrow \mathbb{R}^m$ a continuous function and $\delta : M \longrightarrow \mathbb{R}$ a continuous* positive *function. Suppose that F is smooth on a closed set $N \subseteq M$. Then there exists a smooth function $G : M \longrightarrow \mathbb{R}$ such that $\|F(x) - G(x)\| < \delta(x)$, $\forall x \in U$ and $F(x) = G(x)$ if $x \in N$. Moreover G can be chosen homotopic to F.*

3.10. Remark. Theorem 3.9 holds also if the target space is a differentiable manifold P, modulo interpreting appropriately "approximation". For example if P is a submanifold of some $\mathbb{R}^N$, we can consider the induced metric space structure and define approximation in this context. In the next section we will see that, in fact, any differentiable manifold is a submanifolds of some $\mathbb{R}^N$.

4. Immersions, embeddings and the Theorems of Whitney

In this sections we will prove some results, essentially due to Whitney, on the existence of immersions and embeddings of smooth manifolds into Euclidean spaces. As a consequence we will see that any smooth manifold is diffeomorphic to a submanifold of $\mathbb{R}^N$, for some N sufficiently large. We will follow closely [**?**].

We will start with some definitions.

4.1. Definition. Let M, N be smooth manifolds and $F : M \longrightarrow N$ be a smooth map. We will say that

- F is an *immersion* if $\mathrm{d}F(x) : T_xM \longrightarrow T_{F(x)}N$ is injective, $\forall\ x \in M$,
- F is a *1-1 immersion* if it is an injective immersion,
- F is an *embedding* if F is an immersion and a homeomorphism onto its image (with the relative topology).

4.2. Remark. A few observations and examples.

- By Theorem 1.26, an immersion is locally injective. In fact a little bit more: for any $x \in M$ there exists a neighborhood U of x, such that $F|U : U \longrightarrow N$ is an embedding. But an immersion may not be injective . For example the map $F : \mathbb{R} \longrightarrow \mathbb{R}^2$, $F(t) = (\cos t, \sin t)$ in an immersion which is not injective (see also Remark 1.2).
- An injective immersion may not be an embedding. However, if M is compact, an injective immersion is an embedding (see Exercise 10.14).
- A diffeomorphism is a surjective embedding.

4.3. Definition. A *submanifold* of a smooth manifold N is the image of an embedding $F : M \longrightarrow N$.

4.4. Remark. Sometimes submanifolds are defined, roughly speaking, as images of injective immersion. Such immersions are important because they appear naturally in many contexts, for example in the case of integration of vector fields an distributions (see next section). To be coherent with our definition of submanifolds of $\mathbb{R}^N$, we will stick with Definition 4.3.

A fact that will be useful in few occasions is the following

4.5. Lemma. *Let $F : M \longrightarrow N$ be an immersion and let $N \subseteq M$ be a closed subset. Then, if $F|_N$ is a 1-1 immersion, there exists an open neighborhood U of N such that $F|_U$ is a 1-1 immersion.*

PROOF. ☐

The first result we will prove in this section is that any function $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^p$ can be "approximate" by an immersion, if $p \geq 2n$. We need a few preliminaries.

We will denote by $C(a) \subseteq \mathbb{R}^n$ a cube of side a and define the *volume* of $C(a)$, $v(C(x, a)) := a^n$.

4.6. DEFINITION. A set $A \subseteq \mathbb{R}^n$ has *measure zero* if, given $\epsilon > 0$, there exists a countable family of cubes, $C^n(a_i)$, that cover A, with *total volume* $\sum a_i^n < \epsilon$. If this is the case, we will write $m(A) = 0$.

4.7. REMARK. A few observations are in order.

(1) If $B \subset A \subset \mathbb{R}^n$ and $m(A) = 0$ then $m(B) = 0$.
(2) if $m(A) = 0$ then A does not contain open sets. In particular $\mathbb{R}^n \setminus A$ is *dense*.
(3) if A_i is a countable family of subsets with $m(A_i) = 0$, then $m(\cup_i A_i) = 0$.
(4) if A is a *proper* affine subspace of $\mathbb{R}^n$, then $m(A) = 0$.

We leave to the reader the task of proving the claims above (Exercise 10.10).

4.8. LEMMA. *Let $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth map and $A \subseteq U$ with $m(A) = 0$. Then $m(F(A)) = 0$.*

PROOF. Let $C = C(a)$ be a cube contained in U. Let $b = \sup\{\|\mathrm{d}F(y)\| : y \in C\}$. By the mean value Theorem $\|F(x) - F(y)\| \leq b\|x - y\|$ if $x, y \in C$. In particular $F(C)$ is contained in a cube of side ba.

Let $\epsilon > 0$ be given. Since $m(A \cap C)$ has measure 0, we can cover it with cubes $C(a_i)$ with $\sum a_i^n \leq b^{-n}\epsilon$. Then $F(A \cap C)$ can be covered with cubes of total volume less then ϵ, hence $F(A \cap C)$ has measure zero. Now $F(A)$ is countable union of sets of the type $F(A \cap C)$ and, therefore, has measure zero. ☐

4.9. COROLLARY. *If $F : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a smooth map, $m > n$, $m(F(U)) = 0$.*

PROOF. Consider $G : U \times \mathbb{R}^{m-n} \longrightarrow \mathbb{R}^m$, $G(x, y) = F(x)$. By Lemma 4.8 and Remark 4.7 (4), $0 = m(G(U \times \{0\}) = m(F(U))$. ☐

4.10. DEFINITION. Let M be a smooth manifold and $A \subseteq M$ be a subset contained in the image of a local chart $\phi : U \subseteq \mathbb{R}^n \longrightarrow M$. We say that $m(A) = 0$ if $m(\phi^{-1}A) = 0$.

A subset $A \subseteq M$ has *measure zero* if for every chart $\phi : U \longrightarrow M$, $m(A \cap \phi(U)) = 0$.

4.11. REMARK. By Lemma 4.8, this definition does not depends on the choice of the local chart. Observe also that we have defined sets of measure zero but not the *"measure of a set"*, and we can not really define measure of a set, in our context, since any reasonable definition of "measure of a set" is hardly invariant for diffeomorphisms. To define measures we need some extra structure.

4.12. REMARK. The claims in Remark 4.7 continue to be true in this more general context (substituting affine subspace in claim (4) by submanifold of lower dimension). Also Corollary 4.9 extends immediately to the case of smooth functions between smooth manifolds. More precisely

4.13. COROLLARY. *Let $F : M^n \longrightarrow N^m$ be a smooth map with $n < m$. Then $m(F(M)) = 0$.*

Observe that the condition that a manifold is a second countable topological space plays is essential.

We will prove now a "local version" of the Whitney immersion Theorem.

4.14. LEMMA. *Let $U \subseteq \mathbb{R}^n$ be an open set and $F : U \longrightarrow \mathbb{R}^p$ be a smooth map, $p \geq 2n$. Then, given $\epsilon > 0$, there exist $A \in L(\mathbb{R}^n, \mathbb{R}^p)$ with $\|A\| < \epsilon$ and such that the map $G(x) = F(x) + Ax$ is an immersion.*

PROOF. We want to determine A such that $G = F + A$ is an immersion. Consider the manifold $M(p, n; k)$ of Example 1.20 and the map

$$F_k : M(p, n; k) \times U \longrightarrow M(p, n, \mathbb{R}), \qquad F_k(Q, x) = Q - \mathrm{d}F(x).$$

<u>CLAIM</u>: If $k < n$, the image of F_k has measure zero.

PROOF. We will show that, if $k < n$, the dimension of $M(p, n; k) \times U$ is smaller than $pn = \dim(M(p, n, \mathbb{R}))$ and the conclusion will follow from Corollary 4.13. In fact, as we have seen in Example 1.20, $\dim(M(p, n; k)) = k(p + n - k)$. Hence the domain of F_k has dimension $k(p + n - k) + n$. The latter expression is an increasing function of k as long as $k < n$ (take derivative with respect to k), hence it is, at most, $2n - p + pn - 1$, as long as $k < n$. But we are assuming $p \geq 2n$, hence $2n - p + pn - 1 < pn = \dim(M(p, n, \mathbb{R}))$. □

We conclude now the proof of the Theorem. By the Claim there exists $A \in M(p, n, \mathbb{R})$ with $\|A\| < \epsilon$ and not in the image of F_k, $k < n$. If $\mathrm{d}G(x) = A + \mathrm{d}F(x)$ has rank $k < n$, then $A = F_k(\mathrm{d}G(x), x)$, a contradiction. Hence G is an immersion. □

We want to prove a "global" version of Lemma 4.14.

4.15. THEOREM. [Whitney immersion Theorem] *Let M^n be a smooth manifold and let $F : M^n \longrightarrow \mathbb{R}^p$, be a smooth function, $p \geq 2n$. Given a positive continuous function $\delta : M \longrightarrow \mathbb{R}$, there exists an immersion $G : M \longrightarrow \mathbb{R}^p$ such that $\|F(x) - G(x)\| < \delta(x)$. Moreover, if $\mathrm{d}F$ has rank n on a closed subset $N \subseteq M$, we can choose G such that $G|N = F|N$.*

PROOF. The idea of the proof is to construct a sequence of functions, $F_k : M \longrightarrow \mathbb{R}^p$, by subsequent local modifications of F, using Lemma 4.14, in such a way that $G = \lim_{k\to\infty} F_k$ is a well defined smooth function with the required properties. Let go to work.

If rank $\mathrm{d}F = n$ in N, the same happens in an open neighborhood A of N. We consider the open covering of M given by A and $M \setminus N$. Let $\{\phi_i\}$ be a special atlas dominated by the covering (see Definition 3.5 and Proposition 3.6, also for notations). Without loss of generality we can suppose that $i \in \mathbb{Z}$ and $V_i := \phi_i(B(3)) \subseteq A$ if and only if $i \leq 0$.

Let $F_0 = F$. We will define, inductively, a sequence of functions $F_k : M \longrightarrow \mathbb{R}^p$ whose limit, as $k \to \infty$, will be a well defined function G with the desired properties.

Let ϵ_j be the minimum of δ in $\overline{U_j} = \overline{\phi_j(B(2))}$. Suppose we have defined F_{k-1} such that

1. $\mathrm{d}F_{k-1}(x)$ has rank n on $N_{k-1} = \cup_{j<k}\overline{W_j}$
2. $\|F_{k-1}(x) - F_{k-2}(x)\| < 2^{1-k}\epsilon_{k-1}$.

Let $\lambda : \mathbb{R}^n \longrightarrow [0, 1]$ be a smooth function which is identically 1 on $B^n(1)$, identically zero outside $B^n(2)$.

<u>CLAIM.</u> There exists $A \in L(\mathbb{R}^n, \mathbb{R}^p)$ such that the function

$$F_A : B^n(3) \longrightarrow \mathbb{R}^p, \qquad F_A(x) = F_{k-1} \circ \phi_k(x) + \lambda(x)Ax$$

has the following properties

1. F_A is an immersion in $K = \phi_k^{-1}(N_{k-1}) \cap \overline{B^n(2)}$,

(2) $\|Ax\| \le 2^{-k}\epsilon_k, \quad \forall\, x \in B^n(3)$,
(3) F_A is an immersion on $B^n(1)$.

Proof. Consider the function

$$\Phi : K \times M(p,n,\mathbb{R}) \longrightarrow M(p,n,\mathbb{R}), \qquad \Phi(x,A) = \mathrm{d}F_A(x).$$

Observe that $\Phi(K \times \{0\}) \subseteq M(p,n;n)$. Since $M(p,n;n)$ is open in $M(p,n,\mathbb{R})$, $\mathrm{d}F_A(x) \in M(p,n;n)$, $\forall x \in K$, if $\|A\|$ is sufficiently small. So there exist $\eta > 0$ such that, if $\|A\| < \eta$, F_A verifies the first two conditions. Lemma 4.14 guarantee that there exist $A \in M(p,n,\mathbb{R})$ such that $\|A\| < \eta$ and F_A satisfies the last condition. ☐

Fix $A \in L(\mathbb{R}^n, \mathbb{R}^p)$ that verifies the condition of the claim and define

$$F_k(p) = \begin{cases} F_A \circ \phi_k^{-1}(p) & \text{if } \ p \in V_k \\ F_{k-1}(p) & \text{if } \ p \in M \setminus \overline{U_k} \end{cases}$$

Observe that F_k is a well defined smooth map since the two functions above are smooth and coincide on the intersection of the domains. Moreover

- $\mathrm{d}F_k$ has rank n on N_{k-1} (by the first condition on A).
- $\|F_k(p) - F_{k-1}(p)\| < 2^{-k}\delta(p)\ \forall\, p \in M$ (by the second condition on A).
- $\mathrm{d}F_k$ has rank n on $\overline{W_k}$ (by the third condition on A), hence on N_k.

Define $G(p) = lim_{k\to\infty} F_k(p)$. Observe that, due to the fact that the covering $\{V_j\}$ is locally finite, there exist a neighborhood U_p of p such that $G|U_p = F_k|U_p$ for k sufficiently large. So G is a well defined smooth function and has the required properties. ☐

A natural question to ask is whether we can approximate an immersion by a 1-1 immersion. For example, if we have a regular curve in $\mathbb{R}^2$ with a self intersections point, say p, we can not, in general, approximate it by a regular curve without self intersections. But if we look at the picture as being in $\mathbb{R}^3$ we can "lift" a branch of the curve near p to obtain a new curve in $\mathbb{R}^3$, which is an approximation of the given one, without self intersections. This is, essentially, the content of our next result.

4.16. Theorem. [Whitney injective immersions Theorem] *Let M^n be a differentiable manifold, $F : M \longrightarrow \mathbb{R}^p$ an immersion and $\delta : M \longrightarrow \mathbb{R}$ a positive continuous function. If $p > 2n$, there is a 1-1 immersion $G : M \longrightarrow \mathbb{R}^p$ such that $\|F(p) - G(p)\| < \delta(p)$, $\forall p \in M$. Moreover, if $N \subseteq M$ is a closed set such that $F|N$ is 1-1, we can chose G such that $G|N = F|N$.*

Proof. Since $F|N$ is 1-1, there exist, by Lemma 4.5, an open neighborhood A of N such that $F|A$ is 1-1. Since F is a local embedding there exist an open covering $\{V_\alpha\}$ of M, that refines $\{A, M \setminus N\}$, such that $F|V_\alpha$ is an embedding. Let $\{\phi_i\}$ be a special atlas that refines this covering. We will use the same notations as in the proof of Theorem 4.15. Define $\lambda_k : M \longrightarrow \mathbb{R}$ as the obvious extension of $\lambda \circ \phi_k^{-1} : V_k \longrightarrow \mathbb{R}$.

Let $F_0 = F$. We define, inductively, the sequence of functions

$$F_k(p) = F_{k-1}(p) + \lambda_k(p) b_k,$$

where $b_k \in \mathbb{R}^p$ has to be chosen. To explain this choice we start considering the set $P_k = \{(p,q) \in M \times M : \lambda_k(p) \neq \lambda_k(q)\}$. Then P_k is an open subset of $M \times M$ hence a $2n$-dimensional manifold. Consider the map

$$\Phi_k : P_k \longrightarrow \mathbb{R}^p, \qquad \Phi_k(p,q) = -\frac{F_{k-1}(p) - F_{k-1}(q)}{\lambda_k(p) - \lambda_k(q)}.$$

By Lemma 4.13, the image of Φ_k has measure zero in $\mathbb{R}^p$ (since $p > 2n$).

Therefore we can choose b_k in such a way that

- F_k is an immersion,
- F_k is a $\delta/2^k$ approximation of F_{k-1},
- b_k is not in the image of Φ_k.

It follows that $F_k(p) = F_k(q)$ if and only if $\lambda_k(p) = \lambda_k(q)$ and $F_{k-1}(p) = F_{k-1}(q)$. By a backward induction, the same hold for any $h, 0 < h < k$.

Define $G(p) = \lim_{k\to\infty} F_k(p)$. As in the proof of the preceding Theorem, G is a well defined immersion and $G|N = F|N$. We want to show that G is 1-1. Suppose $G(p) = G(q)$, $p \neq q$. Then, for some k sufficiently large, $F_k(p) = F_k(q)$. Then $F(p) = F(q)$ and p and q can not be in the same U_i. Since $\phi_k(p) = \phi_k(q)$, p, q do not belong to U_k, for $k > 0$. Therefore $p, q \in U$, a contradiction. ☐

The next natural question is if we can approximate an immersion by an embedding. In order to answer this question we need some preliminary ingredients.

4.17. Definition. Let $F : M \longrightarrow \mathbb{R}^p$ be a continuous map. The *limit set* of F, $L(F)$, is the set of points $y \in \mathbb{R}^p$ such that $y = \lim_{k\to\infty} F(x_k)$ where $\{x_k\} \subseteq M$ is a sequence *without limit points.*

We will prove now the basic properties of the limit set that we will need in the proof of the Whitney embedding Theorem.

4.18. Proposition. *Let $F : M \longrightarrow \mathbb{R}^p$ be a continuous map. Then*

1. *$F(M)$ is closed in $\mathbb{R}^p$ if and only if $L(F) \subseteq F(M)$.*
2. *F is a homeomorphism onto its image if and only if it is injective and $L(F) \cap F(M) = \emptyset$.*
3. *If $L(F) = \emptyset$ and $G : M \longrightarrow \mathbb{R}^p$is a continuous function such that $\|F(x) - F(x)\| \leq K \;\; \forall \;\, x \in M$, K a constant, then $L(G) = \emptyset$.*

Proof. Observe that $L(F) \subseteq \overline{F(M)}$. Hence, if $F(M)$ is closed, $L(F) \subseteq F(M)$. Conversely, suppose $L(F) \subseteq F(M)$ and let $y \in \overline{F(M)}$. Then $y = \lim_{k\to\infty} F(x_k)$. If the sequence has no limit points, $y \in L(F) \subset F(M)$. Suppose there is a subsequence x_{k_j} converging to $x \in M$. Then $y = F(x) \in F(M)$. Therefore $F(M)$ is closed.

Suppose now that F is a bijection onto its image. Let $\{x_n\}$ be a sequence in M with $y = \lim_{n\to\infty} f(x_n)$. If $y \in F(M)$ and F^{-1} is continuous, the sequence $\{x_n\}$ converges to $F^{-1}(y)$ hence $y \notin L(F)$. Conversely if $y \in F(M)$ and I and $L(F) \cap F(M) = \emptyset$, then any subsequence of $\{x_n\}$ converges and the limit points coincide, since F is injective. This imply that F^{-1} is continuous.

Let $y \in L(G)$. Then there exist a sequence $\{x_n\}$ in M, without limit points, such that $G(x_n)$ converges to y. Consider $\{F(x_n)\}$. This is a bounded sequence in $\mathbb{R}^p$, hence has a convergent subsequence and the limit is in $L(F)$, a contradiction. Hence $L(G) = \emptyset$. ☐

4.19. Theorem. [Whitney closed embedding Theorem] *Let M be a n-dimensional differentiable manifold. Then there exists an embedding $G : M \longrightarrow \mathbb{R}^{2n+1}$ such that $G(M)$ is closed.*

Proof. We start with the following general fact
Claim. There exists a smooth function $F : M \longrightarrow \mathbb{R}$ with $L(F) = \emptyset$.

Proof. Let $\{\phi_i\}$, $i \in \mathbb{N}$, be a special atlas and $\{\lambda_i\}$ a partition of unity dominated by this atlas. Define $F(x) = \sum_i i\lambda_i(x)$. We claim that $L(F) = \emptyset$. Set $K_i = \cup_{j \leq i}\overline{W_j}$. If $\{x_n\}$ is a sequence without limit points, $\forall i$ there exist $x_{n(i)} \notin K_i$, since K_i is compact. In particular $F(x_{n_i})$ is unbounded and $F(x_n)$ does not converges. ☐

Consider now the function $F : M \longrightarrow \mathbb{R} \subseteq \mathbb{R}^{2n+1}$, F as in the Claim. By Theorem 4.16 there exist a 1-1 immersion $G : M \longrightarrow \mathbb{R}^{2n+1}$ such that $\|G(x) - F(x)\| < 1$. Then $L(G) = \emptyset$, by item (3) of Proposition 4.18. By item (2) of the same Proposition, G is an embedding, and by item (1) $G(M)$ is closed. ☐

4.20. Remark. We have proved that every n-dimensional manifold admit an immersion in $\mathbb{R}^{2n}$ and an embedding in $\mathbb{R}^{2n+1}$. Those results are not sharp. In fact it is possible to prove that there are always immersions of an n-dimensional manifold into $\mathbb{R}^{2n-1}$ and embeddings into $\mathbb{R}^{2n}$. These results, known as the *strong Whitney Theorems*, are sharp. For example the real projective space $\mathbb{R}P^n$ does not immerse into $\mathbb{R}^{2n-2}$ and does not embed into $\mathbb{R}^{2n-1}$, if $n = 2^k$. However a question arise naturally: given n the exist an integer $p(n)$ such that every n-dimensional manifold can be immersed into $\mathbb{R}^{p(n)}$ and there are n-dimensional manifolds that can not be immersed in $\mathbb{R}^{p(n)-1}$? As remarked above, $p(n) = 2n - 1$ if n is "very even", i.e. $n = 2^k$, so there is a suspect that $p(n)$ depends on "haw even" is n. Let be a bit more precise. Let $\alpha(n)$ be the number of 1's in the in the binary expression of n, i.e. if we write $n = 2^{i_1} + \ldots + 2^{i_l}$, $0 \leq i_1 < \ldots < i_l$, then $\alpha(n) = l$. Observe that $\alpha(2^k) = 1$. Using algebraic topological methods (characteristic classes) it can be seen that $p(n) \geq 2n - \alpha(n)$ and it was conjectured, for some times, that $p(n) = 2n - \alpha(n)$. In 1985 R. Cohen proved that a compact n-dimensional manifold can be immersed in $\mathbb{R}^{2n-\alpha(n)}$, giving so a positive answer to the conjecture, at least in the compact case. For the embedding problem it can be shown that a compact *orientable*[12] n-dimensional manifold embeds into $\mathbb{R}^{2n-1}$ (observe the even dimensional real projective spaces are *not* orientable).

5. Integration of vector fields an distributions

5.1. Definition. Let M be a smooth manifold and $U \subseteq M$ be an open set. A *(tangent) vector field* on U is a smooth map $X : U \longrightarrow TM$ such that $\pi \circ X = \mathbb{1}_U$. The space of vector fields on U will be denoted by $\mathcal{H}(U)$.

5.2. Remark. A tangent field is essentially a function that associate to $p \in U$ a derivation of the algebra $\mathcal{F}_p$. It is easy to see that such a vector field define a derivation of the algebra of smooth real valued function defined in U i.e. an element of $\mathcal{D}er(U)$ and, conversely, an element of $\mathcal{D}er(U)$ defines a vector field. So we can identify $\mathcal{H}(U)$ with $\mathcal{D}er(U)$. Therefore we can define the Lie product of vector fields, as the commutator of derivations, Therefore define a Lie algebra structure on $\mathcal{H}(U)$, and prove, using local charts,

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[12] We will define orientability later on.

that if $F : M \longrightarrow N$ is a smooth map, the Lie product of F-related vector fields is F-related to the Lie product of the vector fields (see Proposition **??** of Chapter 0).

5.3. Definition. Let $X \in \mathcal{H}(M)$ and $p \in M$. An integral curve of X with initial condition p is a smooth curve $\gamma : (a, b) \subseteq \mathbb{R} \longrightarrow M$ such that $0 \in (a, b)$, $\gamma(0) = p$ and $\dot{\gamma}(t) = X(\gamma(t))$.

Using local charts and Theorem **??** of Chapter 0, we have

5.4. Theorem. *For all $p \in M$ there exists an integral curve γ of X with initial condition p and such a curve is unique in the sense that two integral curves with the same initial condition agree on the intersection of their domains. In particular there is a unique* maximal *integral curve with initial condition p and this curve depends smoothly on p.*

Then we can define *complete* vector fields as these vector fields whose integral curves are defined in the whole of $\mathbb{R}$ and prove that vector fields with compact support are complete, just like in the case of vector fields on open sets of $\mathbb{R}^n$. For complete vector fields we have the *flow map*

$$\Gamma : M \times \mathbb{R} \longrightarrow M, \qquad \Gamma(p, t) = \gamma_p(t),$$

where γ_p is the integral curve of X with initial condition p.

5.5. Remark. It is now a good time to spend a word on the behavior of the integral curves of a vector field. Let $X \in \mathcal{H}(M)$ and assume, for simplicity, that X is complete. If $X(p) = 0$ then, by unicity, the integral curve of X with initial condition p is the constant curve $\gamma(t) = p$. Suppose now $X(p) \neq 0$ and let γ be the integral curve of X with initial condition p. Then, again by unicity, $\dot{\gamma}(t) \neq 0$, $\forall\, t \in \mathbb{R}$. Therefore γ is an immersion. Then we have two possibilities

- $\gamma(t) \neq \gamma(s), s \neq t$, and γ is a 1-1 immersion of $\mathbb{R}$ into M (it may not be an embedding),
- $\gamma(t) = \gamma(s)$ for some $s > t$. It will be not restrictive to assume $t = 0$, $\gamma(r) \neq \gamma(0)$, $\forall r \in (0, s)$. In this case, again by unicity, $\dot{\gamma}(s) = \dot{\gamma}(0)$ and γ is a periodic curve if period s. In particular γ is a closed curve and induces an embedding $\tilde{\gamma} : S^1 \longrightarrow M$.

Next we consider next the case of *distributions.*

5.6. Definition. Let M be a n-dimensional manifold. A smooth *k-dimensional distribution* $\mathcal{D}$ on M is the choice of a k-dimensional subspace $\mathcal{D}_p \subseteq T_pM$, $\forall\, p \in M$, that depends smoothly on p. This means that, $\forall\, p \in M$, there are smooth vector fields $\{X_1, \dots, X_k\}$, defined in a neighborhood U of p, such that $\mathcal{D}_q$ is spanned by $\{X_1(q), \dots, X_k(q)\}$, $\forall\, q \in U$.

A distribution is a subset of TM, hence we ca talk about vector field (with values) in the distribution.

5.7. Definition.

(1) The distribution $\mathcal{D}$ is *involutive* if the Lie product of two vector fields in the distribution is a vector field in the distribution.
(2) An *integral manifold* (N, ϕ) for the distribution $\mathcal{D}$ is a k-dimensional smooth manifold N and a 1-1 immersion $\phi : N \longrightarrow M$ such that $\mathrm{d}\phi(x)(T_xN) = \mathcal{D}_{\phi(x)}$.

(3) A *maximal integral manifold* for $\mathcal{D}$ is an integral manifold (N, ϕ) such that N is connected and $\phi(N)$ is not properly contained in the image of any connected integral manifold of $\mathcal{D}$.

As we have seen in Chapter 0, Theorem **??**, a necessary and sufficient condition for the existence of integral manifolds for $\mathcal{D}$ is $\mathcal{D}$ to be *involutive*. The delicate question is the existence of maximal integral manifolds. This is the content of the next result.

5.8. THEOREM. [Frobenius Theorem, global version] *Let $\mathcal{D}$ be an involutive k-dimensional distribution on a manifold M and $p_0 \in M$. Then the exist a unique maximal integral manifold of $\mathcal{D}$ containing p_0.*

PROOF. We start by defining the candidate to be the integral manifold we are looking for. Consider

$$N = \{p \in M : \text{there exist a piecewise smooth curve } \gamma : [0,1] \longrightarrow M \ \text{ with } \gamma(0) = p_0, \gamma(1) = p, \ \dot\gamma(t) \in \mathcal{D}_{\gamma(t)}\}.$$

From Theorem **??** in Chapter 0 and the fact that M is second countable it follows that the exists a countable atlas

$$\phi_i : U = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| < 1\} \longrightarrow M,$$

such that the images of the slices $x_{k+j} =$ constant are integral manifolds of $\mathcal{D}$.

We want to define a topology on N. We do this by defining a system of open neighborhoods for $p \in N$. Without loss of generality we can suppose $p = \phi_i(0)$. Observe that $\phi(U \cap \mathbb{R}^k) \subseteq N$ and we declare open neighborhoods of p in N, the images of open sets in $U \cap \mathbb{R}^k$ containing 0. In particular $\phi_i|_{U \cap \mathbb{R}^k}$ is a smooth atlas for N. This topology is Hausdorff since two points in distinct slices have disjoint open neighborhoods. We observe also that a piecewise smooth curve γ, joining p_0 with p and with tangent vector in the distribution, is contained in N and is a continuous curve in this topology. Therefore N is connected. The delicate point in the proof is the following

CLAIM The topology on N defined above is second countable.

PROOF. ☐

☐

6. Lie groups

A very important class of differentiable manifolds is the class of Lie group that we will study in this section. These are object with two structure: a structure of differentiable manifold and a structure of group. We require that the two structures are *compatible*, in the sense that the algebraic operations are smooth maps. More precisely

6.1. DEFINITION. A *Lie group* is a smooth manifold G with a preferred element $e \in G$ and smooth maps:

$$m : G \times G \longrightarrow G, \quad r : G \longrightarrow G,$$

such that the axioms of groups are verified, i.e.

- $m(g, m(h, k)) = m(m(g, h), k)$,
- $m(e, g) = g = m(g, e)$,
- $m(g, r(g)) = e = m(r(g), g)$.

As usual in group theory we will use the notation gh for $m(g,h)$ and g^{-1} for $r(g)$. In the case of Abelian groups, i.e. $m(g,h) = m(h,g)\ \forall\ g,h \in G$, we will also use the notation $m(g,h) := g+h,\ \ r(g) = -g$.

6.2. Definition. Let G be a Lie group. We define the *left translation by* $g \in G$ as the map $L_g : G \longrightarrow G,\ L_g(x) = gx$. Similarly the *right translation by* g is the map $R_g(x) = xg$.

6.3. Lemma. *L_g and R_g are diffeomorphisms of G.*

Proof. L_g is the composition of m, which is smooth, with the smooth map $(c_g, \mathbb{1}) : G \longrightarrow G \times G,\ \ c_g$ being the constant map $c_g(h) = g$. Hence L_g is smooth. Also $(L_g)^{-1} = L_{g^{-1}}$ so L_g has a smooth inverse. Analogously for R_g. □

6.4. Definition. A vector field $X \in \mathcal{H}(G)$ is said to be *left invariant* if $\forall\ g,h \in G,\ X(gh) = \mathrm{d}L_g(X(h))$. Analogously, X is *right invariant* if $\forall\ g,h \in G,\ X(gh) = \mathrm{d}R_h(X(g))$.

We will denote by $\mathcal{L}(G)$ the space of left invariant vector fields. Obviously $\mathcal{L}(G)$ is a (real) vector subspace of $\mathcal{H}(G)$.

Since a left invariant vector field X is L_g-related to itself, we have

6.5. Lemma. *If $X, Y \in \mathcal{L}(G),\ \ [X,Y] \in \mathcal{L}(G)$. In particular $\mathcal{L}G$ is a Lie subalgebra of $\mathcal{H}(G)$.*

6.6. Lemma. *$\mathcal{L}(G)$ is canonically isomorphic to T_eG, as a vector space.*

Proof. Consider the evaluation map $ev : \mathcal{L}(G) \longrightarrow T_eG,\ \ ev(X) = X(e)$. Clearly ev is linear and has a linear inverse $ev^{-1}(X_e)(g) = \mathrm{d}L_g(e)(X_e)$. We should only check that $\mathrm{d}L_g(e)(X_e)$ is a *smooth* vector field. For this we consider a smooth function $f : G \longrightarrow \mathbb{R}$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \longrightarrow G$ such that $\gamma(0) = e, \dot\gamma(0) = X_e$. Then

$$\mathrm{d}L_g(e)(X_e)(f) = X_e(f \circ L_g) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(g\gamma(t)),$$

which is smooth in g. Therefore $\mathrm{d}L_g(e)(X_e)$ is a derivation of $\mathcal{F}(G)$, hence a smooth vector field. □

6.7. Definition. T_eG, endowed with the (Lie) product induced by ev, is a Lie algebra, called the *Lie algebra of* G, and will be denoted by $\widehat{G}$.

6.8. Remark. The Lie algebra of a Lie group is a basic invariant. In fact it is known that there exist a bijection between isomorphism classes of (finite dimensional real) Lie algebras and simply connected Lie groups, up to smooth isomorphisms. We will comment a bit more on this fact later on.

We will study now the integral curves of a left invariant vector field .

6.9. Proposition. *Let $X \in \mathcal{L}(G)$ be a left invariant vector field and $\gamma : (a,b) \longrightarrow G$ be an integral curve of X with initial condition $e \in G$.*

1. *The curve $\gamma_g = L_g \circ \gamma$ is an integral curve of X with initial condition $g \in G$.*
2. *If $t, s, t+s \in (a,b),\ \ \gamma(t+s) = \gamma(t)\gamma(s)$.*
3. *X is complete, i.e. the integral curves are defined on all $\mathbb{R}$.*

PROOF. Since X is left invariant, $\frac{\mathrm{d}}{\mathrm{d}t}L_g \circ \gamma = \mathrm{d}L_g(\dot{\gamma}(t)) = X$, hence $L_g \circ \gamma$ is an integral curve of X. The second Claim follows by observing that, for a fixed s, $\gamma(t+s)$ and $\gamma(t)\gamma(s)$ are integral curves of X with the same initial condition. We will prove now that X is complete.

Let $t_0 = \sup\{t \in \mathbb{R} : \gamma \text{ is defined in } [0,t)\}$, and suppose, by absurd, that $t_0 < \infty$. Let ϵ be a small positive constant and consider the curve

$$\gamma(t) = \begin{cases} \gamma(t), & \text{if } t \le t_0 - \epsilon \\ \gamma(t_0-\epsilon)\gamma(t - t_0 + \epsilon) & \text{if } t_0 - \epsilon \le t < 2t_0 - \epsilon \end{cases}$$

Then γ is an integral curve of X defined in $[0, 2t_0 - \epsilon) \supsetneq [0, t_0)$, a contradiction. □

6.10. EXAMPLE. Consider $GL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) : \det(A) \neq 0\}$. Then $GL(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$, hence a differentiable manifold. The product and inversion are smooth maps (see Example **??**, Chapter 1). Hence $GL(n,\mathbb{R})$ is a n^2-dimensional Lie group. Here some simple observations.

(1) If $A \in GL(n,\mathbb{R})$, $L_A(B) = AB$. In particular L_A is linear and $\mathrm{d}L_A(C) = L_A$, $\forall C \in GL(n,\mathbb{R})$.
(2) If $A \in T_{\mathbb{1}}GL(n,\mathbb{R}) \cong M(n,\mathbb{R})$, the left invariant field (uniquely) determined by A is $\tilde{A}(C) = CA$.
(3) The integral curve of the left invariant extension $\tilde{A}$ of A, with initial condition $\mathbb{1}$, is $\gamma^{\tilde{A}}(t) = \exp(tA) := \sum_{k=0}^{\infty}[k!]^{-1}t^kA^k$. Therefore the integral curve with initial condition C is $C\exp(tA)$.
(4) The flow of $\tilde{A}$ is $\phi_t^{\tilde{A}}(B) = B\exp(tA)$.

We can then compute the Lie algebra structure on $\widehat{GL(n,\mathbb{R})} \cong T_{\mathbb{1}}GL(n,\mathbb{R}) \cong M(n,\mathbb{R})$ using Proposition **??** of Chapter 1. If $A, B \in T_{\mathbb{1}}GL(n,\mathbb{R}) \cong M(n,\mathbb{R})$,

$$[A,B] = [\tilde{A},\tilde{B}](\mathbb{1}) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\exp(tA)B\exp(-tA) =$$

$$= [A\exp(tA)B\exp(-tA) - \exp(tA)BA\exp(-tA)]|_{t=0} = AB - BA.$$

An observation on the topology of $GL(n,\mathbb{R})$. Since $\det : GL(n,\mathbb{R}) \longrightarrow \mathbb{R}\setminus\{0\}$ is a continuous surjective map, $GL(n,\mathbb{R})$ is not connected. Consider $GL^+(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A > 0\}$ and $GL^-(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det A < 0\}$. These are disjoint open submanifolds of $GL(n,\mathbb{R})$, and we claim that they are connected. Let R_i be the diagonal matrix with entries $a_j = 1, j \neq i, a_i = -1$. Multiplication by R_i is a diffeomorphism of $GL^+(n,\mathbb{R})$ onto $GL^-(n,\mathbb{R})$. So it is sufficient to show that $GL^+(n,\mathbb{R})$ is connected. Let $A \in GL^+(n,\mathbb{R})$. We want to show that A belongs to the connected component of $\mathbb{1}$ in $GL^+(n,\mathbb{R})$. By basic linear algebra we know that if $A \in GL(n,\mathbb{R})$ there is an invertible matrix P such that $P^{-1}AP$ is upper triangular. Conjugation by P sends $GL^+(n,\mathbb{R})$ into itself (and fix $\mathbb{1}$), so we can suppose that A is upper triangular. Multiplying by $t \in [0,1]$ the non diagonal elements we get a path joining a diagonal matrix to A, and, along this path, the determinant is constant. So we can suppose that A is diagonal. By convex linear combination we can find a path in $GL^+(n,\mathbb{R})$ between A and a diagonal matrix with entries ± 1. The matrix R_iR_j is in the connected component of $\mathbb{1}$, since the rotation of an angle $\theta \in [0,\pi]$ in the plane $\{e_i, e_j\}$ is a curve in $GL^+(n,\mathbb{R})$ between $\mathbb{1}$ and R_iR_j. So multiplication by R_iR_j sends the connected component of $\mathbb{1}$ into itself. Now we multiply A by the R_i's, for i corresponding to the negative eigenvalues. Since there are an even number of such R_i's, this map sends the connected component of $\mathbb{1}$ into itself and A to $\mathbb{1}$. Hence A belongs to the connected component of $\mathbb{1}$ and this component coincides with $GL^+(n,\mathbb{R})$.

Once we introduce "objects" (Lie groups), it is good strategy to introduce "sub objects" and morphisms, i.e. maps that preserve the structure.

6.11. DEFINITION. Let G, H be Lie groups.

- A *Lie groups homomorphism* $\phi : G \longrightarrow H$ is a smooth map which is a group homomorphism. If ϕ has an inverse, which is a Lie groups homomorphism, we will say that ϕ is a *Lie groups isomorphism.*
- A *Lie subgroup* of H is a pair (G, ϕ) where G is a Lie group and $\phi : G \longrightarrow H$ is a group homomorphism which is a 1-1 immersion.

We will prove some results on uniqueness of subgroups that verify certain condition. So we will make clear what we mean that two subgroups are "equal".

6.12. DEFINITION. Let H be a Lie group and $(G_i, \phi_i),\ i = 1, 2$ be two subgroup. We will say that two subgroups are equivalent if there exists a Lie group isomorphism $\psi : G_1 \longrightarrow G_2$ such that $\phi_2 \circ \psi = \phi_1$.

6.13. PROPOSITION. *If $\phi : G \longrightarrow H$ is a Lie group homomorphism, then* $\mathrm{d}\phi(e) : \widehat{G} \longrightarrow \widehat{H}$ *is a Lie algebras homomorphism.*

PROOF. Let $X \in \widehat{G}$ and $\tilde{X}$ be the left invariant field in H with $\tilde{X}(e) = \mathrm{d}\phi(e)X(e)$ (we are denoting by e the identity element both in G and in H). Since ϕ is a homomorphism, $L_{\phi(g)} \circ \phi = \phi \circ L_g$. Hence

$$\tilde{X}(\phi(g)) = \mathrm{d}L_{\phi(g)}\tilde{X}(e) = \mathrm{d}L_{\phi(g)}\mathrm{d}\phi(e)(X(e)) =$$

$$= \mathrm{d}(L_{\phi(g)} \circ \phi)(e)(X(e)) = \mathrm{d}(\phi \circ L_g)(e)(X(e)) = \mathrm{d}\phi(g)(X(g)).$$

Hence X and $\tilde{X}$ are ϕ-related. In particular, if $X, Y \in \widehat{G}$,

$$\widetilde{[X, Y]}(e) = [\tilde{X}, \tilde{Y}](e) = \mathrm{d}\phi(e)([X, Y]).$$

□

A Lie subgroup $\phi : H \longrightarrow G$, determines a Lie subalgebra $\mathrm{d}\phi(e)(\widehat{H}) \subseteq \widehat{G}$, by Proposition 6.13. We will prove that, conversely, a Lie subalgebra determines a Lie subgroup. We will start with a preliminary result.

6.14. LEMMA. *Let G be a* connected *Lie group and U an open neighborhood of $e \in G$. Then $G = \cup_{n=1}^{\infty} U^n$, where U^n is the set of products on n elements in U.*

PROOF. Consider $U^{-1} := \{g^{-1} : g \in U\}$. Then U^{-1} is also an open neighborhood of e and so is $V := U \cap U^{-1}$. We will show that $G = \cup_{i=1}^{\infty} V^n \subset \cup_{n=1}^{\infty} U^n$. Set $H = \cup_{i=1}^{\infty} V^n$. Then

- H is an abstract subgroup (if $g, h \in H, gh^{-1} \in H$).
- H is open (if $g \in H, gV \subseteq H$).

Now G is covered by the cosets $\{gH\}, g \in G$. Since H is open these cosets are open. In particular $G \backslash H$ is an open set, since is union of the cosets of H, different from H. Since G is connected and $H \neq \emptyset$, $G = H$. □

6.15. THEOREM. *Let G be a Lie group and let $\widetilde{H} \subset \widehat{G}$ be a subalgebra. Then there exists a unique connected subgroup (H, ϕ) of G such that $\mathrm{d}\phi(e)(\widehat{H}) = \widetilde{H}$.*

PROOF. Consider the distribution

$$\mathcal{D}(g) = \mathrm{d}L_g(e)(\widetilde{H}).$$

Since $\widetilde{H}$ is a subalgebra, it follows easily that $\mathcal{D}$ is involutive. Let (H, ϕ) a maximal integral manifold of $\mathcal{D}$ containing e. Set $e = \phi^{-1}(e)$.

<u>CLAIM 1</u>: $\phi(H)$ is an abstract subgroup of G.

PROOF. Let $g \in \phi(H)$. Consider the map $L_{g^{-1}} \circ \phi$. This map is an injective immersion whose image contains e. Moreover

$$\mathrm{d}L_{g^{-1}} \circ \phi(e)(T_e P) = \mathrm{d}L_{g^{-1}}(\widetilde{H}) = \mathcal{D}(g^{-1}),$$

so it is also an integral manifold of $\mathcal{D}$. By maximality $L_{g^{-1}} \circ \phi(H) \subseteq \phi(H)$. Hence, given $g, h \in \phi(H), g^{-1}h \in \phi(H)$ and the Claim is proved. ☐

<u>CLAIM 2</u>: H is a Lie group.

PROOF. We can define an abstract group structure on H by requiring that $\phi : H \longrightarrow \phi(H)$ to be an abstract groups isomorphism. Since H is a smooth manifold, we have only to check that the map $\beta : H \times H \longrightarrow H, \;\; \beta(g, h) := g^{-1}h$, is smooth. This follows from Lemma **??**. ☐

To conclude the proof of the Theorem we need to prove unicity (in the sense of Definition 6.12). Suppose that (K, ψ) is an other such subgroup. Then (K, ψ) must be an other integral manifold of $\mathcal{D}$, with $e \in \psi(K)$. Since (H, ϕ) is maximal, $\psi(K) \subseteq \phi(K)$. From Lemma **??** oh Chapter 2, it follows that we have a well defined smooth map $\kappa : K \longrightarrow H$ such that $\phi \circ \kappa = \phi$. It follows that κ is an injective Lie groups homomorphism. We want to show that κ is onto. Now $\mathrm{d}\kappa(e)$ is an isomorphism an so, by the inverse function Theorem it maps a neighborhood of $e \in H$ diffeomorphically onto a neighborhood of $e \in H$. Now, by Lemma 6.14, both K and H are generated by such neighborhoods, a fact that clearly implies that κ is onto. ☐

6.16. COROLLARY. *Let G be a Lie group. Then there is a 1-1 correspondence between connected Lie subgroups of G and Lie subalgebras of $\widehat{G}$.*

A particularly interesting case is the following

6.17. DEFINITION. A 1-*parameter subgroup* of a Lie group G is a 1-dimensional Lie subgroup.

Let $X \in \widehat{G}$ and $\tilde{X}$ be the left invariant extension of X. By Proposition 6.9 $\tilde{X}$ is complete hence the integral curve of $\tilde{X}$ thought $e \in G$ is a smooth map $\gamma_X : \mathbb{R} \longrightarrow G$ which, again by Proposition 6.9, it is an abstract group homomorphism. Moreover, if $X \neq 0, \;\; \tilde{X}$ is everywhere nonzero and γ is an immersion. Now the integral curves of a nonzero complete vector field are either injective or periodic (see Remark **??** of Chapter 2). In the first case $(\mathbb{R}, \gamma_X)$ is a Lie subgroup. In the second case, γ_X induces an embedding $\overline{\gamma}_X : S^1 \longrightarrow G$. Let p be the period of γ_X. We can define a Lie group structure on S^1, identifying it with $\mathbb{R}/p\mathbb{Z}$ and, with this structure, $(S^1, \overline{\gamma}_X)$ is a Lie subgroup of G.

6.18. DEFINITION. Let G be a Lie group. We define the *exponential map* :

$$\exp : T_e \longrightarrow G, \qquad \exp(X) = \gamma_X(1),$$

where γ_X is the integral curve of the left invariant extension of X with $\gamma_X(0) = e$.

6.19. Lemma. $\exp(tX) = \gamma_X(t)$.

Proof. Fix t an define $\sigma(s) = \gamma_X(st)$. Then σ is an integral curve of tX (which is left invariant) with $\sigma(0) = e$. Hence $\exp(tX) = \sigma(1) = \gamma(t)$. □

6.20. Lemma. *The exponential map is smooth and* $\mathrm{d}\exp(0) : T_0T_eG \cong T_0G \longrightarrow T_0G$ *is the identity map.*

Proof. $\mathrm{d}\exp(0)(X)$ is the tangent vector, at $e \in G$, of the curve $\exp(tX) = \gamma_X(t)$. Hence the conclusion. □

In particular exp is a diffeomorphism of a neighborhood of $0 \in T_eG$ onto a neighborhood of $e \in G$. Composition of this chart with left translations give an atlas for G.

A useful property of the exponential map is that it is *natural* with respect to Lie groups homomorphisms

6.21. Proposition. *Let* $\phi : G \longrightarrow H$ *be a Lie groups homomorphism. Then* $\phi \circ \exp = \exp \circ \mathrm{d}\phi(e)$.

Proof. Let $X \in \widehat{G}$. Then $\gamma(t) = \phi(\exp(tX))$ is a smooth curve in H with $\dot\gamma(0) = \mathrm{d}\phi(e)(X)$. Since ϕ is a homomorphism, γ is also a 1-parameter subgroup, hence $\phi(\exp(tX)) = \exp(t\mathrm{d}\phi(e)(X))$. □

Proposition 6.21 allows us to prove that, under natural conditions, an abstract subgroup of a Lie group is, in fact, a Lie subgroup.

6.22. Theorem. *Let* G *be a Lie group and* $H \subseteq G$ *be an (abstract) subgroup which is a closed subset. Then there exists a unique manifold structure in* H *such that the inclusion* $i : H \longrightarrow G$ *is a Lie subgroup.*

Proof. □

A useful consequence is the following

6.23. Proposition. *Let* $\psi : G \longrightarrow H$ *be a Lie groups homomorphism. Then* $\ker\psi$ *is a Lie subgroup of* G *with Lie algebra* $\ker \mathrm{d}\psi(e)$.

Proof. Set $K = \ker\psi$. Then K is a closed subgroup, hence a Lie subgroup, by Theorem 6.22. Now $X \in \widehat{K}$ if and only if $\exp(tX) \in K$, by Proposition 6.21, and this occurs if and only if $X \in \ker \mathrm{d}\psi(e)$. □

7. The adjoint representation

Let $\mathbb{E}$ be a (finite dimensional) vector space. We will denote by $End(\mathbb{E})$ the space of linear maps of $\mathbb{E}$ into itself and by $Aut(\mathbb{E})$ the subspace of the invertible ones. Once fixed a bases of $\mathbb{E}$ we will identify such spaces with $M(n, \mathbb{R})$ and $GL(n, \mathbb{R})$ respectively.

7.1. Definition. Let G be a Lie group. A *representation* of G in $\mathbb{E}$ is a Lie groups homomorphism $\rho : G \longrightarrow Aut(\mathbb{E})$.

We will introduce now a very important representation. Consider a Lie group G and the map $\kappa_g : G \longrightarrow G$, $\kappa_g(h) = ghg^{-1}$. Clearly $\kappa_g(e) = e \ \ \forall \ \ g \in G$.

7.2. Definition. The *adjoint representation* of G is the representation $Ad : G \longrightarrow Aut(\widehat{G})$, $Ad(g) = \mathrm{d}\kappa_g(e)$.

We also define the map

$$ad : \widehat{G} \longrightarrow End(\widehat{G}) = T_{\mathbb{1}} Aut(\widehat{G}), \qquad ad(X) = \mathrm{d}Ad(e).$$

By Proposition 6.21 we have the commutative diagrams

$$\begin{tikzcd} G \arrow[r, "Ad"] & Aut(G) \\ \widehat{G} \arrow[u, "\exp"] \arrow[r, "ad"] & End(\widehat{G}) \arrow[u, "\exp"'] \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\kappa_g"] & G \\ \widehat{G} \arrow[u, "\exp"] \arrow[r, "Ad(g)"] & \widehat{G} \arrow[u, "\exp"'] \end{tikzcd}$$

In particular

$$\exp(tAd(g)X) = g\exp(tX)g^{-1}.$$

In the case $G = GL(n, \mathbb{R})$ we have

$$Ad(X)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} X\exp(tY)X^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\exp(tXYX^{-1}) = XYX^{-1},$$

$$ad(X)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} Ad(\mathbb{1} + tX)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(\mathbb{1} + tX)Y(\mathbb{1} + tX)^{-1} = XY - YX = [X, Y].$$

The last formula is a general fact.

7.3. PROPOSITION. *If G is a Lie group, $X, Y \in \widehat{G}$,* $ad(X)(Y) = [X, Y]$.

PROOF. First observe that

$$\begin{aligned} ad(X)(Y) = \mathrm{d}Ad(e)(X)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} Ad(\exp(tX))(Y) = \\ = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} Ad(\exp(tX))(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}(\kappa_{\exp(tX)})(Y). \end{aligned}$$

We extend X, Y to left invariant fields $\tilde{X}$, $\tilde{Y}$ and denote by ϕ_t is the flux of $\tilde{X}$. Then

$$\begin{aligned} ad(X)(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}R_{\exp(-tX)} \circ \mathrm{d}L_{\exp(tX)}(Y) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}R_{\exp(-tX)}\tilde{Y}(\exp(tX)) = \\ = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\mathrm{d}\phi_{-t}(\tilde{Y}(\phi_t(e)) = [X, Y]. \end{aligned}$$

□

We will study now some special subgroups. Let G be a group and let $\widehat{G}$ be a Lie algebra, We define

- the *center* of G, $Z(G) = \{g \in G : gh = hg, \ \ \forall \ \ h \in G\}$,
- the *center of* $\widehat{G}$, $Z(\widehat{G}) = \{X \in \widehat{G} : [X, Y] = 0, \ \ \forall \ \ Y \in \widehat{G}\}$.

G is Abelian if and only if $Z(G) = G$. By analogy we will say that $\widehat{G}$ is an *Abelian Lie algebra* if $Z(\widehat{G}) = \widehat{G}$.

If G is a Lie group, $Z(G)$ is a closed subgroup, hence, by Theorem 6.22 a Lie subgroup.

7.4. PROPOSITION. *If G is a connected Lie group,* $\ker Ad = Z(G)$ *and* $Z(\widehat{G}) = \widehat{Z(G)}$.

PROOF. Let $g \in Z(G)$. Then $\exp(tX) = g\exp(tX)g^{-1} = exp(tAd(g)(X))$. If t is sufficiently small, $tX, tAd(g)(X)$ belong to a neighborhood of $0 \in \widehat{G}$ where exp is a diffeomorphism onto its image. Hence $tX = Ad(g)(X)$, i,e $g \in \ker Ad$. Conversely, suppose $g \in KerAd$. Then $\exp(X) = \exp(Ad(g)(X)) = g\exp(H)g^{-1}$. In particular g commutes with elements in the image of exp. But the image of exp contain an open neighborhood U of e. Since G is connected, any element $h \in G$ is product of elements in U, by Lemma 6.14, and $g \in Z(G)$. The last Claim follows from Proposition 6.23 applied to $Ad : G \longrightarrow Aut(\widehat{G})$. ☐

7.5. Corollary. *A connected Lie group is Abelian if and only is its Lie algebra is Abelian.*

7.6. Corollary. *If G is a Lie group and $X, Y \in \widehat{G}$ are commuting tangent vectors (i.e. $[X,Y]=0$), then* $\exp(X+Y) = \exp(X)\exp(Y)$.

Proof. Consider $\widehat{H} = span\{X, Y\}$. Then $\widehat{H}$ is an Abelian subalgebra of $\widehat{G}$ and the associated subgroup H is Abelian. Then the map

$$\alpha : \mathbb{R} \longrightarrow H, \qquad \alpha(t) = \exp(tX)\exp(tY)$$

is a smooth homomorphism. Therefore $\alpha(t) = \exp(t\dot{\alpha}(0)) = \exp(t(X+Y))$. ☐

7.7. Proposition. *Let H be a connected Lie subgroup of a connected Lie group G. Then H is normal if and only if $\widehat{H}$ is an ideal of $\widehat{G}$.*

Proof. Assume that $\widehat{H}$ is an ideal of $\widehat{G}$. We want to prove that H is normal. Let $Y \in \widehat{H},\ X \in \widehat{G}, g = \exp(X)$. Then

$$g\exp(Y)g^{-1} = \exp(Ad(\exp(X))Y) = \exp(\exp(ad(X))Y) = \exp(\sum_{k=0}^{\infty}[k!]^{-1}ad(X)^k(Y).$$

Since $\widehat{H}$ is an ideal, $ad(X)^k(Y) \in \widehat{H}$, as follows from Proposition 7.3. Hence $ghg^{-1} \in H$, for g, h in small neighborhoods of e in G and H respectively. But G, H are connected, hence $ghg^{-1} \in H, \ \forall\, g \in G,\ h \in H$, by Lemma 6.14, and H is normal.

Conversely, suppose H normal in G. Let $Y \in \widehat{H},\ X \in \widehat{G}, t, s \in \mathbb{R}, g = \exp(tX)$. Then

$$g\exp(sY)g^{-1} = exp(Ad(g)(sY)) = \exp(s[\exp((ad(tX)(Y))]) \in H.$$

Hence $\exp((ad(tX)(Y)) \in \widehat{H},\ \ forall\ t \in \mathbb{R}$. Now

$$\exp((ad(tX)(Y)) = \exp(t\ ad(X)(Y)) = Y + t\ ad(X)(Y) + \sum_{k=2}^{\infty}[k!]^{-1}t^k ad(X)^k(Y)$$

is a smooth curve in $\widehat{H}$ with tangent vector, at zero, $ad(X)(Y) = [X,Y]$. Hence $[X,H] \in \widehat{H}$ and $\widehat{H}$ is an ideal. ☐

8. Covering groups

In this subsection we will study coverings of a Lie group. We refer to the Appendix for the basic facts on covering spaces theory.

We will start with a simple fact.

8.1. Proposition. *Let $\phi : G \longrightarrow H$ be a surjective Lie group homomorphism. The following conditions are equivalent*

(1) $\ker\phi$ *is discrete,*
(2) $\mathrm{d}\phi(e) : \widehat{G} \longrightarrow \widehat{H}$ *is an isomorphism,*
(3) ϕ *is a covering map.*

Proof. Since ϕ is smooth and surjective, $\dim(G) \geq \dim(H)$ by Theorem **??** of Chapter 2. Let $K := \ker\phi$. If K is discrete, $\{0\} = \widehat{K} = \ker \mathrm{d}\phi(e)$ and $\mathrm{d}\phi(e)$ is an isomorphism. Conversely, suppose that $\mathrm{d}\phi(e)$ is an isomorphism. Then ϕ is a diffeomorphism of a neighborhood U of $e \in G$ onto an open neighborhood V of $e \in H$. In particular $U \cap K = \{e\}$. Let $g \in K$. Then $gU \cap K = \{g\}$. In fact, if $h \in gU \cap K, g^{-1}h \in U \cap K$ and therefore $g = h$. Therefore K is discrete. Suppose now that ϕ is a covering map. Then $K = \phi^{-1}(e)$ is discrete, being the fibre over e. So we are left to prove that condition (1) (or (2)) implies that ϕ is a covering map. Let U, V be as above. Then $\phi^{-1}(V) = \cup_{g \in K} gU$. Now the (distinct) gU's are disjoint, by the argument used above, and $\phi|gU : gU \longrightarrow V$ is a diffeomorphism, hence V is evenly covered. Let $h \in H$ and consider the open neighborhood hV. Then $\phi^{-1}(hV) = \cup_{g \in \phi^{-1}(h)} gU$. Then these open sets are disjoint, ϕ restricted to each of them is a diffeomorphism onto hV, and the latter is evenly covered. So ϕ is a covering map. ☐

8.2. Corollary. *Let G be a simply connected Abelian Lie group. Then G is isomorphic to the additive group $\mathbb{R}^n$.*

Proof. We consider the (additive) vector group $\widehat{G} \cong \mathbb{R}^n$. Since G is Abelian, $\exp : \widehat{G} \longrightarrow G$ is a group homomorphism, by Corollary 7.6. We will show that it is surjective. Let $g \in G$. Since G is connected, g is product of elements in a small neighborhood of e, which is in the image of exp. Hence $g = \exp(X_1) \cdots \exp(X_k) = \exp(X_1 + \cdots + X_k)$. Since $\mathrm{d}\exp(0) = \mathbb{1}$, *exp* is a covering map by Proposition 8.1. Since G is simply connected, exp is a diffeomorphism and, being a homomorphism, it is a Lie groups isomorphism. ☐

We will classify now all Abelian Lie groups. We need the following

8.3. Lemma. *Let $K \subseteq \mathbb{R}^n$ be a discrete non trivial subgroup. Then there exist linearly independent vectors, $v_1, \ldots, v_k$, that generates K.*

Proof. We proceed by induction on n. Let $n = 1$ and let $g \in K \setminus \{e\}$ be an element minimal norm. Suppose $h \in K \cap (mg, (m+1)g$ then $h - mg \in K$ and $|h - mg| < |g|$, contradicting the minimality of $|g|$. Hence $h \in g\mathbb{Z}$ and K is generated by g. Suppose the Lemma holds for discrete subgroups of $\mathbb{R}^{n-1}$. Take again an element $g \in K \setminus \{e\}$ of minimal norm. Let $\pi : \mathbb{R}^n \longrightarrow g^{\perp}$ be the orthogonal projection. ☐

8.4. Theorem. *An Abelian Lie group G is isomorphic to the direct product of the k-torus $S^1 \times \cdots \times S^1$, ($k$ factors), and $\mathbb{R}^{n-k}$.*

Proof. First observe that, by Corollary 7.6 $\exp : \widehat{G} \longrightarrow G$ is a homomorphism. Also exp is surjective, by the same argument used in Corollary 8.2. Hence $K = \ker\exp$ is a discrete subgroup and $G \cong \widehat{G}/K$. By Lemma 8.3 K is generated by k independent vectors. Consider a linear isomorphism $L : \widehat{G} \longrightarrow \mathbb{R}^n$ which sends these vectors in the first k vectors of the canonical basis. Then L induces a Lie group isomorphism $\widetilde{L} : \widehat{H}/K \longrightarrow \mathbb{R}^n/\Gamma$ where Γ is the subgroup spanned by the first k vectors of the canonical basis. The conclusion follows from the simple fact that $\mathbb{R}^n/\Gamma$ is Lie isomorphic to $S^1 \times \cdots \times S^1 \times \mathbb{R}^{n-k}$. ☐

8.5. Theorem. *Let G be a Lie group and let $\pi : \overline{G} \longrightarrow G$ be a covering map. Then $\overline{G}$ has a (unique) manifold structure such that π is a Lie groups homomorphism.*

Proof. (Sketch) Consider the commutative diagram

☐

In particular, since manifolds are locally semi simply connected, there is a (unique) simply connected Lie group $\widetilde{G}$, that covers G. For simply connected groups we have the following important result

8.6. THEOREM. *Let G, H be Lie groups with G simply connected. If $\psi : \widehat{G} \longrightarrow \widehat{H}$ is a Lie algebras homomorphism, then there exists a unique Lie groups homomorphism $\phi : G \longrightarrow H$ such that $\psi = \mathrm{d}\phi(e)$. In particular simply connected Lie groups with isomorphic Lie algebras are Lie isomorphic.*

PROOF. $\square$

8.7. REMARK. A celebrated result of Ado, that we will not prove here, states that a (finite dimensional) Lie algebra $\widehat{H}$ is a subalgebra of $\widehat{GL(n,\mathbb{R})}$, for some n. In particular there is a Lie group H with Lie algebra $\widehat{H}$, by **??**, and , in particular, a (unique) simply connected one, by Theorem 8.6. Hence

8.8. THEOREM. *There is a 1-1 correspondence between finite dimensional Lie algebras and simply connected Lie groups.*

9. Group actions and homogeneous spaces

Let G be a Lie group, H a subgroup and $\pi : G \longrightarrow G/H$ be the quotient map. If H is not normal, the quotient set G/H does not have a natural group structure. However has a natural topology, the quotient topology. In this topology the open sets are the ones whose inverse image is open.

9.1. THEOREM. *If (H, ϕ) is a Lie subgroup of G, and ϕ is an embedding with closed image*[13] *, then the quotient space $G/\phi(H)$ has a unique smooth structure such that*

1. *π is smooth,*
2. *there are local smooth sections, i.e., for every point in the quotient there exists an open neighborhood U and a smooth map $s : U \longrightarrow G$ such that $\pi \circ s = \mathbb{1}_U$.*

PROOF. Since ϕ is an embedding we will identify H with its image $\phi(H)$. $\square$

The following corollary will be useful to prove connectness of some Lie groups

9.2. COROLLARY. *If $H, G/H$ are connected, then G is connected.*

PROOF. $\square$

9.3. DEFINITION. Manifolds of the type G/H where G is a Lie group and H is a closed subgroup, are called *homogeneous manifolds.*

Let M be a smooth manifold and G a Lie group.

9.4. DEFINITION. A smooth left *action* of G on M is a smooth map $\mu : G \times M \longrightarrow M$ such that

1. $\mu(e, x) = x \ \ \forall \ \ x \in M$,
2. $\mu(g, \mu(h, x)) = \mu(gh, x) \ \ \forall \ \ x \in M, \ g, \ h \in G$.

---

[13] It is known that ϕ is an embedding if and only if it has closed image. We will not prove this fact in those notes.

When the action is fixed we will write gx for $\mu(g,x)$.

We can define the "left translations" $\mu_g : M \longrightarrow M$, $\mu_g(x) = gx$. These are clearly smooth maps an, in fact, diffeomorphisms, since $(\mu_g)^{-1} = \mu_{g^{-1}}$, as follows easily from the definition. So we can think of an action as a homomorphism of the group into the group of diffeomorphism of the manifold (with suitable conditions of differentiability).

9.5. Definition. Given an action of G on M we define

(1) $\mathcal{O}(x) := \{gx : g \in G\}$, the *orbit* of x,
(2) $H(x) := \{g \in G : gx = x\}$, the *isotropy (sub)group* of x.

Moreover we will say that the action is *transitive* is $\mathcal{O}(x) = M$,

9.6. Remark. If the action is transitive then $\mathcal{O}(y) = M, \quad \forall\, y \in M$. In fact, given $z \in M, z = hx, y = kx$ and $z = hk^{-1}y \in \mathcal{O}(y)$.

9.7. Example. Identify $S^1 = SO(2)$ with the group of rotations of $\mathbb{R}^3$ around the z-axis. Then S^1 acts on S^2 in a natural way. If $p \in S^2$ is not on the z-axes, $\mathcal{O}(p)$ is the "parallel" trough p. If p is the north or south pole, then $\mathcal{O}(p) = p$. In the first case the isotropy subgroup is the identity, in the second one is the all S^1.

This action induces an action on $\mathbb{R}P^2$. If we look at $\mathbb{R}P^2$ as the upper hemisphere S^+, modulo the identification $x \sim \pm x, x \in \partial S^+$, we have that S^1 still fixes the north pole, whose isotropy is the all S^1, the other orbits are copies of S^1, with trivial isotropy, except for the orbit of $x \in \partial S^+$ that is a $\mathbb{R}P^1$, which is diffeomorphic to S^1, but the isotropy is $\{\pm 1\} \subseteq S^1$.

9.8. Theorem. *Let M be a differentiable manifold and G a Lie group that acts transitively on M. Fix $p \in M$ an let $H = H(p)$ be the isotropy subgroup. Then the map*

$$h : G/H \longrightarrow M, \qquad h(gH) = gp$$

is well defined and a diffeomorphism.

Proof. □

9.9. Remark. Let G be a Lie group and H a closed subgroup. Then there is a natural *transitive* action

$$G \times G/H \longrightarrow G/H, \qquad g(hH) = (gh)H.$$

In particular a manifold M is (diffeomorphic to) a homogeneous manifold if and only if there is a Lie group G that acts transitively on M. We want to stress the fact that G has to be a Lie group and not only a topological group. In fact, for any connected manifold M, the group of diffeomorphism of M, which is a topological group with a suitable topology, acts transitively on M (see Exercise **??** of Chapter 2). However this group is not a Lie group.

Naturally there may be several groups that acts transitively on the same differentiable manifold.

9.10. Example. Let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. There is a natural action of $O(n+1)$ on S^n, namely $\mu(A,x) = A(x)$. Let $e_1 = (1,0,\ldots,0) \in S^n$. This action is clearly transitive. The isotropy group is the $H(e) = \{[a_{ij}] \in O(n+1) : a_{1i} = a_{i1} = \delta_{1i}$ which is clearly isomorphic to $O(n)$. Then S^n is diffeomorphic to

$O(n+1)/O(n)$. The action above induces a transitive action of $O(n+1)$ on $\mathbb{R}P^n = S^n/\{\pm 1\}$. The isotropy group is $H([e_1]) = \{[a_{ij}] \in O(n+1) : a_{1i} = a_{i1} = \pm\delta_{1i}$.

The subgroup $SO(n+1) \subseteq O(n+1)$ also acts transitively on S^n and $\mathbb{R}P^n$. We leave to the reader the task of determine the isotropy subgroups.

The following result is useful to prove connectness of some Lie groups.

9.11. Proposition. *let G be a Lie group and $H \subseteq G$ a closed subgroup. If H and F/H are connected, then G is connected.*

Proof. $\square$

9.12. Example. As an example we will use the result above to prove to prove that $SO(n)$ is connected. We proceed by induction. $SO(2) = S^1$ hence it is connected. Suppose $SO(n-1)$ connected and consider the homogeneous manifold $S^{n-1} = SO(n)/SO(n-1)$ (see Example 9.10). Since S^{n-1} and $SO(n-1)$ are connected, the latter by the inductive hypothesis, it follows that $SO(n)$ is connected.

10. Exercises

10.1. Prove that a smooth (sub)manifold is connected if and only if it is path-connected.

10.2. Prove that the definitions of tangent space given in Remark 1.11 are equivalent.

10.3. Let $M \subseteq \mathbb{R}^N$ be a submanifold. For $p \in M$ let $\{X_i\},\ \{\xi_j\}$ be bases for T_pM and $[T_pM]^\perp$ respectively. Show that $\{(X_i, 0),\ (0, X_j)\}$ and $\{(X_i, 0),\ (0, \xi_j)\}$ are bases for $T_{(p,0)}TM$ and $T_{(p,0)}\nu M$ respectively.

10.4. Let M be a smooth manifold and $p \in M$. Consider the set $\widetilde{T_pM} = \{\gamma : I \longrightarrow M : \gamma \text{ is smooth and } \gamma(0) = p\}$, where $I \subseteq \mathbb{R}$ is an interval containing 0. In $\widetilde{T_pM}$ define the relation

$$\gamma \sim \sigma \quad \Longleftrightarrow \quad \frac{\mathrm{d}[\phi^{-1} \circ \gamma]}{\mathrm{d}t}(0) = \frac{\mathrm{d}[\phi^{-1} \circ \sigma]}{\mathrm{d}t}(0),$$

for a chart $\phi : \Omega \longrightarrow M$ with $p \in \phi(\Omega)$.

(1) Prove that $\sim$ does not depend on the chart and it is an equivalence relation.
(2) Define a vector space structure on $\widetilde{T_pM}/\sim$ such that $\widetilde{T_pM}/\sim\ \cong T_pM$.

10.5. Prove that $\mathbb{R}P^1$ is diffeomorphic to S^1 and $\mathbb{C}P^1$ is diffeomorphic to S^2. What about the quaternionic and Cayley projective lines?

10.6. Prove that if X is a smooth vector field in $\mathbb{R}P^n$ and $\pi : S^n \longrightarrow \mathbb{R}P^n$ is the projection, then there exist a smooth vector field $\tilde{X}$ on S^n such that $\mathrm{d}\pi(\tilde{X}) = X$. Under which condition a vector field on S^n is induced in this way by a vector field on $\mathbb{R}P^n$?

10.7. Prove that S^3 and S^7 are parallelizable.

10.8. Prove that if $\mathbb{R}P^n$ is parallelizable, then S^n is parallelizible.

10.9. Prove that $\mathbb{R}P^3$ and $\mathbb{R}P^7$ are parallelizable.

10.10. Prove the claims in Remark 4.7.

10.11. Let M, N be smooth manifolds and let $f : M \longrightarrow N$ be a smooth map. Suppose that f is a bijection and $\mathrm{d}f(x)$ is injective, $\forall x \in M$. Prove that f is a diffeomorphism (hint: use Remark 4.12 to prove that $\mathrm{d}f(x)$ is an isomorphism, i.e. M, N have the same dimension).

10.12. Prove that if $F : M \longrightarrow \mathbb{R}^N$ is an embedding, $F(M)$ is a submanifold of $\mathbb{R}^N$ in the sense of Definition 1.1.

10.13. Let M be a compact manifold and N a connected non compact manifold of the same dimension. Prove that there are no immersions of M into N.

10.14. Prove that if M is compact and $F : M \longrightarrow N$ is a 1-1 immersion, then F is an embedding.

10.15. Consider the map $V : S^{n-1} \subseteq \mathbb{R}^n \longrightarrow M(n, \mathbb{R})$, $V(x) = x^t x$, where $x = [x_1, \ldots, x_n]$ is viewed as an $1 \times n$ matrix and x^t is the transpose of x.

(1) Prove that V is an immersion (called the *Veronese immersion*).
(2) Prove that V induces an embedding $\tilde{V} : \mathbb{R}P^{n-1} \longrightarrow M(n, \mathbb{R})$ (called the *Veronese embedding*).
(3) Prove that the image of V is contained in a sphere of an $[\frac{n(n+1)}{2} - 1]$-dimensional affine subspace of $M(n, \mathbb{R})$.

10.16. Consider the sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Define $V : S^{2n-1} \longrightarrow M(n, \mathbb{C})$, $V(z) = z^* z$, where $z = [z_1, \ldots, z_n]$ and z^* is the transpose conjugate of z.

(1) Prove that V induces an embedding $\tilde{V} : \mathbb{C}P^{n-1} \longrightarrow M(n, \mathbb{C})$, the *Veronese embedding.*
(2) Use the same idea to define an embedding of the quaternionic projective space into $M(n, \mathbb{H})$, the space of $n \times n$ matrices with quaternionic entries.

10.17. Give an example of a 1-1 immersion $\phi : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that there exists a smooth curve $\sigma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^2$ with image contained in $\phi(R)$, but $\dot{\sigma}(0)$ not tangent to γ at $\sigma(0)$.

10.18. Let $\pi : M \longrightarrow N$ be a surjective local diffeomorphism. We will say that $\tilde{X} \in \mathcal{H}(M)$ *project* onto $X \in \mathcal{H}(N)$, if $\mathrm{d}\pi(x)(\tilde{X}) = X(\pi(x))$.

(1) Prove that, if π is surjective, $\forall\ X \in \mathcal{H}(N)$ there is a unique $\tilde{X} \in \mathcal{H}(M)$ that project onto X.
(2) Prove that if $\tilde{X}$ projects onto X, the map π sends integral curves of $\tilde{X}$ onto integral curves of X.

10.19. Consider the torus $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$ and the map $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$, $\pi(t, s) = (\cos t, \sin t, \cos s, \sin s)$.

(1) Prove that π is an immersion.
(2) Prove that π, as a map onto T^2, is a local diffeomorphism.
(3) Let $X = (x_0, y_0) \in R^2$ and extend X to a constant vector field in $\mathbb{R}^2$. Prove that X projects, via π, onto a vector field on the torus.
(4) Prove that, if y_0/x_0 is rational, the image of the integral curve $\gamma(t) = tX$ is an (embedded) closed curve.
(5) Prove that, if y_0/x_0 is irrational, the image of the integral curve $\gamma(t) = tX$ is an 1-1 immersed curve with dense image (in T^2). Conclude that such an integral curve is not embedded.

10.20. Prove that, for all $x, y \in \mathbb{R}^n$ there exists a diffeorphism $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\phi(x) = y$ and $\phi(z) = z$ for $\|z\|$ sufficiently large (hint: consider the (constant) vector field $X = y - x$. If γ_t is the flow of X, $\gamma_1(x) = y$. Take now a vector field that coincide with X in a ball containing x, y and vanish outside a bigger ball).

10.21. Let M be a *connected* smooth manifold, $p, q \in M$. Prove that there is a diffeomorfism $\phi : M \longrightarrow M$ such that $\phi(p) = q$ (hint: use the result above for the case in which p and q belong to the image of a given chart and use this fact to show that the set of points q' such that there exist a diffeomorphism taking a fixed p to q' is open and closed in M).

10.22. Let G be a Lie Group. Prove that the connected component of the identity, G_e is a normal open and closed subgroup. in particular, if G is compact, G_e is compact

10.23. Determine, up to isomorphisms, all 1-dimensional Lie groups and 2-dimensional simply connected Lie groups.

10.24. Let $\mathbb{K}$ be the field of real or complex numbers.Consider the canonical scalar product in $\mathbb{K}^n$, $\langle x, y \rangle = \sum_i x_i \overline{y_i}$. Prove that a $\mathbb{K}$-linear map $A : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ preserves the scalar product if and only is $AA^* = \mathbb{1}$, where A^* is the conjugate transpose (with respect to $\langle \cdot, \cdot \rangle$) of A.

10.25. Consider the groups $G = \{A \in L(K^n, \mathbb{K}^n) : AA^* = \mathbb{1}\}$. Define a Lie group structure on G and determine $\widehat{G}$ (as a Lie algebra). These groups are called the *orthogonal* an the *unitary* group, respectively, and are denoted by $O(n)$, $U(n)$.

10.26. Prove that $O(n)$ and $U(n)$ are compact, $U(n)$ is connected and $O(n)$ has two connected components. The connected components of $O(n)$ that contains $\mathbb{1}$ is called the *special orthogonal group* and is denoted by $SO(n)$.

10.27. Consider $SU(n) = \{A \in U(n) : \det(A) = 1\}$.

1. Prove that $SU(n)$ is a Lie subgroup of $U(n)$.
2. Prove that the map $m : S^1 \times SU(n) \longrightarrow U(n)$, $m(z, A) = zA$ is a surjective Lie group homomorphism.
3. Prove that $\ker m$ is a cyclic group of order n.

10.28. Let $\mathbb{Z}_2 = \{\pm 1\}$ be the multiplicative group with two elements.

1. Prove that $O(2n+1)$ is isomorphic, as Lie group, to $SO(2n+1) \times \mathbb{Z}_2$.
2. Prove that $O(2n)$ is diffeomorphic to $SO(2n) \times \mathbb{Z}_2$.
3. Define a Lie group structure on $SO(2n) \times \mathbb{Z}_2$ in such a way that it is isomorphic to $SO(2n)$.

10.29. Prove that $SL(n, \mathbb{R}) := \{A \in M(n, \mathbb{R}) : \det(A) = 1\}$ is a *connected* Lie subgroup of $GL(n, \mathbb{R})$ and compute $\widehat{SL(n, \mathbb{R})}$ (hint: compute $\mathrm{d}\det(A)(A)$). This group is called the *special linear group*.

10.30. Prove directly that $A \in Skw(n, \mathbb{R})$ if and only if $\exp(A) \in O(n)$ and $\det(A) = 1$ if and only if $\operatorname{trace}(A) = 0$.

10.31. Let $A \in M(n, \mathbb{R})$ be a positive symmetric matrix. Prove that $\exp^{-1}(A) \neq \emptyset$. Use this fact to prove that there exists a positive symmetric matrix B such that $B^2 = A$. The matrix B is also denoted by $\sqrt{A}$.

10.32. Let $Sym(n, \mathbb{R})^+$ be the set of positive definite symmetric matrices (observe that it is an open set of the vector space of symmetric matrices, hence a differentiable manifold).

(1) Prove the *polarization formula*: Given $A \in GL(n, \mathbb{R})$, there exist $P \in Sym(n, \mathbb{R})^+$, $R \in O(n)$ such that $A = PR$ (hint set $P = \sqrt{AA^t}$ and $R = P^{-1}A$).
(2) Prove that $m : Sym(n, \mathbb{R})^+ \times O(n) \longrightarrow GL(n, \mathbb{R})$, $m(P, T) = PT$ is a diffeomorphism.

10.33. Let D be the group of real upper triangular matrices with positive diagonal entries.

(1) Prove that, given $A \in GL(n, \mathbb{R})$, there exists a unique pair of matrices $(P, R) \in D \times O(n)$ such that $A = PR$ (hint: this is equivalent to the Gram-Schmidt orthonormalization process).
(2) Prove that that $m : D \times O(n) \longrightarrow GL(n, \mathbb{R})$, $m(P, T) = PT$ is a diffeomorphism. In particular $GL(n, \mathbb{R})$ is diffeomorphic to $\mathbb{R}^{n(n+1)/2} \times O(n)$.

10.34. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Consider $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. Identify K^* with the multiple of $\mathbb{1} \in GL(n, \mathbb{K})$. Show that $\mathbb{K}^*$ is a normal subgroup and define $PGL(n, \mathbb{K}) = GL(n, \mathbb{K})/\mathbb{K}^*$. Prove that $GL(n, \mathbb{K})/\mathbb{K}^*$ is a Lie group. This group is called the *projective linear group.*

10.35. Consider S^3 as the space of unit quaternions (see Remark 2.13 of Chapter 2 for the definition of quaternions).

(1) Prove that S^3 is a (non commutative) Lie group (with the induced product).
(2) Compute $\widehat{S^3}$ and show that it is isomorphic, as a Lie algebra, to $\mathbb{R}^3$ with the usual vector product.
(3) Prove that $\{\pm 1\} \subseteq S^3$ is a normal subgroup.
(4) Define a Lie group structure on $\mathbb{R}P^3$ such that the quotient map $q : S^3 \longrightarrow \mathbb{R}P^3$ is a Lie group homomorphism.

10.36. Consider S^3 as above. If $q \in S^3$, define $L_q : \mathbb{H} \cong \mathbb{R}^4 \longrightarrow \mathbb{H}$, $L_q(x) = qx\bar{q}$.

(1) Prove that $L_q \in O(4)$ and $L_q(1) = 1$.
(2) Define $\tilde{L}_q = L_q|1^\perp (1^\perp = \{p \in \mathbb{H} : p = xi + yj + zk\} \cong \mathbb{R}^3)$. Prove that the map $\pi : S^3 \longrightarrow SO(3)$, $\pi(q) = \tilde{L}_p$ is a Lie group homomorphism.
(3) Prove that $SO(3)$ is isomorphic, as a Lie group to $\mathbb{R}P^3$.

10.37. Consider the map $L_{p,q} : \mathbb{H} \longrightarrow \mathbb{H}$, $L_{p,q}(x) = px\bar{q}$.

(1) Prove that, if $p, q \in S^3$, $L_{p,q} \in SO(4)$.
(2) Prove that the map $L : S^3 \times S^3 \longrightarrow SO(4)$, $L(p, q) = L_{p,q}$ is a surjective Lie group homomorphism.
(3) Compute $\ker L$ and show that L induces a diffeomorphism of $SO(4)$ onto $S^3 \times \mathbb{R}P^3$.
(4) Determine a Lie group structure on $S^3 \times \mathbb{R}P^3$ that makes the diffeomorphism above a Lie group isomorphism.

10.38. Exhibit the Stieffel and Grasmann manifold as homogeneous manifolds (see Chapter 2 for definition). Prove that they are compact and connected.

10.39. Define transitive actions of $PGL(n+1,\mathbb{K})$ on $\mathbb{K}P^n$. Determine the isotropy subgroups.

10.40. Prove that the matrix $\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ is not not in the image of $\exp : M(2,\mathbb{R}) \longrightarrow GL(2,\mathbb{R})$.

Remark: Just for the records. If G is a connected compact Lie group, $\exp : \widehat{G} \longrightarrow G$ is surjective.

CHAPTER 4

The theory of de Rham for differentiable manifolds

In this Chapter we will extend the concepts and results of the first two Chapters to the case of differentiable manifolds. Since we have done most of the hard work in the first two Chapters, this extension will very "smooth."

1. Cohomology and homology for differentiable manifolds

1.1. Definition. Let M be a differentiable manifold. A *differential p-form* is a law that associate to each $x \in M$ an exterior p-form $\omega(x) \in \Lambda^p(T_xM)$ that depends differentiably on x. This means that, given smooth vector fields $X_1, \dots, X_p \in \mathcal{H}(M)$, the function $y \rightsquigarrow \omega(y)(X_1(y), \dots, X_p(y))$ is smooth.

We will denote by $\Omega^p(M)$ the set of differential p-forms on M.

$\Omega^p(M)$ has an obvious structure of real vector space. Moreover we can multiply a differential form by a smooth function and this operation is associative and distributive, in the appropriate sense, i.e. $\Omega^p(M)$ is a *module over* $\mathcal{F}(M)$. Finally, the wedge product induces a wedge product of differential forms. So $\Omega^*(M) := \oplus\Omega^p(M)$ is an associative, graded commutative algebra.

A differential form $\omega \in \Omega^p(M)$ induces a $\mathcal{F}(M)$-multilinear map

$$\tilde{\omega} : \mathcal{H}(M) \times \cdots \times \mathcal{H}(M) \longrightarrow \mathcal{F}(M), \quad \tilde{\omega}(X_1, \dots, X_p)(x) = \omega(x)(X_1(x), \dots, X_p(x)).$$

As for the case of open sets of $\mathbb{R}^n$ we have the simple but important converse

1.2. Theorem. [Tensoriality criterion] *A $\mathbb{R}$-multilinear map*

$$\tilde{\omega} : \mathcal{H}(M) \times \cdots \times \mathcal{H}(M) \longrightarrow \mathcal{F}(M),$$

is induced by a differential form if and only if it is $\mathcal{F}(M)$-multilinear.

1.3. Example. Since $\Lambda^0(\mathbb{R}^n) = \mathbb{R}$, $\Omega^0(M) = \mathcal{F}(M)$.

1.4. Example. If $f \in \mathcal{F}(M)$, $\mathrm{d}f$ is the differential 1-form $\mathrm{d}f(X) = X(f)$.

Let $F : M \longrightarrow N$ be a smooth map. Then, for all $x \in M$ we have a linear map $\mathrm{d}F(x) : T_xM \longrightarrow T_{F(x)}N$ and an induced linear map

$$F^* : \Omega^p(N) \longrightarrow \Omega^p(M), \qquad F^*(\omega)(X_i, \dots, X_p) = \omega(\mathrm{d}F(X_1), \dots, \mathrm{d}F(X_p)).$$

It is easily seen that such map commutes with multiplications by functions in $\mathcal{F}(N)$, in the sense that $F^*(f\omega)(x) = (f \circ F)F^*(\omega)$. Also, $F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau)$, i.e. $F^* : \Omega^*(N) \longrightarrow \Omega^*(M)$ is an algebras homomorphism.

Finally we have the *functorial properties*

- $\mathbb{1}_M^* = \mathbb{1}_{\Omega^p(M)}$,
- If $F_1 : M_1 \longrightarrow M_2$ and $F_2 : M_2 \longrightarrow M_3$ are smooth maps, $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

In particular, if F is a diffeomorphism, F^* is an isomorphism.

Also we have thAs for the case of open sets of $\mathbb{R}^n$ we have the *exterior differential* of a differential form.

1.5. Theorem. *There exists a unique sequence of* $\mathbb{R}$*-linear operators* $\mathrm{d}^p : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ $p = 0, \ldots, n$, *such that:*

1. $\mathrm{d}^0 = \mathrm{d}$ *(the usual differential, see Example 1.4).*
2. $\mathrm{d}^{p+1} \circ \mathrm{d}^p = 0$.
3. *If* $\omega \in \Omega^p(U)$, $\tau \in \Omega^q(U)$, $\mathrm{d}^{p+q}\omega \wedge \tau = \mathrm{d}^p\omega \wedge \tau + (-1)^p \omega \wedge \mathrm{d}^q\tau$.

Moreover, if $F : M \longrightarrow N$ *is a smooth map and* $\omega \in \Omega^p(N)$, $\mathrm{d}^p F^* \omega = F^* \mathrm{d}^p \omega$.

When clear from the context we will write simply d *for* d^p.

Proof. We will sketch a couple of possible approach to the proof, leaving the details to the reader. We can use the hint given by Proposition 2.11 in Chapter 1 and define, for $\omega \in \Omega^p(M)$,

$$\mathrm{d}\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \cdot \omega(X_0, \ldots, \hat{X}_i, \ldots X_p) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

Alternatively, we can proceed as follows: if ω has support in the image of a chart $\phi : U \subseteq \mathbb{R}^n \longrightarrow M$, we can define $\mathrm{d}\omega = (\phi^{-1})^* \mathrm{d}\phi^* \omega$ ($\mathrm{d}\phi^*\omega$ is the exterior differential of a form defined in $U \subseteq \mathbb{R}^n$). Again two ways to prove that this definition does not depend on the choice of the chart. Either we can prove that this definition coincides with the one given above, or we can use the unicity part of Theorem 2.9 of Chapter 1. Finally observe that any form is a (locally finite) sum of forms supported in images of charts. ☐

1.6. Remark. Just like in the case of open sets in $\mathbb{R}^n$, d is a local operator (Lemma 2.10 of Chapter 1).

So we have a sequence of vector spaces and $\mathbb{R}$-linear maps:

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}^0} \Omega^1(M) \xrightarrow{\mathrm{d}^1} \cdots \xrightarrow{\mathrm{d}^{n-1}} \Omega^n(M) \longrightarrow 0$$

which is a *cochain complex*, called the *de Rham complex of* M. We define, as in Chapter 1,

- $Z^p(M) := \ker\ \mathrm{d}^p$, *the space of* p*-cocycles* or *closed* p*-forms.*
- $B^p(M) := \mathrm{Im}\ \mathrm{d}^{p-1}$, *the space* p*-coboundaries* or *exact* p*-forms.*
- $H^p(M) := Z^p(M)/B^p(M)$, *the* p*-dimensional (de Rham) cohomology of* M.

Let $F : M \longrightarrow N$ be a smooth function and $F_p^* : \Omega^p(N) \longrightarrow \Omega^p(M)$ the induced map. It follows from the last claim in Theorem 1.5, that F^* sends closed forms to closed forms and exact forms to exact forms. Hence it induces a linear map, denoted by the same symbol,

$$F^* : H^p(N) \longrightarrow H^P(M), \qquad F^*([\alpha]) = [F^*\alpha],$$

where $[\tau]$ denotes the class of the form τ in the quotient space.

The basic functorial properties

- $\mathbb{1}_M^* = \mathbb{1}_{H^p(M)}$,
- If $F_1 : M_1 \longrightarrow M_2$ and $F_2 : M_2 \longrightarrow M_3$ are smooth maps, then $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$,

are obvious. In particular, if F is a diffeomorphism, F^* is an isomorphism. So the de Rham cohomology is a (differential) *topological invariant* of M.

The basic properties of the de Rham cohomology proved in Chapter 1 hold true with essentially the same proofs.

1.7. PROPOSITION.

- *If M is a point, $H^0(M) \cong \mathbb{R}$, $\ H^p(M) = \{0\}$, $\ p > 0$.*
- *M is connected if and only if $H^0(M) \cong \mathbb{R}$.*
- *If $M = \coprod M_\alpha$ (disjoint union) then $H^p(M) = \prod H^p(M_\alpha$ (direct product).*
- *If M, N are connected and $F : M \longrightarrow N$ is a smooth map, $F^* : H^0(N) \longrightarrow H^0(M)$ is an isomorphism.*

1.8. THEOREM. [Homotopy invariance for cohomology] *Let $F, G : M \longrightarrow N$ be a smooth maps between smooth manifolds. If F is homotopic to G, then $F^* = G^* : H^p(N) \longrightarrow H^p(M)$, $\ \forall\ p$. In particular, if $F : M \longrightarrow N$ is a homotopy equivalence, $F^* : H^p(N) \longrightarrow H^p(M)$ is an isomorphism.*

PROOF. It is enough to show, just like in the case of open sets of $\mathbb{R}^n$, that the inclusions $j_i : M \longrightarrow M \times \mathbb{R}$, $\ j_i(x) = (x, i)$, $\ i = 0, 1$, induce the same homomorphism in cohomology. For this is enough to produce an algebraic homotopy $\tilde{H} : \Omega^p(M \times \mathbb{R}) \longrightarrow \Omega^{p-1}(M)$ between the maps induced by the inclusions. We fix an atlas $\{U_i, \phi_i\}$ and a partition of unity λ_i dominated by the atlas. Given $\omega \in \Omega^p(M \times \mathbb{R})$ set $\omega_i = \lambda_i \omega$. Then ω_i is supported in $\phi_i(U_i) \times \mathbb{R}$ and $\omega = \sum \omega_i$. For fixed i, we have defined an algebraic homotopy $\tilde{H}_i : \Omega^p(U_i \times \mathbb{R}) \longrightarrow \Omega^{p-1}(U_i)$. We define $\tilde{H}(\omega_i) = (\phi_i^{-1})^* \tilde{H}_i(\phi_i \times \mathbb{1})^* \omega_i$ and $\tilde{H}(\omega) = \sum \tilde{H}(\omega_i)$. It is easy to see that $\tilde{H}$ is a well defined algebraic homotopy between the maps induced by the inclusions. ☐

1.9. THEOREM. [Mayer Vietoris sequence for de Rham cohomology] *Let M be a differentiable manifold, U_1, U_2 open sets such that $M = U_1 \cup U_2$. Let $V = U_1 \cap U_2$ and $k_i : V \longrightarrow U_i$, $\ j_i : U_i \longrightarrow M$ be the inclusions. Then there exists a sequence of linear maps $\Delta^* : H^p(V) \longrightarrow H^{p+1}(M)$, such that the sequence below is exact:*

$$\cdots \longrightarrow H^p(M) \xrightarrow{(j_1^*, j_2^*)} H^p(U_1) \oplus H^p(U_2) \xrightarrow{(k_1^* - k_2^*)} H^p(V) \xrightarrow{\Delta^*} H^{i+1}(M) \longrightarrow \cdots$$

PROOF. The result is a purely algebraic consequence of the exactness of the sequence

$$\{0\} \longrightarrow \Omega^p(U) \xrightarrow{(j_1^*, j_2^*)} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{(k_1^* - k_2^*)} \Omega^p(V) \longrightarrow \{0\}.$$

The exactness of this sequence is proved exactly as in the case of open sets of $\mathbb{R}^n$. ☐

1.10. EXAMPLE. Let S^n be the unit sphere. The map $r : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n$, $\ r(x) = \dfrac{x}{\|x\|}$ is a homotopy equivalence. Hence (see Example 4.15 and Exercise 7.28 in Chapter 1)

$$H^p(S^n) = \begin{cases} \mathbb{R} & \text{if } p = 0, n \\ \{0\} & \text{if } p \neq 0, n \end{cases}$$

1.11. EXAMPLE. Consider the complex projective space $\mathbb{C}P^n$. $\mathbb{C}P^1 = S^2$ and its cohomology was computed in the example above. We will prove, by induction, that

$$H^p(\mathbb{C}P^n) = \begin{cases} \mathbb{R} & \text{if } p = 2k, \quad 0 \leq k \leq n \\ \{0\} & \text{otherwise} \end{cases}$$

We can look at $\mathbb{C}P^n$ in two different ways

- as the quotient of the (closed) unit disk $D \subseteq \mathbb{C}^n$ modulo the equivalence relation $x \sim y \iff x = y$ or $\|x\| = \|y\| = 1$ and $x = \lambda y, \lambda \in \mathbb{C}$,
- as the quotient of the unit sphere $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ modulo the equivalence relation $x \sim y \iff x = \lambda y, \lambda \in \mathbb{C}$.

Look at the first model and define $U_1 = \{[x] \in \mathbb{C}P^n : \|x\| < 1\}$, $U_2 = \mathbb{C}P^n \setminus \{0\}$. Then we have

- U_1 is diffeomorphic to the open ball, hence contractible.
- The map $r : D \setminus \{0\} \longrightarrow S^{2n-1}$, $r(x) = \dfrac{x}{\|x\|}$, is a homotopy equivalence which induces an homotopy equivalence between U_2 and $\mathbb{C}P^{n-1}$ (second model)
- $U_1 \cap U_2 = \{x \in D : 0 < \|x\| < 1\}$, hence homotopy equivalent to S^{2n-1}.

For this decomposition the Mayer Vietoris sequence is

$$\cdots H^{p-1}(S^{2n-1}) \longrightarrow H^p(\mathbb{C}P^n) \longrightarrow H^p(U_1) \oplus H^p(\mathbb{C}P^{n-1}) \longrightarrow H^p(S^{2n-1}) \longrightarrow H^{p+1}(\mathbb{C}P^n) \longrightarrow \cdots.$$

Hence, if $0 < p \leq 2n-2$, $H^p(\mathbb{C}P^n) \longrightarrow H^p(U_1) \oplus H^p(\mathbb{C}P^{n-1}) = H^p(\mathbb{C}P^{n-1})$ is an isomorphism. For $p = 2n-1$ we have

$$\{0\} = H^{2n-2}(S^{2n-1}) \longrightarrow H^{2n-1}(\mathbb{C}P^n) \longrightarrow \{0\},$$

hence $H^{2n-1}(\mathbb{C}P^n) = \{0\}$. For $p = 2n$ we have

$$H^{2n-1}(U_1) \oplus H^{2n-1}(\mathbb{C}P^{n-1}) = \{0\} \longrightarrow H^{2n-1}(S^{2n-1}) \longrightarrow H^{2n}(\mathbb{C}P^n) \longrightarrow H^{2n}(U_1) \oplus H^{2n}(\mathbb{C}P^{n-1}) = \{0\},$$

hence $H^{2n}(\mathbb{C}P^n) \cong \mathbb{R}$. The conclusion follows by the inductive hypothesis (observe that $H^0(\mathbb{C}P^n) \cong \mathbb{R}$, by connectedness).

1.12. Example. Essentially the same argument used in the example above allow us to compute the cohomology of the quaternionic projective space $\mathbb{H}P^n$. We obtain

$$H^p(\mathbb{H}P^n) = \begin{cases} \mathbb{R} & \text{if } p = 4k, \quad 0 \leq k \leq n \\ \{0\} & \text{otherwise} \end{cases}$$

For the singular homology we also have very little new. If M is a differentiable manifold, a smooth p-simplex is a smooth map $\sigma : \Delta^p \longrightarrow M$, Δ^p being the standard p-simplex, and the space of p-chains of M, $C_p(M)$, is the real vector space with basis the singular simplices. We can define the boundary operator $\partial_p : C_p(M) \longrightarrow C_{p-1}(M)$ just like in Chapter 2 and obtain the singular chain complex of M

$$\cdots \longrightarrow C_{p+1}(M) \xrightarrow{\partial_{p+1}} C_p(M) \xrightarrow{\partial_p} C_{p-1}(M) \longrightarrow \cdots$$

whose homology, $H_p(M)$, is the *singular homology of* M.

olso, if $F : M \longrightarrow N$ is a smooth manifold, we have an induced morphism $F_* : C_p(M) \longrightarrow C_p(N)$ (composing a singular simplex with F) and this morphism induces a linear map (denoted by the same symbol) F_*, $F_* : H_P(M) \longrightarrow H_p(N)$. This construction verifies the *functorial properties*

- $\mathbb{1}_M = \mathbb{1}_{H_p(M)}$,
- $(F \circ G)_* = F_* \circ G_*$.

The basic properties of homology also hold true, with the same proofs as in the case of open sets of $\mathbb{R}^n$.

1.13. Proposition.

- *If M is a point,* $H_0(M) \cong \mathbb{R}, \quad H_p(M) = \{0\}, \quad p > 0.$
- *M is connected if and only if* $H_0(M) \cong \mathbb{R}$.
- *If* $M = \coprod M_\alpha$ *(disjoint union) then* $H_p(M) = \oplus H_p(M_\alpha$ *(direct sum).*
- *If M, N are connected and* $F : M \longrightarrow N$ *is a smooth map,* $F_* : H_0(N) \longrightarrow H_0(M)$ *is an isomorphism.*

1.14. Theorem. [Homotopy invariance for homology] *Let* $F, G : M \longrightarrow N$ *be a smooth maps between smooth manifolds. If F is homotopic to G, then* $F_* = G_* : H_p(N) \longrightarrow H_p(M), \quad \forall\ p$. *In particular, if* $F : M \longrightarrow N$ *is a homotopy equivalence,* $F_* : H_p(N) \longrightarrow H_p(M)$ *is an isomorphism.*

1.15. Theorem. [Mayer Vietoris sequence for singular homology] *Let M be a differentiable manifold,* $U_1,\ U_2$ *open sets such that* $M = U_1 \cup U_2$. *Let* $V = U_1 \cap U_2$ *and* $k_i : V \longrightarrow U_i,\ j_i : U_i \longrightarrow M$ *be the inclusions. Then there exists a sequence of linear maps* $\Delta_* : H_p(M) \longrightarrow H_{p-1}(M)$, *such that the sequence below is exact*

$$\cdots \longrightarrow H_p(V) \overset{[(k_1)_*,(k_2)_*]}{\longrightarrow} H_p(U_1) \oplus H_p(U_2) \overset{[(j_1)_*-(j_2)_*]}{\longrightarrow} H_p(M) \xrightarrow{\Delta_*} H_{p-1}(V) \longrightarrow \cdots .$$

2. The Theorems of de Rham and Künnet for differentiable manifolds

As in the case of open sets of $\mathbb{R}^n$ we can define, for a differentiable manifold M, integration of p-forms over p-chains of M. This construction induces, via the Theorem of Stokes, a de Rham map

$$dR_M : H^p(M) \longrightarrow [H_p(M)]^*,$$

which is *natural* with respect to smooth maps.

The de Rham Theorem, proved in Chapter 2 for open sets of $\mathbb{R}^n$, states that, in the latter case, dR is an isomorphism. In this section we will extend this result, and the Künnet formula, to the case of differentiable manifolds.

We can try to reproduce the proofs given for the case of open sets of $\mathbb{R}^n$. The careful reader understand quickly the this reproduction is immediate once we have a result of the type of Lemma 3.1 of Chapter 2. For the latter we encounter a couple of problems.

- We need a proper function.
- We need a concept of *convex sets*

The first problem is easily solved. In fact, by Whitney's Theorem we can assume that M is a submanifold of $\mathbb{R}^N$, which is a closed subset. Then the the restriction to M of the function $\phi(x) = \|x\|^2$ is a smooth proper function.

For the second we need a family of subsets of M, $\{C_\alpha, \alpha \in \mathcal{A}\}$ with the following properties

- each C_α is contractible,
- the intersection of two sets in the family is still in the family,
- $\forall\ x \in M$ and neighborhood $U \subseteq M$ of x, there exist α such that $C_\alpha \subseteq U$.

It can be proved that such a family indeed exists, the geodetically convex sets for example, and we will comment on this class in Chapter 5.

In this section we will take a different approach, which is interesting in itself. In fact we will prove that a smooth n-dimensional manifold M is homotopy equivalent to an open subset of $\mathbb{R}^N$, N sufficiently large, and use directly the case of open sets of $\mathbb{R}^N$ to obtain the desired generalizations.

Let M be a n-dimensional manifold. By Whitney's Theorem we can assume, without loss of generality, that M is a submanifold of $\mathbb{R}^N$ and a closed subset. Consider the *normal bundle* (see Example 1.19 of Chapter 3) and the *endpoint map*

$$\nu M = \{(p,\xi) \in M \times \mathbb{R}^N : \xi \in [T_pM]^{\perp}\}, \quad E : \nu M \longrightarrow \mathbb{R}^N, \quad E(p,\xi) = p + \xi.$$

Observe that E is a smooth map between manifolds of the same dimension. Let $\epsilon : M \longrightarrow \mathbb{R}$ be a *positive* continuous function. We set

$$\nu_\epsilon M = \{(p,\xi) \in M \times \mathbb{R}^N : \xi \in [T_pM]^{\perp}, \|\xi\| < \epsilon(p)\}$$

2.1. Theorem. [Tubular neighborhood Theorem] *There exists a positive continuous function* $\epsilon : M \longrightarrow \mathbb{R}$ *such that* $E|\nu_\epsilon M$ *is a diffeomorphism onto an open set* $Tub(M) := Tub(M)_\epsilon \subseteq \mathbb{R}^N$*.*

Proof. We first compute $\mathrm{d}E(p,0),\ p \in M$. Let $\{X_i\}, \{\xi_j\}$ be bases for T_pM and $[T_pM]^{\perp}$ respectively. Then $T_{(p,0)}\nu M$ is spanned by $\{(X_i,0),(0,\xi_j)\}$ (see Exercise 10.3 of Chapter 3). Let $\gamma_i : (-\epsilon,\epsilon) \longrightarrow M$ be curves such that $\dot\gamma_i(0) = X_i$. Then

$$\mathrm{d}E(p,0)(0,\xi_j) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} t\xi_j = \xi_j, \qquad \mathrm{d}E(p,0)(X_i,0)\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\gamma_i(t) = X_i.$$

Therefore $\mathrm{d}E(p,0)$ is an isomorphism and, by the inverse map Theorem, E maps a neighborhood $\tilde U$ of $(p,0)$ diffeomorphically onto an open neighborhood of $p \in \mathbb{R}^N$. $\tilde U$ contains a neighborhood of the type $U_{r(p)} = \{(q,\xi) \in \nu M : q \in U', \|\xi\| < r(p)\}$ where U' is a neighborhood of $p \in M$ and $r(p) > 0$.

We will assume, for the moment, that M is *compact*. Then we can cover $\{(p,0) : p \in M\} \subseteq \nu M$ with a finite number of neighborhoods $U_{r(p_i)}$. Let $r < r(p_i)$. Then $E|\nu_s M$ is a local diffeomorphism for $s < r$, in particular an open map. Then the Theorem will follows from the following fact

Claim 1. There exists a constant $\epsilon \in (0,r)$ such that $E|\nu_\epsilon M$ is a diffeomorphism onto the open set $Tub(M) = E(\nu_\epsilon M) \subseteq \mathbb{R}^N$.

Proof. It is sufficient to prove that there exists $\epsilon > 0$ such that $E|\nu_\epsilon M$ is injective. Suppose that this not the case. Then, for all n there are distinct points $(p_n,\xi_n),\ (q_n,\eta_n)$ with $\|\xi_n\|, \|\eta_n\| < 1/n$ and $E(p_n,\xi_n) = E(q_n,\eta_n)$. By compactness, there are subsequences converging, for $n \to \infty$, to points $(p,0)$ and $(q,0)$ respectively and, by continuity $E(p,0) = E(q,0)$. But this implies $p = q$. Then, for n sufficiently large, $(p_n,\xi_n),\ (q_n,\eta_n)$ are distinct points in the same $U_{r(p_i)}$, a contradiction since $E|U_{r(p_i)}$ is injective. ☐

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We consider now the case when M is not necessarily compact. We will start with some notations. Given $p \in M$ and $\epsilon > 0$ we define the *normal ball* at p of radius ϵ as

$$B_p^{\perp}(\epsilon) = \{p + \xi \in \mathbb{R}^N : \xi \in [T_pM]^{\perp}.\ \|\xi\| < \epsilon\}.$$

Given $K \subseteq M$ we will say that ϵ_K is an *admissible radius* if $B_p^{\perp}(\epsilon_K) \cap B_q^{\perp}(\epsilon_K) = \emptyset$ if $p,q \in K,\ p \neq q$. We will also set $T_K = \cup_{p\in K} B_p^{\perp}(\epsilon_K)$.

As in the proof of Theorem 3.2 of Chapter 3 we have a sequence of compact sets $K_1 \subset \cdots \subset K_i \subset \cdots$ with $M = \cup K_i$ and $K_i \subseteq Int(K_{i+1})$. Then, as in the proof of Claim 1., we can find constant $\epsilon_i > 0$ such that $E|\{(p,\xi) : p \in Int(K_i), \|\xi\| < \epsilon_i\}$ is a diffeomorphism onto an open set $U_i \subseteq \mathbb{R}^N$. We can suppose, without loss of generality, $\epsilon_i \geq \epsilon_{i+1}$.

<u>Claim 2.</u> There is a sequence of constants $\alpha_i \geq \alpha_{i+1}$ such that if $p \in K_i,\ q \in K_j, p \neq q,\quad B_p^{\perp}(\alpha_i) \cap B_q^{\perp}(\alpha_j) = \emptyset$.

Proof. Set $\alpha_1 = \epsilon_2, \alpha_2 = \epsilon_3$. Suppose to have defined $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s$ and the Claim holds for $i, j \leq s$. Define

$$0 < \alpha_{s+1} < min\{\epsilon_{s+2}, \alpha_s, d(K_{s+1} \setminus intK_s, \cup_{i=1}^{s-1} T_{K_i}\alpha_i)\},$$

where d is the distance in $\mathbb{R}^N$.

We want to show that the Claim holds for $i, j \leq s+1$. For $i, j \leq s$ this is the inductive hypothesis. For $i = s, j = s+1$ the Claim follows from the fact that $\alpha_s \leq \epsilon_{s+1}$. Finally, suppose $p \in K_{s+1} \setminus IntK_s,\ q \in K_i,\ i < s$.

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2.2. Definition. The set $Tub(M)$ is called a *tubular neighborhood* of M.

2.3. Corollary. *The inclusion $i : M \longrightarrow Tub(M)$ is a homotopy equivalence.*

Proof. Define $r : Tub(M) \longrightarrow M, r(E(p,\xi)) = p$. Then $r \circ i = 1\!\!1_M$ and $i \circ r \sim 1\!\!1_{Tub(M)}$ via the homotopy $H(p,\xi) = E(p, s\xi)$. ☐

At this point we can prove the announced results

2.4. Theorem. [de Rham Theorem for smooth manifolds] *Let M be a smooth manifold. The de Rham map $dR_M : H^p(M) \longrightarrow [H_p(M)]^*$ is an isomorphism.*

Proof. Since integration commutes with induced maps, we have a commutative diagram

$$\begin{tikzcd}
H^p(Tub(M)) \arrow[r, "dR_{Tub(M)}"] \arrow[d, "i^*"'] & {[H_p(Tub(M))]^*} \arrow[d, "(i_*)^*"] \\
H^p(M) \arrow[r, "dR_M"] & {[H_p(M)]^*}
\end{tikzcd}$$

Since the vertical maps, as well as the top horizontal one, are isomorphisms, the same is true for the bottom horizontal one. ☐

Using essentially the same argument we have

2.5. Theorem. [Künnet Theorem for smooth manifolds] *Let $M_i,\ i = 1, 2$ be smooth submanifolds of $\mathbb{R}^{N_i}$. Then*

$$H^p(M_1 \times M_2) \cong \oplus_{k+l=p} H^k(M_1) \otimes H^l(M_2).$$

3. The Jordan separation Theorem revisited

As a further application of the existence of a tubular neighborhood, we prove now a version of the Theorem of Jordan. The proof is interesting since it put light on the topological properties that make the Theorem work.

3.1. Theorem. [Jordan Theorem for smooth hypersurfaces] *Let* $M^n \subseteq \mathbb{R}^{n+1}$ *be a* connected *submanifold which is a closed subset. Suppose that there exist a smooth function* $\eta : M \longrightarrow \mathbb{R}^{n+1}$ *such that* $\eta(p) \in [T_pM]^\perp,\ \|\eta(p)\| = 1$. *Then* $\mathbb{R}^{n+1} \setminus M$ *has exactly two connected components.*

Proof. We start observing that the map $t : M \times \mathbb{R} \longrightarrow \nu M,\ \ t(p,t) = (p, t\eta)$ is a diffeomorphism. Then $\nu M \setminus M \times \{0\}$ has exactly two connected components $\nu_\pm = \{(p,\xi) \in \nu M : \langle \xi, \eta\rangle > 0 \ \ (\text{respectively} < 0)\}$.

Consider a tubular neighborhood $E|_{\nu_\epsilon M} : \nu_\epsilon M \longrightarrow Tub_\epsilon(M)$. It is easily seen that $\nu_\epsilon M \setminus M \times \{0\}$ has also exactly two connected components which are the intersection of $\nu_\epsilon M$ with $\nu_\pm$. Since $E|\nu_\epsilon M$ is a diffeomorphism onto its image and $E|_{\{(p,0)\in \nu M\}}$ is a diffeomorphism onto M, $Tub_\epsilon(M) \setminus M$ has also exactly two connected components, say $T_\pm$. The Theorem will follows from the Claims below.

<u>Claim 1.</u> Any connected component of $\mathbb{R}^n \setminus M$ contains either T_+ or T_-. In particular $\mathbb{R}^n \setminus M$ has, at most, two connected components.

Proof. Let C be a connected component of $\mathbb{R}^n \setminus M$ and $x \in C$. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^n$ a curve with $\gamma(0) = x, \gamma(1) \in M$. Let $t_0 = sup\{t \in [0,1] : \gamma(s) \notin Tub_\epsilon(M)\ \forall\ s \leq t\}$. Then, for r sufficiently small $\gamma|[0, t_0 + r]$ does not intersect M and $\gamma(t_0 + r) \in T_\pm$ Let say $\gamma(t_0 + r) \in T_+$. Then $C \cap T_+ \neq \emptyset$. Since T_+ is connected, $T_+ \subseteq C$. Finally, given three connected components of $\mathbb{R}^n \setminus M$ two of them, by the argument above, contains T_+ or T_-, hence have non empty intersection and therefore they coincide. ☐

<u>Claim 2.</u> $\mathbb{R}^n \setminus M$ is not connected.

Proof. We will prove that there exists a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that f assumes positive and negative values and $f(x) = 0$ if and only if $x \in M$. This will prove the claim since a continuous real valued function defined on a connected set and assuming positive and negative values must vanish somewhere.

On $Tub_\epsilon(M)$ we have such a function, namely $\tilde{g} = \pi_2 \circ t^{-1} \circ [E|\nu_\epsilon M]^{-1}$, where $\pi_2 : M \times \mathbb{R} \longrightarrow \mathbb{R}$ is the projection on the second factor. The problem is how to extend $\tilde{g}$ to a function defined on all $\mathbb{R}^n$. We could be tempted to use general results on existence of extensions of continuous functions but we can not guarantee that such an extension does not vanish outside M. The strategy we will use is to modify the function $\tilde{g}$ near the boundary of $Tub_\epsilon(M)$ in such a way that the new function, call it g, still has the properties we want and it is locally constant near $\partial Tub_\epsilon(M)$.

Suppose we have constructed such a function g. Again we have problem extending it. In order to construct such an extension we will proceed as follows: consider the 1-form $\omega = \mathrm{d}g \in \Omega^1(Tub_\epsilon(M)$. Since g is locally constant near in a neighborhood U of $\partial\, Tub_\epsilon(M)$, $\omega = 0$ in U and we can extend ω to a form defined in all $\mathbb{R}^n$ setting $\omega = 0$ outside $Tub_\epsilon(M)$. This form is clearly closed and therefore, since $H^1(\mathbb{R}^n) = \{0\}$, there exists a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $\omega = \mathrm{d}f$. Moreover this function is unique up to a constant and therefore unique if we require $f(p) = 0$ for some $p \in M$. Observe that, with this choice, f agree with g on $Tub_\epsilon M$. In particular f assumes positive and negative values and, on $Tub_\epsilon(M)$, does not vanish outside M. We want to show that f does not vanish outside $Tub_\epsilon(M)$. Let $x \in \mathbb{R}^n \setminus Tub_\epsilon(M)$ and let $\gamma : [0,1] \longrightarrow \mathbb{R}^n$

be a curve with $\gamma(0) = x, \gamma(1) \in M$. As in Claim 1., let $t_0 = sup\{t \in [0,1] : \gamma(s) \notin Tub_\epsilon(M) \ \forall s \leq t\}$. Then, for r sufficiently small $\gamma|[0, t_0 + r]$ does not intersect M and $\gamma(t_0 + s) \in U$ for $s \leq r$. Hence

$$f(x) = f(\gamma(t_0 + r)) - \int_{\gamma|[0,t_0+r]} \omega = f(\gamma(t_0 + r)) \neq 0,$$

since $\omega = 0$ along $\gamma([0, t_0 + r])$.

We will prove now that a function g with the properties required above exists. We will identify $\nu_\epsilon M$ with $Tub_\epsilon(M)$. For $p \in M$ we consider the piecewise affine function $\lambda_p : \mathbb{R} \longrightarrow \mathbb{R}$ which is the identity in $(-\epsilon(p)/2, \epsilon(p)/2)$, it is identically 1 for $t > 2\epsilon(p)/3$ and identically -1 for $t < -2\epsilon(p)/3$. Then $\tilde{\lambda}(p,t) := \lambda_p(t)$ is a continuous function, smooth for $(p,t) \notin V = \{(q,s) : s \in (-2\epsilon(q)/3, -\epsilon(q)/2) \cup (\epsilon(q)/2, 2\epsilon(q)/3)\}$. Using Theorem 3.9 of Chapter 3, we can approximate $\tilde{\lambda}$ by a smooth function λ which coincide with $\tilde{\lambda}$ in $M \times \mathbb{R} \setminus V$. We define $g : M \times \mathbb{R} \longrightarrow \mathbb{R}, \ \ g(p,\xi) = \lambda(p, \tilde{g}(p,\xi))$. In this way we have a smooth function on $Tub_\epsilon(M)$, which assume positive and negative values, vanishes only on M and is constant un a neighborhood of $\partial Tub_\epsilon(M)$.

□

□

3.2. Remark. The condition of the existence of the function η, equivalent to the fact that M is orientable (see next section), is a consequence of the fact that M is closed in $\mathbb{R}^{n+1}$. We will prove this fact in the next section. With respect to the situation discussed in Remark 5.9 of Chapter 1, we observe that we do not need a “model” for our conclusion. However we need differentiability hypothesis, which were no needed in Remark 5.9.

A natural question is if the Theorem holds for a connected closed hypersurface M of a connected differentiable manifold N. We will assume that N is embedded as a closed subset of some $\mathbb{R}^N$. For this situation we can define the normal bundle of M in N as

$$\nu M = \{(p,\xi) \in TN : \xi \in [T_pM]^\perp\}.$$

Using a suitable endpoint map[1] we can define tubular neighborhood also in this context, and try to reproduce the proof given above. There are two essential points needed

- the fact that the tubular neighborhood are diffeomorphic to $M \times \mathbb{R}$,
- the fact that any closed 1-form in N is exact.

For the first point it is sufficient to assume that both M and N are orientable. For the second one we need to assume that $H^1(N) = \{0\}$. With these two assumption is possible to reproduce the proof given above and conclude that $N \setminus M$ has exactly two components (if M, N are connected). The two conditions are essential as the following two examples show.

• Consider the real projective plane $\mathbb{R}P^2$ as the quotient of the closed disk $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ modulo the equivalence relation $x \sim y \Leftrightarrow x = y$ or $x = \pm y, \|x\| = 1$. Consider the map $\gamma : [0,1] \longrightarrow \mathbb{R}P^2, \gamma(t) = [\cos \pi t, \sin \pi t]$ (where $[\cdot]$ is the equivalence class). Then γ induces an embedding of the circle into $\mathbb{R}P^2$ whose complement is the open disk, hence a connected set. The problem here is that $\mathbb{R}P^2$ is not orientable.

---

[1] The Riemannian exponential map that will be defined in the next chapter

• Consider the torus $T^2 = S^1 \times S^1$ and the embedded circle $\Gamma = S^1 \times \{(0,1)$. Then $T^2 \setminus \Gamma$ is connected. The problem here is that $H^1(T^2) = \mathbb{R} \oplus \mathbb{R} \neq \{0\}$, by the Künnet formula.

Observe that in both examples, the complement of the submanifold is connected. This a general fact since the simple existence of the tubular neighborhoods implies that the complement of the submanifold has, *at most*, two connected components (see the proof of Claim 1).

4. Orientable manifolds, integration of n-forms and the Theorem of Stokes

We want to define the integral of a differential n-form on an n-dimensional manifold. We will start recalling the change of coordinate formula for multiple integrals in $\mathbb{R}^n$.

4.1. Theorem. *Let F be a diffeomorphism of an open set $A \subseteq \mathbb{R}^n$ onto an open subset of $\mathbb{R}^n$. We will denote by $J[F] : A \longrightarrow \mathbb{R}$ the* Jacobian detrminant*, $J[F] := \det[\mathrm{d}F]$. Then, if $D \subseteq A$ is the closure of an open bounded set and $f : \phi(D) \longrightarrow \mathbb{R}$ is a continuous function,*

$$\int_{F(D)} f = \int_D f \circ F|J[F]|.$$

Let now $\omega \in \Omega^n(M)$ be a differential form and let $\phi : U \subseteq \mathbb{R}^n \longrightarrow M$ be a local chart. Suppose first that ω has compact support contained in $\phi(U)$. It is natural to (try to) define

$$\int_M \omega = \int_U \phi^*\omega = \int_U f(x_1,\dots,x_n)\ \mathrm{d}x_1 \cdots \mathrm{d}x_n, \quad \phi^*\omega = f\ \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

We would like to prove to prove that the definition does not depend on the chart. If $\psi : V \longrightarrow M$ is an other chart containing the support of ω, we should have

$$\int_U \phi^*\omega = \int_V \psi^*\omega$$

that is, using the formula of change of variables in the multiple integrals, $J[\psi^{-1} \circ \phi] = |J[\psi^{-1} \circ \phi]|$, and this is not, in general, the case. However these considerations lead to the concept of *orientable manifolds.*

We recall that an orientation on a vector space is a choice of an equivalence class of equioriented bases (see Definition 1.26 in Chapter 1).

4.2. Definition. Let M be a differentiable manifold. An *orientation* on M is the choice of an orientation on each T_xM that depends differentiably on x. This means that for all $x \in M$ there is a neighborhood U of x and smooth vector fields $X_1,\dots,X_n \in \mathcal{H}(U)$ such that $\{X_1(y),\dots,X_n(y)\}$ is a positive bases, $\forall\ y \in U$.

We will say that M is *orientable* if such an orientation exists, and that it is *oriented* if an orientation has been fixed. If $F : M \longrightarrow N$ is a diffeomorphism, we will say that f *preserve the orientation* or is *orientation preserving* if $\mathrm{d}F$ sends positive bases onto positive bases. Otherwise we will say that F *invert the orientation.*

4.3. Remark. $\mathbb{R}^n$ comes with a canonical orientation, the one defined by the canonical basis. If F is a diffeomorphism between open sets of $\mathbb{R}^n$, it preserves the orientation if and only if $J[F] > 0$.

4.4. Example. A parallelizable manifold, in particular a Lie group, is orientable.

4.5. Example. The sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is orientable. In fact $T_pS^n = \{x \in \mathbb{R}^{n+1} : \langle x,p\rangle = 0\}$. Define $\{X_1,\dots X_n\} \subseteq T_pS^n$ to be positive if and only if $\{X_1,\dots,X_n,p\}$ is a positive bases of $\mathbb{R}^{n+1}$. More in general a submanifold $M^n \subseteq \mathbb{R}^{n+1}$ is orientable if and only if there exist a smooth map $N : M \longrightarrow \mathbb{R}^{n+1}$ such that $N(p) \in [T_pM]^\perp$ and $N(p) \neq 0$, $\forall\ p \in M$.

The example above is a particular case of a much more general fact mentioned in Remark 3.2

4.6. THEOREM. *Let $M^n \subseteq \mathbb{R}^{n+1}$ be a submanifold which is a closed subset. Then M is orientable.*

PROOF. □

4.7. THEOREM. *Let M be a n-dimensional differentiable manifold. Then the following conditions are equivalent*

(1) *M is orientable.*
(2) *There is a differential form $\omega \in \Omega^n(M)$ such that $\omega(x) \neq 0, \quad \forall \ \ x \in M$.*
(3) *There is a spacial atlas $\{\psi_i\}$ such that $J[\psi_i^{-1} \circ \psi_j]$ is positive, where defined.*

PROOF. By Whitney's Theorem we may assume that M is a submanifold of $\mathbb{R}^N$. Let $p \in M$ and let $\{X_1, \dots, X_n\}$ ba a local *positive* frame of vector fields in a neighborhood of p. Using the orthonormalization process we may assume that the frame is orthonormal. If $\{\omega_1, \dots, \omega_n\}$ is the dual frame, we define $\omega = \omega_1 \wedge \cdots \wedge \omega_n$. It is sufficient to prove that ω is well defined, i.e. does not depend on the chosen frame. In fact, if $\{Y_1, \dots, Y_n\}$ is an other *orthonormal positive* frame and $\overline{\omega}$ is the associated n-form, $\overline{\omega} = \det A \ \omega$ where A is the matrix of the change of bases. Since both bases are orthonormal and positive, $A \in SO(n)$, hence $\det A = 1$. So (1) ⇒ (2).

We will show now that (2) ⇒ (3). Let ω be a nowhere zero n-form. Start with a special atlas $\{\phi_i, \mathbb{R}^n\}$. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad F(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots x_n)$. So F reverses the orientation. Now $\phi_i^*\omega = f_i \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a *nowhere vanishing* function. Define $\psi_i = \phi_i$ if $f_i > 0$ and $\psi_i = \phi_i \circ F$ if $f_i < 0$. It is easily seen that the (special) atlas $\{\psi_i\}$ has the required property.

Finally we will show that (3) ⇒ (1). Start with a special atlas $\{\psi_i\}$ as constructed above. Consider the local orientation on M defined by the local frame $\{X_j^i = \mathrm{d}\psi_i(e_j)\}$ where $\{e_j\}$ is the canonical basis of $\mathbb{R}^n$. In $\psi_i(\mathbb{R}^n) \cap \psi_j(\mathbb{R}^n)$ the frames are related by the matrix $[\mathrm{d}(\psi_i^{-1} \circ \psi_j)]$ which has, by hypothesis, positive determinant. So the two local orientations agree on the intersection. □

At this point we can define the integral of n-form on an *oriented* differentiable n-dimensional manifold. Let M be such a manifold and $\omega \in \Omega^n(M)$. To avoid convergence problem we will assume that ω has compact support. Consider a special atlas $\{\psi_k\}$ with $J[\psi_i^{-1} \circ \psi_j] > 0$. Let $\{\lambda_i\}$ be a partition of unity dominated by such a covering. Then, by the considerations above, the integral of $\lambda_i\omega$ over M is well defined.

4.8. DEFINITION. With the notations above we define

$$\int_M \omega = \sum_i \int_M \lambda_i \omega.$$

It is easily seen that the definition does not depends on the partition of unity.

At this point we would like to have, in this context, a suitable version of the Theorem of Stokes. For this we need to define the concept of *manifold with boundary* or ∂-manifold. Consider the (closed) half space and its boundary,

$$\mathbb{H}^n = \{(x_1, \dots x_n) \in \mathbb{R}^n : x_n \geq 0\}, \qquad \partial\mathbb{H}^n = \{(x_1, \dots x_n) \in \mathbb{H}^n : x_n = 0\} \cong \mathbb{R}^{n-1}.$$

The following lemma will be useful

4.9. Lemma. *Let $F : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ be a homeomorphism. If $p \in \partial\mathbb{H}^n$, then $F(p) \in \partial\mathbb{H}^n$.*

Proof. Suppose $F(p) \notin \partial\mathbb{H}^n$. Then $\mathbb{H}^n \setminus \{F(p)\}$ is homotopy equivalent to S^{n-1} while $\mathbb{H}^n \setminus \{p\}$ is contractible. ☐

4.10. Definition. Let M be a topological (metric is enough) space.

(1) A *∂-chart* is a map $\phi : U \subseteq \mathbb{H}^n \longrightarrow M$, U open in $\mathbb{H}^n$, such that $\psi(U)$ is open and ψ is a homeomorphis onto its image.
(2) A topological ∂-atlas is a collection of ∂-charts whose images cover M. If such an atlas exists, we will say that M is a *topological ∂-manifold.*

Just as in the case of smooth manifolds we have the associated "differentiable concepts" for ∂-manifolds.

4.11. Definition. Let M be a topological ∂-manifold.

(1) A *differentiable ∂-atlas* is a topological ∂-atlas such that the change of coordinates are smooth.
(2) Two differentiable ∂-atlases are *equivalent* if their union is a differentiable ∂-atlas.
(3) A *differentiable ∂-structure* is an equivalence class of differentiable ∂-atlases (or a maximal one).
(4) *A differentiable (or smooth) ∂-manifold* is a topological ∂-manifold together with a differentiable ∂-structure.

Smooth maps between (smooth) ∂-manifolds and the tangent space of a ∂-manifold are defined as in the case of empty boundary. Therefore we can talk, also in this context, of smooth curves and their tangent vectors, vector field on ∂-manifolds, differential of a smooth map etc.. Let M be a ∂-manifold and $p \in \partial M$. A smooth curve $\gamma : [0, \epsilon) \longrightarrow M$ with $\gamma(0) = p$ is called an *inner curve* if $\dot\gamma(0) \notin T_p\partial M$. In this case we will say that $\dot\gamma(0)$ is an *inner vector.* In particular T_pM is spanned by $T_p\partial M$ and an inner vector. It is easy to see that if $\gamma : [0, \epsilon) \longrightarrow M$ is an inner curve, $\gamma((0, \eta)) \subseteq M \setminus \partial M$ if η is sufficiently small.

4.12. Example. Let M be a ∂-manifold, $p \in \partial M$ and let $\psi : U \longrightarrow M$ be a chart with $x = \psi^{-1}(p)$. Then $\psi(x + te_n), t \geq 0$ is an inner curve and $\mathrm{d}\psi(x)(e_n) = \dot\gamma(0)$ is an inner vector.

4.13. Lemma. *Let M be a smooth ∂-manifold. Define the* boundary *of M as*

$$\partial M = \{p \in M : \text{there exist a chart } \phi : U \subseteq \mathbb{H}^n \longrightarrow M \text{ such that } \phi^{-1}(p) \in \partial\mathbb{H}^n\}.$$

Then ∂M is well defined, it is a differentiable manifold with $\partial(\partial M) = \emptyset$, and, if M is orientable, ∂M is orientable.

Proof. The fact that ∂M is well defined follows from Lemma 4.9. Also, given a chart $\psi : U \subseteq \mathbb{H}^n \longrightarrow M$, $\psi|U \cap \partial\mathbb{H}^n : U \cap \partial\mathbb{H}^n \longrightarrow \partial M$ is a chart for ∂M. It is also clear that, given two charts for M such that the change of coordinates is smooth, so happens for the induced charts for ∂M. Finally, if M is orientable and $p \in \partial M$ we can define a basis $\{X_1, \dots, X_{n-1}\}$ of $T_p\partial M$ to be positive if $\{X_1, \dots, X_{n-1}, X\}$ is a positive basis for T_pM, where X is an inner vector. We leave to the reader the task of proving the differentiability condition in the definition of orientability. ☐

The concept of special atlas and partition of unity dominated by an open covering are the same as in the case of manifolds without boundary. It is worth to observe that a partition of unity for M, restricted to

∂M is a partition of unity for ∂M. Also we can define integration of a n-form with compact support on a ∂-manifold, if it is oriented.

At this point we can state the Theorem of Stokes in our context.

4.14. THEOREM. [Stokes Theorem] *Let M be a ∂-manifold of dimension n and $\omega \in \Omega^{n-1}(M)$ a form with compact support. Then*

$$\int_M \mathrm{d}\omega = \int_{\partial M} i^*\omega,$$

where $i : \partial M \longrightarrow M$ is the inclusion. In particular if $\partial M = \emptyset$, $\displaystyle\int_M \mathrm{d}\omega = 0$.

PROOF. We start proving the Theorem "locally".

CLAIM 1. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ be a form with compact support. Then

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = 0.$$

PROOF. We can suppose, without loss of generality, that $\mathrm{supp}(\omega) \subseteq Int(\Delta^n)$, where Δ^n is the standard n-simplex. Then, by Stokes Theorem for chains (see Theorem **??** of Chapter 2),

$$\int_{\mathbb{R}^n} \mathrm{d}\omega = \int_{\Delta^n} \mathrm{d}\omega = \int_{\partial\Delta^n} \omega = 0$$

since $\omega = 0$ on $\partial\Delta^n$. □

CLAIM 2. Let $\omega \in \Omega^n(\mathbb{H}^n)$ be a form with compact support. Then

$$\int_{\mathbb{H}^n} \mathrm{d}\omega = \int_{\partial\mathbb{H}^n} \omega.$$

PROOF. Again, without loss of generality, we can suppose that $\mathrm{supp}(\omega) \subseteq Int(\Delta^n) \cup Int(\Delta^{n-1})$. Then, by Stokes Theorem for chains,

$$\int_{\mathbb{H}^n} \mathrm{d}\omega = \int_{\Delta^n} \mathrm{d}\omega = \int_{\partial\Delta^n} \omega = \int_{\Delta^{n-1}} \omega = \int_{\partial\mathbb{H}^n} \omega,$$

since $\omega = 0$ on all faces of Δ^n having e_n as a vertex. □

At this point we can "glue together" the local information using a partition of unity. Fix a special atlas $\{U, \phi_i\}$ with $U = \mathbb{R}^n$ or $\mathbb{H}^n$ and let $\{\lambda_i\}$ be a partition of unity dominated by $\phi_i(U_i)$. Set $\omega_i = \lambda_i\omega$. Since integration is additive, we have to prove the Theorem just for ω_i. From the Claims and the definition of integral, we have

$$\int_M \mathrm{d}\omega_i = \int_{U_i} \mathrm{d}\phi_i^*\omega_i = \int_{\partial U_i} \phi_i^*\omega_i = \int_{\partial M} \omega_i.$$

□

5. Invariant cohomology and the cohomology of compact Lie groups

Let G be a Lie group, M a smooth manifold and $\mu : G \times M \longrightarrow M$ a smooth action.

5.1. Definition. A differential form $\omega \in \Omega^p(M)$ is *μ-invariant*, or simply *invariant* when clear from the context, if $\mu_g^* \omega = \omega,\ \forall\, g \in G$. We will denote by $\Omega_\mu(M)$ the space of invariant differential p-forms.

Here two simple facts whose proof is left to the reader.

- The exterior product of invariant forms is invariant.
- If $\omega \in \Omega^p_\mu(M), \quad \mathrm{d}\omega \in \Omega^{p+1}_\mu(M)$

In particular the sequence

$$\cdots \longrightarrow \Omega^{p-1}_\mu(M) \xrightarrow{\mathrm{d}^{p-1}} \Omega^p_\mu(M) \xrightarrow{\mathrm{d}^p} \Omega^{p+1}_\mu(M) \longrightarrow \cdots$$

is well defined and it is a subcomplex of the de Rham complex of M, the *invariant de Rham complex.*

5.2. Definition. The cohomology of the invariant de Rham complex is called the *invariant cohomology* of M and will be denoted by $H^p_\mu(M)$.

5.3. Example. A particular but important case is when $M = G$ and the action is by left translations. In this case an invariant form is called a *left invariant form* and the space of left invariant p-forms will be denoted by $\Omega^p_L(G)$. We can also consider the action of G on itself by right translations, $\mu : G \times G \longrightarrow G, \quad \mu(g,h) = hg^{-1}$ (the inverse is needed for the map satisfy the conditions to be an action). Differential p-forms invariant for this action will be called *right invariant forms* and the space of such forms will be denoted by $\Omega^p_R(G)$.

A left (resp. right) invariant p-form is uniquely determined by its value at the identity. More precisely the evaluation map $ev : \Omega^p_L(G) \longrightarrow \Lambda^p(\widehat{G}), \quad ev(\omega) = \omega(e)$ is an isomorphism of algebras (with respect to the exterior products). Since a left invariant form is constant on left invariant vector fields, the differential of $\omega \in \Omega^p_L(G)$ is given by

$$\mathrm{d}\omega(X_0, \ldots, X_p) = \sum_{i<i} \omega([\tilde{X}_i, \tilde{X}_j], \tilde{X}_0, \ldots, \widehat{\tilde{X}_i}, \ldots, \widehat{\tilde{X}_j}, \ldots, \tilde{X}_p)$$

where $\tilde{X}_k$ is the left invariant extention of X_k (observe that, since $\mathrm{d}\omega$ is a form, $\mathrm{d}\omega(X_0, \ldots, X_p)$ does not depend on the extensions of the X_k's). If we identify $\Omega^p_L(G)$ with $\Lambda^p(\widehat{G})$, the invariant de Rham complex becomes

$$\cdots \longrightarrow \Lambda^{p-1}(\widehat{G}) \xrightarrow{\mathrm{d}^{p-1}} \Lambda^p(\widehat{G}) \xrightarrow{\mathrm{d}^p} \Lambda^{p+1}(\widehat{G}) \longrightarrow \cdots$$

where

$$\mathrm{d}^p\omega(X_0, \ldots, X^p) = \sum_{i<i} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p).$$

The cohomology of the complex above is called the *cohomology of $\widehat{G}$.*

We want to compare the invariant cohomology with the usual de Rham cohomology. For this we need to introduce integration of functions on a Lie group.

Let G be a n-dimensional Lie group. A left invariant n-form is uniquely determined up to a multiplicative constant. Let $\omega \neq 0$ be such a form, fixed once for all. For a smooth function $f : G \longrightarrow \mathbb{R}$, with compact support, we define the *Haar integral* of f,

$$\int_G f := \int_G f\omega.$$

The Haar integral is well defined since a Lie group is orientable and it is linear on the space of compactly supported smooth real valued functions. Moreover, as it easily seen, the integral is *left invariant*, i.e.

$$\int_G f = \int_G f \circ L_g.$$

We will be interested in the case of *compact* Lie groups. For such a group we can choose ω such that

$$\int_G \omega = 1.$$

5.4. LEMMA. *If G is a compact Lie group, the Haar integral is right invariant.*

PROOF. Consider the right action $R_g(x) = xg^{-1}$. Since the left and right actions commutes, we have

$$L_h^*(R_g^*\omega) = R_g^*(L_h^*\omega) = R_g^*\omega$$

since ω is left invariant. Then $R_g^*\omega$ is left invariant, hence, by unicity of left invariant n-forms (up to constant), $R_g^*\omega = c(g)\omega$, where $c : G \longrightarrow \mathbb{R} \setminus \{0\}$. Since $R_{gh}^* = R_h^* \circ R_g^*$, c is a homomorphism into the multiplicative group $\mathbb{R} \setminus \{0\}$. But the only compact subgroups of this group are $\{1\}, \{\pm 1\}$. Since G is connected, $c(G) = 1$. Hence

$$\int_G f \circ R_h = \int_G (f \circ R_h) R_h^*\omega = \int_G R_h^*(f\omega) = \int_G f\omega = \int_G f.$$

□

Let $\mu : G \times M \longrightarrow M$ be a smooth action, G compact. Using integration we can "average" a differential p-form to obtain an invariant one.For $g \in G$, $X \in \mathcal{H}(M)$, we set $X^g := \mathrm{d}\mu_g(X)$. Consider the operator

$$\mathrm{I} : \Omega^p(M) \longrightarrow \Omega^p(M), \quad \mathrm{I}(\omega)(X_1, \ldots, X_p) := \int_G \omega(X_1^g, \ldots, X_p^g).$$

5.5. PROPOSITION.

(1) *$I(\omega) \in \Omega_\mu^p(M)$.*
(2) *If $\omega \in \Omega_\mu^p(M)$, $I(\omega) = \omega$.*
(3) *$\mathrm{d}I(\omega) = I(\mathrm{d}\omega)$, i.e I is a morphism of the complex $\Omega^*(M)$ into the complex $\Omega_\mu^*(M)$.*

PROOF. For the first claim we have

$$\mu_g^*(I(\omega))(X_1, \ldots, X_p) = I(\omega)(X_1^g, \ldots, X_p^g) = \int_G \omega(X_1^{gh}, \ldots X_p^{gh}) = \int_G \omega(X_1^h, \ldots, X_p^h) = I(\omega)(X_1, \ldots, X_p).$$

The second claim is immediate and for the last one we have

$$\mathrm{d}(I(\omega))(X_0, \ldots, X_p) = \sum_{i=0}^{p}(-1)^i X_i I(\omega)(X_0, \ldots, \widehat{X_i}, \ldots X_p) + \sum_{i<j}(-1)^{i+j} I(\omega)([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p) =$$

$$= \sum_{i=0}^{p}(-1)^i X_i \int_G \omega(X_0^g, \ldots, \widehat{X_i^g}, \ldots X_p^g + \sum_{i<j}(-1)^{i+j} \int_G \omega([X_i, X_j]^g, \ldots, \widehat{X_i^g}, \ldots, \widehat{X_j^g}, \ldots, X_p^g) =$$

$$= \int_G \mathrm{d}\omega(X_0^g, \ldots, X_p^g) = I(\mathrm{d}\omega)(X_0, \ldots, X_p).$$

□

The group G acts on $H^p(M)$ in a natural way, $g[\omega] = [\mu_g^*\omega]$. We will denote by $[H^p(M)]^\mu$ the fixed points set of this action, i.e. $[H^p(M)]^\mu = \{[\omega] \in H^p(M) : g[\omega] = [\omega]\ \forall\ g \in G\}$.

5.6. Theorem. *The inclusion* $J\colon \Omega^p_\mu(M) \longrightarrow \Omega^p(M)$ *induces an isomorphism*

$$J^* : H^p_\mu(M) \longrightarrow [H^p(M)]^\mu.$$

Proof. First notice that J is a chain map, i.e. $\mathrm{d} \circ \mathrm{J} = \mathrm{J} \circ \mathrm{d}$ and hence induces a morphism $\mathrm{J}^* : H^p_\mu(M) \longrightarrow [H^p(M)]$. Since $\mathrm{I} \circ \mathrm{J} = \mathbb{1}$, $\mathrm{I}^* \circ \mathrm{J}^* = \mathbb{1}$ and J^* is injective. We will show that $Im\mathrm{J}^* = [H^p(M)]^\mu$.

Let $[\alpha] \in [H^p(M)]^\mu$. Then $[\alpha] = [\mu_g^*\alpha]$ and hence $\alpha - \mu_g^*\alpha = \mathrm{d}\beta$ for some β depending on g. For any cycle $c \in Z_p(M)$ and any $g \in G$ we have

$$\int_c \alpha - \mu_g^*\alpha = \int_c \mathrm{d}\beta = \int_{\partial c} \beta = 0,$$

by Stokes Theorem. Hence

$$\int_c I(\alpha) = \int_c \int_G \mu_g^*\alpha = \int_G \int_c \mu_g^*\alpha = \int_G \int_c \alpha = \int_c \alpha \int_G 1 = \int_c \alpha.$$

Therefore, by the Theorem of de Rham, $[I(\alpha)] = [\alpha]$. Observe now that $JI(\omega) = I(\omega)$. Hence $J^*([I(\omega)]) = [I\omega] = [\omega]$. □

5.7. Lemma. *Il G is* connected, $[H^p(M)]^\mu = H^p(M)$.

Proof. Let $g \in G$ and let $\gamma : [0,1] \longrightarrow G$ be a curve with $\gamma(0) = g$, $\gamma(1) = e$. Then $H(x,t) := \mu_{\gamma(t)}(x)$ is a homotopy between μ_g and $\mathbb{1}$. Hence $\mu_g^* = \mathbb{1}$, in cohomology, and the action of G on $H^p(M)$ is trivial. □

5.8. Corollary. *If G is a compact* connected *Lie group, its cohomology is isomorphic to the cohomology of its Lie algebra.*

Proof. Let L denote the action by left translations. Since G is connected, $[H^p(G)]^L = H^p(G)$ and $H^p_L(G)$ is the cohomology of $\widehat{G}$. □

In particular, *the cohomology of a compact connected Lie group can be computed by purely algebraic means.* In fact we can do even better. Consider the action

$$\kappa : (G \times G) \times G \longrightarrow G, \qquad \kappa((g_1, g_2), h) = g_1 h g_2^{-1}.$$

If G is compact and connected, the cohomology of G is the cohomology of the complex of κ-invariant forms. Such forms are left and right invariant and therefore are determined by their values at $e \in G$. We extend the adjoint representation to act on forms

$$Ad : G \times \Lambda^p(\widehat{G}) \longrightarrow \Lambda^p(\widehat{G}), \qquad Ad(g)\omega(X_1, \ldots, X_p) = \omega(Ad(g^{-1})X_1, \ldots, Ad(g^{-1})X_p).$$

It is easily seen that $\omega \in \Lambda^p(\widehat{G})$ is κ-invariant if and only if it is Ad-invariant. We want a characterization of Ad-invariant forms. First a Lemma of general character.

5.9. Lemma. *Let $\mathbb{E}$ be a vector space, G be a Lie group and $\theta : G \longrightarrow End(\mathbb{E})$ be a representation. Then $\theta(g)v = v, \ \ \forall\, g \in G$ if and only if $\mathrm{d}\theta(e)(X)v = 0 \ \ \forall X \in \widehat{G}$.*

Proof. Fix $X \in \widehat{G}$. Then $\theta(\exp tX)$ is a 1-parameter subgroup of $End(\mathbb{E})$, hence $\theta(\exp tX) = \exp tA := \sum_{k=0}^{\infty} t^k [k!]^{-1} A^k$ for some $A \in End(\mathbb{E})$. Therefore

$$\mathrm{d}\theta(e)(X)v = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\theta(\exp tX)v = Av.$$

If v is fixed by $\theta(g), \ \ \forall\, g \in G, \ \ \mathrm{d}\theta(e)(X)v = \dfrac{\mathrm{d}}{\mathrm{d}t}v = 0, \ \forall X \in \widehat{G}$. Conversely, if $Av = 0, \ \exp(tA)v = v$ which implies $\theta(g)v = v, \ \forall g \in G$. ☐

5.10. Theorem. *$\omega \in \Lambda^p(\widehat{G})$ is Ad-invariant if and only if*

$$\sum_{i=1}^{p} \omega(X_1, \ldots, X_{i-1}, [Y, X_i], X_{i+1}, \ldots, X_p) = 0.$$

Proof. Let θ be the adjoint action on $\mathbb{E} = \Lambda^p(\widehat{G})$. Then

$$\mathrm{d}\theta(e)Y\omega(X_1, \ldots, X_p) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\theta(\exp Y)\omega(X_1, \ldots, X_p) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\omega(Ad(\exp(-tY)X_1, \ldots, Ad(\exp(-tY)X_p) =$$

$$= \sum_{i=1}^{p} \omega(X_1, \ldots, \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}Ad(\exp(-tY)X_i, \ldots, X_p) = \sum_{i=1}^{p} \omega(X_1, \ldots, ad(-Y)X_i, \ldots, X_p) =$$

$$= \sum_{i=1}^{p} \omega(X_1, \ldots, [X_i, Y], \ldots, X_p),$$

where the third equality follow from the fact that ω is multilinear. Then the Theorem follows from the preceding Lemma.

☐

To compute the cohomology of G from the complex of Ad-invariant forms we need to compute the differential of such a form.

5.11. Proposition. *If $\omega \in \Lambda^p(\widehat{G})$ is Ad-invariant, then $\mathrm{d}\omega = 0$.*

Proof. Set $\epsilon_{ii} = 0, \ \ \epsilon_{ij} = (-1)^j, \ i < j, \ \epsilon_{ij} = (-1)^{j+1}, i > j$. Then

$$\mathrm{d}\omega(X_0, \ldots, X_p) = 2^{-1} \sum_{i \neq j} (-1)^i \epsilon_{ij} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots \widehat{X_j}, \ldots, X_p) =$$

$$= 2^{-1} \sum_{i} (-1)^i \sum_{j} \epsilon_{ij} \omega(([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots \widehat{X_j}, \ldots, X_p) = 0,$$

by Theorem 5.10. ☐

5.12. Corollary. *The cohomology of a compact connected Lie group G is isomorphic to the space of Ad-invariant forms.*

5.13. Remark. If G is not connected, we can apply the Corollary to the connected component of the identity, G_e (which is a compact connected subgroup, see Exercise 10.22 of Chapter 3). Since all connected components are diffeomorphic, the cohomology of G is the direct product of as many copies of the cohomology of G_e as many connected components there are.

We will give now an application of Corollary 5.12. First a Definition.

5.14. Definition. Let G be a Lie group. The *derived algebra* is the subalgebra D_G of $\widehat{G}$ defined by

$$D_G = \{Z \in \widehat{G} : Z = [X, Y], \quad X, Y \in \widehat{G}\}.$$

5.15. Proposition. *Let G be a compact, connected Lie group. Then*

(1) $H^1(G) = \{0\}$ *if and only if* $D_G = \widehat{G}$.
(2) *If* $H^1(G) = \{0\}$*, then* $H^2(G) = \{0\}$.

Proof. If $D_G \neq \widehat{G}$ there is a nonzero 1-form $\omega \in \Lambda^1(\widehat{G})$ vanishing on D_G. By Proposition 5.11, ω is *Ad*-invariant and defines a nonzero cohomology class. Conversely, if there is a nonzero *Ad*-invariant form, this form vanishes on D_G, hence $D_G \neq \widehat{G}$.

Suppose now $D_G = \widehat{G}$ and let ω be an *Ad*-invariant 2-form. Then

$$0 = \mathrm{d}\omega(X, Y, Z) = -\omega([X, Y], Z) - \{\omega([Z, X], Y) + \omega(X, [Z, Y])\} = -\omega([X, Y], Z),$$

where the last equality follows from Proposition 5.11. Since $D_G = \widehat{G}$, $\omega = 0$ and $H^2(G) = \{0\}$. □

In the next Chapter we will give a "geometric" characterization of *Ad*-invariant forms (see Remark **??**).

6. Exercises

6.1. Prove that if M is a compact manifold, $\dim(H^p(M))$ is finite.

6.2. Prove the Künnet formula for smooth manifolds (Theorem 2.5).

6.3. Prove that, if a manifold can be covered by two local charts such that the intersection of the domains is connected, then it is orientable. Use this fact to give an alternative proof that S^n, $n > 1$, is orientable.

6.4. The *(open) Möebius band* M is defined as the quotient of the square $\{(t, s) \in \mathbb{R}^2 : t \in [0, 1], s \in (0, 1)\}$ modulo the equivalence relation generated by $(0, s) \sim (1, 1 - s)$. Prove that M admits a smooth atlas consisting of two charts, but it is *not* orientable. Find an embedding $i : M \longrightarrow \mathbb{R}^3$.

6.5. Let $U, V \subseteq \mathbb{R}^n$ be connected open set and $F : U \longrightarrow V$ be a diffeomorphism. Let $\omega \in \Omega^n(V)$ be a form with compact support. Prove that:

$$\int_{F(U)} \omega = \pm \int_U F^*\omega,$$

with the sign + (resp. −) if F preserves (resp. inverts) the orientation. Extend the result to the case of smooth oriented manifolds.

6.6. Let M be a compact, orientable n-dimensional manifold and $p \in M$. Let $\phi : \mathbb{R}^n \longrightarrow M$, $\phi(0) = p$ be a local chart. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth non negative function with compact support and $f(0) \neq 0$. Consider the form $\omega = f\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \in \Omega^n(\mathbb{R}^n)$. Prove that $(\phi^{-1})^*\omega$ extends tho a smooth form in $\Omega^n(M)$ which represent a nonzero element of $H^n(M)$.

Remark It can be shown that, if M is a compact connected smooth manifold, $H^n(M) \cong \mathbb{R}$ if M is orientable and $H^n(M) = \{0\}$ if M is not orientable. The first claim follows from Poincaré duality, a fact that we will prove in the next Chapter (Theorem 3.9).

6.7. Prove that the *antipodal map* $A : S^n \longrightarrow S^n$, $A(x) = -x$ is orientation preserving if and only if n is odd. Prove that $A^* = (-1)^{n+1} : H^n(S^n) \cong \mathbb{R} \longrightarrow H^n(S^n) \cong \mathbb{R}$.

6.8. Consider the vector field $X : S^{2k-1} \longrightarrow TS^{2k-1}$, $X(x_1, \ldots, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$ (see Example 1.32). Prove that $H(x,t) = (\cos \pi t x + \sin \pi t X(x))$ is a homotopy between A and $\mathbb{1}$. Conclude that $A \sim \mathbb{1}$ if and only if n is odd, and, if n is even, every vector field on S^n has a zero.

CHAPTER 5

Harmonic forms and the Theorem of Hodge

In this Chapter we will introduce the Laplace-Beltrami operator, a generalization at level of differentiable p-forms of the classical Laplacian of functions. Hence we will have the concept of *harmonic forms*, i.e. forms in the kernel of this operator and we will give a very rough idea of the proof of the classical Theorem of Hodge which states that in each cohomology class of a *compact* Riemannian manifold, there is a unique harmonic form. We will give also some applications of this Theorem to the topology of manifold with "positive curvature".

1. Some basic facts in Riemannian geometry

In this section we will describe the basic concepts and results in Riemannian geometry *that we will use in this Chapter.* It is not intended to be an introduction to the subject.

1.1. The Levi-Civita connection of a submanifold. For vector fields $X, Y \in \mathcal{H}(\mathbb{R}^N)$, we will denote by $\overline{\nabla}_Y X$ the usual directional derivative, i.e. if $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^N$ is a smooth curve with $\gamma(0) = p$, $\dot{\gamma}(0) = Y(p)$,

$$\overline{\nabla}_Y X(p) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} X(\gamma(t)).$$

1.1. Remark. It is well known that $\overline{\nabla}_Y X(p)$ depends only on $Y(p)$ and the values of X along any curve $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^N$ with $\dot{\gamma}(0) = Y(p)$.

Let $M \subseteq \mathbb{R}^N$ be an n-dimensional submanifold and $X, Y \in \mathcal{H}(M)$. By Remark 1.1 we can compute $\overline{\nabla}_Y X(p)$, since we only have to know X along a curve γ with $\dot{\gamma}(0) = Y(p)$ and this curve can be taken in M. This is not, in general, a tangent vector to M. But we can project it on T_pM.

1.2. Definition. We define the *covariant derivative*, at p, of X in the direction Y as

$$\nabla_Y X(p) = P_p^\top(\overline{\nabla}_Y X(p)),$$

where $P_p^\top : \mathbb{R}^N \longrightarrow T_pM$ is the orthogonal projection.

1.3. Remark. $\nabla_Y X$ is smooth, as a function of p, hence $\nabla_Y X \in \mathcal{H}(M)$. Hence we have a $\mathbb{R}$-bilinear map

$$\nabla : \mathcal{H}(M) \times \mathcal{H}(M) \longrightarrow \mathcal{H}(M), \qquad \nabla(Y, X) := \nabla_Y X$$

.

1.4. Definition. The operator ∇ is called the *Levi-Civita (or Riemannian) connection* of M

1.5. Proposition. *The Levi-Civita connection verify the following properties:*

(1) ∇ *is* $\mathbb{R}$*-bilinear and* $\mathcal{F}(M)$ *linear in the first variable,*
(2) *if* $f \in \mathcal{F}(M)$ *then* $\nabla_Y fX = Y(f)X + f\nabla_Y X$*,*
(3) $\nabla_Y X - \nabla_X Y = [X, Y]$*,*
(4) $Y\langle X, Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle$*.*

PROOF. The properties are obvious for $\overline{\nabla}$ and are preserved by $P_p^{\top} : \mathbb{R}^N \longrightarrow T_pM$. ☐

Those properties characterize the Levi-Civita connection in view of the following result, sometimes called the *Fundamental Theorem of Riemannian Geometry*:

1.6. THEOREM. *There exist a unique operator* $\nabla : \mathcal{H}(M) \times \mathcal{H}(M) \longrightarrow \mathcal{H}(M)$ *with the properties of Proposition 1.5*

PROOF. Let $\tilde{\nabla} : \mathcal{H}(U) \times \mathcal{H}(U) \longrightarrow \mathcal{H}(U)$ be an operator satisfying the properties above. For $X, Y, Z \in \mathcal{H}(M)$ we have:

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \tilde{\nabla}_X Y, Z\rangle + \langle Y, \tilde{\nabla}_X Z\rangle, \\ Y\langle Z, X\rangle &= \langle \tilde{\nabla}_Y Z, X\rangle + \langle Z, \tilde{\nabla}_Y X\rangle, \\ Z\langle X, Y\rangle &= \langle \tilde{\nabla}_Z X, Y\rangle + \langle X, \tilde{\nabla}_Z X\rangle. \end{aligned}$$

Adding the first two and subtracting the last one we have

$$2\langle \tilde{\nabla}_X Z, Y\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle Y, X\rangle + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle + \langle [Z, Y], X\rangle.$$

On the other hand the same formula holds for the Levi-Civita connection ∇, hence $\tilde{\nabla} = \nabla$. ☐

1.7. REMARK. The formula obtained in the proof of Theorem 1.6 is called the *Kozul formula*. An important observation is that the right hand side of the Kozul formula depends only on the fact that we have a scalar product on each tangent space and on the differential topological properties of M (Lie product of vector fields).

1.8. LEMMA. ∇_Y *is a local operator, i.e. if two vector fields* X, Z *coincide on an open set* $U \subseteq M$ *then* $\nabla_Y X = \nabla_Y Z$ *in* U*.*

PROOF. The statement is true for $\overline{\nabla}$ and preserved by $P^{\top}$, hence is true for ∇. ☐

1.9. REMARK. Lemma 1.8 allows us to take covariant derivatives of locally defined vector fields.

Let $\phi : \Sigma \subseteq \mathbb{R}^n \longrightarrow U = \phi(\Sigma) \subseteq M$ be a local chart and $\dfrac{\partial}{\partial x_i}$ the associated coordinate vector fields. Define the functions $\Gamma_{ij}^k : U \longrightarrow \mathbb{R}$ by the condition:

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

1.10. DEFINITION. The functions Γ_{ij}^k are called the Christoffel symbols of the connection.

It is convenient to define the covariant derivative of vector fields along a curve. Let $\gamma : (a, b) \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve. We recall that a vector field along γ is a smooth map $X : (a, b) \longrightarrow \mathbb{R}^N$ such that $X(t) \in T_{\gamma(t)}M, \ \forall\ t \in (a, b)$. We will denote by $\mathcal{H}(\gamma)$ the space of such vector fields.

1.11. Theorem. *There is a unique* $\mathbb{R}$*-linear operator* $\dfrac{\nabla}{\mathrm{d}t} : \mathcal{H}(\gamma) \longrightarrow \mathcal{H}(\gamma)$ *such that:*

(1) *if* $X \in \mathcal{H}(\gamma),\ f \in \mathcal{F}((a,b)),\ \dfrac{\nabla}{\mathrm{d}t} fX = [\dfrac{\mathrm{d}}{\mathrm{d}t} f]X + f\dfrac{\nabla}{\mathrm{d}t}X,$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}t}\langle X(t), Y(t)\rangle = \langle \dfrac{\nabla}{\mathrm{d}t}X(t), Y(t)\rangle + \langle X(t), \dfrac{\nabla}{\mathrm{d}t}Y(t)\rangle,\ \forall X,\ Y \in \mathcal{H}(\gamma),$

(3) *if* X *extends to a vector field in a neighborhood of* $\gamma(t)$*, then* $\dfrac{\nabla}{\mathrm{d}t}X = \nabla_{\dot\gamma(t)}X$.

Proof. It is clear the the operator $\dfrac{\nabla}{\mathrm{d}t}X(t) = P^{\top}_{\gamma(t)}(\dfrac{\mathrm{d}}{\mathrm{d}t}X(t))$ verifies the properties in question. Suppose that $\dfrac{\tilde\nabla}{\mathrm{d}t}$ is an other operator with the same properties. It is easily seen that it is a local operator in the sense that if two vector fields along γ coincide in an open neighborhood of t_0, so do their images. So we can work in a coordinate neighborhood. Let $\{x_1, \ldots, x_n\}$ be local coordinates in that neighborhood, and $\gamma(t) = \{x_1(t), \ldots, x_n(t)\}$ the local coordinate expression of γ. Then

$$X(t) = \sum_{i=1}^{n} X_i(t)\frac{\partial}{\partial x_i} \quad \text{and} \quad \frac{\tilde\nabla}{\mathrm{d}t}X = \sum_{i=1}^{n}[\frac{\mathrm{d}}{\mathrm{d}t}X_i]\frac{\partial}{\partial x_i} + X_i\nabla_{\dot\gamma(t)}\frac{\partial}{\partial x_i}.$$

Since the same holds for $\dfrac{\nabla}{\mathrm{d}t}X$, we have $\dfrac{\tilde\nabla}{\mathrm{d}t}X = \dfrac{\nabla}{\mathrm{d}t}X$. □

1.12. Remark. We stress the difference between $\nabla_{\dot\gamma}$ and $\dfrac{\nabla}{\mathrm{d}t}$. In fact a vector field along γ may not be, even locally, the restriction of a vector field in M. The extreme case is when γ is a constant curve and X is a non constant vector field along γ. Then $\dot\gamma = 0$ but, in general, $\dfrac{\nabla}{\mathrm{d}t}X \neq 0$.

1.13. Remark. We can take also covariant derivatives of exterior forms. For a 1-form $\omega \in \Omega^1(M)$, and $X \in T_pM$, we have the dual vector field $Y = \sharp\ \omega$ ($\sharp$ is the inverse of the "musical isomorphism" $\flat : T_pM \longrightarrow [T_pM]^*$) and it is natural to define $\nabla_X\omega$ to be dual of ∇_XY. This lead to the formula:

$$[\nabla_X\omega](W) = (\sharp\nabla_XY)(W) = \langle\nabla_XY, W\rangle = X\langle Y, W\rangle - \langle Y, \nabla_XW\rangle = X\omega(W) - \omega(\nabla_XW).$$

Next we observe that any $\omega \in \Omega^p(M)$ is, locally, sum of decomposable forms, i.e. exterior products of 1-form. On such decomposable form we can extend ∇_X to act as a derivation, i.e. $\nabla_X(\omega_1 \wedge \cdots \wedge \omega_p) = \sum_{k=1}^{p} \omega_1 \wedge \cdots \wedge \nabla_X\omega_k \wedge \cdots \omega_p$. A direct calculation gives the expression:

$$\nabla_X\omega(X_1, \ldots, X_p) = X(\omega(X_1, \ldots, X_p)) - \sum_{i=1}^{p}\omega(X_1, \ldots, X_{i-1}, \nabla_XX_i, X_{i+1}, \ldots, X_p).$$

It is easy to check that $\nabla_X\omega$, given by the formula above, is an alternating $\mathcal{F}(M)$ multilinear map, hence a differential form. So the formula above can be taken as definition of $\nabla_X\omega$.

1.2. The curvature tensor.

1.14. Definition. The *curvature* of the Levi-Civita connection is the map

$$R(X,Y)Z := \nabla_X\nabla_YZ - \nabla_Y\nabla_XZ - \nabla_{[X,Y]}Z, \quad X, Y, Z \in \mathcal{H}(M).$$

1.15. Remark. Some authors use the opposite sign in the definition of R.

The basic property is that R is a *tensor*, i.e.:

1.16. Lemma. *R is $\mathcal{F}(M)$-trilinear.*

Proof. R is clearly $\mathbb{R}$-trilinear. The linearity with respect to $\mathcal{F}(M)$ follows by a direct calculation using the properties of the connection and of the Lie product. ☐

1.17. Remark. Lemma 1.16 implies that the value of $R(X,Y)Z$ at a given point $p \in M$ depends only on the value of X, Y, Z at that point, i.e. to compute $[R(X,Y)Z](p)$ we can use arbitrary extensions of $X(p)$, $Y(p)$ and $Z(p)$ (see Theorem 1.2 of Chapter 1). For example, if M is an open set of $\mathbb{R}^n$ we can extends vectors to constant vector fields and therefore we have the expected fact that the curvature of an open subset of $\mathbb{R}^n$ is zero.

The curvatures tensors have interesting symmetries:

1.18. Proposition. *For $X, Y, Z, W \in \mathcal{H}(M)$ we have:*

1. $R(X,Y)Z = -R(Y,X)Z$,
2. $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,
3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(first Bianchi identity),*
4. $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Proof. (1) follows directly from the definition. (2) is equivalent to $\langle R(X,Y)Z, Z\rangle = 0$. The latter follows from a direct calculations taking in account

$$\langle \nabla_X\nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle = \frac{1}{2}XY\langle Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle, \quad \langle \nabla_{[X,Y]}Z, Z\rangle = \frac{1}{2}[X,Y]\langle Z, Z\rangle.$$

Next we prove (3).

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \\
&+\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y = \\
&= \nabla_Y[Z,X] + \nabla_Z[X,Y] + \nabla_X[Y,Z] + \nabla_{[X,Z]}Y + \nabla_{[X,Y]}Z + \nabla_{[Z,Y]}X = \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0,
\end{aligned}$$

where the last equality is the Jacobi identity for the Lie product of vector fields (see ? Chapter ?).

We will sketch now the proof of (4), leaving the details to the reader. Consider the Bianchi identity

$$\langle R(X,Y)Z, W\rangle + \langle R(Y,Z)X, W\rangle + \langle R(Z,X)Y, W\rangle = 0.$$

Permuting cyclically the vectors X, Y, Z and W, we obtain three more equations. Summing the four equations we obtain

$$\langle R(X,Z)Y, W\rangle + \langle R(Y,W)Z, X\rangle = 0$$

and the conclusion follows from (2). ☐

It will be useful, later on, to look at the curvature tensor in a slight different way. For $p \in M$ we define

$$\rho_p : \Lambda^2(T_pM^*) \longrightarrow \Lambda^2(T_pM^*), \qquad \langle \rho(X \wedge Y), Z \wedge W\rangle = \langle R(X,Y)W, Z\rangle.$$

By Proposition 1.18, ρ_p is a well defined linear *symmetric* endomorphism of $\Lambda^2(T_pM^*)$.

1.19. Definition. The operator ρ is called the *curvature operator of M*.

Associated to the curvature tensor there are several "measures of curvature". We will describe three of the most important.

• The *sectional curvature*, K. This is a function defined, in principle, on pairs of linearly independent vectors tangent at the same point

$$K(X,Y) := \frac{\langle \rho(X \wedge Y), X \wedge Y\rangle}{\|X \wedge Y\|^2} = \frac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}.$$

It follows from Exercise 7.6 of Chapter 1, that $K(X,Y)$ depends only on the plane spanned by X and Y.

The sectional curvature K is then a real valued function defined on 2-dimensional tangent subspaces.

• The *Ricci curvature*, *Ricc*. For fixed $X, Y \in T_pM$ consider the linear map $L_{X,Y} : T_pM \longrightarrow T_pM,\ L_{X,Y}(Z) = R(Z,X)Y$. The *Ricci tensor* is defined as

$$\tilde{Q}(X,Y) = \text{trace } L_{X,Y}.$$

By Proposition 1.18, $\tilde{Q}$ is a symmetric tensor. The Ricci curvature is defined as the quadratic form

$$Ricc(X) = \tilde{Q}(X,X).$$

A couple of observations will be useful. Fix an orthonormal basis $\{X_1, \dots, X_n\}$. Then the map

$$Q : T_pM \longrightarrow T_pM, \qquad Q(X) = \sum R(X, X_i)X_i$$

is well defined and $\tilde{Q}(X,Y) = \langle Q(X), Y\rangle$. Also, if $X = X_1,\ \ Ricci(X) = \sum_2^n K(X, X_i)$.

The Ricci curvature is then a real valued funtion defined on the tangent bundle.

• The *scalar curvature* S. This is a function defined on the manifold M as the average of the Ricci curvatures. If $X_1, \dots X_n$ is an orthonormal basis, then

$$S = \text{trace} Q = \sum_1^n Ricc(X_i).$$

The scalar curvature is a real valued function defined on the manifold.

1.20. Remark. If $n = 2$ the sectional curvature coincides with the scalar curvature and is usually called the *Gaussian curvature* of the surface.

1.21. Remark. Some authors define the Ricci and scalar curvatures dividing our expressions by $n-1$ and $n(n-1)$ respectively.

1.3. Parallel tranlation and geodesics. Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve. We have introduced the operator: $\dfrac{\nabla}{\mathrm{d}t} : \mathcal{H}(\gamma) \longrightarrow \mathcal{H}(\gamma)$ of covariant derivative along γ. When no confusion will arise, we will set $\dfrac{\nabla}{\mathrm{d}t}X = X'$.

1.22. Definition. Let $\gamma : [a,b] \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve. A vector field $X \in \mathcal{H}(\gamma)$ is *parallel along* γ if $X' := \dfrac{\nabla}{\mathrm{d}t}X = 0$.

1.23. Remark. If M is an open set of $\mathbb{R}^n$, a vector field is parallel if and only if it is constant. But in general we should not confuse parallel fields in the manifold with parallel field in the ambient Euclidean space. For example, if $M = S^n$, the unit sphere in $\mathbb{R}^{n+1}$, and $\gamma(t) = \sin(t)e_1 + \cos(t)e_2$, the tangent vector $\dot{\gamma}(t)$ is parallel along γ, in the sense of the definition, but not parallel in $\mathbb{R}^{n+1}$. However parallelism has a "mechanical interpretation". Suppose, for simplicity, we have a surface $M \subseteq \mathbb{R}^3$, a curve γ in M and a vector field X along γ. We can "roll" the surface on a plane along the curve obtaining a curve in the plain and a vector field along this curve. Then the original vector field is parallel if and only if the the vector field obtained by this process is parallel in the sense of the plain, i.e. is constant. The reader is invited to think in the example above: we can "roll" the sphere S^2 on a plane in such a way that the curve γ correspond to a straight line and the tangent vector field to the tangent to the line. Then the latter is parallel in the sense of the plain. These intuitive arguments can be made precise.

1.24. Remark. It is convenient to have an expression of the equation of parallelism in terms of local coordinates. Let $\gamma(t) = (x_1(t), \dots, x_n(t))$ be the expression of the curve in local coordinates and let $X(t) = \sum X_i(t)\frac{\partial}{\partial x_i} \in \mathcal{H}(\gamma)$. Then

$$X'(t) = \sum_k \dot{X}_k \frac{\partial}{\partial x_k} + \sum_i X_i \nabla_{(\sum_j \dot{x}_j \frac{\partial}{\partial x_j})} \frac{\partial}{\partial x_i} = \sum_k [\dot{X}_k + \sum_{i,j} \dot{x}_i X_j \Gamma^k_{ij}] \frac{\partial}{\partial x_k}.$$

The equation $X' = 0$ is a system of ordinary differential equations. For such systems we have we have (local) existence and unicity[1] for the solutions with given initial conditions. We also have smooth dependence on the initial conditions. In our case, since the system is linear, the solutions are defined on the entire interval of definition of the coefficients.

1.25. Proposition. *Let $\gamma : [a,b] \longrightarrow M$ be a smooth curve and $X_a \in T_{\gamma(a)}M$, The there exists a unique vector field $X \in \mathcal{H}(\gamma))$ such that $X' = 0$ and $X(a) = X_a$.*

Proof. Cover $\gamma([a,b])$ with a finite number of charts $\{U_i, \phi_i\}$. Then $\gamma^{-1}\phi_i(U_i)$ is an open covering of $[a,b]$. Then there exist a subdivision $t_0 = a < t_1 < \cdots < t_k = b$ such that $\gamma([t_i, t_{i+1}]) \subseteq \phi_i(U_i)$. By Remark 1.24 there is a unique parallel vector field $X_1 \in \mathcal{H}(\gamma|[t_0, t_1 + \epsilon))$, $\epsilon > 0$, with $X(a) = X_a$. By the same reason there is a parallel vector field $X_2 \in \mathcal{H}(\gamma|(t_1 - \epsilon, t_2 + \epsilon))$ with $X_2(t_1) = X_1(t_1)$. The two vector fields coincide in $(t_1 - \epsilon, t_1 + \epsilon)$ since in this interval they are parallel and coincide at t_1. So we have a well definite smooth parallel vector field in $[a, t_2 + \epsilon)$. Proceeding in the same way, we obtain, after a finite number of steps, a vector field $X \in \mathcal{H}(\gamma)$ with the required properties. ☐

Proposition 1.25 allows us to define a map

$$P_\gamma : T_{\gamma(a)}M \longrightarrow T_{\gamma(b)},$$

which associate to a given vector $X_a \in T_{\gamma(a)}M$ the value of the parallel translation of X_a at the value $t = b$ of the parameter. This map is called *parallel translation along* γ. P_γ is linear, since the equation $X' = 0$ is linear, and it is an isometry since, if $X, Y \in \mathcal{H}(\gamma)$ are parallel vector fields,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X(t), Y(t)\rangle = \langle X'(t), Y(t)\rangle + \langle X(t), Y'(t)\rangle = 0.$$

1.26. Remark. The word "connection" came from the fact that, using parallel translation, we can "connect" tangent spaces at different points. We should point out, however, that this "connection" depends strongly on the curve.

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[1] Unicity has to be intended as the fact that two solutions with the same initial condition coincide in the intersection of the domains of definition.

The next results link parallel translation with curvature and will be of use later on. First some preliminaries. Let $U \subseteq \mathbb{R}^2$ and $F : U \longrightarrow M$ a smooth map (sometimes called a surface). A vector field along F is a smooth map $X : U \longrightarrow TM$ such that $X(t,s) \in T_{F(t,s)}M$. We will denote by $\mathcal{H}(F)$ the vector space of vector fields along F. We have the operators $\frac{\nabla}{\mathrm{d}t}, \frac{\nabla}{\mathrm{d}s}$, the covariant derivative along the curves $F(t, s_0)$, $F(t_0, s)$.

1.27. Definition. A vector field $X \in \mathcal{H}(F)$ is *parallel along F* if

$$\frac{\nabla}{\mathrm{d}t}X = 0 = \frac{\nabla}{\mathrm{d}s}X.$$

The existence of parallel vector fields along surfaces is not at all obvious since the equations involved are not any more *ordinary* differential equations, but they are *partial* differential equations. The results we are after are *integrability conditions* for the existence of solutions for such differential equations.

1.28. Lemma. *If* $X \in \mathcal{H}(F)$*,*

$$\frac{\nabla}{\mathrm{d}t}\frac{\nabla}{\mathrm{d}s}X - \frac{\nabla}{\mathrm{d}s}\frac{\nabla}{\mathrm{d}t}X = R(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s})X.$$

Proof. Compute both sides in local coordinates. □

Suppose that $X \in \mathcal{H}(F)$ is parallel. Then, by Lemma 1.28, $R(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s})X = 0$. So the vanishing of some curvatures is a necessary condition for the existence of parallel vector fields alon a surface. We will show next that, in some sense, they are also sufficient.

Let $F : \mathbb{R}^2 \longrightarrow M$ be a surface and let X_0 be a vector in $T_{F(0,0)}M$. The natural way to try to construct a parallel vector field along F, with initial condition X_0 is to parallel translate X_0 along the curve $\sigma(t) = F(t, 0)$ obtaining a vector field $X(t, 0)$, parallel along σ. Then parallel translate $X(t, 0)$ along the curve $\gamma_t(s) = F(t, s)$ (t fixed). In such a way we obtain a vector field $X \in \mathcal{H}(F)$ with $X(0, 0) = X_0$.

1.29. Theorem. *The vector field X defined above is parallel along F if*

$$R(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s})X = 0.$$

Proof. We have seen that the condition is necessary. We will show that it is sufficient. By construction $\frac{\nabla}{\mathrm{d}s}X = 0$ so we have to show that $\frac{\nabla}{\mathrm{d}t}X = 0$. This is true, again by construction, along σ. Moreover

$$\frac{\nabla}{\mathrm{d}s}\frac{\nabla}{\mathrm{d}t}X = \frac{\nabla}{\mathrm{d}t}\frac{\nabla}{\mathrm{d}s}X - R(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial s})X = \frac{\nabla}{\mathrm{d}t}\frac{\nabla}{\mathrm{d}s}X = 0.$$

Therefore $\frac{\nabla}{\mathrm{d}t}X$ is parallel along the curves γ_t. But, along these curves, $\frac{\nabla}{\mathrm{d}t}X$ vanishes at $s = 0$, hence $\frac{\nabla}{\mathrm{d}t}X = 0$ along the all curve γ_t. Therefore $\frac{\nabla}{\mathrm{d}t}X = 0$. □

A concept very much related to the one of parallel translation is the one of *geodesic.*

1.30. Definition. A curve $\gamma : (a, b) \subseteq \mathbb{R} \longrightarrow M$ is a *geodesic* if $\dot{\gamma}$ is parallel along γ.

1.31. Remark. If γ is a geodesic, $\dot{\gamma}$ is parallel along γ and therefore has constant norm, since parallel translation is an isometry. It follows that the geodesics came with a special parametrization, a parametrization proportional to arc length, i.e. $\|\dot{\gamma}(t)\|$ is constant.

1.32. Remark. It is useful to have a local coordinate expression for the geodesic equation $\dot{\gamma}' = 0$. Let $\gamma(t) = (x_1(t), \dots x_n(t))$ be the expression of γ in some local coordinate system. Then, from Remark 1.24, the equation $\dot{\gamma}' = 0$ takes the form

$$\ddot{x}_k + \sum_{i,j} \Gamma^k_{ij} \dot{x}_i \dot{x}_j = 0, \qquad k = 1, \dots, n.$$

The equation above is a system of second order ordinary differential equations, easily reduced to a system of the first order by introducing the auxiliary variable $y = \dot{x}$ (in this way we double the number of equations). Then we have existence, unicity and smooth dependence on the initial conditions. However the system *is not* linear and we do not have existence in the large. A simple example is the following.

1.33. Example. Consider $M = \mathbb{R}^2 \setminus \{0\}$. The geodesic with initial condition $p = (-\epsilon, 0),\ X(p) = (1, 0)$ is $\gamma_X(t) = (-\epsilon + t, 0)$ and is defined for $t < \epsilon$.

1.34. Example. Consider the unit sphere in $\mathbb{R}^{n+1}$. Then the curve $\gamma(t) = \sin(t)e_1 + \cos(t)e_2$, is a geodesic (see Remark 1.23).

1.35. Remark. In general the integration of the geodesic equations is a quite hard task even for very simple surfaces.

Let $\gamma_X : [0, a) \longrightarrow M$ be the geodesic with initial conditions $\gamma(0) = p, \quad \dot{\gamma}(0) = X$. Fix $s \in \mathbb{R}$ and consider the curve $\sigma(t) = \gamma_X(st)$. Then σ is a solution of the geodesics equation with initial condition sX, and it is defined in the interval $[0, a/s)$. This observation leads to the following

1.36. Lemma. *The exists $\epsilon > 0$ such that, if $\|X\| < \epsilon$, the geodesic γ_X is defined for $t = 1$.*

Proof. For $Z \in T_pM$, with $\|Z\| = 1$, define $r(Z) = \sup\{t \in [0, \infty) : \gamma_Z \text{ is defined in } [0, t)\}$. The r is a well defined function on the unit sphere of T_pM. Moreover, by the smooth dependence from the initial data, r is continuous, hence has a positive minimum $\epsilon = \epsilon(p)$. □

1.37. Definition. Let $p \in M$ and ϵ as in Lemma 1.36. We define

$$\exp_p : \{X \in T_pM : \|X\| < \epsilon\} \longrightarrow M, \qquad \exp(X) = \gamma_X(1).$$

1.38. Remark. Observe that $\exp_p(tX) = \gamma_X(t)$ (when defined). This means, geometrically, that $\exp_p$ sends the ray $\{tX\} \subseteq T_pM$, "linearly", onto the geodesic γ_X.

Again, by the smooth dependence, $\exp_p$ is a smooth function.

1.39. Lemma. $\mathrm{d}\exp_p(0) = \mathbb{1}$ *(modulo the identification $T_0T_pM \cong T_pM$).*

Proof. $\mathrm{d}\exp_p(0)X$ is the tangent vector to $\exp(tX)$ at $t = 0$, i.e. X. □

In particular, by the inverse function Theorem, $\exp_p$ is a diffeomorphism of a small ball $V = \{Y \in T_pM : \|Y\| < \delta\}$ onto an open neighborhood $\exp_p(V) \subset M$. In other words, $(V, \exp_p|V)$ is a chart for M.

1.40. Definition. The set $\exp_p(V)$ is called a *normal neighborhood* of p and the local coordinates are called *normal coordinates.*

1.41. Remark. The name exponential come from the fact that, in the case of $O(n) \subseteq \mathbb{R}^{n^2}$ (see Example ? of Chapter ?), $exp_{1\!\!1}(X) = \sum_0^\infty X^k$ $X \in Skw(n, \mathbb{R}) = T_{1\!\!1} O(n)$. This is a particular case of a more general fact that we will discuss in subsection 1.5

The following result is often useful to simplify computations.

1.42. Proposition. *Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis for T_pM. Then there exist local orthonormal vector fields $X_1, \ldots, X_n$ such that $X_i(p) = E_i$, and*

$$\nabla_{X_i} X_j(p) = 0.$$

Proof. Let U be a normal neighborhood of p. At $q = \exp_p(Y)$ we define $X_i(q)$ as the parallel translated of E_i along the curve $\exp_p(tY)$. The vector fields X_i are smooth since geodesics and parallel translation depends smoothly on the initial conditions. Moreover, since parallel translation is an isometry, they are orthonormal. Finally

$$\nabla_Y X_i(p) = \frac{\nabla}{\mathrm{d}t} X_i(p) = 0,$$

since X_i is parallel along $\exp(tY)$. $\square$

1.4. Riemannian manifolds. The careful reader has probably realized that all we have done up to now depend only on the existence of a scalar product in each tangent space. This suggest the following

1.43. Definition. A *Riemannian manifold* is a differentiable manifold together with a scalar product $\langle \cdot, \cdot \rangle_p$ on each tangent space T_pM, that depends smoothly on p. This means that, given smooth vector fields $X,\ Y \in \mathcal{H}(M)$, the function that associate to $p \in M$ the number $\langle X(p), Y(p) \rangle_p$ is smooth. When clear from the context we will omit the subindex p.

We will call this assignment of scalar products a *Riemannian metric*

In a Riemannian manifold, we can define the Levi-Civita connection using the Kozul formula (see Remark 1.7), prove the basic properties stated in Proposition 1.5 and show the uniqueness as in Theorem 1.6.

Once we have the Levi-Civita connection, we can define the curvature tensor, parallel translation, geodesics etc.. The proofs of the respective properties go just as in the case of submanifolds.

A priori the “universe” of Riemannian manifold is larger than the one of submanifolds of Euclidean space. It turns out that this is not the case. We will discuss shortly this fact now.

1.44. Definition. Let M, N be Riemannian manifolds and $f : M \longrightarrow N$ a smooth map.

• We will say that f is an *isometry* if it is a diffeomorphism and

$$\langle \mathrm{d}f(X), \mathrm{d}f(Y) \rangle = \langle X, Y \rangle \quad \forall\ X, Y \in \mathcal{H}(M),$$

(we use the same symbol for the Riemannian metrics of M and N).

• We will say that f is a *local isometry* if $\forall\ p \in M$ there exist an open neighborhood $U \subseteq M$ of p such that $f(U)$ is open and $f|_U$ is an isometry onto $F(U)$.

• We will say that f is an *isometric immersion (resp. an isometric embedding)* if f is an immersion (resp. an embedding) and the formula above holds.

It follows from the unicity of the Levi-Civita connection that a isometry sends covariant derivative of vector fields into the covariant derivative of the images via the differential. In particular it preserve curvature and sends geodesics to geodesics. For the case of isometric immersions the image of a geodesic is a *geodesic of the image* which is not, in general, a geodesic of the ambient space.

1.45. REMARK. It is worthwhile to observe, for the case of submanifolds, that the existence of an isometry between two submanifolds does not means that they sit similarly in the ambient space[2]. The classical example is the strip $S = \{(t, \theta, 0) \in \mathbb{R}^3 : \theta \in (0, \pi)\}$ and the cylinder $C = \{(\cos\theta, \sin\theta, t) \in \mathbb{R}^3 : \theta \in (0, \pi)\}$. The map $F(t, \theta, 0) = (\cos\theta, \sin\theta, t)$ is an isometry, but the two submanifolds do not differ by an isometry of $\mathbb{R}^3$. The way that a submanifold sits inside the ambient manifold is described by the so called *second fundamental form* which is the difference between the connection in the ambient space and the connection of the submanifold. If the second fundamental form vanish, the submanifold is called *totally geodesic* and, in this case, the geodesics of the submanifold are geodesic of the ambient manifold (and viceversa).

In principle the category of Riemannian manifolds, up to isometries, seems larger that the one of submanifolds of Euclidean spaces. However, as in the case of differentiable manifolds, this is not the case, due to the following result of J. Nash (that we will not prove here):

1.46. THEOREM. [Nash] *Let M be a Riemannian manifold. Then there exist an isometric embedding $F : M \longrightarrow \mathbb{R}^N$, for N sufficiently large.*

1.47. REMARK. If $F : M \longrightarrow N$ is an isometric embedding, $F(M)$ is a submanifold of N and $F : M \longrightarrow F(M)$ is an isometry.

1.5. Lie groups. Let G be a Lie group.

1.48. DEFINITION. A Riemannian metric on G is *left (resp. right) invariant* if the left (resp. right) translations are isometries. A Riemannian metric is *bi-invariant* if it is left *and* right invariant.

Fix an inner product $\langle \cdot, \cdot \rangle_e$ on $\widehat{G} = T_eG$. The we can define a Riemannian metric on G setting

$$\langle X, Y \rangle_g = \langle \mathrm{d}L_{g^{-1}}X, \mathrm{d}L_{g^{-1}}Y \rangle_e.$$

This metric is, essentially by definition, a left invariant metric. Let now $\{E_1, \ldots, E_n\}$ be an orthonormal basis for $\widehat{G}$. Then the left invariant translations are an orthonormal basis of vector fields. If $\{\omega_1, \ldots, \omega_n\}$, $n = dim\ G$ is the dual basis, $\omega = \omega_1 \wedge \cdots \wedge \omega_n$ is a left invariant form, called the *volume form* (see ?). Naturally the same considerations work for right invariant objects. However, in general does not exist a bi-invariant metric. A particular interesting case when such a metric exists is the following.

1.49. PROPOSITION. *Let G be a* compact *Lie group. Then there exist a bi-invariant metric on G.*

PROOF. Start with a left invariant metric and "average" it under right translations, i.e. define a new metric by

$$\langle\langle X, Y \rangle\rangle = \int_G \langle \mathrm{d}R_g X, \mathrm{d}R_g X \rangle \mathrm{d}\omega.$$

□

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[2] i.e. there exist an isometry of the ambient space that takes one onto the other.

For Lie groups with bi-invariant metric, the relevant geometric objects as connection, geodesics and curvature may be described in algebraic terms. For the rest of this subsection, G *will denote a Lie group with a bi-invariant metric* (that we will denote by $\langle \cdot, \cdot \rangle$).

1.50. Lemma. *Let* $X, Y, Z \in \mathcal{L}(G)$. *Then*

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle.$$

Proof. Let $\gamma(t) = \exp(tY)$. Since the metric is bi-invariant

$$\langle X, Z \rangle = \langle \mathrm{d}R_{\gamma(-t)} \circ \mathrm{d}L_{\gamma(t)} X, \mathrm{d}R_{\gamma(-t)} \circ \mathrm{d}L_{\gamma(t)} Z \rangle = \langle Ad(\gamma(t))X, Ad(\gamma(t))Z \rangle = constant.$$

Differentiating with respect to t, and computing for $t = 0$, we have

$$0 = \langle [\mathrm{d}Ad](e)(Y)X, Ad(e)Z \rangle + \langle Ad(e)X, [\mathrm{d}Ad](e)(Y)Z \rangle = \langle ad(Y)X, Z \rangle + \langle X, ad(Y)Z \rangle = \langle [Y, X], Z \rangle + \langle X, [Y, Z] \rangle.$$

☐

1.51. Remark. It is not difficult to show that a left invariant metric for which the conclusion of Lemma 1.50 holds is actually a bi-invariant metric.

1.52. Theorem. *Let* $X, Y, Z \in \mathcal{L}(G)$. *Then*

1. $2\nabla_X Y = [X, Y]$,
2. $4R(X, Y)Z = [Z, [X, Y]]$.

Proof. First we prove that $\nabla_X X = 0$. In fact

$$2\langle \nabla_X X, Y \rangle = \langle X, [Y, X] \rangle = -\langle X, [X, Y] \rangle = 0,$$

where the first equality is the Kozul formula (see Remark 1.7) and the last one is by Lemma 1.50. In particular

$$0 = \nabla_{X+Y}(X + Y) = \nabla_X Y + \nabla_Y X.$$

On the other hand

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Adding the two equation we have (1). In order to prove (2) we compute

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

using (1). Thus we obtain

$$R(X, Y)Z = \frac{1}{4}[X, [Y, Z]] - \frac{1}{4}[Y, [X, Z]] - \frac{1}{2}[[X, Y], Z]$$

and the conclusion follows from the Jacobi identity. ☐

1.53. Corollary. *The geodesics of* G *are left translated of 1-parameter subgroups. In particular they are defined for all values of the parameter*[3].

Proof. Let $X \in \mathcal{L}(G)$. Then the integral curves of X, which are translated of the 1-parameter subgroup $\exp(tX)$, are geodesics since, by Theorem 1.52, $\nabla_X X = 0$. ☐

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[3] Riemannian manifolds for which the geodesics are defined for all values of the parameter are called *complete*.

1.54. Corollary. *Let* $X, Y \in \mathcal{L}(G)$ *be orthonormal. Then the sectional curvature of the plane spanned by* X, Y *is given by* $K(X,Y) = \|[X,Y]\|^2$. *In particular*

(1) $K(X,Y) \geq 0$ *and* $K(X,Y) = 0$ *if and only if* $[X,Y] = 0$.

(2) *The Ricci curvature is non negative and positive if and only if the center of* $\widehat{G}$ *is trivial.*

Proof. The claims for the sectional curvature come directly from Theorem 1.52 (2). Let $X \in \widehat{G}$ be a unit vector and let $\{X_1 = X, \dots, X_n\}$ be an orthonormal basis for $\widehat{G}$. Then

$$Ricc(X) = \sum_2^n K(X, X_i) = \sum_2^n \|[X, X_i]\|^2 \geq 0.$$

Moreover $Ricc(X) = 0$ if and only if X commutes with all the X_i's, i.e. X is in the center of $\widehat{G}$. ☐

2. The Laplace-Beltrami Operator

We want to define the (geometers) Laplacian of a function on a Riemannian manifold. We proceed as in the case of $\mathbb{R}^n$ (see Exercise 7.29 of Chapter 1). We have the *gradient operator*, $\nabla f = \sharp \mathrm{d} f$, and we will define the *divergence* of a vector field.

Let $x \in M$ and $X \in \mathcal{H}(M)$. Consider the linear transformation $L_X : T_xM \longrightarrow T_xM$ given by $L_X(Y) = \nabla_Y X$ and define

$$\operatorname{div}(X) = \operatorname{trace} L_X.$$

In particular, if $\{X_1, \dots, X_n\}$ is an orthonormal basis,

$$\operatorname{div} X = \sum_{i=1}^n \langle \nabla_{X_i} X, X_i \rangle,$$

2.1. Definition. The*Laplacian* is the operator

$$\Delta : \Omega^0(M) = \mathcal{F}(M) \longrightarrow \Omega^0(M) = \mathcal{F}(M), \qquad \Delta f = -\operatorname{div} \mathrm{d} f.$$

2.2. Remark. Let $\{X_1, \dots, X_n\}$ be orthonormal vector fields. Then

$$\Delta f = -\sum_1^n X_s X_s(f) + \nabla_{X_s} X_s(f).$$

In particular if the X_i's are like in Lemma 1.42, $\Delta f(p) = -\sum_1^n X_s X_s(f)(p)$. Also the product formula in Exercise 7.29 of Chapter? extend to our case.

2.3. Lemma. *Given* $f, g \in \mathcal{F}(M), \Delta(fg) = g\Delta f + f\Delta g - 2\langle \nabla f, \nabla g \rangle$.

We want to extend the Laplacian to an operator on p-forms. Since we have the differential we need to define the divergence of a p-form.

For 1-forms ω, the divergence is defined as the divergence of the dual vector field $\sharp\omega$ (which is a function, that is a 0-form). For p-forms we give the following

2.4. Definition. Let $\omega \in \Omega^p(M)$, $X_1, \dots, X_{p-1} \in T_pM$. Consider the bi-linear map $L_{X_1,\dots,X_{p-1}}(X,Y) = [\nabla_X \omega](Y, X_1, \dots, X_{p-1})$. The *divergence* of ω is defined as the $(p-1)$-form

$$\operatorname{div}\omega(X_1, \dots, X_{p-1}) = \operatorname{trace} L_{X_1,\dots,X_{p-1}}.$$

In particular, if $\{X_1, \dots, X_n\}$ is an orthonormal basis of T_xM, we have the formula:

$$(\operatorname{div}\omega)(X_{i_1}, \dots, X_{i_{p-1}}) = \sum_{j=1}^{n} (\nabla_{X_j}\omega)(X_j, X_{i_1}, \dots, X_{i_{p-1}}).$$

It is useful to have an expression of the differential in terms of the connection.

2.5. Lemma. *If* $\omega \in \Omega^p(M)$,

$$\mathrm{d}\omega(X_0, \dots X_p) = \sum_{0}^{p} (-1)^i \nabla_{X_i}\omega(X_0, \dots, \hat{X}_i, \dots, X_p).$$

Proof. Let $p \in M$. Choosing suitable normal coordinates, we can extend the X_i's to vector fields such that, *at* p, $\nabla_{X_i}X_j(p) = 0$. In particular, $[X_i, X_j](p) = 0$. Since we are working with differential forms, the values, at p, do not depend on the extensions. Hence, at p, formula (2.5) coincide with the one that defines d (see Definition **??** of Chapter 3). □

2.6. Definition.

- The *codifferential* is the operator:

$$\delta : \Omega^p(M) \longrightarrow \Omega^{p-1}(M) \qquad \delta\omega = -\operatorname{div}\omega.$$

- The *Laplace-Beltrami operator*, or simply the *Laplacian*, is the operator:

$$\Delta : \Omega^p(M) \longrightarrow \Omega^p(M) \qquad \Delta\omega = \mathrm{d} \circ \delta\omega + \delta \circ \mathrm{d}\omega.$$

- A form $\omega \in \Omega^p(M)$ is *coclosed* if $\delta\omega = 0$ and *harmonic* if $\Delta\omega = 0$.

2.7. Remark. If $\omega \in \Omega^0$, $\delta\omega = 0$ and the definitions of the Laplacian in 2.6 and 2.1coincide.

Suppose now that M is *oriented* and let $*\colon \Omega^p(M) \longrightarrow \Omega^{n-p}(M)$ be the Hodge operator (see **??** of Chapter 1).

2.8. Proposition. $\operatorname{div}\omega = (-1)^{n(p-1)} * d * \omega \quad \forall \omega \in \Omega^p(M)$.

Proof. Let $\{X_1, \dots, X_n\}$ be a positive orthonormal basis of T_pM and extend the X's to local vector fields such that $(\nabla_{X_i}X_j)(p) = 0$ (see Remark 1.42). Then, up to terms vanishing at p,

$$(* \operatorname{div}\omega)(X_1, \dots, X_{n-p+1}) = \operatorname{div}\omega(X_{n-p+2}, \dots X_n) = \sum_{k=1}^{n-p+1} X_k\omega(X_k, X_{n-p+2}, \dots, X_n),$$

$$\begin{aligned}(\mathrm{d} * \omega)(X_1, \dots, X_{n-p+1}) &= \sum_{k=1}^{n-p+1} (-1)^{k+1} \nabla_{X_k}(*\omega)(X_1, \dots, \hat{X}_k, \dots, X_{n-p+1}) \\ &= \sum_{k=1}^{n-p+1} (-1)^{k+1+n-p+1-k} X_k \cdot \omega(X_k, X_{n-p+2}, \dots, X_n).\end{aligned}$$

Then $* \operatorname{div}\omega = (-1)^{n-p}\mathrm{d} * \omega$. Applying $*$ to both sides we get the conclusion. □

2.9. Remark. In principle, to make sense to the statement of Proposition 2.8, we need M to be oriented. However, we observe that $*$ appears twice and therefore the formula does not depend on the choice of the orientation.

2.10. Corollary. *A form $\omega \in \Omega^p(M)$ is closed (resp. coclosed) if and only if $*\omega$ is coclosed (resp. closed).*

Proof. Suppose $\mathrm{d}\omega = 0$. Then, up to sign, $\delta * \omega = * * \delta * \omega = *\mathrm{d}\omega = 0$. The other statement is proved in a similar way. ☐

We will suppose, from now on, that M is a *compact, oriented* [4] Riemannian manifold. We will define the L^2 inner product in $\Omega^p(M)$. We recall that the *volume form* of M is the n-form $\mathrm{d}M := *1$, i.e. the form that takes the value 1 on orthonormal positive basis. We define:

$$(\omega_1, \omega_2) = \int_M \langle \omega_1(x), \omega_2(x) \rangle \mathrm{d}M = \int_M \omega_1 \wedge *\omega_2. \tag{4}$$

Now $(\cdot, \cdot)$ is a symmetric, positive definite bi-linear map, i.e. a scalar product in $\Omega^p(M)$.

2.11. Remark. We observe explicitly that $\Omega^p(M)$ is not complete, in relation to the induced metric structure, if $\dim(M) > 0$. In particular, with this scalar product, $\Omega^p(M)$ is a *pre-Hilbert space* and not an Hilbert space. This is the main difficulty in the proof of the Theorem of Hodge that we will discuss in the next section.

2.12. Proposition. *For $\omega \in \Omega^{p-1}(M), \tau \in \Omega^p(M)$ we have:*

$$(d\omega, \tau) = (\omega, \delta\tau).$$

In particular $(\Delta\omega_1, \omega_2) = (\omega_1, \Delta\omega_2)$, i.e Δ is self adjoint with respect to the L^2 inner product.

Proof. We observe that $\mathrm{d} * \tau \in \Omega^{(n-p+1)}(M)$. Then

$$\mathrm{d} * \tau = (-1)^{(p-1)(n-p+1)} * * \mathrm{d} * \tau = (-1)^{(p-1)(n-p+1)+n(p-1)+1} * \delta\tau = (-1)^{(p-1)^2+1} * \delta\tau.$$

Therefore

$$\mathrm{d}(\omega \wedge *\tau) = \mathrm{d}\omega \wedge *\tau + (-1)^{(p-1)}\omega \wedge \mathrm{d} * \tau = \mathrm{d}\omega \wedge *\tau - \omega \wedge *\delta\tau.$$

and, from Stokes Theorem,

$$0 = \int \mathrm{d}(\omega \wedge *\tau) = \int \mathrm{d}\omega \wedge *\tau - \int \omega \wedge *\delta\tau = (\mathrm{d}\omega, \tau) - (\omega, \delta\tau).$$

☐

2.13. Corollary. *If M is compact, a p-form ω is harmonic if and only if it is closed and coclosed.*

Proof. It is clear tat if $\mathrm{d}\omega = 0 = \delta\omega$, the $\Delta\omega = 0$. Conversely, if $\Delta\omega = 0$, we have

$$0 = (\Delta\omega, \omega) = (\mathrm{d}\delta\omega, \omega) + (\delta\mathrm{d}\omega, \omega) = (\mathrm{d}\omega, \mathrm{d}\omega) + (\delta\omega, \delta\omega) = \|\mathrm{d}\omega\|^2 + \|\delta\omega\|^2.$$

Hence o ω is closed and coclosed. ☐

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[4] Orientability is not really needed for most of what we will do. We will comment on this point later on.

2.14. Remark. If M in not compact, still a closed and coclosed form is harmonic. However the converse is not true. For example the 0-form $f : \mathbb{R} \longrightarrow \mathbb{R},\ f(x) = x$ is harmonic but not closed.

2.15. Corollary. [Hopf's Lemma] *If M is compact and $f \in \Omega^0(M)$, then*

$$\int_M (\Delta f) * 1 = 0.$$

In particular, if $\Delta f \geq 0$, then $\Delta f = 0$ and f is locally constant.

Proof.

$$\int_M (\Delta f) * 1 = (\Delta f, 1) = (f, \Delta 1) = 0.$$

Moreover, if $\Delta f \geq 0$, $\Delta f = 0$. Hence f is harmonic, hence closed, and therefore locally constant. □

3. The Theorem of Hodge

Let M be a compact Riemannian manifold. We will denote by $\mathbb{H}^p(M)$ the space of harmonic p-forms on M. Since a harmonic form is closed, we have a map:

$$h^p : \mathbb{H}^p(M) \longrightarrow H^p(M), \qquad h^p(\omega) = [\omega].$$

The aim of this section is to give a very rough idea of the following result, known Hodge's Theorem.

3.1. Theorem. [Hodge] *The map h^p is an isomorphism.*

3.2. Remark. Clearly $\mathbb{H}^p(M)$ is a vector space and h^p is linear. But, in general, the wedge product of harmonic forms is not harmonic, so does not induces a graded algebra structure in $\mathbb{H}^*(M) := \oplus \mathbb{H}^p(M)$. Hence does not make sense to ask if $h^* := \oplus h^p$ is an algebras morphism.

The geometric idea behind the proof 3.1 is quite simple.

3.3. Proposition. *A closed form ω is harmonic if and only if it is a minimul of the fuction $g(\tau) = (\tau, \tau) = \|\tau\|^2$ restricted to the cohomology class $[\omega] = \{\omega + \mathrm{d}\beta : \beta \in \Omega^{(p-1)}(M)\}$.*

Proof. Let $\omega \in \Omega^p(M)$ be a closed form. If $\beta \in \Omega^{(p-1)}(M)$, we define:

$$g_\beta : \mathbb{R} \longrightarrow \mathbb{R}, \qquad g_\beta(t) = (\omega + t\mathrm{d}\beta, \omega + t\mathrm{d}\beta) = \|\omega + t\mathrm{d}\beta\|^2.$$

Then:

$$\frac{\mathrm{d}g_\beta}{\mathrm{d}t}(0) = 2(\omega, \mathrm{d}\beta) = 2(\delta\omega, \beta).$$

If ω is a minimum of g in $[\omega]$, $\dfrac{\mathrm{d}g_\beta}{\mathrm{d}t}(0) = 0$, $\forall \beta \in \Omega^{(p-1)}(M)$ and, therefore, $\dfrac{\mathrm{d}g_\beta}{\mathrm{d}t}(0) = 2(\delta\omega, \beta) = 0\ \forall \beta \in \Omega^{(p-1)}(M)$. In particular $\delta\omega = 0$ and ω is coclosed, hence harmonic, being closed. Conversely, if ω is coclosed, then it is a critical point of g restricted to $[\omega]$. But $[\omega]$ is convex and the only critical points of g are minimums. □

3.4. Corollary. *The map h^p is injective.*

Proof. It follows from the general fact that in an inner product space a convex subset has, at most, one point of minimal norm [5]. □

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[5] This fact follows essentially from the parallelogram low and is not true, in general, for normed spaces.

3.5. COROLLARY. *$\mathbb{H}^p(M)$ is finite dimensional.*

PROOF. It follows from the fact that $H^p(M)$ is finite dimensional (see Exercise **??** of Chapter 3), an the injectivity of h^p. $\square$

To show that h^p is surjective we have to show that there exist a form of minimal norm in $[\omega]$. This is not clear since $\Omega^p(M)$ is not complete. The natural idea is to complete $\Omega^p(M)$ with respect to a suitable scalar product, and look, in the completition $\overline{\Omega^p(M)}$, for the minimum of the L^2 norm in $\overline{[\omega]}$. This minimum exists and is unique by general arguments. However it may not be in $\Omega^p(M)$ hence may not be a harmonic p-form. The fact that the minimum is in $\Omega^p(M)$ is a highly non trivial result and came from the *regularity theory for elliptic partial differential equations.*

We have an apparently different, but essentially equivalent approach to the Theorem of Hodge, in particular to the regularity problem.

From the fact that $\mathbb{H}^p$ is finite dimensional (see Corollary 3.5), it follows that is closed in $\Omega^p(M)$ [6]. Then we have an orthogonal decomposition:

$$\Omega^p(M) = \mathbb{H}^p(M) \oplus \mathbb{H}^p(M)^\perp. \tag{5}$$

Moreover $\Delta(\Omega^p(M)) \subseteq \mathbb{H}^p(M)^\perp$, since Δ is self adjoint.

It is well known, from basic linear algebra, that if $A : \mathbb{R}^n \to \mathbb{R}^n$ is a self adjoint linear operator, the equation $Ax = y$ has a solution if and only if $y \in [\ker A]^\perp$. The regularity theorem is essentially equivalent to the fact that the same result holds for Δ. More precisely:

3.6. THEOREM. *If $\beta \in \Omega^p(M)$, the equation $\Delta\omega = \beta$ has a solution if and only if $\beta \in \mathbb{H}^p(M)^\perp$.*

We have an immediate but important consequence known as the Hodge's decomposition Theorem:

3.7. THEOREM.

$$\Omega^p(M) = \mathbb{H}^p(M) \oplus \Delta(\Omega^p(M)) = \mathbb{H}^p(M) \oplus \delta(\Omega^{(p+1)}(M)) \oplus \mathrm{d}(\Omega^{(p-1)}(M)).$$

PROOF. Theorem 3.6 implies $\Delta(\Omega^p(M)) = \mathbb{H}^p(M)^\perp$, hence the conclusion. $\square$

3.8. REMARK. Theorem 3.7 gives also a proof of the Theorem of Hodge. In fact, if ω is a closed form we have an orthogonal decomposition $\omega = \omega_H + \omega_\delta + \omega_{\mathrm{d}}$. Since ω is closed, $(\omega, \delta\tau) = (\mathrm{d}\omega, \tau) = 0\ \forall \tau \in \Omega^{(p+1)}(M)$. Hence $\omega = \omega_H + \mathrm{d}\beta$, i.e. $[\omega] = [\omega_H]$.

As a first application of the Theorem of Hodge we will give now a fast proof of the Poincaré dualty Theorem.

3.9. THEOREM. [Poincaré duality] *Let M be a compact oriented differentiable manifold. Then there exist an isomorphism, called* Poincaré duality, *$P^p : H^p(M) \longrightarrow H^{(n-p)}(M)$.*

PROOF. Fix a Riemannian metric on N [7]. If $\omega \in \Omega^p(M)$ is an harmonic form, Corollary 2.10 implies that $*\omega$ is harmonic. Hence $P^p = * : \mathbb{H}^p(M) \longrightarrow \mathbb{H}^{(n-p)}(M)$ is an isomorphism. $\square$

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[6] Since any finite dimensional normed space is complete, $\mathbb{H}^p(M)$ is complete, hence closed in $\Omega^p(M)$.

[7] For example look at M as a submanifold of $\mathbb{R}^N$.

3.10. Corollary. *Let M be a compact, connected, orientable smooth manifold of dimension n. Then $H^n(M) \cong \mathbb{R}$.*

Proof. Put a Riemannian metric on M. Then $H^n(M) \cong H^0(M) \cong \mathbb{R}$ (since M is connected). $\square$

3.11. Remark. In the arguments related to the Theorem of Hodge orientability allows the use the $*$-operator and integration of n-forms (or functions). But this is not essential since we can integrate a function on a Riemannian manifold with respect to the *Riemannian density* (whatever this mens) and the $*$-operator appears twice in the formula of the Laplacian, so a change of orientation does not affect the formulas. However, in Theorem 3.9 the $*$-operator appears only once. It is a fact that this Theorem does not holds for non orientable manifolds. For example it is possible to prove that a connected *non orientable* n-dimensional manifold has vanishing n-dimensional cohomology. In particular, for compact manifolds, orientability is detected by the topological invariant H^n.

4. The Weitzenbök formula

We want to describe the Laplacian of a p-form in terms of geometric object like curvature and covariant derivative. We start observing that, since we can take covariant differentiation of form, we can define an action of the curvature tensor on forms.

4.1. Definition. Let $X, Y \in \mathcal{H}(M), \omega \in \Omega^p(M)$. We define

$$R(X,Y)\omega = \nabla_X \nabla_Y \omega - \nabla_Y \nabla_X \omega - \nabla_{[X,Y]}\omega.$$

We state some facts whose proofs are left to the reader:

4.2. Proposition.

1. *The map $R : \mathcal{H}(M) \times \mathcal{H}(M) \times \Omega^p(M) \longrightarrow \Omega^p(M)$, $R(X,Y,\omega) = R(X,Y)\omega$ is is $\mathcal{F}(M)$-trilinear. In particular it is induced by a $\mathbb{R}$-trilinear map $R_q : T_pM \times T_qM \times \Lambda^P(T_qM) \longrightarrow \Lambda^p(T_qM)$, $q \in M$.*
2. *If $\omega \in \Omega^1(M)$ and $Z = \sharp\omega$, then $R(X,Y)\omega = \flat R(X,Y)Z$ ($\flat : T_qM \longrightarrow [T_qM]^* = \Lambda^1(T_qM)$ and $\sharp = \flat^{-1}$ are the “musical isomorphisms”).*
3. *If $\omega = \phi_1 \wedge \cdots \wedge \phi_p$, $\phi_i \in \Omega^1(M)$, $R(X,Y)\omega = \sum \phi_1 \wedge \cdots \wedge R(X,Y)\phi_k \wedge \cdots \wedge \phi_p$.*
4. *The map $R(X,Y) : \Omega^p(M) \longrightarrow \Omega^p(M)$ is antisymmetric.*

We will be interested in a “curvature” that generalizes the Ricci curvature.

4.3. Definition. The *generalized Ricci curvature* is the map

$$Q^p : \Omega^p(M) \longrightarrow \Omega^p(M), \qquad Q^p(\omega)(X_{i_1}, \ldots, X_{i_p}) = \sum_{s=1}^{n} \sum_{k=1}^{p} R(X_s, X_{i_k})\omega(X_{i_1}, \ldots, \hat{X}_{i_k}, X_s, \ldots X_{i_p}),$$

where $\{X_1, \ldots, X_p\}$ is an orthonormal basis of T_pM.

4.4. Remark. Observe that Q^p is a “trace”, in particular the definition does not depends on the choice of the basis.

We will describe $\Delta\omega$, $\omega \in \Omega^p(M)$, in terms of Levi-Civita connection and the generalized Ricci curvature.

4.5. Theorem. *Let* $\{X_1,\dots,X_n\}$ *be orthonormal vector fields defined in a neighborhood of* $q\in M$ *and* $\omega\in\Omega^p(M)$. *Then*

$$\Delta\omega(q)=-\sum_{s=1}^{n}\nabla_{X_s}\nabla_{X_s}\omega(q)+Q^p(\omega)(q).$$

Proof. Since we are interest in the value of $\Delta\omega$ at $q\in M$, we can assume that the X_i's are like in Proposition 1.42. Using the expressions of d,δ in 2.4 and 2.5 we have, up to terms vanishing at q,

$$\begin{aligned}&\mathrm{d}\delta\omega(X_1,\dots,X_p)=\textstyle\sum_{k=1}^{p}(-1)^{k+1}(\nabla_{X_k}\delta\omega)(X_1,\dots,\hat{X}_k,\dots,X_p)=\\ &=\textstyle\sum_{k=1}^{p}(-1)^{k+1}X_k[\delta\omega(X_1,\dots,\hat{X}_k,\dots,X_p)]=\sum_{k=1}^{p}(-1)^{k}X_k[\sum_{s=1}^{n}\nabla_{X_s}\omega(X_s,X_1,\dots,\hat{X}_k,\dots,X_p)]=\\ &=\textstyle\sum_{k=1}^{p}(-1)^{k}\sum_{s=1}^{n}\nabla_{X_s}\nabla_{X_s}(X_s,X_1,\dots,\hat{X}_k,\dots,X_p).\end{aligned}$$

$$\begin{aligned}&\delta\mathrm{d}\omega(X_1,\dots,X_p)=-\textstyle\sum_{s=1}^{n}\nabla_{X_s}\mathrm{d}\omega(X_s,X_1,\dots,X_p)=\\ &=-\textstyle\sum_{s=1}^{n}X_s[\nabla_{X_s}\omega(X_1,\dots,X_p)+\sum_{k=1}^{p}(-1)^k\nabla_{X_k}\omega(X_s,X_1,\dots,\hat{X}_k,\dots,X_p)]=\\ &=-\textstyle\sum_{s=1}^{n}\nabla_{X_s}\nabla_{X_s}\omega(X_1,\dots,X_p)-\sum_{s=1}^{n}\sum_{k=1}^{p}(-1)^k\nabla_{X_s}\nabla_{X_k}\omega(X_s,X_1,\dots,\hat{X}_k,\dots,X_p).\end{aligned}$$

The conclusion follows adding the two equations above.

☐

4.6. Corollary. [Weitzenbök's formula] *If* $\omega\in\Omega^p(M)$,

$$\langle\Delta\omega(p),\omega(p)\rangle=\frac{1}{2}\Delta(\|\omega\|^2)+\sum_{k=1}^{n}\|\nabla_{X_s}\omega\|^2+\langle Q^p(\omega),\omega\rangle,\qquad(\Delta\omega,\omega)=\sum_{k=1}^{n}\|\nabla_{X_s}\omega\|^2+(Q^p(\omega),\omega).$$

Proof. Consider vector fields $\{X_1,\dots,X_n\}$ as above. Then

$$\langle\nabla_{X_s}\nabla_{X_s}\omega,\omega\rangle=X_s\langle\nabla_{X_s}\omega,\omega\rangle-\|\nabla_{X_s}\omega\|^2=\frac{1}{2}X_sX_s\langle\omega,\omega\rangle-\|\nabla_{X_s}\omega\|^2=-\frac{1}{2}\Delta(\|\omega\|^2)-\|\nabla_{X_s}\omega\|^2,$$

where, in the last equality we use Remark 2.2. The first equation follows from Theorem 4.5. The second equation is obtained from the first one by integration, remembering Hopf's Lemma 2.15. ☐

5. Some applications

From Corollary 4.6 and the Theorem of Hodge it follows that, if M is a compact Riemannian manifold with Q^p positive definite, $H^p(M)=\{0\}$. Since the argument is an integral argument, it follows that the same is true if $Q^p\geq 0$ and $Q^p>0$ for some point $q\in M$. We want similar results under "more usual" curvature conditions.

We will start with the case $p=1$. In this case, Q^1 is, essentially, the Ricci curvature (see Exercise 6.6).

5.1. Theorem. *Let* M *be a Riemannian manifold with Ricc* ≥ 0 *and Ricc*$(p)>0$, *for some* $p\in M$, *then* $H^1(M)=\{0\}$.

5.2. Remark. An observation is due for the reader familiar with the fundamental group. A classical result in Riemannian geometry is the Bonnet-Myers Theorem that states that a compact manifold M with *positive* Ricci curvature has finite fundamental group. From this fact it follows, by standard algebraic topology, that $H^1(M)=\{0\}$. So, on one side the result above is weaker the the Bonnet-Myers Theorem but, on the other hand, we need only that the Ricci curvature is non negative and positive at *some point* to conclude that H^1 vanishes.

If we require that that the Ricci curvature is non negative, then the conclusion does not follows (in fact it is false). However we still have some relevant informations. Given a harmonic form $\omega \in \Omega^1(M)$, it follows from Corollary 4.6, that it is parallel, or, equivalently, the dual vector field $X = \sharp\omega$ is *globally* parallel, i.e $\nabla_Z X = 0$, $\forall Z \in TM$. The existence of parallel vector fields is a very strong restriction on the *geometry* of the manifold as we have already seen in Theorem 1.29. In this situation we have

5.3. COROLLARY. *Let M be a compact connected n-dimensional manifold with non negative Ricci curvature. Then* $\dim H^1(M) \leq n$.

PROOF. A parallel vector field (or 1-form) is uniquely determined by its value at a point. Hence the space of parallel 1-forms has dimension at most n. ☐

5.4. REMARK. It is possible to show, in the hypothesis of the Corollary, that if $\dim H^1(M) = n$, M is isometric to the product of n circles. This is simple to see in the case that M is a Lie group since the existence of n linearly independent parallel vector fields implies that the curvature tensor vanishes identically, hence the group is Abelian , and therefore isomorphic the product of n circles.

We consider now a compact connected Lie group G with a bi-invariant metric. We have seen that the cohomology of such group is isomorphic to the space of *Ad*-invariant forms. Such formas have the following "geometric" interpretation

5.5. PROPOSITION. *A form $\omega \in \Omega^p(G)$ is Ad-invariant if and only if it is parallel, if and only if it is harmonic.*

PROOF. The first equivalence follows from Theorem 1.52 and Theorem 5.10 of Chapter 4. For the second we observe that clearly a parallel form is harmonic and conversely if ω is harmonic, there is an *Ad*-invariant, hence parallel, form in $[\omega]$ which, by unicity is ω. ☐

5.6. THEOREM. *Let G be a compact connected Lie group G. Then $\widehat{G}$ has trivial center if and only if $H^1(G) = \{0\}$. In this case $H^2(G) = \{0\}$, $H^3(G) \neq \{0\}$.*

PROOF. Consider a bi-invariant Riemannian metric on G. If $\widehat{G}$ has trivial center, this metric has positive Ricci curvature, by Theorem 1.54, hence $H^1(G) = \{0\}$, by Theorem 5.1. If $H^1(G) = \{0\}$, $D_G = G$, by Proposition 5.15 of Chapter 4, hence $\widehat{G}$ has trivial center. In this case $H^2(G) = \{0\}$, again by Proposition 5.15 of Chapter 4. For the last assertion we consider the tri-linear map $\omega : \widehat{G} \times \widehat{G} \times \widehat{G} \longrightarrow \mathbb{R}$, $\omega(X, Y, Z) := \langle [X, Y], Z\rangle$. This map is an *Ad*-invariant exterior form in $\Lambda^3(\widehat{G})$, by Lemma 1.50. Observe that $\omega \neq 0$, since $\widehat{G}$ is non Abelian, hence ω defines a non zero element in $H^3(G)$. ☐

5.7. REMARK. The last assertion in Theorem 5.6 depends only on the fact that $\widehat{G}$ is not Abelian. In particular a compact connected two dimensional Lie group must be Abelian, hence isomorphic to $S^1 \times S^1$.

5.8. COROLLARY. *S^n admits a Lie group structure if and only if $n = 0, 1, 3$.*

PROOF. For $n = 0, 1, 3$, S^n is the unit sphere of the real, complex and quaternionic lines respectively, so it admits a Lie group structure. For $n \neq 0, 1, 3$, $H^i(S^n) = \{0\}$, $i = 1, 3$, contraddincting Theorem 5.6. ☐

5.9. Definition. A *Kähler manifold* is a Riemannian manifold M together with a smooth map $J : TM \longrightarrow TM$ such that

(1) $\pi \circ J = \pi$ and, $\forall p \in M$ and $J_p := J|T_pM : TpM \longrightarrow T_pM$ is a linear isometry,
(2) $J^2 = -\mathbb{1}$,
(3) J is parallel in the sense that, if $X, Y \in \mathcal{H}(M)$, $\;J(\nabla_X Y) = \nabla_X(J(Y))$.

A Kähler manifold has even dimension, say $2n$, and $\omega(X, Y) := \langle X, JY\rangle$ is a *parallel* 2-form, called the *Kähler form* or *fundamental form.* Moreover $\omega^n \neq 0$ (see Exercise 7.33 in Chapter 1). In particular we have

5.10. Proposition. *If M is a compact Kähler manifold $H^{2k}(M)$ contains the non zero element $[\omega]^k, k \leq n$.*

5.11. Example. The main example of a compact Kähler manifold is the complex projective space $\mathbb{C}P^n$. This space is the orbit space of the action

$$\mu : S^1 \times S^{2n+1} \longrightarrow S^{2n+1}, \quad \mu(\theta, p) = e^{i\theta}p.$$

The orbit $\mathcal{O}(p)$ is the great circle $\{e^{i\theta}p\}$. We will denote by $\xi(p)$ the unit tangent vector to the orbit at p. The tangent space T_pS^{2n+1} splits as the orthogonal sum of the *real* line $V_p = \{t\xi(p) : t \in \mathbb{R}\}$ and its orthogonal complement H_p in T_pS^{2n+1}. The space V_p (resp. H_p) is called the *vertical* (resp. *horizontal*) space. Consider the quotient map and its differential,

$$\pi : S^{2n+1} \longrightarrow \mathbb{C}P^n, \quad \pi(p) = [p], \qquad \mathrm{d}\pi(p) : T_pS^{2n+1} \longrightarrow T_{[p]}\mathbb{C}P^n.$$

The kernel of $\mathrm{d}\pi(p)$ is the vertical subspace V_p and $\mathrm{d}\pi(p)$ maps H_p isomorphically onto $T_{[p]}\mathbb{C}P^n$. Given $X \in T_{[p]}\mathbb{C}P^n$, the *horizontal lift of X, at p,* is the vector $\tilde{X}(p) = [\mathrm{d}\pi(p)|H_p]^{-1}(X)$. The fact that $\mu_\theta : S^{2n+1} \longrightarrow S^{2n+1}$ is an isometry allows us to define a Riemannian metric on $\mathbb{C}P^n$. Given $X_1, X_2 \in T_{[p]}\mathbb{C}P^n$ we define $\langle X_1, X_2\rangle := \langle \tilde{X_1}(p), \tilde{X_2}(p)\rangle$, where the second scalar product is the one in T_pS^{2n+1}. This scalar product is well defined, since if $q = \mu_\theta(p)$, $\;\tilde{X}_i(q) = \mathrm{d}\mu_\theta(p)(\tilde{X}_i)$ and $\mathrm{d}\mu_\theta(p)$ is an isometry.

We will define now a smooth map $J : T\mathbb{C}P^n \longrightarrow T\mathbb{C}P^n$ that will define a structure of Kähler manifold on $\mathbb{C}P^n$. Given $X \in T[p]\mathbb{C}P^n$ we define $JX := \mathrm{d}\pi(i\tilde{X}(p))$. We have to check, first of all, that J is well defined and this follows from the fact that $\mathrm{d}\mu_\theta(p)|H_p : H_p \longrightarrow H_{e^{i\theta}p}$ is $\mathbb{C}$-linear. Clearly J is smooth and $J^2 = -\mathbb{1}$. So we are left to prove that J is parallel. This is consequence of more general facts that we will discuss in Exercise 6.7.

In particular we can compute the product in the algebra $H^*(\mathbb{C}P^n) = \oplus H^k(\mathbb{C}P^n)$. The Kähler form ω generates $H^2(\mathbb{C}P^n)$ and its powers, ω^k generates $H^{2k}(\mathbb{C}P^n), k \leq n$. So $H^*(\mathbb{C}P^n)$ is isomorphic, as an algebra, to the algebra of polynomials in one real variable t divided by the ideal generated by t^{n+1}.

6. Exercises

6.1. product metric and connections.

6.2. geodesics of $O(n)$.

6.3. Prove that if $X \in T_pM$, and ω_1, ω_2 are forms, $\nabla_X(\omega_1 \wedge \omega_2) = \nabla_X\omega_1 \wedge \omega_2 + \omega_1 \wedge \nabla_X\omega_2$.

6.4. Prove that for $\omega, \tau \in \Omega^p(M), X \in T_qM$, $\;X\langle\omega, \tau\rangle = \langle\nabla_X\omega, \tau\rangle + \langle\omega, \nabla_X\tau\rangle$.

6.5. Prove Proposition 4.2

6.6. Prove that, if $\omega \in \Omega^1(M)$ and $X = \sharp\omega$, then $Q^1(\omega) = \flat Q(X)$, where $\flat : T_pM \longrightarrow [T_pM]^*$ is the "musical" isomorphism and Q is the Ricci tensor defined in Section 1.

6.7. Riemannian subimmersion sand $\mathbb{C}P^n$

6.8. Prove that $H^k(S^2 \times S^4) \cong H^k(\mathbb{C}P^3)$ (as vector spaces). Use the Künnet formula to compute the algebra structure of $H^*(S^2 \times S^4)$ and Example 5.11 to conclude that $S^2 \times S^4$ is not homotopy equivalent to $\mathbb{C}P^3$.

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