

WIEDERSEHEN METRICS AND EXOTIC INVOLUTIONS OF EUCLIDEAN SPHERES

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ABSTRACT. We provide explicit, simple, geometric formulas for free involutions ρ of Euclidean spheres that are not conjugate to the antipodal involution. Therefore the quotient S^n/ρ is a manifold that is homotopically equivalent but not diffeomorphic to $\mathbb{R}P^n$. We use these formulas for constructing explicit non-trivial elements in $\pi_1\text{Diff}(S^5)$ and $\pi_1\text{Diff}(S^{13})$ and to provide explicit formulas for non-cancellation phenomena in group actions.

1. INTRODUCTION

A smooth free involution ρ on a sphere S^n is called *exotic* if it is not conjugate by a diffeomorphism to the standard antipodal involution $\alpha(x) = -x$. The quotient S^n/ρ is then a manifold that is homotopically equivalent but not diffeomorphic to the standard real projective space $\mathbb{R}P^n$.

There are several methods of constructing exotic involutions. The first examples of such involutions were constructed by Hirsch and Milnor ([20]) in S^5 and S^6 , as restrictions to invariant (standard) spheres of certain free involutions on exotic spheres. Then there are examples constructed via surgery, e.g. [1, 8, 14]. The reader can see also the basic reference [25] for topological and differentiable invariants of involutions, and the classification and discussion using analytical methods in [2, 28].

A different path in the construction of exotic involutions is given by simple involutions that restrict to involutions of Brieskorn spheres ([2, 6, 21, 22]). These have been used, for example, in the work of Grove and Ziller [16] to construct metrics of non-negative sectional curvature on exotic real projective spaces of dimension 5, and by Boyer, Galicki, and Nakamaye [5] to construct Sasakian metrics of positive Ricci curvature on exotic real projective spaces of dimension $4m + 1$.

In this paper we construct free exotic involutions of Euclidean spheres S^n , for $n = 5, 6, 13, 14$. The origin of the formulas is quite geometric: Recall that a Riemannian metric is called *wiedersehen* with respect to points N and S in a manifold M if every geodesic emanating from N reaches S at a fixed length ℓ and vice versa. The *wiedersehen* property at a point implies that M is homeomorphic to the sphere (see the book [4] for a complete discussion). Lifting these geodesics to total spaces of bundles over spheres, one gets many results and explicit formulas in differential and algebraic topology ([11, 12, 29, 30]).

In dimensions 5 and 6, our involutions are essentially geometric formulas for the Hirsch-Milnor involutions. These involutions are given by restrictions of a natural

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involution of the Milnor exotic sphere $\Sigma_{2,-1}^7$ to certain invariant submanifolds \mathcal{S}_6 and \mathcal{S}_5 that, by Morse theory, turn out to be spheres.

Using the geodesics of a wiedersehen metric of $\Sigma_{2,-1}^7$ constructed in [11], we transfer the Hirsch-Milnor involutions from the invariant 6-sphere \mathcal{S} , now realized as the “equator” with respect to the metric, to the Euclidean 6-sphere contained in the tangent space at a point of $\Sigma_{2,-1}^7$. The geometric origin of these involutions is reflected in several features:

- In contrast to the works above, our exotic involutions are described by simple explicit formulas on the respective standard Euclidean spheres (as opposed to Brieskorn spheres, or spheres inside of exotic spheres); the simplicity of the formulas also translates to an elementary pictorial description of the involutions (see Figures 1,2,3 in Section 2).
- We can extend the constructions by substituting quaternions by Cayley numbers everywhere, thus getting exotic involutions of spheres in dimensions 13 and 14. These extensions are not immediate, since there is no Cayley analog of the Gromoll-Meyer fibration $Sp(2) \rightarrow \Sigma_{2,-1}^7$. This kind of phenomenon – the non-trivial extensions to the Cayley case – already appears in [12].
- The proof of the exoticity of the involutions is rather different from the usual ones: for S^6 and S^{14} , we describe precisely the \mathbb{Z}_2 -action on $\text{Diff}^+(S^n)$ of conjugation by the antipodal map. This leads to several interesting open questions regarding the structure of the relevant diffeomorphisms groups (see section 3). For S^5 and S^{13} , the involutions are shown to be exotic using the fact that gluing diffeomorphisms σ in $\pi_0 \text{Diff}^+(S^6)$ and $\pi_0 \text{Diff}^+(S^{14})$ have explicit lifts under the boundary map $\pi_1 \text{Diff}^+(S^n) \rightarrow \pi_0 \text{Diff}^+(S^{n+1})$. This is done in Section 4.
- The fact that these constructions admit lots of symmetries is exploited in [13], where in particular we provide an explicit cohomogeneity one diffeomorphism between a Brieskorn sphere and the standard sphere S^5 which relates our constructions to the usual Brieskorn ones. Again, extensions to the Cayley case are given there. From these computations and known results [25, 32] it follows that our S^5/ρ and S^{13}/ρ are not homeomorphic to the standard real projective spaces $\mathbb{R}P^5$ and $\mathbb{R}P^{13}$.
- Actually the discovery of the formulas for the involutions was somewhat serendipitous, while we looked for geometric models of bundles over exotic spheres. This is reflected by an application of these exotic involutions, the construction of a very explicit example of non-cancellation phenomenon in group actions: concretely, we will give non-conjugate actions r_1, r_2 of $\mathbb{Z}_2 \times S^3$ on $X = S^6 \times S^3$ such that the restricted \mathbb{Z}_2 - and S^3 -actions are conjugate, that is, “neither factor can be cancelled” (see Section 6 for details). Again, the formulas for these actions come from trivializations of bundles using the geodesics of wiedersehen metrics.

Preliminaries. We summarize here the fundamental topological facts that we need throughout the paper (see e.g. [23, 24]): Let $\text{Diff}^+(S^{n-1})$ and $\text{Diff}^+(D^n)$ denote the group of orientation-preserving diffeomorphisms of S^{n-1} and D^n , respectively. Via the restriction homomorphism, $\text{Diff}^+(D^n)$ can be regarded as a normal subgroup

of $\text{Diff}^+(S^{n-1})$. The quotient group

$$\Gamma_n = \text{Diff}^+(S^{n-1}) / \text{Diff}^+(D^n) = \pi_0 \text{Diff}^+(S^{n-1})$$

is abelian and consists of the equivalence classes of isotopic orientation preserving diffeomorphisms of S^{n-1} .

The group Θ_n is the abelian group of h-cobordism classes of homotopy n -spheres under the connected sum operation. For $n \geq 5$, every homotopy n -sphere is homeomorphic to S^n and two homotopy n -spheres are h-cobordant if and only if they are orientation preservingly diffeomorphic. Thus, Θ_n can be regarded as the group of all diffeomorphism classes of differentiable structures on the topological n -sphere. For $n \geq 5$, Θ_n is isomorphic to Γ_n . The isomorphism $\Gamma_n \rightarrow \Theta_n$ is given by using $\sigma \in \text{Diff}^+(S^n)$ to glue a twisted n -sphere from two disks.

Note that the group Θ_n (and thus the group Γ_n for $n \geq 5$) contains an important normal subgroup bP^{n+1} . This group consists of all h-cobordism classes of homotopy n -spheres that bound parallelizable manifolds.

2. EXPLICIT INVOLUTIONS OF EUCLIDEAN SPHERES

Let \mathbb{H} and \mathbf{Ca} denote the quaternions and the Cayley numbers, respectively, and let \Re and \Im denote the real and imaginary parts. Moreover, write

$$S^6 = \{(p, w) \in \mathbb{H} \times \mathbb{H} \mid \Re(p) = 0, |p|^2 + |w|^2 = 1\},$$

and similarly

$$S^{14} = \{(p, w) \in \mathbf{Ca} \times \mathbf{Ca} \mid \Re(p) = 0, |p|^2 + |w|^2 = 1\}.$$

Consider the map $b : S^6 \rightarrow S^3$ (resp. $b : S^{14} \rightarrow S^7$) given by

$$b(p, w) = \begin{cases} \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}, & w \neq 0 \\ -1, & w = 0, \end{cases}$$

where e^x denotes the exponential map of the group S^3 of unit quaternions; thus $e^{\pi p} = \cos(\pi|p|) + \sin(\pi|p|)(p/|p|)$. The map b is a real analytic map whose homotopy class generates $\pi_6(S^3)$ (resp. $\pi_{14}(S^7)$, see [12]). We call these maps *Blakers-Massey elements*. The map b in the 6-dimensional case is found using the wiedersehen metric to explicitly represent the boundary map of the homotopy sequence of the fibration $S^3 \cdots Sp(2) \rightarrow S^7$. This method has also been used in [29, 30] in order to produce explicit representatives of several homotopy groups of the classical groups along the borderline between the stable and unstable range.

Consider now $\sigma : S^6 \rightarrow S^6$ (resp. $S^{14} \rightarrow S^{14}$) given by

$$\sigma(p, w) = (b(p, w)p \overline{b(p, w)}, b(p, w)w \overline{b(p, w)}).$$

The map σ is a real analytic, orientation-preserving diffeomorphism that is *not* isotopic to the identity. Therefore the union of two 7-disks (resp 15-disks) by σ is an exotic sphere Σ . This map is also found using the pointed wiedersehen metric (in the 7-dimensional case): $\sigma = \exp_S^{-1} \circ \exp_N$, where N and S denote the wiedersehen points of the metric (see [11] for details). In the 7-dimensional case, Σ generates the group $\Gamma_7 \cong \mathbb{Z}_{28}$, and in the 15-dimensional case, Σ generates the first factor in $\Gamma_{15} \cong bP^{16} \times \mathbb{Z}_2 \cong \mathbb{Z}_{8128} \times \mathbb{Z}_2$ (see [11, 12]).

Let us consider now the map $\rho = \alpha\sigma$, where α is the antipodal map of the sphere. We have

Theorem 1. *The map ρ is a free involution of S^6 (resp. S^{14}).*

Proof. Let us first write some remarkable properties of the map b (see also [12]):

- The map b is equivariant under automorphisms of \mathbb{H} (resp. \mathbf{Ca}). In the quaternionic case this means that $b(qp\bar{q}, qw\bar{q}) = qb(p, w)\bar{q}$ for unit q . In the case of Cayley numbers, the latter property holds provided that q lies in the subalgebra generated by p and w .
- $b(-p, -w) = \overline{b(p, w)}$.
- $\sigma^k(p, w) = (b(p, w)^k p \overline{b(p, w)^k}, b(p, w)^k w \overline{b(p, w)^k})$.

With these properties the fact that $\rho \circ \rho$ is the identity is easy to establish.

In order to prove that ρ is free, first note that $\rho(p, w) = (p, w)$ means that both p and w anticommute with $b(p, w)$. In particular this implies that the real parts of w and $b(p, w)$ are zero (and the real part of p is zero by definition). Note that $e^{\pi p} = \cos(\pi|p|) + \sin(\pi|p|)\frac{p}{|p|}$; thus $\operatorname{Re}(b(p, w)) = 0$ if and only if $|p| = 1/2$ and thus $|w| = \sqrt{3}/2$. Using the substitution $p \mapsto qp\bar{q}$, $w \mapsto qw\bar{q}$, we can assume that $p = \mathbf{i}/2$ and $w = \frac{\sqrt{3}}{2}(\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j})$ for some $\theta \in [0, \pi]$. Then $b(p, w) = \cos(2\theta)\mathbf{i} + \sin(2\theta)\mathbf{j}$. From the fact that $b(p, w)$ anticommutes with p we get $\cos(2\theta) = 0$ and hence $\sin(2\theta) = \pm 1$. Thus $\theta = \pi/4$ or $\theta = 3\pi/4$. In none of these two case w anticommutes with $b(p, w)$. This argument generalizes to the Cayley case since we are dealing with the algebra generated by p and w and therefore everything happens in a copy of \mathbb{H} inside of \mathbf{Ca} . \square

Note that the map ρ restricts to the 5-sphere (resp. 13-sphere) given by the condition $\Re(w) = 0$. In the next two sections we will show that all of these involutions are exotic; we finish this section by giving a pictorial sequence in Figures 1 to 3 showing the involution on S^5 , done by translating the fact that the conjugation $x \mapsto qxq^{-1}$ by a unit quaternion q acting on a purely imaginary x is given by rotating x along the axis $\Im(q)$ with angle θ , where $\cos(\theta) = 2\Re(q)^2 - 1$.

It can be read easily from Figures 1 to 3 that the map constructed is a fixed point free involution of S^5 :

- If $w \rightarrow 0$ then $|p| \rightarrow 1$. Hence in the second step we get close to a rotation by 360° , i.e., to the identity of S^5 . The map therefore extends at least continuously to the case where $w = 0$.
- If w and p are linearly dependent, the image of w and p is just $-w$ and $-p$. If w and p are linearly independent the axis of rotation constructed in the first step is still contained in the plane spanned by w and p . Hence, they cannot be mapped to $-w$ and $-p$ in the second step. This shows that the map is fixed point free.
- When the map is applied to w' and p' the axis constructed in the first step is given by the central direction of the cones opening downwards. In the second step one rotates around this axis by an angle of $|p'| \cdot 360^\circ$ (observe the right hand rule). Since $|p'| = |p|$ this gives $-w$ and $-p$. Hence in the final step one is back at w and p . This shows that the map is an involution.

Of course, it would be nice to see from the pictorial description of the involution why it is exotic (without going up in dimension to $\Sigma_{2,-1}^7$).

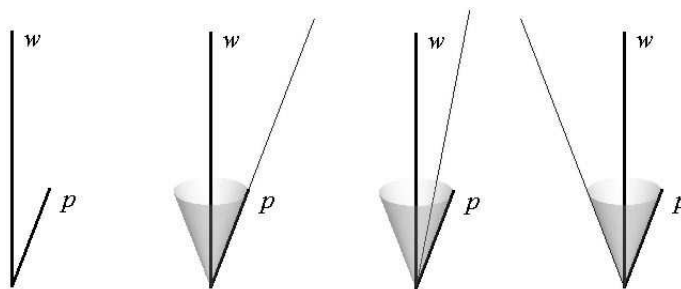


FIGURE 1. Consider two vectors $p, w \in \mathbb{R}^3$ with $|p|^2 + |w|^2 = 1$. The set of all such vectors forms the sphere S^5 . Suppose that $w \neq 0$. First rotate p 180° around w and obtain an oriented axis.

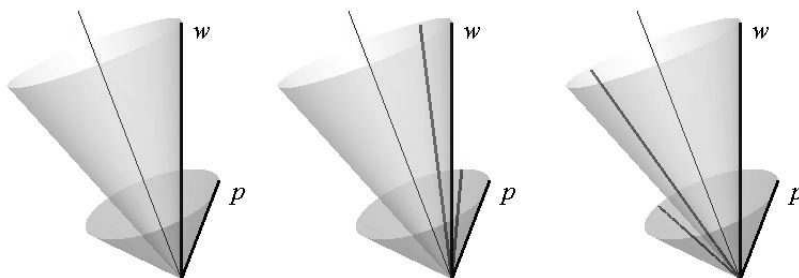


FIGURE 2. Second, rotate p and w around this new axis by an angle of $|p| \cdot 360^\circ$.

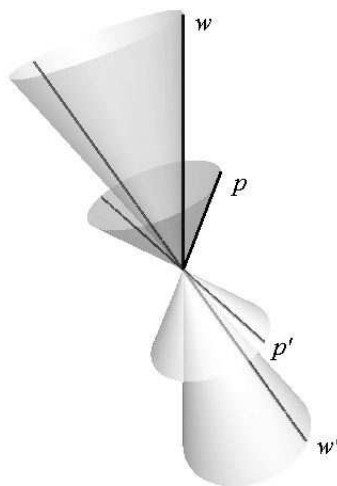


FIGURE 3. Finally map the resulting vectors to their antipodes. This process extends analytically to the case where $w = 0$ (in this case $w' = 0$, $p' = -p$) and gives a visual description of an exotic involution of S^5 .

3. AN INVOLUTION OF $\text{Diff}^+(S^{n-1})$

Now let us prove that ρ is not equivalent (in the conjugation sense) to the antipodal involution α . In order to accomplish that, consider the following \mathbb{Z}_2 -action \mathcal{A} on $\text{Diff}^+(S^n)$,

$$\mathcal{A}(h) = \alpha \circ h \circ \alpha^{-1} = \alpha \circ h \circ \alpha,$$

or, more simply,

$$\mathcal{A}(h)(x) = -h(-x).$$

Note that, being conjugation, \mathcal{A} is a group isomorphism; $\mathcal{A}(hf) = \mathcal{A}(h)\mathcal{A}(f)$ in the group $\text{Diff}^+(S^{n-1})$. Note also that if two diffeomorphisms h_0 and h_1 are joined by a curve of diffeomorphism h_t , then $\mathcal{A}(h_t)$ joins $\mathcal{A}(h_0)$ to $\mathcal{A}(h_1)$ through diffeomorphisms. Therefore \mathcal{A} descends to an action on $\pi_0 \text{Diff}^+(S^{n-1}) = \Gamma_n$; which by abuse of notation we also denote by \mathcal{A} .

In the particular case of $n = 7$ or $n = 15$, we can explicitly compute how this action behaves in $\Gamma_7 \cong \mathbb{Z}_{28}$ and in the index 2 subgroup $bP^{16} \cong \mathbb{Z}_{8128}$ of Γ_{15} by acting on the known representative σ . In fact, Brumfiel [7] has shown that $\Gamma_{15} \cong \Theta_{15} \cong bP^{16} \oplus (\Theta_{15}/bP^{16}) \cong \mathbb{Z}_{8128} \oplus \mathbb{Z}_2$. A computation shows the following:

Main commutation relation: $\alpha\sigma = \sigma^{-1}\alpha$, or equivalently,

$$\mathcal{A}(\sigma) = \sigma^{-1}.$$

Thus the action on Γ_7 and on the subgroup $bP^{16} \subset \Gamma_{15}$ is given by

$$\mathcal{A}(n) = -n.$$

It would be very interesting to find how \mathcal{A} acts on the complement of $bP^{16} \subset \Gamma_{15}$. Clearly, it maps the elements of order 2 into themselves. So the question boils down to whether \mathcal{A} interchanges $(0, 1)$ and $(4064, 1)$ or not. Alas, we do not have any explicit representative to decide this.

We are now ready to show that ρ is not conjugate to the identity:

Theorem 2. *The map ρ is not conjugate to the antipodal map, i.e., there is no diffeomorphism $h : S^6 \rightarrow S^6$ (resp $h : S^{14} \rightarrow S^{14}$) that satisfies $h\rho h^{-1} = \alpha$.*

Proof. Let us assume that such a diffeomorphism exists. Without loss of generality, we can suppose that h is orientation preserving, since if there exists an orientation-reversing diffeomorphism j such that $j\rho j^{-1} = \alpha$, the diffeomorphism $h = \alpha j$ satisfies the same equation and is orientation preserving.

Now, we have,

$$h\rho h^{-1} = \alpha \Leftrightarrow \sigma = \alpha h^{-1} \alpha h \Leftrightarrow \sigma = \mathcal{A}(h^{-1})h.$$

Taking isotopy classes on both sides, we have

$$[\sigma] = [\mathcal{A}(h^{-1})h] = [\mathcal{A}(h^{-1})][h] = 2[h] + \tau,$$

where the term τ only appears in the 14-dimensional situation; τ is zero except in the case that $[h] = (n, 1) \in \mathbb{Z}_{8128} \oplus \mathbb{Z}_2 \cong \Gamma_{15}$, and \mathcal{A} interchanges $(0, 1)$ and $(4064, 1)$. Then, $\tau = (4064, 0)$. In any case, σ is even inside of $bP_8 \cong \mathbb{Z}_{28}$ (resp. $bP^{16} \cong \mathbb{Z}_{8128}$), which contradicts the fact that σ is a generator. \square

Let us give some remarks on the proof. Notice that, with the same computations as in Proposition 1, the maps $\rho_k = \alpha\sigma^k$ are also free involutions. However, the proof of Theorem 2 only shows that ρ_k is exotic for odd k . Indeed this must be so, since we have

$$\sigma^{-\ell}\rho_k\sigma^\ell = \sigma^{-\ell}\alpha\sigma^k\sigma^\ell = \alpha\sigma^{k+2\ell} = \rho_{k+2\ell},$$

and therefore all the ρ_k for k even are conjugate among themselves (in particular conjugate to the antipodal map a), and all the odd ρ_k are conjugate to $\rho_1 = \alpha\sigma$.

In fact, what we have done amounts to an analysis of the combinatorics of the path-connected components of $\text{Diff}(S^6)$ (and half of the components of $\text{Diff}(S^{14})$, but for simplicity we just discuss the S^6 -case). Indeed, $\text{Diff}(S^6)$ has 56 connected components, half of which are orientation preserving and the other half orientation reversing. The orientation preserving components are represented by the classes $[Id], [\sigma], [\sigma^2], \dots, [\sigma^{27}]$ whereas the orientation reversing components are represented by the classes $[\alpha], [\alpha\sigma], [\alpha\sigma^2], \dots, [\alpha\sigma^{27}]$. We have that conjugation by the antipodal map acts, on each of these halves, by $n \mapsto -n$ in \mathbb{Z}_{28} , that is, by fixing $[Id]$ (resp. $[\alpha]$) and the component of σ^{14} (resp. $[\alpha\sigma^{14}]$), and permuting symmetrically the rest $[\sigma^n] \mapsto [\sigma^{28-n}]$ (resp. $[\alpha\sigma^n] \mapsto [\alpha\sigma^{28-n}]$). As an additional by-product of this proof, we also get (compare [26])

Theorem 3. *Every orientation reversing diffeomorphism of S^6 is isotopic to a free involution.*

Let us close this section with several remarks and questions from the global analysis point of view: The identity homeomorphism is a fixed point of the \mathbb{Z}_2 -action \mathcal{A} . Any orientation preserving fixed point of this action must lie either on the component of the identity or in the component of σ^{14} . Note that $\mathcal{A}(\sigma^{14}) = \sigma^{-14}$, which is not the same map; it just lies in the same path connected component.

Question. Is there an odd map $f \in \text{Diff}^+(S^6)$ in the isotopy class of σ^{14} ? That is, a fixed point of \mathcal{A} .

Question. Is there $f \in \text{Diff}^+(S^6)$ in the isotopy class of σ^{14} that satisfies $f \circ f = Id$? That is, a fixed point of the inverse involution $\mathcal{B} : \text{Diff}^+(S^6) \rightarrow \text{Diff}^+(S^6)$ given by $\mathcal{B}(f) = f^{-1}$. Again, such a fixed point can only be isotopic either to the identity, or to σ^{14} .

The relevance of such an involution is that it would be a diffeomorphism that realizes the isotopy $\sigma^{28} \cong Id$ “on the nose”, thus greatly helping in the understanding of exotic diffeomorphisms and exotic spheres.

We can broaden these questions as follows:

Question. Which isotopy classes of orientation-preserving diffeomorphisms can be realized by maps of finite order?

For example, find a diffeomorphism $\eta : S^6 \rightarrow S^6$ representing $4 \in \mathbb{Z}_{28} \cong \Gamma_7$ such that $\eta^7 = Id$.

Explicit answers to these questions would provide exotic diffeomorphisms that improve upon the diffeomorphism σ , since they would express the group structure of $\Gamma_7 \cong \mathbb{Z}_{28}$ in a direct way.

Also, note that the main commutation relation can be expressed as the statement that the powers σ^k of σ are contained in the subset of $\text{Diff}^+(S^6)$ where the \mathcal{A} -orbit

and the \mathcal{B} -orbit coincide. It would be interesting to study the structure of this subset; in particular, to find some other elements.

4. RESTRICTION TO INVARIANT SPHERES

As we have remarked, the exotic involution ρ of S^6 (resp. S^{14}) has an invariant 5-sphere (resp. invariant 13-sphere) given by the $\Re(w) = 0$. We have

Theorem 4. *The restriction of ρ to S^5 (resp. S^{13}) is a free involution that is not conjugate to the antipodal map.*

The proof for S^5 could be copied from [20] in the 5-dimensional case. However, their proof breaks down in S^{13} since there are exotic 13-spheres. In the rest of this section we prove both cases at once using the boundary map ∂ in the classical exact sequence

$$\pi_1 \text{Diff}^+(S^{n-1}) \xrightarrow{\partial} \pi_0 \text{Diff}^+(S^n) \rightarrow \pi_0 \text{Diff}^+(D^n) \rightarrow \pi_0 \text{Diff}^+(S^{n-1}) \rightarrow \Gamma_n \rightarrow 0.$$

By a theorem of Cerf [9], $\pi_0 \text{Diff}^+(D^n) = 0$ for $n \geq 6$. In our context, Cerf's result implies that there are loops $\hat{\sigma}$ in $\text{Diff}^+(S^5)$ and $\text{Diff}^+(S^{13})$ which map under ∂ to diffeomorphisms of S^6 and S^{14} that are isotopic to our exotic diffeomorphisms σ . It is not a priori clear that such loops $\hat{\sigma}$ can be given explicitly.

However, the concrete formula for σ is of such a form, as we shall see in Lemma 1 below. In line with the spirit of this paper, we shall use these explicit loops $\hat{\sigma}$ to prove Theorem 4; they give identities of maps at several instances that induce very concrete homotopy identities. Moreover, our proof becomes independent of Cerf's theorem this way.

Let us now recall a concrete definition of the boundary homomorphism ∂ . The elements of $\pi_1 \text{Diff}^+(S^{n-1})$ can be represented by paths $\hat{\beta} : [0, 1] \rightarrow \text{Diff}^+(S^{n-1})$ that map a neighborhood of $\{0, 1\}$ to the identity map. By standard approximation results, it can be assumed that such a path $\hat{\beta}$ induces a smooth map $S^{n-1} \times [0, 1] \rightarrow S^{n-1}$, which is, by abuse of notation, again denoted by $\hat{\beta}$. The image of $\hat{\beta}$ under the boundary map ∂ is now given by

$$\begin{aligned} \partial(\hat{\beta}) : S^{n-1} \times [0, 1] &\rightarrow S^{n-1} \times [0, 1] \\ (x, t) &\mapsto (\hat{\beta}(x, t), t). \end{aligned}$$

Clearly, $\partial(\hat{\beta})$ is an orientation preserving diffeomorphism of the cylinder $S^{n-1} \times [0, 1]$ which coincides with the identity in a neighborhood of the boundary. Thus $\partial(\hat{\beta})$ induces an orientation preserving diffeomorphism $\partial(\hat{\beta})$ of the sphere S^n .

Lemma 1. *The exotic diffeomorphisms σ of S^6 and S^{14} naturally define explicit loops $\hat{\sigma}$ in $\text{Diff}^+(S^5)$ and $\text{Diff}^+(S^{13})$ with $\partial(\hat{\sigma}) = \sigma$.*

Proof. Recall that for (p, w) satisfying $\Re(p) = 0$, one has

$$\sigma(p, w) = (\text{b}(p, w)p \overline{\text{b}(p, w)}, \text{b}(p, w)w \overline{\text{b}(p, w)}).$$

Separating $w = w_0 + \omega$, where $w_0 = \Re(w)$, we have that

$$\sigma(p, w_0 + \omega) = (\text{b}(p, w_0 + \omega)p \overline{\text{b}(p, w_0 + \omega)}, w_0, \text{b}(p, w_0 + \omega)\omega \overline{\text{b}(p, w_0 + \omega)}),$$

since conjugation preserves the real part. Thus we can think of $w_0 \in [-1, 1]$ as the parameter of a curve of diffeomorphisms of the standard S^5 (resp. S^{13}). Note

that for $w_0 = \pm 1$, $b(p, w) = 1$, and thus at these levels the diffeomorphisms are the identity. \square

Let T_n denote the characteristic subgroup in $\pi_0 \text{Diff}^+(S^n) \cong \Gamma_{n+1}$ generated by all elements of order 2, and let $N_{n-1} := \partial^{-1}(T_n)$ be the corresponding normal subgroup in $\pi_1 \text{Diff}^+(S^{n-1})$.

Lemma 2. *The loops $\hat{\sigma}$ introduced in Lemma 1 generate the cyclic subgroups*

$$\pi_1 \text{Diff}^+(S^5)/N_5 = \mathbb{Z}_{14} \quad \text{and} \quad \pi_1 \text{Diff}^+(S^{13})/N_{13} = \mathbb{Z}_{4064}.$$

Proof. Because of the definition of N_{n-1} , the boundary maps ∂ induce isomorphisms

$$\bar{\partial}: \pi_1 \text{Diff}^+(S^{n-1})/N_{n-1} \rightarrow \pi_0 \text{Diff}^+(S^n)/T_n.$$

Now observe that by definition T_6 is the subgroup $\mathbb{Z}_2 = \langle 14 \rangle$ in $\pi_0 \text{Diff}^+(S^6)$ and that T_{14} is the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (4064, 0), (0, 1) \rangle$ in $\pi_0 \text{Diff}^+(S^{14}) = bP^{16} \times \mathbb{Z}_2 = \mathbb{Z}_{8128} \times \mathbb{Z}_2$. To conclude the argument, recall that the diffeomorphisms $\sigma = \partial(\hat{\sigma})$ generate \mathbb{Z}_{28} and bP^{16} , respectively. \square

Before we employ this structural information in the proof of Theorem 4 we need to introduce some notation: Let σ_0 be the restriction of σ to the equator of S^6 (resp. S^{14}) given by $w_0 = \mathfrak{R}(w) = 0$ and let α_0 denote the antipodal map of the equator. In order to be consistent with the standard convention for concatenation of loops we will now define all loops and paths on the unit interval $[0, 1]$. In particular, we assume that $\hat{\sigma}$ is parametrized on $[0, 1]$ such that $\hat{\sigma}(\frac{1}{2}) = \sigma_0$ and such that $\hat{\sigma}(0)$ gives the identity of S^5 at the north pole of S^6 .

The first information that we get from Lemma 1 is how the actions \mathcal{A} of the previous section transfer by the boundary map ∂ to the cyclic subgroups of $\pi_1 \text{Diff}^+(S^5)$ and $\pi_1 \text{Diff}^+(S^{13})$ generated by $\hat{\sigma}$. We evidently have

$$\widehat{\alpha\sigma\alpha^{-1}} = \alpha_0(-\hat{\sigma})\alpha_0^{-1}$$

where $-\hat{\sigma}$ denotes the reverse loop, i.e., $(-\hat{\sigma})(t) = \hat{\sigma}(1-t)$. Thus, the commutation identity $\alpha\sigma\alpha^{-1} = \sigma^{-1}$ turns into

$$\alpha_0(-\hat{\sigma})\alpha_0^{-1} = (\hat{\sigma})^{-1}.$$

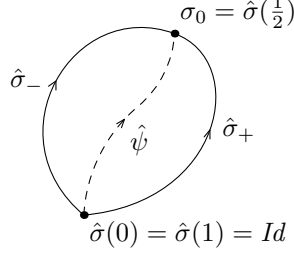
Suppose now that – in contrast to the claim of Theorem 4 – the involution $\rho_0 = \alpha_0\sigma_0$ is conjugate to the antipodal map α_0 , i.e., there is a diffeomorphism h of S^5 or S^{13} such that $\rho_0 = \alpha_0\sigma_0 = h\alpha_0h^{-1}$. Since α_0 commutes with hyperplane reflections, we may assume that h is orientation preserving. Solving for σ_0 , we obtain

$$\sigma_0 = \alpha_0^{-1}h\alpha_0h^{-1}.$$

Using a path A in $SO(6) \subset \text{Diff}^+(S^5)$ (resp. in $SO(14) \subset \text{Diff}^+(S^{13})$) from the identity to the antipodal map α_0 (such a path exists for all odd dimensional spheres), we get a path $\hat{\psi}$ from the identity to σ_0 given by

$$\hat{\psi}(t) = A(t)^{-1}hA(t)h^{-1}.$$

The idea is to use this path to cut $\hat{\sigma}$ in half and to show that the element $[\hat{\sigma}] \in \pi_1 \text{Diff}(S^{n-1})$ is the product of a square and a correction factor contained

FIGURE 4. Configuration of paths in $\text{Diff}^+(S^5)$ and $\text{Diff}^+(S^{13})$

in N_{n-1} , a factorization that contradicts Lemma 2. In order to make this idea concrete, we decompose σ into two paths $\hat{\sigma}_+$ and $\hat{\sigma}_-$ defined on $[0, 1]$ by

$$\begin{aligned}\hat{\sigma}_+(t) &= \hat{\sigma}(\tfrac{1}{2}t) \\ \hat{\sigma}_-(t) &= \hat{\sigma}(1 - \tfrac{1}{2}t)\end{aligned}$$

Thus, $\sigma = \hat{\sigma}_+ \sqcup (-\hat{\sigma}_-)$ where \sqcup denotes juxtaposition of paths. We concatenate each of these two paths with $-\hat{\psi}$ and get loops

$$\hat{\phi}_\pm = \hat{\sigma}_\pm \sqcup (-\hat{\psi}).$$

(see figure 4). Clearly, we have

$$[\hat{\sigma}] = [\hat{\sigma}_+ \sqcup (-\hat{\sigma}_-)] = [\hat{\phi}_+ \sqcup (-\hat{\phi}_-)] = [\hat{\phi}_+][(-\hat{\phi}_-)] = [\hat{\phi}_+][\hat{\phi}_-]^{-1}.$$

We claim that $[\hat{\phi}_+]$ equals $[\hat{\phi}_-]^{-1}$ up to some correction factor in N_{n-1} . This implies that $[\hat{\sigma}] \equiv [\hat{\phi}_+]^2 \pmod{N_{n-1}}$, which is the contradiction that we are striving for.

Decomposing the identity $(\hat{\sigma})^{-1} = \alpha_0(-\hat{\sigma})\alpha_0^{-1}$ with respect to the components of $\hat{\sigma} = \hat{\sigma}_+ \sqcup (-\hat{\sigma}_-)$, we find that on the level of paths the following holds

$$(\hat{\sigma}_-)^{-1} = \alpha_0\hat{\sigma}_+\alpha_0^{-1}.$$

Combining these identities we obtain

$$\begin{aligned}(\hat{\phi}_-)^{-1} &= (\hat{\sigma}_-)^{-1} \sqcup (-\hat{\psi})^{-1} \\ &= \alpha_0\hat{\sigma}_+\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1} \\ &\simeq \alpha_0(\hat{\sigma}_+ \sqcup (-\hat{\psi}) \sqcup \hat{\psi})\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1} \\ &= \alpha_0\hat{\phi}_+\alpha_0^{-1} \sqcup \alpha_0\hat{\psi}\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1} \quad \text{rel}\{0, 1\}.\end{aligned}$$

Clearly, the map $(s, t) \mapsto A(s)\hat{\phi}_+(t)A(s)^{-1}$ provides a homotopy $\hat{\phi}_+ \simeq \alpha_0\hat{\phi}_+\alpha_0^{-1}$. Since $\hat{\phi}_+(t) = Id$ in a neighborhood of $\{0, 1\}$, the preceding map is actually a homotopy $\text{rel}\{0, 1\}$, and so we conclude that in $\pi_1 \text{Diff}^+(S^{n-1})$ the following identity holds

$$[\hat{\phi}_-]^{-1} = [\hat{\phi}_+] \cdot [\alpha_0\hat{\psi}\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1}].$$

Thus the proof of Theorem 4 is completed by the following lemma:

Lemma 3. *The path $\alpha_0\hat{\psi}\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1}$ represents an element in the normal subgroup N_{n-1} of $\pi_1 \text{Diff}^+(S^{n-1})$.*

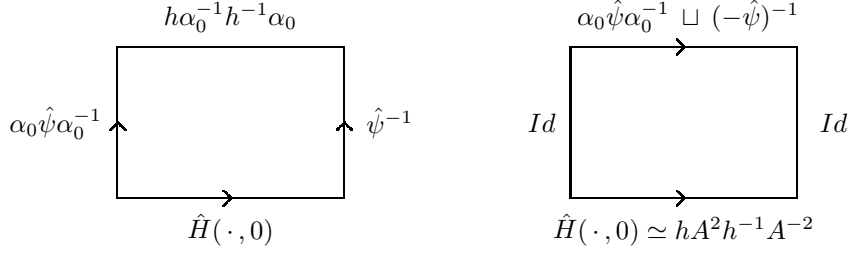


FIGURE 5. The homotopy \hat{H} induces a homotopy $\text{rel}\{0, 1\}$ between $\alpha_0\hat{\psi}\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1}$ and $hA^2h^{-1}A^{-2}$.

Proof. For the purpose of this argument, we find it convenient to assume that the path A in $SO(n) \subset \text{Diff}^+(S^{n-1})$ connecting the identity to α_0 is the 1-parameter subgroup obtained by exponentiating some almost complex structure $J \in \mathfrak{so}(n)$. On the path level, this choice yields the identity

$$(-A)^{-1} = A\alpha_0^{-1} = \alpha_0^{-1}A,$$

where we remind the reader that the minus sign in this equation represents the reverse path, $-\gamma(t) = \gamma(1-t)$.

Now consider the map $\hat{H} : [0, 1] \times [0, 1] \rightarrow \text{Diff}^+(S^{n-1})$ given by

$$\hat{H}(s, t) := \alpha_0 A(s+t-st)^{-1} h A(s+t-st) \alpha_0^{-1} A(1-s+ts)^{-1} h^{-1} A(1-s+ts).$$

Recall that $\alpha_0^2 = \text{Id}$. Thus with the help of the preceding identity, it is easy to verify that

$$\begin{aligned} \hat{H}(0, t) &= \alpha_0 A(t)^{-1} h A(t) \alpha_0^{-1} A(1)^{-1} h^{-1} A(1) = \alpha_0 \hat{\psi}(t) \alpha_0^{-1}, \\ \hat{H}(1, t) &= \alpha_0 A(1)^{-1} h A(1) \alpha_0^{-1} A(t)^{-1} h^{-1} A(t) = \hat{\psi}(t)^{-1}, \\ \hat{H}(s, 1) &= \alpha_0 A(1)^{-1} h A(1) \alpha_0^{-1} A(1)^{-1} h^{-1} A(1) = h \alpha_0^{-1} h^{-1} \alpha_0. \end{aligned}$$

Thus \hat{H} induces a homotopy $\text{rel}\{0, 1\}$ of the concatenation $\alpha_0\hat{\psi}\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1}$ to the path

$$\begin{aligned} s \mapsto \hat{H}(s, 0) &= \alpha_0 A(s)^{-1} h A(s) \alpha_0^{-1} A(1-s)^{-1} h^{-1} A(1-s) \\ &= \alpha_0 A(s)^{-1} h A(s)^2 h^{-1} A(s)^{-1} \alpha_0^{-1} \\ &= \alpha_0 A(s)^{-1} (h A(s)^2 h^{-1} A(s)^{-2}) (\alpha_0 A(s)^{-1})^{-1}. \end{aligned}$$

Note that A^2 itself is a loop in $SO(n) \subset \text{Diff}^+(S^{n-1})$ based at the identity and so is $hA^2h^{-1}A^{-2}$. Thus the map

$$(s, t) \mapsto \alpha_0 A(s+t-st)^{-1} (h A(s)^2 h^{-1} A(s)^{-2}) (\alpha_0 A(s+t-st)^{-1})^{-1}$$

provides a homotopy of the path $s \mapsto \hat{H}(s, 0)$ to the path $hA^2h^{-1}A^{-2} \text{ rel}\{0, 1\}$. Hence

$$[\alpha_0\hat{\psi}\alpha_0^{-1} \sqcup (-\hat{\psi})^{-1}] = [hA^2h^{-1}][A^2]^{-1} \in \pi_1 \text{Diff}^+(S^{n-1}).$$

Being the image of an element in $\pi_1 SO(n)$ under the canonical inclusion, it is evident that $[A^2] \in \pi_1 \text{Diff}^+(S^{n-1})$ is an element of order at most 2. Conjugation by h induces an automorphism of $\pi_1 \text{Diff}^+(S^{n-1})$, and so $[hA^2h^{-1}]$ is also an element

of order at most 2. Hence $[A^2]$ and $[hA^2h^{-1}]$ must both map into the 2-torsion group $T_n \subset \pi_0 \text{Diff}^+(S^n)$ under the boundary map, and therefore

$$[\alpha_0 \hat{\psi} \alpha_0^{-1} \sqcup (-\hat{\psi})^{-1}] \in \partial^{-1}(T_n) = N_{n-1}$$

as claimed. \square

5. THE GEOMETRY OF HIRSCH-MILNOR INVOLUTIONS

The fact that $\rho = \alpha\sigma$ is an exotic free involution seems like a huge coincidence at first, stemming from the peculiar algebraic properties of the Blakers-Massey elements b . However, a more careful study explains why ρ has these properties. The understanding comes from the interplay between three constructions:

- (1) The Hirsch-Milnor construction of exotic involutions of S^5 and S^6 ([20]), based on the Milnor exotic sphere $\Sigma_{2,-1}^7$ ([27]).
- (2) The Gromoll-Meyer description of the Milnor exotic sphere as a quotient Σ_{GM}^7 of the Lie group $Sp(2)$ ([15]).
- (3) The study of the geometry of geodesics of certain metrics on the Gromoll-Meyer exotic sphere carried out in [11], and [12], which in particular produces partial sections and trivializations of the bundle $S^3 \cdots Sp(2) \rightarrow \Sigma_{GM}^7$.

In [20], Hirsch and Milnor constructed involutions of S^6 and S^5 that are not conjugate to the antipodal map. These involutions are constructed as follows: first consider the Milnor exotic sphere $\Sigma_{2,-1}^7$ ([27]). This sphere is a S^3 bundle over S^4 with structure group $SO(4)$, and the bundle description is given by taking two copies of $\mathbb{R}^4 \times S^3$ and identifying $\mathbb{R}^4 - 0 \times S^3$ with $\mathbb{R}^4 - 0 \times S^3$ via the map

$$(u, v) \rightarrow (u', v') = (u/|u|^2, \frac{1}{|u|}u^2vu^{-1}).$$

The antipodal map in the fibers of the fibration $S^3 \cdots \Sigma_{2,-1}^7 \rightarrow S^4$ (that is, in bundle coordinates, the well-defined map $(u, v) \mapsto (u, -v)$) is a free involution τ on $\Sigma_{2,-1}^7$. This involution supports invariant spheres $S^5 \subset S^6 \subset \Sigma_{2,-1}^7$, described by certain subsets of the bundle coordinates, namely,

$$\begin{aligned} S^6 &= \{(u, v) \mid \Re(uv) = 0\}, \\ S^5 &= \{(u, v) \mid \Re(uv) = \Re(v) = 0\}. \end{aligned}$$

Then Hirsch and Milnor show that the involutions are exotic by showing that, if they were not exotic, then $\Sigma_{2,-1}^7$ would have even order in the cyclic group $\Gamma_7 \cong \mathbb{Z}_{28}$, which contradicts the fact that $\Sigma_{2,-1}^7$ is a generator. A proof in the same spirit shows that τ restricted to S^5 is also exotic.

Next in line of this exploration is the Gromoll-Meyer expression for the sphere $\Sigma_{2,-1}^7$ as the quotient of the group $Sp(2)$ of 2×2 quaternionic matrices A satisfying $A^*A = AA^* = I_{2 \times 2}$ by the S^3 -action

$$q \star \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} qa\bar{q} & qc \\ qb\bar{q} & qd \end{pmatrix}$$

The explicit identification between the Gromoll-Meyer sphere Σ_{GM}^7 and the Milnor exotic sphere $\Sigma_{2,-1}^7$ is given in [15], e.g., if $\mathcal{U} \subset \Sigma_{GM}^7$ is the set of classes

$$\mathcal{U} \left\{ \left[\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right] \text{ such that } d \neq 0 \right\}$$

we have $f : \mathcal{U} \rightarrow \mathbb{R}^4 \times S^3$ given by

$$\left[\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right] \mapsto (u, v) = |d|^{-2}(\bar{c}d, \bar{d}ad|a|^{-1});$$

see [15] for the other charts and their inverses; however we warn the reader that the matrices in [15] are transposes of ours (we write them that way since we use the projection onto the first column a lot).

From the Gromoll-Meyer formulas, we see that

Proposition 1. *The Hirsch-Milnor exotic involutions are induced by the antipodal free involution m on $Sp(2)$ given by*

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \xrightarrow{m} \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}.$$

Moreover, the invariant spheres S^6 (resp S^5) are given by the projection of the sets \mathcal{S}_6 (resp. \mathcal{S}_5) given by

$$\begin{aligned} \mathcal{S}_6 &= \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in Sp(2) \mid \Re(a) = 0 \right\}, \\ \mathcal{S}_5 &= \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in Sp(2) \mid \Re(a) = \Re(b) = 0 \right\}. \end{aligned}$$

Then we have the construction of a pointed wiedersehen metric on Σ_{GM}^7 given in [11]. The horizontal lift of geodesics from Σ^7 to $Sp(2)$ provides a section and therefore a trivialization of the bundle $S^3 \cdots (Sp(2) \setminus S^3) \rightarrow (\Sigma_{GM}^7 \setminus \{\text{south pole}\})$.

More precisely, let $N, S \in \Sigma_{GM}^7$ be given by $N = [I_{2 \times 2}], S = [-I_{2 \times 2}]$. The geodesics from N are the projections of the horizontal geodesics through the identity of $Sp(2)$, which are given by

$$\gamma_{(p,w)}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma'_{(p,w)}(0) = \begin{pmatrix} p & -\bar{w} \\ w & 0 \end{pmatrix},$$

where p is a pure quaternion and $|p|^2 + |w|^2 = 1$. Then

$$\gamma_{(p,w)}(t) = \begin{pmatrix} \cos(t) + \sin(t)p & -\sin(t)e^{tp}\bar{w} \\ \sin(t)w & \frac{w}{|w|}(\cos(t) - \sin(t)p)e^{tp}\frac{\bar{w}}{|w|} \end{pmatrix},$$

in the “generic” case $w \neq 0$. In the case $w = 0$,

$$\gamma_{(p,w)}(t) = \begin{pmatrix} e^{tp} & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that the set \mathcal{S}_6 is the set of midpoints of the horizontal geodesics from the identity, given by $t = \pi/2$, and therefore the invariant $S^6 \subset \Sigma_{GM}^7$ is the “equator” of the wiedersehen metric on Σ_{GM}^7 given by points equidistant from the north and south poles, and the invariant sphere S^5 is given by $t = \pi/2, \Re(w) = 0$.

All we need to construct a formula for the Hirsch-Milnor involution of S^6 is therefore to take advantage of the section given by the geodesics. Let us restrict

the bundle $S^3 \cdots Sp(2) \rightarrow \Sigma_{GM}^7$ to $S^3 \cdots \mathcal{S}_6 \rightarrow S^6$. But this bundle is trivial, and the midpoints of the geodesics from the north pole produce a trivialization; we have

$$\psi : S^6 \times S^3 \rightarrow \mathcal{S}_6 \subset Sp(2)$$

given by

$$\begin{aligned} \psi((p, w), q) &= \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \gamma_{(p, w)}(\pi/2) \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} p & -e^{\frac{\pi}{2}p} \bar{w} \\ w & -\frac{w}{|w|} p e^{\frac{\pi}{2}p} \frac{\bar{w}}{|w|} \end{pmatrix} \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and the inverse of ψ is given by

$$\psi^{-1} \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = ((\bar{q} a q, \bar{q} b q), q)$$

where

$$q \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = -\frac{b}{|b|} e^{-\frac{\pi}{2}a} \frac{c}{|c|}.$$

The following is then clear:

Proposition 2. *The map ψ is a bundle trivialization, that is, it is an S^3 -equivariant map from $S^6 \times S^3 \rightarrow \mathcal{S}_6$, where the action on the left hand side is the left multiplication action on the S^3 -factor and the action on the right hand side is the Gromoll-Meyer action restricted to \mathcal{S}_6 .*

The involution m on $Sp(2)$ pulls back as

$$\psi^{-1} \circ m \circ \psi((p, w), q) = (\alpha \sigma^{-1}(p, w), q b(p, w)).$$

Projecting onto the first component of $S^6 \times S^3$, we get our exotic involution $a\sigma^{-1} = \rho_{-1}$, which is conjugate to ρ by σ^{-1} .

Note that the exotic projective space $\mathbb{R}P_\rho^6 = S^6/\rho$ is also the quotient of $S^6 \times S^3$ under the $\mathbb{Z}_2 \times S^3$ -action \star given by

$$\begin{aligned} (0, \theta) \star ((p, w), q) &= ((p, w), \theta q), \\ (1, \theta) \star ((p, w), q) &= (\rho_{-1}((p, w)), \theta q b(p, w)). \end{aligned}$$

since in the quotient, the S^3 -action by θ just kills the S^3 -factor of $S^6 \times S^3$ just leaving $S^6/\rho_{-1} \cong S^6/\rho$.

6. NON-CANCELLATION PHENOMENA IN GROUP ACTIONS

The usual form of non-cancellation phenomena, e.g. [17, 18, 19] is expressed by manifolds M_1, M_2, N where M_1 and M_2 are not homotopy equivalent but $M_1 \times N$ is diffeomorphic to $M_2 \times N$ (see also [3]).

Here we are interested in explicit formulas for subtler *differentiable* non-cancellation phenomena, exemplified for instance by the fact that for any 7-dimensional exotic sphere Σ^7 , $\Sigma^7 \times S^3$ is diffeomorphic to $S^7 \times S^3$ (see [31, 33]). Here we use the exotic involution $\rho : S^6 \rightarrow S^6$ constructed above to give some differentiable non-cancellation phenomena of *group actions*.

Theorem 5. *There exist explicit actions r_1, r_2 of $\mathbb{Z}_2 \times S^3$ on $X = S^6 \times S^3$ such that neither factor can be cancelled. More precisely,*

- the restrictions of the actions r_1 and r_2 to the subgroup $\{0\} \times S^3$ are differentially conjugate,
- the restrictions of the actions r_1 and r_2 to the subgroup $\mathbb{Z}_2 \times \{1_{S^3}\}$ are differentially conjugate,
- the full actions r_1, r_2 of $\mathbb{Z}_2 \times S^3$, however, are not differentially conjugate.

The construction is based on the following consideration: in addition to the Gromoll-Meyer action, the group S^3 also acts freely in $Sp(2)$ as follows:

$$q \bullet \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{q} \end{pmatrix},$$

producing a principal fibration $S^3 \cdots Sp(2) \rightarrow S^7$, where S^7 is the standard 7-sphere. In fact, the projection of $Sp(2)$ onto S^7 is just $A \rightarrow 1^{st}$ column of A .

The canonical wiedersehen metric on S^7 produces also a partial section, and therefore a trivialization of the bundle $S^3 \cdots Sp(2) \setminus S^3 \rightarrow S^7 \setminus \{\text{south pole}\}$. Let us remark that the fiber over the south pole of S^7 is the same as the fiber over the south pole of Σ_{GM}^7 .

Let us list the following trivial fact as a proposition:

Proposition 3. *The antipodal involution m on $Sp(2)$ above descends under the \bullet -action to the canonical involution α on S^7 .*

Thus we have two different S^3 -principal fibrations with $Sp(2)$ as total space:

$$\begin{array}{ccccc} & & S^3 & & \\ & & \star \downarrow & & \\ S^3 & \xrightarrow{\bullet} & Sp(2) & \longrightarrow & S^7 \\ & & \downarrow & & \\ & & \Sigma^7 & & \end{array}$$

Note that *the same* involution m on $Sp(2)$, when restricted to \mathcal{S}_6 , descends to the non-conjugate involutions α and ρ on S^6 , depending on whether one uses the \bullet - or the \star - projections, respectively. This observation is the basis of our particular non-cancellation phenomenon.

Let us proceed with the computation. Consider the trivialization analogous to the one considered in the previous section, but now for the bundle $S^3 \cdots Sp(2) \rightarrow S^7$ restricted to $S^3 \cdots \mathcal{S}_6 \rightarrow S^6$. In other words, consider the map

$$\begin{aligned} \phi : S^6 \times S^3 &\rightarrow \mathcal{S}_6 \subset Sp(2), \\ \phi((p, w), q) &= \gamma_{(p,w)}(\pi/2) \begin{pmatrix} 1 & 0 \\ 0 & \bar{q} \end{pmatrix} = \begin{pmatrix} p & -e^{\frac{\pi}{2}p}\bar{w} \\ w & -\frac{w}{|w|}pe^{\frac{\pi}{2}p}\frac{\bar{w}}{|w|} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{q} \end{pmatrix}. \end{aligned}$$

The inverse of ϕ is given by

$$\phi^{-1} \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = ((a, b), q), \quad \text{where} \quad q \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = -\frac{\bar{c}}{|c|}e^{\frac{\pi}{2}a}\frac{\bar{b}}{|b|}.$$

Similarly, we have

Proposition 4. *The map ϕ is a bundle trivialization, that is, it is an S^3 -equivariant map from $S^6 \times S^3 \rightarrow \mathcal{S}_6$, where the action on the left hand side is the left multiplication action on the S^3 -factor and the action on the right hand side is the standard \bullet -action above action restricted to \mathcal{S}_6 .*

Then a computation shows that

$$\phi^{-1} \circ m \circ \phi((p, w), q) = (\alpha(p, w), q \overline{b(p, w)}),$$

and projecting onto the first component of $S^6 \times S^3$ we get the standard antipodal involution on S^6 .

The standard projective space $\mathbb{R}P^6 = S^6/\alpha$ is then the quotient of the $\mathbb{Z}_2 \times S^3$ -action \circ on $S^6 \times S^3$ given by

$$\begin{aligned} (0, \theta) \circ ((p, w), q) &= ((p, w), \theta q), \\ (1, \theta) \circ ((p, w), q) &= (\alpha(p, w), \theta q \overline{b(p, w)}). \end{aligned}$$

Let r_1, r_2 now be, respectively, the \star - and \circ -actions defined above. Then clearly r_1 and r_2 are not conjugate, since a conjugacy between r_1 and r_2 would imply that $\mathbb{R}P_\rho^6$ and $\mathbb{R}P_\alpha^6$ are diffeomorphic, and we know they are not.

Observe that the restrictions of r_1 and r_2 to $\{0\} \times S^3$ coincide and so they are trivially differentiably conjugate. On the other hand, the restrictions of r_1 and r_2 to $\mathbb{Z}_2 \times \{1_{S^3}\}$ are given by

$$\begin{aligned} (1, 1_{S^3}) \star ((p, w), q) &= (\rho_{-1}(p, w), q b(p, w)), \\ (1, 1_{S^3}) \circ ((p, w), q) &= (\alpha(p, w), q \overline{b(p, w)}). \end{aligned}$$

These two actions are equivalent; they are really the same involution m on $Sp(2)$ restricted to the set $\mathcal{S}_6 \cong S^6 \subset Sp(2)$. The fact that they look different comes from the different trivializations of the bundle $S^3 \cdots \mathcal{S}_6 \rightarrow S^6$. In fact, the diffeomorphism conjugating them is given by

$$F((p, w), q) = ((qp\bar{q}, qw\bar{q}, \bar{q}).$$

Note that the conjugating diffeomorphism F is also an involution!

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