Markov Processes Relat. Fields **1**, 473–490 (1995)

Markov Processes and **Related Fields**

Stochastic Model of Massively Parallel Computation

A. Greenberg [1] , **V.A. Malyshev** [2] and **S.Yu. Popov** [3]

[1] Math. Sciences Research Center, AT&T Bell Labs, Murray Hill, NJ 07974, USA
[2] INRIA, Domaine de Voluceau, Rocquencourt, B.P. 105–78153, Le Chesnay Cedex, France
[3] Laboratory of Large Random Systems, Faculty of Mechanics and Mathematics, Moscow State University, 119899, Moscow, Russia

Received May 5, 1995, revised October 1, 1995

Abstract. A problem of massive parallelism is considered, where N processor units are used for large-scale simulation or computation. Processor unit i has its accumulated local time variable $z_i(t)$. At Poisson time moments t_k^i, it gets a job and chooses randomly I other units. If its local time does not exceed the local times of the chosen I units, then $z_i(t_k^i)$ is augmented by an independent random variable η_{ik}. In the large N limit we obtain a deterministic nonlinear PDE for the density of local times. Subsequent corollaries are a travelling wave solution, the linear time growth of the mean local time etc.

Keywords: parallel computation, asymptotic independence

AMS Subject Classification: Primary 60J75, 60K35

1. Origin of the problem

We consider a synchronisation problem in massive parallel computations [1–3, 5, 6]. Assume that we have N processor units which are used simultaneously for some large-scale simulation or computation. Each of these processor units simulates or/and computes some local characteristics, and during this simulation it can use data from other units which are produced in the process. This provides the interaction between these processor units. This interaction can slow down the process compared to when the processor units work independently.

This slowing-down is caused by the fluctuations in the time that the different processors need to accomplish their local tasks. Thus some processors have to stop and wait until others (from which they need the data) will finish their work.

If the arrival rate of problems to a processor unit is one and if the mean time to solve each of them is one as well, then during a large time interval t, approximately t problems will arrive and will be solved. In the interacting case, however, t problems will arrive but only $m(t)$ of these will be solved. For some reasonable mathematical model we will determine $m(t)$ explicitly. It turns out to be a linear function of t. Thus we get a speed that is less than one.

2. Mathematical model and main results

We consider a system consisting of N nodes, enumerated by $i = 1, \ldots, N$. The state of node i at time t, $t \geq 0$, is a nonnegative random variable $z_i^N(t)$, called the local time in this node at time t. As initial conditions we assume that the $z_i^N(0)$, $i = 1, \ldots, N$, are independent, identically distributed non-negative random variables with density $p_0(x)$, such that

$$\int_0^{+\infty} x^2 p_0(x)\, dx < \infty.$$

Let $J < N$ be a fixed integer and $v \in \mathbf{R}_+$ a given constant. The exogenous arrivals of customers at node i occur at the successive moments $t_1^i < t_2^i < \cdots$. These arrival streams at the different nodes are assumed to be mutually independent Poisson streams with intensity μ. At moment t_k^i node i chooses J other nodes at random. Let us denote this random set by Y_k^i, such that $i \notin Y_k^i$, and assume that all these random sets are also independent. If $z_j^N(t_k^i) \geq z_i^N(t_k^i)$ for all $j \in Y_k^i$, then the local time at node i jumps instantaneously, so that

$$z_i^N(t_k^i + 0) = z_i^N(t_k^i) + \xi_k^i,$$

where ξ_k^i are independent, exponentially distributed random variables with parameter λ. The random variables $z_i^N(t)$ are thus completely defined. We may assume that with probability one at each node we have a left continuous piecewise constant nondecreasing random process on $\mathbf{R}_+$.

We get a more general model if for some $v \geq 0$ we put for all i and for all $t \neq t_k^i$

$$z_i'(t) = \begin{cases} -v, & \text{if } z_i(t) > 0, \\ 0, & \text{if } z_i(t) = 0, \end{cases}$$

(here we mean the right derivative, if $z_i(t) = 0$), so all functions z_i are piecewise linear. We call $z_i(t)$ the local time of the ith particle at time t.

In the sequel we shall investigate the behaviour of this system for large values of N. Let us denote by $\#_t^N[x, x + \Delta x]$ the number of particles z_i in the interval $[x, x + dx]$ at time t, that is $x \leq z_i(t) \leq x + \Delta x$.

We will formulate the main results of this paper.

Theorem 2.1. *For all x, Δx, t the following limit*

$$\lim_{N \to \infty} \frac{\#_t^N[x, x + \Delta x]}{N}$$

exists in probability, and there is a non-negative density function $\tilde{p}(x,t)$, *such that*

$$\lim_{N\to\infty} \frac{\#_t^N[x, x+\Delta x]}{N} = \tilde{p}(x,t)\Delta x + o(\Delta x). \tag{2.1}$$

We will derive an integro-differential equation for the density $\tilde{p}(x,t)$ and investigate some properties of the system using this equation.

Theorem 2.2. *The function* $\tilde{p}(x,t)$ *defined above, is a solution to the following integro-differential equation*

$$\frac{\partial \tilde{p}(x,t)}{\partial t} = -\mu G^J(x,t)\tilde{p}(x,t) + \lambda\mu e^{-\lambda x}\int_0^x \tilde{p}(y,t)G^J(y,t)e^{\lambda y}\,dy, \tag{2.2}$$

where

$$G(x,t) = \int_x^{+\infty} \tilde{p}(y,t)\,dy. \tag{2.3}$$

Remark 2.1. If we take the lower limit of the integration in (2.2) to be equal to $-\infty$, then (2.2) has a *travelling wave* solution (see [4])

$$\tilde{p}(x,t) = \lambda\Big(1+\exp\Big\{J\Big(\lambda x - \frac{\mu t}{J+1}\Big)\Big\}\Big)^{\frac{J+1}{J}}\Big(1+\exp\Big\{J\Big(\lambda x - \frac{\mu t}{J+1}\Big)\Big\}\Big).$$

As a next step we prove in Section 6 that this equation has a unique solution.

Theorem 2.3. *There is a unique solution of* (2.2) *among the functions that are continuous in* x *and continuously differentiable in* t.

Define the mean coordinates

$$m(t) = \int_0^{+\infty} x\tilde{p}(x,t)\,dx \quad \text{and} \quad m_0 = \int_0^{+\infty} xp_0(x)\,dx.$$

In Section 7 we shall finally prove the following important proposition.

Theorem 2.4. *The function* $m(t)$ *is the following linear function of time*

$$m(t) = \frac{\mu t}{\lambda(J+1)} + m_0. \tag{2.4}$$

3. Asymptotic independence

In this section we prove that any fixed number of nodes becomes independent, if the total number of nodes N tends to infinity. First we introduce some notation. Clearly, the random variable $z_1(t)$ has a "positive density" $p^{(N)}(x,t)$, i.e.

$$\mathbf{P}\{z_1(t) \in A\} = \int_A p^{(N)}(x,t)\,dx. \tag{3.1}$$

Let us write $Z(t) = (z_1(t), \ldots, z_N(t))$, and let $X = [0,\infty)^N$. The process $Z(t)$ is a pure jump Markov process with state space X. We will write $Z(t) = x = (x_1, \ldots, x_N)$, if $z_1(t) = x_1, \ldots, z_N(t) = x_N$.

Let us introduce some more notation. Let for $t > s$

$$P(s,x,t,B) = \mathbf{P}\{Z(t) \in B \mid Z(s) = x\}.$$

Our assumptions imply the existence of a positive density function $p(x_1, \ldots, x_N, t)$ such that for any set $A \subset X$

$$P(0,0,t,A) = \int_A p(x_1, \ldots, x_N, t)\,dx_1 \ldots dx_N. \tag{3.2}$$

For any set $\{i_1, \ldots, i_k\}$, $k < N$, we introduce correlation functions

$$p^{(i_1,\ldots,i_k)}(x_1, \ldots, x_k, t)$$

in a natural way:

$$p^{(i_1,\ldots,i_k)}(a_1, \ldots, a_k, t) = \int_B p(x_1, \ldots, x_N, t)\,dx_{j_1} \ldots dx_{j_{N-k}},$$

where $\{j_1, \ldots, j_{N-k}\} = \{1, \ldots, N\} \setminus \{i_1, \ldots, i_k\}$, and

$$B = \bigl\{x \in X \bigm| x_i = a_j, \text{ if } \ i = i_j \in \{i_1, \ldots, i_k\}\bigr\}.$$

Clearly,

$$p^{(i_1,\ldots,i_k)}(x_1, \ldots, x_k, t) = p^{(1,\ldots,k)}(x_1, \ldots, x_k, t) = p(x_1, \ldots, x_k, t), \tag{3.3}$$

since the random variables $z_i(t)$ are symmetrically distributed.

Let us first formulate and prove the following lemma.

Lemma 3.1. *Let $k \in \mathbf{Z}_+$ be a fixed number and $A_1, \ldots, A_k$ measurable sets with $A_i \subset \mathbf{R}_+$. Then for all t (note that in fact all $z_i(t)$ depend on N)*

$$\mathbf{P}\{z_1(t) \in A_1, \ldots, z_k(t) \in A_k\} - \prod_{i=1}^{k} \mathbf{P}\{z_i(t) \in A_i\} \to 0 \tag{3.4}$$

as $N \to \infty$, uniformly in $A_1, \ldots, A_k$, i.e. any fixed number of particles are asymptotically independent.

Next suppose that $p_0(x)$ is continuous and that there exists some $K > 0$ such that $p_0(x) < K$ for all x. Then the following corollary easily follows from the lemma.

Corollary 3.1. *For the density functions $p(x_1, \ldots, x_k, t)$ defined above, a similar assertion holds, namely*

$$p(x_1, \ldots, x_k, t) - \prod_{i=1}^{k} p(x_i, t) \to 0 \tag{3.5}$$

as $N \to \infty$, uniformly in $x_1, \ldots, x_k$.

Proof of Lemma 3.1. For the sake of brevity we consider the case $k = 2$ only. This proof can be generalised for any fixed $k \geq 3$. So, we want to prove that for any fixed t

$$\mathbf{P}\{z_1^N(t) \in A_1,\ z_2^N(t) \in A_2\} \to \mathbf{P}\{z_1^N(t) \in A_1\}\mathbf{P}\{z_2^N(t) \in A_2\} \tag{3.6}$$

as $N \to \infty$. For the proof we need the following definition.

Definition 3.1. For any node i and any t we define a random set $D_i(t)$, which we shall call *dependency set* or *D-set*, in the following way:

1. If $t < t_1^i$, we set $D_i(t) = \{i\}$.

2. Otherwise, let $0 < t_1^i < t_2^i < \cdots < t_l^i < t$, and let $Y_1^i, \ldots, Y_l^i$ be the corresponding random sets that were introduced above. Then we set

$$D_i(t) = \{i\} \bigcup \Big(\bigcup_{j=1}^{l} \bigcup_{m \in Y_j^i} D_m(t_j^i) \Big). \tag{3.7}$$

Note, that if for two particles i and j their D-sets ($D_i(t)$ and $D_j(t)$) do not intersect, then the random variables $z_i(t)$ and $z_j(t)$ are conditionally independent, since we assumed independence of $z_i(0)$ and $z_j(0)$.

Let us continue the proof. Note that $z_1(0)$ and $z_2(0)$ are independent, i.e.

$$\mathbf{P}\{z_1(0) \in A_1,\ z_2(0) \in A_2\} = \mathbf{P}\{z_1(0) \in A_1\}\mathbf{P}\{z_2(0) \in A_2\}.$$

We can therefore write

$$\begin{aligned}
&\mathbf{P}\{z_1(t) \in A_1,\ z_2(t) \in A_2\} \\
&\quad = \mathbf{P}\{z_1(t) \in A_1,\ z_2(t) \in A_2 \mid D_1(t) \cap D_2(t) \neq \emptyset\}\mathbf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} \\
&\quad + \mathbf{P}\{z_1(t) \in A_1,\ z_2(t) \in A_2 \mid D_1(t) \cap D_2(t) = \emptyset\}\mathbf{P}\{D_1(t) \cap D_2(t) = \emptyset\} \\
&\quad \stackrel{\text{def}}{=} A + B.
\end{aligned}$$

Note, that if

$$\mathbf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} \to 0$$

as $N \to \infty$, then it easily follows that $A \to 0$ and

$$B - \mathbf{P}\{z_1(t) \in A_1\}\mathbf{P}\{z_2(t) \in A_2\} \to 0$$

as $N \to \infty$, thus implying (3.6). Consequently, the proof of Lemma 3.1 is completed by proving the following lemma.

Lemma 3.2. *We have*

$$\mathbf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} \to 0, \quad N \to \infty. \tag{3.8}$$

Proof. We shall prove the stronger result that there exists $C_0 > 0$, such that

$$\mathbf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} \leq C_0 N^{-1/3}. \tag{3.9}$$

It is evident, that

$$\begin{aligned}
&\mathbf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} \\
&\leq \mathbf{P}\{D_1(t) \cap D_2(t) \neq \emptyset \mid |D_1(t)| \leq M,\ |D_2(t)| \leq M\} + 2\mathbf{P}\{|D_1(t)| > M\} \\
&\stackrel{\text{def}}{=} F + G.
\end{aligned}$$

Using some elementary considerations, we can obtain

$$F < \frac{CM^2}{N}. \tag{3.10}$$

Let us estimate G. To this end, define random variables $d_i(t)$ in the following way:

1. If $t < t_1^i$, we set $d_i(t) = 1$.
2. Otherwise, let $0 < t_1^i < t_2^i < \cdots < t_l^i < t$, and let $Y_1^i, \ldots, Y_l^i$ be the corresponding random sets that were introduced above. Set

$$d_i(t) = 1 + \sum_{j=1}^{l} \sum_{m \in Y_j^i} d_m(t_j). \tag{3.11}$$

Clearly, $|D_i(t)| \leq d_i(t)$. Majorisation by a branching process gives the following estimate:

$$\mathbf{E}d_i(t) \leq e^{(J-1)\mu t},$$

and thus, by Chebyshev's inequality,

$$G = 2\mathbf{P}\{|D_i(t)| > M\} \leq \frac{e^{(J-1)\mu t}}{M} = \frac{C}{M}. \tag{3.12}$$

Combining (3.12) with (3.10) and choosing $M = N^{1/3}$ we get (3.9). This completes the proof. □

4. Existence of the asymptotic density

In this section we shall prove the existence of a function $p(x,t)$ satisfying (2.1), and we shall also prove that this function coincides with the limiting density of the first particle. (The existence of this limiting density will be proved afterwards.)

For the proof of Theorem 2.1, we have to prove the existence of a (non-random) function $\tilde{p}(x,t)$ satisfying (2.1) and coinciding with the limiting density $p(x,t)$ of the first particle.

Proof of Theorem 2.1. We shall prove that our system obeys some law of large numbers. Let us introduce some notation. We take the time t to be fixed, and so we shall sometimes write z_i instead of $z_i(t)$. Let $\Delta = [x, x+dx]$,

$$I_k(\Delta) = \begin{cases} 1, & \text{if } z_k \in \Delta; \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

and

$$S_N = \sum_{i=1}^{N} I_i(\Delta).$$

Clearly,

$$\frac{S_N}{N} = \frac{1}{N}\sum_{k=1}^{N} I_k(\Delta) = \frac{\#_t[x, x+\Delta x]}{N}. \tag{4.2}$$

Denote $m_N = \mathbf{E} I_1(\Delta)$ under the condition, that the system consists of N particles.

Obviously,

$$m_N \to p(x,t)\,dx, \quad N \to \infty. \tag{4.3}$$

Indeed,

$$m_N = \mathbf{P}\{z_1 \in \Delta \mid N \text{ particles}\} = \int_x^{x+dx} p^{(N)}(y,t)\,dy \to p(x,t)\,dx$$

as $N \to \infty$, since $p^{(N)}(x,t) \to p(x,t)$ by virtue of Theorem 5.1 in the next section, with $p(x,t)$ satisfying (2.2).

Denote

$$c_N = \operatorname{cov}(z_i, z_j) = \operatorname{cov}(z_1, z_2) = \mathbf{E}(I_1(\Delta) I_2(\Delta)) - \mathbf{E} I_1(\Delta)\mathbf{E} I_2(\Delta)$$

under the condition, that the system consists of N particles.

Since any finite number of particles are asymptotically independent, $c_N \to 0$ as $N \to \infty$.

Denote

$$\hat{S}_N = \sum_{i=1}^{N} (I_i(\Delta) - m_N).$$

Hence, we have to prove that

$$U_N = \frac{\hat{S}_N}{N} \xrightarrow{\mathbf{P}} 0. \tag{4.4}$$

For (4.4) it is sufficient to prove that

$$\mathbf{E}U_N^2 \to 0, \quad N \to \infty. \tag{4.5}$$

Indeed,

$$\begin{aligned} \mathbf{E}U_N^2 &= \frac{1}{N^2}\mathbf{E}\Big(\sum_{i=1}^{N}(I_i(\Delta) - m_N)\Big)^2 \\ &= \frac{1}{N^2}\Big(\sum_{i=1}^{N}\mathbf{E}(I_i(\Delta) - m_N)^2 + (N^2 - N)c_N\Big) \le 2c_N \to 0, \quad N \to \infty. \end{aligned}$$

Hence, (4.4) holds, thus implying the validity of (2.1) and so the proof of Theorem 2.1 is complete. □

5. The main equation

We fix N and omit it in our notation until otherwise stated.

In this section we shall obtain a limiting integro-differential equation for $p(x,t)$ as $N \to \infty$. In Section 4 we have shown that this function coincides with the asymptotic density $\tilde{p}(x,t)$ defined by (2.1).

Note, that

$$P(0,0,t,A) = p(x,t)\,dx \tag{5.1}$$

for $A = [x, x+dx] \times [0, +\infty)^{N-1}$. Letting $y \in X$, we denote

$$G_y^i = \mathbf{P}\{z_i(t) \le z_{j_k}(t),\ k = 1,\dots,J \mid z_i(t) = y_i\}, \tag{5.2}$$

with the set $Y_i = \{j_1, \dots, j_J\}$ chosen at random such, that $i \notin Y_i$.

We will use the second Kolmogorov equation for this process to derive an equation for $p(x,t)$ as $N \to \infty$ (here $p(x,t)$ is the density of the first particle).

Let us denote

$$\hat{a}(s,x,B) = \lim_{t \searrow s} \frac{P(s,x,t,B) - I_{\{x \in B\}}}{t - s} \tag{5.3}$$

and

$$a(s,x) = -\hat{a}(s,x,\{x\}), \quad a(s,x,B) = \hat{a}(s,x,B \setminus \{x\}). \tag{5.4}$$

Then the Kolmogorov equation for our process is given by

$$\frac{\partial P(s,x,t,B)}{\partial t} = -\int\limits_B a(t,y)P(s,x,t,dy) \ + \int\limits_X a(t,y,B)P(s,x,t,dy). \quad (5.5)$$

Let $K^{(i)}(y,B)$ be the probability of jumping from y to the set B under the condition, that the local time at the ith particle changes. One easily gets, that

$$K^{(1)}(y,B) = \lambda e^{-\lambda(x-y_1)}dx \quad (5.6)$$

for $B = [x, x+dx] \times [0, +\infty)^{N-1}$.

Let us now calculate $a(s,x)$ and $a(s,x,B)$. Using the properties of the Poisson process we can write

$$\begin{aligned} a(s,x) &= \lim_{t \searrow s} \frac{1 - P(s,x,t,\{x\})}{t-s} \\ &= \lim_{t \searrow s} \frac{1}{t-s} \mathbf{P}\{\text{at least one jump occured during time } \ t-s\} = \mu \sum_{i=1}^{N} G_x^i. \end{aligned}$$

In the same way we obtain

$$a(s,x,B) = \lim_{t \searrow s} \frac{P(s,x,t,B \setminus \{x\})}{t-s} = \mu \sum_{i=1}^{N} G_x^i K^{(i)}(x,B).$$

Let us choose $B = [x, x+dx] \times [0, +\infty)^{N-1}$. We can rewrite the right-hand side of (5.5) as follows:

$$\begin{aligned} -\int\limits_B a(t,y)P(s,x,t,dy) \ &+ \int\limits_X a(t,y,B)P(s,x,t,dy) \\ = \quad & -\mu \int\limits_B G_y^1 P(0,0,t,dy) + \mu \int\limits_X G_y^1 K^{(1)}(y,B)P(0,0,t,dy) \\ & + \sum_{i=2}^{N} \Big(-\mu \int\limits_B G_y^i P(0,0,t,dy) + \mu \int\limits_X G_y^i K^{(i)}(y,B)P(0,0,t,dy) \Big) \\ = \quad & -\mu \int\limits_B G_y^1 P(0,0,t,dy) + \mu \int\limits_X G_y^1 K^{(1)}(y,B)P(0,0,t,dy), \end{aligned}$$

since $K^{(i)}(y,B) = \mathbf{1}_B$, for $i \geq 2$ and for B chosen like this.

Letting N tend to infinity we will obtain the limiting equation. Let us denote for any set $S = \{i_1, \ldots, i_J\}$ and any $y \in X$

$$I_{S,y} = \begin{cases} 1, & \text{if } y_1 \leq y_{i_1}, \ \ldots, \ y_1 \leq y_{i_J}; \\ 0, & \text{otherwise.} \end{cases} \quad (5.7)$$

Denote $B' = [x, x + dx] \times [0, +\infty)^J$. Using the fact that the $z_i(t)$ are symmetrically distributed as well as Lemma 3.1 and Corollary 3.1 we find

$$\begin{aligned}
\int_B G_y^1 P(0,0,t,dy) &= \int_B G_y^1 p(y_1,\dots,y_N,t)\,dy_1\dots dy_N \\
&= \frac{1}{C_{N-1}^J} \sum_{S=\{i_1,\dots,i_J\}} \int_B I_{S,y}\, p(y_1,\dots,y_N,t)\,dy_1\dots dy_N \\
&= \int_B I_{\{2,\dots,J+1\},y}\, p(y_1,\dots,y_N,t)\,dy_1\dots dy_N \\
&= \int_{B'} I_{\{2,\dots,J+1\},y}\, p(y_1,\dots,y_{J+1},t)\,dy_1\dots dy_{J+1} \\
&\xrightarrow{N\to\infty} \int_x^{x+dx} p(y_1,t)\,dy_1 \prod_{i=2}^{J+1} \int_0^{+\infty} I_{\{y_1\le y_i\}} p(y_i,t)\,dy_i = p(x,t)G^J(x,t)\,dx.
\end{aligned}$$

In a similar way we obtain that the second term in the right-hand side of (5.5) is (asymptotically) equal to

$$\lambda\mu e^{-\lambda x} dx \int_0^x p(y,t)G^J(y,t)e^{\lambda y}\,dy.$$

Using this, together with the fact that $P(0,0,t,B) = p(x,t)\,dx$, we see that the equation for $p^{(N)}(x,t)$ is asymptotically equal to (2.2). In the next section we shall prove the existence of a unique solution of (2.2). The following proposition then holds by virtue of Lemma 6.2 of the next section. To avoid confusion we will append index N whenever necessary.

Theorem 5.1. *We have*

$$p^{(N)}(x,t) \to p(x,t), \quad N \to \infty, \tag{5.8}$$

with $p^{(N)}(x,t)$ defined by (3.1), and $p(x,t)$ the solution of (2.2).

6. Existence and uniqueness of the solution of the main equation

Now we shall prove the existence and uniqueness of the solution of equation (2.2), which describes the evolution of the system. We consider the following problem:

$$\left\{\begin{array}{rcl}
\dfrac{\partial p(x,t)}{\partial t} &=& -\mu G^J(x,t)p(x,t) + \lambda\mu e^{-\lambda x} \int\limits_0^x p(y,t)G^J(y,t)e^{\lambda y}\,dy, \\
G(x,t) &=& \int\limits_x^{+\infty} p(y,t)\,dy, \\
p(x,0) &=& p_0(x),
\end{array}\right. \tag{6.1}$$

where $p_0(x)$ is a density and $\operatorname{supp} p_0(x) \subset [0, +\infty)$.

Proof of Theorem 2.3. Let us first rewrite (6.1) in an equivalent form. Note, that

$$\int_0^{+\infty} p(x,t)\,dx = 1 \tag{6.2}$$

for all $t \geq 0$, provided a solution exists. Indeed, taking the integral $\int_0^\infty dx$ on both sides of our equation we get

$$\frac{d}{dt}\int_0^{+\infty} p(x,t)\,dx = 0. \tag{6.3}$$

Since $p_0(x)$ is a density, we immediately get (6.2). It is not difficult to show that $p(x,t) \geq 0$. So, provided it exists, $p(x,t)$ is a density for all $t \geq 0$. Thus

$$G(x,t) = 1 - \int_0^x p(y,t)\,dy,$$

and we have the following equivalent formulation of our problem:

$$\begin{cases} \dfrac{\partial p(x,t)}{\partial t} &= -\mu G^J(x,t)p(x,t) + \lambda\mu e^{-\lambda x}\int_0^x p(y,t)G^J(y,t)e^{\lambda y}\,dy, \\ G(x,t) &= 1 - \int_0^x p(y,t)\,dy, \\ p(x,0) &= p_0(x). \end{cases} \tag{6.4}$$

Integrating (6.4) from 0 to t, we get

$$p(x,t) = \mu\int_0^t \Big(- G^J(x,\tau)p(x,\tau) + \lambda e^{-\lambda x}\int_0^x p(y,\tau)G^J(y,\tau)e^{\lambda y}\,dy\Big)\,d\tau. \tag{6.5}$$

Note, that the value of $p(x,t)$ depends only on $p_0(y)$, $0 \leq y \leq x$. So in order to prove the existence and uniqueness of the solution of (6.1), it is sufficient to prove the following lemma.

Lemma 6.1. *The following problem has a unique solution for any* $x_0 > 0$.

$$\begin{cases} \dfrac{\partial p(x,t)}{\partial t} &= -\mu G^J(x,t)p(x,t) + \lambda\mu e^{-\lambda x}\int_0^x p(y,t)G^J(y,t)e^{\lambda y}\,dy, \\ G(x,t) &= \int_x^{+\infty} p(y,t)\,dy, \\ p(x,0) &= p_0(x), \\ 0 \leq x \leq x_0, & t \geq 0. \end{cases} \tag{6.6}$$

Proof. First we shall prove that (6.6) has a unique solution in the rectangle

$$R = \{(x,t) : 0 \le x \le x_0,\ 0 \le t \le t_0\},$$

for some $t_0 > 0$.

Let us consider the metric space X of consisting of continuous functions $p(x,t)$, $(x,t) \in R$, and the distance function

$$\rho(p,q) = \sup_{\substack{0\le x\le x_0\\ 0\le t\le t_0}} |p(x,t) - q(x,t)|.$$

Consider the operator A defined by

$$A(p)(x,t) = p_0(x) + \mu \int_0^t \Big(- G^J(x,\tau)p(x,\tau) + \lambda e^{-\lambda x} \int_0^x p(y,\tau)G^J(y,\tau)e^{\lambda y}\, dy\Big)\, d\tau.$$

We shall show that A is a contraction operator on X for some $t_0 > 0$. As in Section 3, let us assume the existence of a positive constant K such that $p_0(x) \le K$ for all x. Using (6.1), it is then not difficult to see that

$$\frac{\partial p(x,t)}{\partial t} \le \lambda\mu\ ,$$

and so

$$p(x,t) \le K + \lambda\mu t. \tag{6.7}$$

Denote

$$H(x,t) = 1 - \int_0^x q(y,t)\, dy.$$

Then we have

$$\begin{aligned}\rho(A(p),A(q)) = \mu \sup_{\substack{0\le x\le x_0\\ 0\le t\le t_0}} \Big| \int_0^t \Big(& H^J(x,\tau)q(x,\tau) - G^J(x,\tau)p(x,\tau) \\ & + \lambda \int_0^x e^{-\lambda(x-u)} \left(G^J(u,\tau)p(u,\tau) - H^J(u,\tau)q(u,\tau)\right) du\Big) d\tau \Big| \\ & \le \mu t_0(1+\lambda x_0)((K+\lambda\mu t_0)(J+1)x_0 + 1)\rho(p,q),\end{aligned}$$

where we have used the estimate

$$\begin{aligned} & \left|H^J(x,t)q(x,t) - G^J(x,t)p(x,t)\right| \\ \le\ & \left|H^J(x,t) - G^J(x,t)\right| q(x,t) + G^J(x,t)\,|q(x,t) - p(x,t)| \\ \le\ & ((K+\lambda\mu t_0)(J+1)x_0 + 1)\,|p(x,t) - q(x,t)|\,,\end{aligned}$$

which follows from (6.7) and

$$|H^{J+1}(x,t) - G^{J+1}(x,t)| \leq (J+1)|H(x,t) - G(x,t)|.$$

Choosing t_0 such that

$$\mu t_0(1+\lambda x_0)((K+\lambda\mu t_0)(J+1)x_0+1) < 1,$$

we obtain that $\rho(A(p), A(q)) \leq \theta\rho(p,q)$, for some constant $\theta < 1$. Hence, the operator A has a unique fixed point, and this is precisely the solution of the problem on the set R. Note that our choice of t_0 does not depend on the initial condition $p_0(x)$. So, taking $p(x,t_0)$ as a new initial condition and choosing t_1 in the same way, we can extend our solution to the set $\{(x,t) : 0 \leq x \leq x_0,\ 0 \leq t \leq t_0 + t_1\}$, and so on. As it is not difficult to show that

$$\sum_{i=0}^{+\infty} t_i = +\infty,$$

the existence and uniqueness of the solution of (6.1) immediately follow. □

Next we formulate and prove a supplementary assertion.

Lemma 6.2. *Suppose that we have a sequence of functions $p_N(x,t)$ with $p_N(x,0) = p_0(x)$ and*

$$\left|\frac{\partial p(x,t)}{\partial t} - \mu\Big(-G^J(x,t)p(x,t) + \lambda\int\limits_0^x G^J(u,t)p(u,t)e^{-\lambda(x-u)}du\Big)\right| \leq \varphi_N \quad (6.8)$$

for some $\varphi_N \equiv \varphi_N(x,t) > 0$ with the following property: for any x_0 there exists $t_0 > 0$ such that

$$\rho_N = \sup_{\substack{0\leq x\leq x_0\\ 0\leq t\leq t_0}} |\varphi_N(x,t)| \to 0, \quad N \to \infty$$

Let $p(x,t)$ be the unique solution of (6.1). Then

$$p_N(x,t) \to p(x,t), \quad N \to \infty, \qquad (6.9)$$

uniformly on any compact set.

Proof. Fix arbitrary x_0 and t_0. Denote

$$r_N(t) = \sup_{0\leq x\leq x_0} |p_N(x,t) - p(x,t)|.$$

Clearly, $r_N(0) = 0$. It is not difficult to prove (by integrating (6.1) and (6.8) from 0 to t) that there exist positive constants C_1 and C_2, not depending on N, such that

$$r_N(t) \leq C_1\rho_N + C_2 \int_0^t r_N(\tau)\,d\tau.$$

Using Gronwall's inequality, we get

$$r_N(t) \leq C_1\rho_N e^{C_2 t}$$

and letting $N \to \infty$, we obtain (6.9). □

Remark 6.1. In the general situation, i.e. when the density $p(x,t)$ may be singular, propositions resembling Theorem 2.3 and Lemma 6.2 can be proved for the function

$$G(x,t) = 1 - \int_0^x p(y,t)\,dy.$$

The proof is analogous to the proofs of Theorem 2.3 and Lemma 6.2, except for some non-important details.

7. Some properties of the system

Consider the random variable η^t with density $p_(x,t)$. In this section we shall calculate the expectation of η^t, and we will obtain bounds on the variance $D(t)$ and the second moment $b(t)$ of η^t.

Clearly, for the proof of Theorem 2.4 it is sufficient to prove the following lemma.

Lemma 7.1. *The following equality holds*

$$\frac{dm(t)}{dt} = \frac{\mu}{\lambda(J+1)}, \tag{7.1}$$

where

$$m(t) = \mathbf{E}\eta^t = \int_0^{+\infty} xp(x,t)\,dx \tag{7.2}$$

is the expectation of η^t.

Proof. Multiplying (2.2) by x and taking the integral from 0 to $+\infty$, we get

$$\frac{dm(t)}{dt} = -\mu \int_0^{+\infty} xG^J p\,dx + \lambda\mu \int_0^{+\infty} xe^{-\lambda x} dx \int_0^x pG^J e^{\lambda y}\,dy. \tag{7.3}$$

Changing the integration order in the second term in the right-hand side of (7.3) we obtain

$$\begin{aligned}\frac{dm(t)}{dt} &= -\mu \int_0^{+\infty} xG^J p\,dx + \lambda\mu \int_0^{+\infty} p(y,t)G^J(y,t)e^{\lambda y}dy \int_y^{+\infty} xe^{-\lambda x}dx \\ &= -\mu \int_0^{+\infty} xG^J p\,dx + \mu \int_0^{+\infty} yG^J p\,dy + \frac{\mu}{\lambda}\int_0^{+\infty} pG^J dx \\ &= \frac{\mu}{\lambda(J+1)} \int_0^{+\infty} d(-G^{J+1}) = \frac{\mu}{\lambda(J+1)},\end{aligned}$$

so that (7.1) holds. This proves Theorem 2.4. □

Next we shall derive bounds for the second moment and the variance of η^t. Let us write

$$b(t) = \mathbf{E}(\eta^t)^2 = \int_0^{+\infty} x^2 p(x,t)dx$$

and $D(t) = b(t) - m^2(t)$. Let $\eta_1^t, \ldots, \eta_{J+1}^t$ be independent, identically distributed random variables with density $p(x,t)$, and let

$$\rho(t) = \mathbf{E}\min\{\eta_1^t, \ldots, \eta_{J+1}^t\}. \tag{7.4}$$

We will prove the following lemma.

Lemma 7.2. *Under the above conditions $b(t)$ satisfies the following differential equation*

$$\frac{db(t)}{dt} = \frac{2\mu}{\lambda^2(J+1)} + \frac{2\mu}{\lambda(J+1)}\rho(t). \tag{7.5}$$

Proof. Similarly to the proof of Theorem 7.1 we can write

$$\begin{aligned}\frac{db(t)}{dt} &= -\mu \int_0^{+\infty} x^2G^J p\,dx + \lambda\mu \int_0^{+\infty} p(y,t)G^J(y,t)e^{\lambda y}dy \int_y^{+\infty} x^2e^{-\lambda x}dx \\ &= -\mu \int_0^{+\infty} x^2G^J p\,dx + \mu \int_0^{+\infty} y^2G^J p\,dy + \frac{2\mu}{\lambda}\int_0^{+\infty} yG^J p\,dy + \frac{2\mu}{\lambda^2}\int_0^{+\infty} G^J p\,dy \\ &= \frac{2\mu}{\lambda}\int_0^{+\infty} yG^J p\,dy + \frac{2\mu}{\lambda^2(J+1)}.\end{aligned}$$

However,

$$\begin{aligned}
\int_0^{+\infty} yG^J p\,dy &= \frac{1}{J+1}\int_0^{+\infty} y\,d(1-G^{J+1}) \\
&= \frac{1}{J+1}\int_0^{+\infty} y\,d\mathbf{P}\{\min\{\eta_1^t,\dots,\eta_{J+1}^t\}<y\} \\
&= \frac{1}{J+1}\mathbf{E}\min\{\eta_1^t,\dots,\eta_{J+1}^t\}.
\end{aligned}$$

Hence,

$$\frac{db(t)}{dt} = \frac{2\mu}{\lambda^2(J+1)} + \frac{2\mu}{\lambda(J+1)}\rho(t),$$

thus proving Lemma 7.2. □

Theorem 7.1. *The variance $D(t)$ satisfies the following inequality*

$$D(t) \le \frac{2\mu t}{\lambda^2(J+1)}. \tag{7.6}$$

Proof. Evidently $\mathbf{E}\min\{\eta_1^t,\dots,\eta_{J+1}^t\} \le \mathbf{E}\eta^t$. Hence

$$\frac{db(t)}{dt} \le \frac{2\mu}{\lambda^2(J+1)} + \frac{2\mu}{\lambda(J+1)}m(t) = \frac{2\mu}{\lambda^2(J+1)} + \frac{2\mu^2 t}{\lambda^2(J+1)^2},$$

and consequently

$$b(t) \le \frac{2\mu t}{\lambda^2(J+1)} + \frac{\mu^2 t^2}{\lambda^2(J+1)^2}.$$

Then (7.6) follows, since $D(t) = b(t) - m^2(t)$, and so Theorem 7.1 is proved. □

As a consequence, the deviation of η^t from its mean value is not larger than $O(\sqrt{t})$.

8. The case $v \neq 0$

Here we briefly consider the case that $v \neq 0$. Assume that λ, μ, J are fixed, and let N tend to infinity. Let us denote by $p(x,t)$ the density of the first particle and let $\tilde{p}(x,t)$ be defined by

$$\tilde{p}(x,t)\,dx = \lim_{N\to\infty} \frac{\#_t[x,x+dx]}{N}. \tag{8.1}$$

In this case $p(x,t)$ can be shown to satisfy

$$p(x,t) = \alpha_t\delta(x) + (1-\alpha_t)p_0(x,t), \tag{8.2}$$

where $p_0(x,t)$ is a regular density.

Similarly to the case $v=0$, we can prove that $p(x,t)=\tilde{p}(x,t)$ as $N\to\infty$, and that the following equation holds

$$\begin{aligned}
&-\frac{d\alpha_t}{dt}p_0(x,t)-v(1-\alpha_t)\frac{\partial p_0(x,t)}{\partial x}+(1-\alpha_t)\frac{\partial p_0(x,t)}{\partial t}\\
&= \quad -\mu(1-\alpha_t)^{J+1}G^J(x,t)p_0(x,t) \qquad (8.3)\\
&+ \quad \mu\lambda e^{-\lambda x}(1-\alpha_t)^{J+1}\int_0^x G^J(y,t)p_0(y,t)e^{\lambda y}\,dy+\lambda\mu\alpha_t e^{-\lambda x},
\end{aligned}$$

where

$$G(x,t)=\int_x^{+\infty} p_0(x,t)\,dx.$$

An immediate consequence of this is the following corollary.

Corollary 8.1. *The function α_t satisfies the following differential equation*

$$\frac{d\alpha_t}{dt}=-\mu\alpha_t+v(1-\alpha_t)p_0(0,t). \qquad (8.4)$$

Proof. Equation (8.4) follows by integrating (8.3) from 0 to ∞. □

Denote $v_{\text{cr}}=\mu/(\lambda(J+1))$. We have seen before, that the whole process goes off to infinity with velocity v_{cr}, if $v=0$. It is therefore plausible that the whole system goes to infinity with speed $v_{\text{cr}}-v$, if $v<v_{\text{cr}}$, and that there exists a stationary regime, i.e. $p(x,t)\equiv p(x)$, if $v>v_{\text{cr}}$.

For the stationary regime we have

$$\begin{aligned}
-v(1-\alpha)\frac{\partial p_0(x)}{\partial x}&=-\mu(1-\alpha)^{J+1}G^J(x)p_0(x)\\
&+\mu\lambda e^{-\lambda x}(1-\alpha)^{J+1}\int_0^x G^J(y)p_0(y)e^{\lambda y}\,dy+\lambda\mu\alpha e^{-\lambda x}, \qquad (8.5)
\end{aligned}$$

since $\alpha_t\equiv\alpha$.

Let us denote

$$m_0(t)=\int_0^{+\infty} xp_0(x,t)\,dx \quad \text{and} \quad m(t)=\int_0^{+\infty} xp(x,t)\,dx.$$

We shall calculate α and $m\equiv m(t)$ for the stationary regime.

Theorem 8.1. *For the stationary regime, $\alpha\equiv\alpha_t$ is the unique solution of the following equation*

$$v(1-\alpha)=\frac{\mu(1-\alpha)^{J+1}}{\lambda(J+1)}+\frac{\mu}{\lambda}\alpha. \qquad (8.6)$$

Proof. Multiplying (8.5) by x, integrating from 0 to $+\infty$ and subsequently changing the integration order in the second term of the right-hand side, we get (8.6). We have to prove the existence of a solution of (8.6). Indeed, denote

$$f(\alpha) = -v(1-\alpha) + \frac{\mu(1-\alpha)^{J+1}}{\lambda(J+1)} + \frac{\mu}{\lambda}\alpha.$$

Then

$$f(0) = \frac{\mu}{\lambda(J+1)} - v < 0, \qquad f(1) = \frac{\mu}{\lambda} > 0,$$

and one easily gets, that $f'(\alpha) \geq v > 0$. Hence, (8.6) has a unique solution and so the proof of Theorem 8.1 is finished. □

Theorem 8.2. *For the stationary regime, $m \equiv m(t)$ satisfies the following inequality*

$$m \leq \frac{1}{2(v - v_{\mathrm{cr}})}\Big(2\mu\frac{(1-\alpha)^{J+1}}{\lambda^2(J+1)} + \mu\alpha\Big(\frac{1}{\lambda} + \frac{1}{\lambda^2}\Big)\Big). \tag{8.7}$$

Proof. We obtain this by multiplying (8.5) by x^2, taking the integral from 0 to $+\infty$, and then simplifying the right-hand side similarly to the case $v = 0$ in the calculation of $b(t)$. □

Remark 8.1. Generally speaking, for the particle there are two possible rules for making the decision to jump or not to jump. We have assumed that the ith particle jumps if $z_i(t) \leq z_j(t)$, but here we may substitute $\leq$-sign by $<$-sign. In the case $v > v_{\mathrm{cr}}$ these two situations differ essentially, because $\alpha_t \not\to 0$. The case "$\leq$" we have considered before. For the case "$<$" one can write down a similar equation as (8.3), and it seems likely that $\alpha_t \to 1$ as $t \to +\infty$, if $v > v_{\mathrm{cr}}$.

References

[1] B.D. Lubachevsky (1988) Efficient distributed event-driven simulations of multiple-loop networks. In: *ACM Sigmetrics conference on measurement and modeling of computer systems*, 12–21.

[2] B.D. Lubachevsky (1987) Efficient parallel simulations of asynchronous cellular arrays. *Complex Syst.* **1**, 1099–1123.

[3] A.G. Greenberg, B.D. Lubachevsky, D.M. Nikol and P.E. Wright (1994) Efficient massively parallel simulation of dynamic channel assignment schemes for wireless cellular communications. In: *Proc. of the 8th Workshop on Parallel and Distributed Simulations (PADS' 94).*

[4] A.G. Greenberg, S. Shenker, A.L. Stolyar (1996) Asynchronous updates in large parallel systems. To appear in *Proc. of the 1996 ACM SIGMETRICS Conf.*, Philadelphia, PA, USA.

[5] R.M. Fugimoto (1990) Parallel discrete event simulation. *Commun. ACM* **16** (9), 31–53.

[6] R.M. Fugimoto and D.M. Nicol (1994) Parallel simulation today. *Ann. Oper. Res.*