On Second-Order Optimality Conditions in Nonlinear Optimization

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Abstract

In this work we present new weak conditions that ensure the validity of necessary second-order optimality conditions (SOC) for nonlinear optimization. We are able to prove that weak and strong SOCs hold for all Lagrange multipliers using Abadie-type assumptions. We also prove weak and strong SOCs for at least one Lagrange multiplier imposing the Mangasarian-Fromovitz constraint qualification and a weak constant rank assumption.

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1 Introduction

Optimality conditions play a central role in the study and solution of nonlinear optimization problems. Among them, the KKT conditions are arguably the most celebrated, ensuring first-order stationarity [9, 10, 13, 17, 18, 21]. Their main objective is to assert that there is no descent direction for the objective that remains feasible up to first-order. Second-order conditions try to complete this picture, guaranteeing that the directions that are not of ascent nature are not directions of negative curvature either. This paper studies conditions that ensure the validity of second-order conditions at local minimizers, *i.e.* we are interested in situations where the second-order conditions are necessary.

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Given a local minimizer x^*, the definition of the second-order conditions starts with the identification of a cone of critical directions for which the first-order information is not enough to assert optimality. This cone is called the (strong) critical cone and denoted $\mathrm{T}(x^*)$. In fact, $d \notin \mathrm{T}(x^*)$ if it either is a direction of ascent for the objective or it is a direction that leads directly to infeasible points. See details in Definition 2.1 below. The (strong) second-order condition (SOC) then states that these (critical) directions are not, up to the second-order, descent directions of the Lagrangian $L(\cdot, \lambda, \mu)$ starting from x^*. In other words, the second-order condition states that x^* looks like a local minimizer, up to the second-order, of the Lagrangian with fixed multipliers in all directions of the critical cone. There is also a weak version of the second-order necessary condition that appears naturally in the context of analysis of algorithms [16, 3, 14]. See again Definition 2.1.

These SOCs are stated using multipliers (λ, μ), that form together with x^* a KKT triple. Hence, they depend on the validity of KKT at x^*. This in turn can be guaranteed by a (first-order) constraint qualification. The first, and still most used, constraint qualification is regularity, which states that the gradients of the active constraints are linearly independent at x^*. Even though it is quite restrictive, regularity is still widely used due to its simplicity and special properties, like the uniqueness of the multiplier.

There are many more first-order constraint qualifications in the literature, two of which play an important role in this work. Mangasarian-Fromovitz constraint qualification (MFCQ) is an extension of regularity that is better suited for inequality constraints [19]. It asks that the gradients of the active constraints must be positively linearly independent, with positive multipliers associated with the inequalities [22]. Another important and very general constraint qualification was introduced by Abadie [1]. It states that the cone associated to the linearized constraints coincides with the (geometrical) tangent cone to the feasible set.

In the context of second-order conditions, the usual constraint qualification is regularity. One of its advantages is that it ensures the existence of a unique multiplier, which simplifies the definition of SOCs. In fact, most nonlinear optimization books only define second-order conditions under this assumption [9, 10, 13, 17, 18, 21]. A natural question that arises is what conditions the constraints must satisfy to ensure the validity of a second-order necessary condition. The main objective would be to find conditions that are less stringent than regularity.

A counter-example by Arutyunov, later rediscovered by Anitescu, shows that the natural extension of regularity, Mangasarian-Fromovitz constraint qualification, does not imply either the strong or the weak second-order optimality condition [6, 5]. The research on SOCs has since been performed under two main lines of reasoning: imposing constant rank assumptions and proving that strong SOC holds for every Lagrange multiplier [2, 4, 20], or imposing MFCQ and some additional condition to show that there exists at least one Lagrange multiplier for which strong SOC holds [7, 8].

Another line of research on second-order optimality conditions deals with necessary conditions, without constraint qualifications, that can be made sufficient simply by requiring strict positive semidefiniteness on the same critical cone. Those conditions are typically based on Fritz-John multipliers, where the objective function may be ignored. They also depend on the whole set of Lagrange multipliers, since a verification of positive semidefiniteness should be performed on the maximum of a quadratic form over the set of multipliers. See, for instance, [11].

However it is usually difficult to compute the full set of multipliers. More important, numerical algorithms can only guarantee to approximate a single minimizer-multiplier pair. Therefore, they can only try to enforce the validity of a second-order condition if it holds with a single multiplier. The main objective of this work is then to present new, weaker, conditions that can assert the validity of SOCs with a single multiplier. We prove first that if Abadie CQ holds for a subsystem of the constraints viewed as equalities, a condition weaker than the usual constant rank assumptions, then strong SOC holds for all Lagrange multipliers. This generalizes a result from [2]. As a consequence, we prove that if only equality constraints are present, Abadie CQ is sufficient to ensure strong SOC for all multipliers. As for systems that conform to MFCQ, we show that if a generalized complementarity condition plus a new constant rank condition hold then strong SOC can be asserted for at least one multiplier, this improves a result from [8]. Finally, we also show that the weak SOC is valid for all multipliers whenever Abadie CQ holds for the full set of active constraints, considered as a system of equalities.

The rest of this paper is organized as follows: Section 2 presents the formal definition of the second-order conditions. Section 3 presents definitions and results concerning second-order under Abadie-type assumptions. In Section 4 we present the results under MFCQ. Section 5 presents some concluding remarks.

2 Basic definitions

Let us introduce the second-order optimality conditions below. We start by formally defining the problem of interest.

$$\begin{aligned} \min\ & f_0(x), \\ \text{s.t. } & f_i(x) = 0,\ i = 1, \ldots, m, \\ & f_j(x) \leq 0,\ j = m+1, \ldots, m+p, \end{aligned} \tag{1}$$

where $f_\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell = 0, \ldots, m+p$ are twice continuously differentiable. If x is feasible, we denote as $\mathcal{A}(x)$ the index set of active inequalities at x and as $\mathcal{I}$ the index set of equality constraints. All the equality constraints are, naturally, also said to be active at x. We also use the convention $g_\ell = \nabla f_\ell(x^*)$ and $H_\ell = \nabla^2 f_\ell(x^*)$, for $\ell = 0, \ldots, m+p$, where x^* is a particular feasible point of interest. Finally, given a pair $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$, the function $L(\cdot, \lambda, \mu)$ given by

$$L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=m+1}^{m+p} \mu_j f_j(x)$$

is called the Lagrangian associated to (1).

Now we can state formally the second-order conditions analysed in this paper:

Definition 2.1. *Assume that* $(x^*, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p_+$ *is a KKT triple. The cone*

$$\mathrm{T}(x^*) := \left\{ d \in \mathbb{R}^n \mid g_0' d \leq 0;\ g_i' d = 0,\ i \in \mathcal{I};\ g_j' d \leq 0,\ j \in \mathcal{A}(x^*) \right\}$$

is called the (strong) critical cone *at* x^* *while the smaller cone*

$$\tau(x^*) := \left\{ d \in \mathbb{R}^n \mid g_0' d = 0;\ g_i' d = 0,\ i \in \mathcal{I};\ g_j' d = 0,\ j \in \mathcal{A}(x^*) \right\},$$

is called the weak critical cone *at* x^**.*

The (strong) second-order optimality condition (SSOC) holds at x^* *with multiplier* (λ, μ) *if*

$$\forall d \in \mathrm{T}(x^*),\ d' \left(H_0 + \sum_{i \in \mathcal{I}} \lambda_i H_i + \sum_{j \in \mathcal{A}(x^*)} \mu_j H_j \right) d \geq 0.$$

Similarly, the weak second-order optimality condition (WSOC) holds at x^* *with multiplier* (λ, μ) *if*

$$\forall d \in \tau(x^*),\ d' \left(H_0 + \sum_{i \in \mathcal{I}} \lambda_i H_i + \sum_{j \in \mathcal{A}(x^*)} \mu_j H_j \right) d \geq 0.$$

Observe that the matrix that appears in both conditions above is exactly the Hessian, with respect to x, of the Lagrangian at x^*. Moreover, it is well known that if the strict complementarity holds, *i.e.*, if there exists a multiplier that is strictly positive for all active inequality constraints, then the strong and weak cones are the same and hence both the strong and weak second-order condition are equivalent [3].

3 Abadie-type Conditions

Recently, assumptions based on constant rank that have been used to ensure the validity of second-order conditions for every Lagrange multiplier [2, 4, 20]. In this section we show that such conditions can be naturally replaced by a much weaker condition based on Abadie's CQ. The results are rather simple once we identify which is the correct set of the constraints that must be taken into account.

Let us start this by showing that constant rank implies a weaker constraint qualification for system of equalities and that in turn implies Abadie's condition.

Definition 3.1. *Let* $\Omega = \{x \mid h_i(x) = 0,\ i = 1, \ldots, m'\} \subset \mathbb{R}^n$ *be a system of continuously differentiable equalities such that* $x^* \in \Omega$*. The* Kuhn-Tucker constraint qualification (KTCQ) *holds for* Ω *at* x^* *if, for each* $d \in \mathbb{R}^n$ *where* $\nabla h_i(x^*)'d = 0,\ i = 1, \dots, m'$, *there exists* $T > 0$ *and a differentiable curve* $\alpha : (-T, T) \to \mathbb{R}^n$ *such that*

1. $\alpha(0) = x^*$, $\dot\alpha(0) = d$.
2. $h_i(\alpha(t)) = 0, \forall t \in (-T, T)$ *and* $i = 1, \dots, m'$.

If this curve is also twice continuously differentiable at 0 *we say that* $\mathcal{C}^2$*-KTCQ holds.*

Definition 3.2. *Given a feasible point* x^* *of Problem (1), we define the* Tangent Cone at x^* *as the set of directions* $d \in \mathbb{R}^n$ *such that* $d = 0$ *or* $d = \lim \frac{x^k - x^*}{\|x^k - x^*\|}$ *for some feasible sequence* $x^k \to x^*$. *We say that* x^* *fulfills the* Abadie constraint qualification *when the Tangent Cone at* x^* *is equal to the set of linearized directions, that is the set of directions* $d \in \mathbb{R}^n$ *such that* $g_i'd = 0,\ i \in \mathcal{I};\ g_j'd \le 0,\ j \in \mathcal{A}(x^*)$.

Now we can present the relation with constant rank conditions.

Lemma 3.1. *Consider* Ω *and* x^* *as in Definition 3.1. If the gradients* $\{\nabla h_i(x),\ i = 1, \dots, m'\}$ *have constant rank around* x^*, *then* $\mathcal{C}^2$*-KTCQ holds at* x^*. *In particular Abadie's CQ with respect to* Ω *holds at* x^*.

Proof. It suffices to follow the proof of Bazaraa [9, Theorem 4.3.3]. In particular, let us define the differential equation

$$\dot\alpha(t) = P(\alpha)d, \qquad \alpha(0) = x^*,$$

where $P(x)$ is the matrix that projects onto the subspace orthogonal to $\{\nabla h_i(x),\ i = 1, \dots, m'\}$. Peano's Theorem says that this system must have a solution, since all data is continuous. It is easy then to check that this solution has the properties 1 and 2 from Definition 3.1. Moreover, the solution is twice continuously differentiable because the matrix function $P(x)$ is differentiable under the constant rank assumption [15]. $\square$

We move on to the second-order results. In order to do so let us introduce a technical lemma that will be the key in the proofs.

Lemma 3.2. *Let* $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ *be a multiplier pair associated to a local minimizer* x^* *and a non-zero* $d \in \mathbb{R}^n$. *If there is a feasible sequence* x^k *such that*

$$\frac{x^k - x^*}{\|x^k - x^*\|} \to \frac{d}{\|d\|}$$

and such that for $\ell = 1, \dots, m+p$, *either* $f_\ell(x^k) = o(\|x^k - x^*\|^2)$ *or the respective multiplier is zero, then*

$$d' \left(H_0 + \sum_{i \in \mathcal{I}} \lambda_i H_i + \sum_{j \in \mathcal{A}(x^*)} \mu_j H_j \right) d \ge 0.$$

Proof. First, observe that the complementarity assumption between $f_\ell(x^k)$ and the respective multiplier implies that

$$L(x^k, \lambda, \mu) = f_0(x^k) + o(\|x^k - x^*\|^2).$$

Therefore, we can use the minimality of x^* to see that for large k

$$\begin{aligned}
0 &\le f_0(x^k) - f_0(x^*) \\
&= L(x^k, \lambda, \mu) - L(x^*, \lambda, \mu) + o(\|x^k - x^*\|^2) \\
&= \nabla_x L(x^*, \lambda, \mu)'(x^k - x^*) + \frac{1}{2}(x^k - x^*)'\nabla^2_{xx} L(\bar{x}^k, \lambda, \mu)(x^k - x^*) + o(\|x^k - x^*\|^2) \\
&= \frac{1}{2}(x^k - x^*)'\nabla^2_{xx} L(\bar{x}^k, \lambda, \mu)(x^k - x^*) + o(\|x^k - x^*\|^2),
\end{aligned}$$

where $\bar{x}^k$ belongs to the segment joining x^* and x^k and the last equality follows from the fact that $\nabla_x L(x^*, \lambda, \mu) = 0$.

Dividing the inequality above by $\|x^k - x^*\|^2$ and taking limits in k, it follows that

$$d'\nabla^2_{xx} L(x^*, \lambda, \mu) d \ge 0.$$

□

We can now present the first second-order result: a simple condition that ensures that the weak second-order condition holds at x^*.

Theorem 3.1. *Let x^* be a local minimizer of* (1) *associated to Lagrange multipliers* $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$*. If the system*

$$f_\ell(x) = 0, \;\; \ell \in \mathcal{I} \cup \mathcal{A}(x^*)$$

conforms to Abadie's constraint qualification at x^*, then the weak second-order optimality condition holds with multiplier (λ, μ).

Proof. First observe that $\tau(x^*)$ is just the cone of linearised feasible directions associated to the system of equalities above. Hence, Abadie's condition states that for any non zero $d \in \tau(x^*)$ there is $x^k \to x^*$ that conforms to all equalities and such that

$$\frac{x^k - x^*}{\|x^k - x^*\|} \to \frac{d}{\|d\|}.$$

The result follows now directly from Lemma 3.2. □

This result is a clear generalization of [2, Theorem 3.2], since Lemma 3.1 shows that the weak constant rank condition implies Abadie's CQ, for the relevant system of equalities.

An immediate corollary, which in turn is a generalization of [2, Theorem 3.3], is:

Corollary 3.1. *Consider the case where the minimization problem* (1) *only has equality constraints. Let x^* be a local minimizer of this problem where Abadie's constraint qualification holds. Then, x^* conforms to the KKT conditions and the (strong) second-order optimality condition holds for every Lagrange multiplier.*

Proof. Since there are no inequalities, $\mathcal{A}(x^*) = \emptyset$ and the Abadie assumption of the previous result applies to the original feasible set. Moreover, in the absence of inequalities, the strong and weak critical cones are clearly the same. □

Note the result from the corollary was already known, see the discussion in the end of Chapter 5 of [9]. We will revisit this discussion below.

For now, let us turn our attention to the (strong) second-order optimality condition in the presence of inequalities. Once again, the main assumption will be related to Abadie's CQ for a special subset of the constraints when viewed as equalities. To identify such constraints we introduce some notations and prove a few auxiliary results.

Definition 3.3. *The index set of positive inequality multipliers at x^*, denoted $\mathcal{A}^+(x^*)$, is the set of indexes $j \in \mathcal{A}(x^*)$ for which there exists $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ such that (x^*, λ, μ) is a KKT triple and $\mu_j > 0$. We will denote $\mathcal{A}^0(x^*) = \mathcal{A}(x^*) \setminus \mathcal{A}^+(x^*)$.*

We already know that for $d \in \mathrm{T}(x^*)$ and $j \in \mathcal{A}^+(x^*)$ the inequality appearing in the definition of the critical cone holds as an equality [2]. Hence, this cone can be rewritten as

$$\mathrm{T}(x^*) = \left\{ d \in \mathbb{R}^n \mid g_\ell' d = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*),\ g_j' d \le 0,\ j \in \mathcal{A}^0(x^*) \right\} \tag{2}$$

where the objective function gradient can be omitted because we already assumed that x^* is a KKT point.

Using this fact we can present an interesting characterization of the index set $\mathcal{A}^0(x^*)$.

Lemma 3.3.

$$\mathcal{A}^0(x^*) = \{ j \in \mathcal{A}(x^*) \mid \exists d \in \mathrm{T}(x^*) \text{ s.t. } g_j' d < 0 \}.$$

Proof. From (2) we already know that

$$\mathcal{A}^0(x^*) \supset \{ j \in \mathcal{A}(x^*) \mid \exists d \in \mathrm{T}(x^*) \text{ s.t. } g_j' d < 0 \}.$$

On the other hand, we know that $j \in \mathcal{A}^0(x^*)$ if and only if the linear problem

$$\begin{aligned} \max_{\lambda,\mu} \quad & \mu_j, \\ \text{s.t.} \quad & \sum_{i \in \mathcal{I}} \lambda_i g_i + \sum_{k \in \mathcal{A}(x^*)} \mu_k g_k = -g_0, \\ & \mu_k \ge 0,\ k \in \mathcal{A}(x^*) \end{aligned}$$

has optimal value 0. Hence 0 is also the optimal value of the dual problem

$$\begin{aligned} \min_{d} \quad & g_0' d, \\ \text{s.t.} \quad & g_i' d = 0, \;\; i \in \mathcal{I}, \\ & g_k' d \leq 0, \;\; k \in \mathcal{A}(x^*) \setminus \{j\}, \\ & g_j' d \leq -1. \end{aligned}$$

In particular, the system

$$\begin{aligned} & g_0' d \leq 0, \\ & g_i' d = 0, \;\; i \in \mathcal{I}, \\ & g_k' d \leq 0, \;\; k \in \mathcal{A}(x^*) \setminus \{j\} \\ & g_j' d \leq -1. \end{aligned}$$

has a solution, that is, $j \in \{j \in \mathcal{A}(x^*) \mid \exists d \in \mathrm{T}(x^*) \text{ s.t. } g_j' d < 0\}$. □

Corollary 3.2. *There is* $h \in \mathrm{T}(x^*)$ *s.t.*

$$\begin{aligned} & g_i' h = 0, \;\; i \in \mathcal{I} \cup \mathcal{A}^+(x^*), \\ & g_j' h < 0, \;\; j \in \mathcal{A}^0(x^*). \end{aligned}$$

Proof. As $\mathrm{T}(x^*)$ is a convex cone, it is closed by addition. Hence, it is sufficient to add the vectors given by Lemma 3.3 for each $j \in \mathcal{A}^0(x^*)$. □

Finally we can present the new condition for the validity of the (strong) second-order condition. It is a direct generalization of [2, Theorem 3.1] and [20, Theorem 6], where we clearly identify the set of gradients that need to be well behaved instead of looking at all the subsets that involve active inequalities.

Theorem 3.2. *Let* x^* *be a local minimizer of* (1) *associated to Lagrange multipliers* $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$. *If the Tangent cone of*

$$F_+ := \{x \mid f_\ell(x) = 0, \;\; \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)\}$$

at x^* *contains the critical cone* $\mathrm{T}(x^*)$*, then the (strong) second-order optimality condition holds at* x^* *with multiplier* (λ, μ).

Proof. Let d be any non-zero direction in $\mathrm{T}(x^*)$, and consider without loss of generality that $\|d\| = 1$. Let h be as in Corollary 3.2. For any $k = 1, 2, \ldots$, define

$$d^k := \frac{d + (1/k)h}{\|d + (1/k)h\|}.$$

It follows that $g_\ell' d^k = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$, $g_j' d^k < 0$, for all $j \in \mathcal{A}^0(x^*)$, $\|d^k\| = 1$, and $d^k \to d$. In particulr $d^k \in \mathrm{T}(x^*)$.

The assumption of the theorem implies then that d^k belongs to the tangent cone of the system of equalities at x^*. That is, there must be $x^l \to x^*$, with $f_\ell(x^l) = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$, such that

$$\frac{x^l - x^*}{\|x^l - x^*\|} \to_l d^k.$$

We show now that x^l is feasible in (1) for l large enough. In fact, for $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$ the constraints hold as equalities. For $j \in \mathcal{A}^0(x^*)$ we get

$$f_j(x^l) = f_j(x^*) + \nabla f_j(\bar{x}^l)'(x^l - x^*) = \nabla f_j(\bar{x}^l)'(x^l - x^*),$$

for some $\bar{x}^l$ in the line segment joining x^* and x^l. Then, $\nabla f_j(\bar{x}^l) \to_l g_j$. Since $(x^l - x^*)/\|x^l - x^*\| \to_l d^k$ and $g_j' d^k < 0$, it follows that, for l large enough, $f_j(x^l) < 0$. Finally, continuity of the constraints imply that all inactive constraints hold in x^l for large l.

Since $f_\ell(x^l) = 0$, for all $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$, Lemma 3.2 shows then that

$$(d^k)' \nabla^2_{xx} L(x^*, \lambda, \mu) d^k \geq 0.$$

The result follows taking limits in k. □

Theorem 3.2 above may be seen as a variation of the results described in the discussion of Chapter 5 of [9]. There, the authors define the following second-order constraint qualification.

Theorem 3.3. *(Bazaraa, Sherali, and Shetty [9]) Let x^* be a local minimizer of* (1) *and $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ an associated Lagrange multiplier pair. Let $\mathcal{A}^+_\mu := \{j \in \mathcal{A}(x^*) \mid \mu_j > 0\}$ and $A^0_\mu := \mathcal{A}(x^*) \setminus A^+_u$. If the system*

$$F_\mu := \{x \mid f_\ell(x) = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+_\mu;\ f_j(x) \leq 0,\ j \in \mathcal{A}^0_\mu\} \tag{3}$$

conforms to Abadie's constraint qualification at x^*, then the (strong) second-order optimality condition holds at x^* with multiplier (λ, μ).

Observe that this theorem has a different assumption for each multiplier. Hence it can only ensure an SOC for all multiplier if all the associated systems conform to Abadie's condition.

In order to better understand this result and see the relationship between Theorems 3.3 and 3.2, let us prove two auxiliary lemmas.

Lemma 3.4. *The linearized cones associated to the systems appearing in* (3) *are all the same and coincide with the strong critical cone* $\mathrm{T}(x^*)$*.*

Proof. This is a simple consequence of direct algebraic manipulations of the definitions of the cones and the KKT conditions.

□

This result allows us to interpret the condition from Bazaraa et al. as a family of inclusions indexed by the multiplier pairs. It asserts the validity of SSOC for a specific multiplier pair (λ, μ) whenever

$$\text{Tangent of } F_\mu \supset \mathrm{T}(x^*). \tag{4}$$

It follows immediately that if one of these inclusions holds for (λ, μ), it also holds for all other multiplier pairs $(\tilde{\lambda}, \tilde{\mu})$ where $\mathcal{A}^+_{\tilde{\mu}} \subset \mathcal{A}^+_\mu$. This happens because in this case clearly $F_{\tilde{\mu}} \supset F_\mu$ and this inclusion is inherited by the tangent cones. In particular, if (λ, μ) is a multiplier pair where $\mathcal{A}^+_\mu = \mathcal{A}^+(x^*)$, which always exists since convex combinations of multiplier pairs are multiplier pairs, the strong second-order condition will hold for every multiplier. This fact is summarized in the next theorem.

Theorem 3.4. *Let x^* be a local minimizer of* (1) *and $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ an associated multiplier pair such that $\mathcal{A}^+_\mu = \mathcal{A}^+(x^*)$. If the system*

$$F_\mu = \{x \mid f_\ell(x) = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*);\ f_j(x) \le 0,\ j \in \mathcal{A}^0(x^*)\}$$

conforms to Abadie's constraint qualification at x^*, then the (strong) second-order optimality condition holds at x^* for all multiplier pairs.

Note that the hypothesis of this last result is equivalent to the inclusion (4). Hence, at first sight, Theorem 3.2 may seem to be a generalization of Theorem 3.4, where the critical feasible set F_μ is replaced by the potentially larger set F_+, making the inclusion easier to hold. However, both results are actually equivalent.

Lemma 3.5. *Using the assumptions and notation of Theorems 3.2 and 3.4, then*

$$\textit{Tangent of } F_\mu \supset \mathrm{T}(x^*) \iff \textit{Tangent of } F_+ \supset \mathrm{T}(x^*).$$

Hence, Theorems 3.2 and 3.4 are equivalent.

Proof. It follows directly from the definitions of F_+ and F_μ that $F_+ \supset F_\mu$, hence the direct implication is obvious.

As for the reverse implication, we can follow the proof of Theorem 3.2 to see that given a non-zero $d \in \mathrm{T}(x^*)$, we can find a sequence $d^k \to d$ such that for each k there is a sequence x^l feasible for F_μ where

$$\frac{x^l - x^*}{\|x^l - x^*\|} \to d^k.$$

Hence d^k must also belong to the tangent cone of F_μ. The result follows taking limits in k as tangent cones are closed. □

Actually, the same line of arguments allow us to give a similar variation of Theorem 3.3 where the constraints with index in $\mathcal{A}^0(x^*)$ are omitted. This result encompasses as special cases Theorems 3.2-3.4.

Theorem 3.5. *Let x^* be a local minimizer of* (1) *and* $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ *an associated Lagrange multiplier pair. Let* $\mathcal{A}^+_\mu = \{j \in \mathcal{A}(x^*) \mid \mu_j > 0\}$*. If the tangent cone of*

$$\{x \mid f_\ell(x) = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+_\mu, \quad f_j(x) \leq 0,\ j \in \mathcal{A}^+(x^*) \setminus \mathcal{A}^+_\mu\} \tag{5}$$

at x^* contains the (strong) critical cone $\mathrm{T}(x^*)$*, then the (strong) second-order optimality condition holds at x^* for all multiplier pairs $(\tilde{\lambda}, \tilde{\mu})$ such that* $\mathcal{A}^+_{\tilde{\mu}} \subset \mathcal{A}^+_\mu$*.*

We close this section by showing a simple example where the assumptions of the Theorem above fail for the multipliers with the largest number of strictly positive entries. In particular, Theorems 3.2 and 3.4 can not be applied. However, it is still possible to find a special multiplier for which its assumptions hold and hence where SOC is fulfilled.

Consider the optimization problem

$$\begin{aligned} \min \quad & x_2, \\ \text{s.t.} \quad & -x_1^2 - x_2 \leq 0, \\ & -x_2 \leq 0, \\ & x_1 \leq 0. \end{aligned} \tag{6}$$

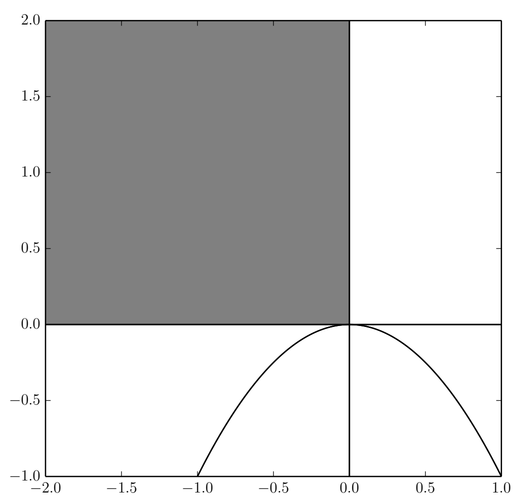

Figure 1: Feasible set of problem (6).

The point $x^* = (0, 0)$ is clearly a solution, which is associated to many possible multipliers. In particular, the multipliers associated to the first two constraints can be strictly positive, while the multiplier associated to the last constraint is always 0. That is, $A^+(x^*) = \{1, 2\}$ and $A^0(x^*) = \{3\}$. The critical cone is only the negative portion of the first axis, $\mathrm{T}(x^*) = \{d \mid d_1 \leq 0, d_2 = 0\}$. If we consider a multiplier where the first two coordinates are not zero, for example $\mu = (1/2, 1/2, 0)$, it follows that the sets F_+, that appears in Theorem 3.2, and F_μ, that appears in Theorems 3.3 and 3.4, coincide and are equal to $\{(0, 0)\}$. Clearly its tangent cone does not contain $\mathrm{T}(x^*)$. On the other hand, if we consider $\tilde{\mu} = (0, 1, 0)$, the set $F_{\tilde{\mu}} = \{x \mid x_1 \leq 0, x_2 = 0\}$, appearing in Theorem 3.3, is exactly $\mathrm{T}(x^*)$. Hence, SSOC holds. The set appearing in Theorem 3.5 is even larger, consisting of the whole first axis.

4 MFCQ-type Conditions

Another approach on the (strong) second-order condition was pioneered by Baccari and Trad [8]. In this paper, the authors show that there is at least one Lagrange multiplier pair such that the second-order condition holds if there is at most one inequality in $\mathcal{A}^0(x^*)$, an assumption called generalized strict complementarity slackness (GSCS), and if a modified version of Mangasarian-Fromovitz constraint qualification holds.

Definition 4.1. *We say that the Modified Mangasarian-Fromovitz (MMF) holds at x^* if MFCQ holds and the rank of the active gradients is deficient of at most one.*

The proof technique is very interesting. They first show that there are two multiplier pairs (λ^1, μ^1) and (λ^2, μ^2) for which

$$\max\left(d'\nabla^2_{xx}L(x^*, \lambda^1, \mu^1)d,\ d'\nabla^2_{xx}L(x^*, \lambda^2, \mu^2)d\right) \geq 0.$$

Then, using the fact that the critical cone $\mathrm{T}(x^*)$ is a first-order cone whenever GSCS holds, it is possible to conclude, using Yuan's Lemma [23, Lemma 2.3] (see also [12]), that there exists at least one multiplier pair for which SSOC holds.

Now, it is simple to see that the GSCS assumption is only used to allow for the use of Yuan's Lemma. However, if one is interested in the weak second-order condition, the cone $\tau(x^*)$ is always a subspace regardless of $\mathcal{A}^0(x^*)$. Hence, Yuan's result can be applied and we can see that:

Corollary 4.1. *Let x^* be a local minimizer of* (1). *If x^* conforms to MMF then WSOC holds for at least one multiplier pair.*

These results are not special cases of the previous second-order results, based only on Abadie's condition for the right set of constraints viewed as equalities. For example consider the problem

$$\begin{aligned} \min \quad & x_2 \\ s.t. \quad & -x_2 \leq 0 \\ & x_1^2 - x_2 \leq 0 \end{aligned}$$

at its global minimizer $(0, 0)$.

However, we can still use the ideas presented in the previous section to extend the corollary above. In particular, we will show that the constraints with indexes in $\mathcal{A}^0(x^*)$ do not play an important role and hence their rank should not be taken into account.

Theorem 4.1. *Let x^* be a local minimizer of (1). Suppose that MFCQ holds at x^* and that all the systems with the form*

$$f_\ell(x) = 0, \;\; \ell \in \mathcal{I}',$$

where $\mathcal{I}' \subset \mathcal{I} \cup \mathcal{A}^+(x^*)$, $\#\mathcal{I}' = \#(\mathcal{I} \cup \mathcal{A}^+(x^*)) - 1$, conform to $\mathcal{C}^2$-KTCQ . Then, WSOC holds at x^* for at least one multiplier pair.

We will prove this result in a series of lemmas below. This proof can also be adapted to give an alternative proof of Baccari and Trad's result.

Lemma 4.1. *Under MFCQ, if the gradients of constraints with index in $\mathcal{I} \cup \mathcal{A}^+(x^*)$ are linearly dependent, then there are two active inequalities j_1 and j_2 such that*

1. *$j_1, j_2 \in \mathcal{A}^+(x^*)$.*
2. *There exists $\gamma_{j_1}, \gamma_{j_2} > 0$ and $\gamma_\ell \in \mathbb{R}$, $\ell \in \mathcal{I} \cup \mathcal{A}^+(x*), \ell \neq j_1, j_2$, such that*
$$\gamma_{j_1} g_{j_1} = \gamma_{j_2} g_{j_2} + \sum_{\substack{\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*) \\ \ell \neq j_1, j_2}} \gamma_\ell g_\ell. \tag{7}$$
3. *It is possible to find two multiplier pairs $(\mu^1, \lambda^1), (\mu^2, \lambda^2) \in \mathbb{R}^m \times \mathbb{R}^p_+$ such that $\lambda^1_{j_1} = \lambda^2_{j_2} = 0$.*

Proof. If the constraints with index in $\mathcal{I} \cup \mathcal{A}^+(x^*)$ are linearly dependent, there must exist $\beta_\ell, \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$, not all zero, such that

$$0 = \sum_{\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)} \beta_\ell g_\ell.$$

We extend these coefficients to $\mathbb{R}^m \times \mathbb{R}^p$ by defining $\beta_j = 0$ for the remaining indexes.

Now, consider a multiplier pair $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^p_+$ such that $\mu_j > 0, \; \forall j \in \mathcal{A}^+(x^*)$. The line that passes through this multiplier with direction β must intercept the set of all possible multipliers in a non-trivial segment. The extremes (λ^1, μ^1) and (λ^2, μ^2) of this segment are clearly associated to two indexes j_1, j_2 for which $\lambda^1_{j_1} = \lambda^2_{j_2} = 0$. This happens because β_{j_1} and β_{j_2} have opposite signs. Now, define $\alpha_{j_1} = |\beta_{j_1}|$, $\alpha_{j_2} = |\beta_{j_2}|$, and $\alpha_\ell = \beta_\ell$, $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$, $\ell \neq j_1, j_2$. □

We now state and prove an auxiliary lemma.

Lemma 4.2. *Consider the assumptions of Theorem 4.1 and of Lemma 4.1 and let j_1, j_2 be the special active inequalities given by this Lemma. If $d \in \mathrm{T}(x^*)$, then there exists two twice continuously differentiable curves $\alpha_k : (-T_k, T_k) \to \mathbb{R}^n$, $T_k > 0$, $k = 1, 2$, such that:*

1. $\alpha_k(0) = x^*$, $\dot\alpha_k(0) = d$.
2. $f_\ell(\alpha_k(t)) = 0, \forall t \in (-T_k, T_k)$ *and* $\ell \in \mathcal{I} \cup \mathcal{A}^+(x),\ \ell \neq j_k$.

Proof. For each $k = 1, 2$, simply apply Lemma 3.1 for the systems

$$f_\ell(x) = 0,\ \ell \in \mathcal{I} \cup \mathcal{A}^+(x),\ \ell \neq j_k.$$

□

This result is complemented by the lemma below, that gives hints on what happens in constraint j_k when one follows the curve $\alpha_k(t)$, $t \in (-T, T)$.

Lemma 4.3. *Consider the assumptions and notation of Lemma 4.2. Fix a direction $d \in \mathrm{T}(x^*)$ and the respective curves α_k, $k = 1, 2$. Define $\varphi^k_\ell(t) = f_\ell(\alpha_k(t))$, $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$. These functions are twice continuously differentiable, $\varphi^k_{j_k}(0) = \dot\varphi^k_{j_k}(0) = 0,\ k = 1, 2$, and*

$$\ddot\varphi^1_{j_1}(0) = -\frac{\gamma_{j_2}}{\gamma_{j_1}} \ddot\varphi^2_{j_2}(0).$$

Proof. Using standard calculus rules, since $j_k \in \mathcal{A}^+(x^*)$, it is easy to see that

$$\begin{aligned} \varphi^k_{j_k}(0) &= f_{j_k}(\alpha_k(0)) = f(x^*) = 0, \\ \dot\varphi^k_{j_k}(0) &= \nabla f_{j_k}(\alpha_k(0))' d = g'_{j_k} d = 0. \end{aligned}$$

Now let us compute the second derivative. For $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$, $\ell \neq j_k$, we get that $\ddot\varphi^k_\ell(0) = 0$, because the function is constantly 0 in $(-T_k, T_k)$. Hence, standard calculus rules shows that

$$0 = \ddot\varphi^k_\ell(0) = d' H_\ell d + g'_\ell \ddot\alpha_k(0), \quad \ell \in \mathcal{I} \cup \mathcal{A}^+(x^*),\ \ell \neq j_k. \tag{8}$$

Finally, for $\ell = j_1$, we get

$$\begin{aligned} \ddot\varphi^1_{j_1}(0) &= d' H_{j_1} d + g'_{j_1} \ddot\alpha_1(0) \\ &= \frac{\gamma_{j_1}}{\gamma_{j_1}} d' H_{j_1} d + \frac{\gamma_{j_2}}{\gamma_{j_1}} g'_{j_2} \ddot\alpha_1(0) + \sum_{\substack{\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*) \\ \ell \neq j_1, j_2}} \frac{\gamma_\ell}{\gamma_{j_1}} g'_\ell \ddot\alpha_1(0) \quad \text{[Using (7)]} \\ &= \frac{\gamma_{j_1}}{\gamma_{j_1}} d' H_{j_1} d - \frac{\gamma_{j_2}}{\gamma_{j_1}} d' H_{j_2} d - \sum_{\substack{\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*) \\ \ell \neq j_1, j_2}} \frac{\gamma_\ell}{\gamma_{j_1}} d' H_\ell d \qquad \text{[Using (8)]}. \end{aligned} \tag{9}$$

Analogously, for $\ell = j_2$,

$$
\begin{aligned}
\ddot{\varphi}^2_{j_2}(0) &= d' H_{j_2} d + g'_{j_2} \ddot{\alpha}_2(0) \\
&= \frac{\gamma_{j_2}}{\gamma_{j_2}} d' H_{j_2} d + \frac{\gamma_{j_1}}{\gamma_{j_2}} g'_{j_1} \ddot{\alpha}_2(0) - \sum_{\substack{\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*) \\ \ell \neq j_1, j_2}} \frac{\gamma_\ell}{\gamma_{j_2}} g'_\ell \ddot{\alpha}_2(0) \quad \text{[Using (7)]} \\
&= \frac{\gamma_{j_2}}{\gamma_{j_2}} d' H_{j_2} d - \frac{\gamma_{j_1}}{\gamma_{j_2}} d' H_{j_1} d + \sum_{\substack{\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*) \\ \ell \neq j_1, j_2}} \frac{\gamma_\ell}{\gamma_{j_2}} d' H_\ell d \qquad \text{[Using (8)]}. \quad (10)
\end{aligned}
$$

Comparing (9) and (10), the result follows. □

We are now able to prove Theorem 4.1

Proof. (Theorem 4.1) If the gradients of $\mathcal{I} \cup \mathcal{A}^+(x^*)$ are linearly independent, the result follows from Theorem 3.2. Otherwise, let $d \in \mathrm{T}(x^*)$ be a direction of norm 1 such that $g'_j d < 0$, $j \in \mathcal{A}^0(x^*)$. We show first that

$$
\max\left(d' \nabla^2_{xx} L(x^*, \lambda^1, \mu^1) d,\, d' \nabla^2_{xx} L(x^*, \lambda^2, \mu^2) d\right) \geq 0.
$$

We start by recalling that $d' g_\ell = 0$, for all $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*)$ and $g'_j d < 0$, for all $j \in \mathcal{A}^0(x^*)$. Let j_1 and j_2 be the special indexes appearing in the lemmas above and consider the respective curves α_1 and α_2. As in Lemma 4.3, define $\varphi^k_\ell(t) := f_\ell(\alpha_k(t)),\ k = 1, 2,\ \ell \in \mathcal{I} \cup \mathcal{A}(x^*)$. We already know that, for $j \in \mathcal{A}^0(x^*)$,

$$
\varphi^k_j(0) = 0, \quad \dot{\varphi}^k_j(0) = g'_j d < 0.
$$

Hence, the curves α_k are feasible for these constraints and small t. Now, for $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*), \ell \neq j_k$, $\varphi^k_\ell(t) = 0$, for all $t \in [0, T_k]$, $k = 1, 2$. So, these constraints are also satisfied. The only constraints that may fail is f_{j_1} along curve α_1 and f_{j_2} along α_2. Considering Lemma 4.3, there are only two possibilities:

1. $\ddot{\varphi}^1_{j_1}(0), \ddot{\varphi}^2_{j_2}(0) \neq 0$. Using again Lemma 4.3, we can see that exactly one of the functions $\ddot{\varphi}^k_{j_k},\ k = 1, 2$ has strictly negative second derivative at $t = 0$. Hence, this function has to be negative for small t and the respective curve must be feasible. Choosing the respective multiplier pair (λ^k, μ^k), we can now use Lemma 3.2 to see that

$$
d' \nabla^2_{xx} L(x^*, \mu^k, \lambda^k) d \geq 0.
$$

2. $\ddot{\varphi}^1_{j_1}(0) = \ddot{\varphi}^2_{j_2}(0) = 0$. In this case, along α_1 all constraints but f_{j_1} are satisfied. If f_{j_1} is also satisfied along α_1, then this curve is feasible and we proceed as above. If α^1 does not satisfy f_{j_1}, as $\varphi^1_{j_1}(0) = \dot{\varphi}^1_{j_1}(0) = \ddot{\varphi}^1_{j_1}(0) = 0$, this infeasibility is of order two. In particular, there is a sequence $x^k \to x^*$ such that

$$
\frac{x^k - x^*}{\|x^k - x^*\|} \to d, \quad 0 < f_{j_1}(x^k) = o(\|x^k - x^*\|^2). \tag{11}
$$

Now, as the full feasible set conforms to Mangasarian-Fromovitz condition, it conforms to an error bound. Therefore, there is a feasible sequence $\{\bar{x}^k\}$ and a constant $M > 0$ such that $\|\bar{x}^k - x^k\| \leq M f_{j_1}(x^k) = o(\|x^k - x^*\|^2)$. Let us study $\{\bar{x}^k\}$.

First observe that,

$$\begin{aligned}
\frac{\bar{x}^k - x^*}{\|\bar{x}^k - x^*\|} &= \frac{\bar{x}^k - x^*}{\|\bar{x}^k - x^*\|} \\
&= \frac{\bar{x}^k - x^k}{\|\bar{x}^k - x^*\|} + \frac{x^k - x^*}{\|\bar{x}^k - x^*\|} \\
&= \frac{o(\|x^k - x^*\|^2)}{\|\bar{x}^k - x^*\| + o(\|x^k - x^*\|^2)} + \frac{x^k - x^k}{\|x^k - x^*\| + o(\|x^k - x^*\|^2)} \\
&\to d.
\end{aligned}$$

Moreover, for $\ell \in \mathcal{I} \cup \mathcal{A}^+(x^*), \ell \neq j_1$

$$\begin{aligned}
f_\ell(\bar{x}^k) &= f_\ell(x^k) + \nabla f_\ell(x^k)(\bar{x}^k - x^k) \\
&= 0 + o(\|x^k - x^*\|^2).
\end{aligned}$$

And for j_1,

$$\begin{aligned}
f_{j_1}(\bar{x}^k) &= f_{j_1}(x^k) + \nabla f_{j_1}(x^k)(\bar{x}^k - x^k) \\
&= o(\|x^k - x^*\|^2) + o(\|x^k - x^*\|^2) \\
&= o(\|x^k - x^*\|^2).
\end{aligned}$$

Then, we can use again Lemma 3.2 to see that

$$d' \nabla^2_{xx} L(x^*, \mu^k, \lambda^k) d \geq 0.$$

Finally, any direction $d \in \tau(x^*)$ can be approximated by directions like the ones considered above, hence the continuity of the functions involved implies that

$$\forall d \in \tau(x^*), \ \max\left(d' \nabla^2_{xx} L(x^*, \lambda^1, \mu^1) d, \, d' \nabla^2_{xx} L(x^*, \lambda^2, \mu^2) d\right) \geq 0.$$

As $\tau(x^*)$ is a subspace, Yuan's Lemma shows that there is a multiplier (λ, μ) which is a convex combination of (λ^1, μ^1) and (λ^2, μ^2) such that WSOC holds. □

A very similar proof can also be used to demonstrate a direct generalization of the main result in [8], which involves a strong second-order condition. Here, the generalized strict complementarity assumption can not be dropped as it is essential to apply Yuan's Lemma to the strong critical cone, which is not necessarily a subspace. This result is also related to the conjecture in the end of [3, Section 5] where instead of using an assumption of the kind "full rank minus 1" we employ an assumption with weaker flavour, that is implied by "constant rank of the full set of gradients minus 1".

Theorem 4.2. *Let x^* be a local minimizer of* (1)*. Suppose that MFCQ and GSCS hold at x^*, and that all the systems with the form*

$$f_\ell(x) = 0, \ell \in \mathcal{I}',$$

where $\mathcal{I}' \subset \mathcal{I} \cup \mathcal{A}^+(x^*)$, $\#\mathcal{I}' = \#(\mathcal{I} \cup \mathcal{A}^+(x^*)) - 1$, conform to $\mathcal{C}^2$-KTCQ . Then, SSOC holds at x^* for at least one multiplier pair.

Proof. Just follow the proof of Theorem 4.1. At the last part we can still use Yuan's Lemma since GSCS condition implies that the critical cone is a first-order cone as shown in the proof of [8, Theorem 5.1]. □

We end this section with an interesting example that shows the usefulness of the results above. Consider the following optimization problem:

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & \frac{1}{2}x_1^2 - x_2 \le 0 \\ & x_1^2 - x_2 \le 0 \\ & (x_1 - x_2)^2 - x_1 - x_2 \le 0 \\ & (x_1 + x_2)^2 + x - x_2 \le 0. \end{aligned} \tag{12}$$

Its feasible set is displayed in Fig. 2.

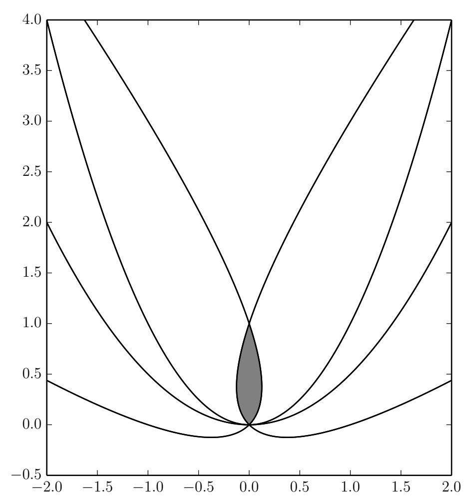

Figure 2: Feasible set of problem (12).

The minimizer is, clearly, the origin, where the feasible set is very well behaved. It conforms to the Mangasarian-Fromovitz constraint qualification. The second-order condition also holds. Actually, the critical cones are composed only by the origin, so the second-order condition holds trivially.

In spite of its simple nature, the result from Baccari and Trad can not be used to ensure the validity of a second-order condition. The reason for this is that the assumptions of [8, Theorem 5.1 and 7.7] require the existence of 3, the total number of active constraints minus one, linearly independent gradients. This is impossible in $\mathbb{R}^2$. On the other hand, Theorems 4.2 and 4.1 both can be applied, as the last two gradients are linearly independent and span the whole plane.

5 Conclusions

In this paper we proved the validity of the classical weak and strong second-order necessary optimality conditions under assumptions weaker than regularity. Abadie-type assumptions yield SOCs that hold for every Lagrange multiplier pair, while conditions based on MFCQ-type assumptions ensure SOCs for at least one Lagrange multiplier pair. In our future research, we plan to study the possibility of using such conditions, or other related ideas, to extend the convergence theory of algorithms specially tailored to find second-order stationary points as the methods described in [3, 14].

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