Erratum to: New Constraint Qualifications and Optimality Conditions for Second Order Cone Programs

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Erratum to: New Constraint Qualifications and Optimality Conditions for Second Order Cone Programs

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Abstract

In this note we show with a counter-example that all conditions proposed in [Y. Zhang, L. Zhang, New Constraint Qualifications and Optimality Conditions for Second Order Cone Programs. Set-Valued Var. Anal (2019) 27:693–712] are not constraint qualifications for second-order cone programming.

Keywords: Constraint qualifications; Optimality conditions; Second-order cone programming; Global convergence. We

consider the (nonlinear) second-order cone programming problem

$$\begin{aligned} \text{Minimize} \quad & f(x), \\ \text{s.t.} \quad & g_j(x) \in K_{m_j}, \quad j = 1, \ldots, \ell, \end{aligned} \tag{1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g_j\colon \mathbb{R}^n \to \mathbb{R}^{m_j}$, $j = 1, \ldots, \ell$ are continuously differentiable and the second-order cone K_m is defined as $K_m := \{z := (z_0, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid z_0 \geq \|\bar{z}\|\}$ if $m > 1$ and $K_1 := \{z \in \mathbb{R} \mid z \geq 0\}$. Here $\|\cdot\|$ is the Euclidean norm.

Given a feasible point x^*, we denote by $I_0(x^*) := \{j \in \{1, \ldots, \ell\} \mid g_j(x^*) = 0\}$ the index set of constraints at the vertex of the corresponding second-order cone and by $I_B(x^*) := \{j \in \{1, \ldots, \ell\} \mid [g_j(x^*)]_0 = \|\overline{g_j(x^*)}\| > 0\}$ the index set of constraints at the non-zero boundary of the corresponding second-order cone. For $j \in I_B(x^*)$ we define $\phi_j(x) := \frac{1}{2}([g_j(x)]_0^2 - \|\overline{g_j(x)}\|^2)$, with $\nabla \phi_j(x) = J_{g_j}(x)^T R_{m_j} g_j(x)$, where $J_{g_j}(x)^T$ is the $n \times m_j$ transposed Jacobian of g_j and R_m is the $m \times m$ diagonal matriz with 1 at the first position and -1 at the remaining positions.

In [11], the authors present an extension of the classical constant rank constraint qualification (CRCQ, [9]) for the second-order cone programming problem (1). It reads as follows:

Definition 1. *The Constant Rank Constraint Qualification (CRCQ) as defined in [11] holds at a feasible point x^* of (1) if there exists a neighborhood V of x^* such that for any index sets $J_1 \subseteq I_0(x^*)$ and $J_2 \subseteq I_B(x^*)$, the family of matrices whose rows are the union of $J_{g_j}(x), j \in J_1$ and the vector rows $\nabla \phi_j(x)^T, j \in J_2$ has the same rank for all $x \in V$.*

When $j \in I_B(x^*)$, the conic constraint $g_j(x) \in K_{m_j}$ can be locally replaced by the nonlinear constraint $\phi_j(x) \geq 0$, which is active at x^* (see e.g. [7, Section 4] for more details). Note also that for $j \in I_0(x^*)$ such that K_{m_j} is one-dimensional, the constraint $g_j(x) \in K_{m_j}$ is also a standard nonlinear constraint. Hence, the particularity of a second-order cone lies on the fact that one may have a "multi-dimensionally active" constraint $g_j(x^*) = 0$, which must be treated accordingly since these are tipically the constraints that are hard to tackle. The first impression one has when reading Definition 1 is that there is no special treatment for these active constraints. In particular, one would expect some regularity to be assumed for each constraint $g_j(x) \in K_{m_j}$ when $j \in I_0(x^*)$. To emphasize this last point, let us consider problem (1) with a single second-order cone, that is, $\ell = 1$, with constraint $g(x) \in K_{m_1}$. Let x^* be a feasible point such that $g(x^*) = 0$. According to Definition (1), CRCQ holds at x^* when the set of vectors given by all rows of $J_g(x)$ has constant rank, i.e., the full set of gradients $\{\nabla g_0(x), \ldots, \nabla g_{m_1-1}(x)\}$ has constant rank, and no subset of these vectors is considered. However, it is well

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known that the classical CRCQ for nonlinear programming requires that all subsets of active constraints possesses the constant rank property.

Despite these considerations, the example given below shows that even a strengthen definition of CRCQ, that takes all these subsets into account, is not a constraint qualification. This thus invalidates all the results proved in [11]. Therein, the authors also propose a definition for the relaxed-CRCQ (RCRCQ, [10]) and for the Constant Rank of the Subspace Component (CRSC, [6]), which, being weaker than their definition of CRCQ, are not constraint qualifications either. In particular, the definition of RCRCQ is done in such a way that only the full set of *all* gradients in $I_0(x^*)$ is considered, while every subset $J_2 \subseteq I_B(x^*)$ is considered (namely, J_1 is taken to be fixed and equal to $I_0(x^*)$ in Definition 1). However, it is easy to see that this is not a constraint qualification, since when one considers only one-dimensional cones, and consequently (1) reduces to a nonlinear programming problem, RCRCQ reads identical to the so-called Weak Constant Rank property from [1], which is not a constraint qualification. Our counter-example is discussed in the sequel.

Consider the following problem of one-dimensional variable:

$$\begin{aligned} \text{Minimize} \quad & f(x) := -x, \\ \text{s.t.} \quad & g(x) \in K_2, \end{aligned} \tag{2}$$

with

$$g(x) = \begin{pmatrix} g_0(x) \\ g_1(x) \end{pmatrix} := \begin{pmatrix} x \\ x+x^2 \end{pmatrix}.$$

The unique feasible point is $x^* = 0$, thus, it is a global solution. Since $g(x^*) = 0$, the Karush-Kuhn-Tucker conditions for this problem are given by the existence of $\mu \in K_2$ such that $\nabla f(x^*) - J_g(x^*)^T \mu = 0$, that is

$$-1 - \mu_0 - \mu_1 = 0, \tag{3}$$

with $\mu = (\mu_0, \mu_1)^T \in K_2$, or, equivalently, $\mu_0 \geq |\mu_1|$. Thus, (3) can not hold and the Karush-Kuhn-Tucker conditions fail. On the other hand $J_g(x) = \begin{pmatrix} 1 \\ 1+2x \end{pmatrix}$ for all x. In particular, $\nabla g_0(x) = 1$ and $\nabla g_1(x) = 1 + 2x$ for all x. Thus, all subsets of gradients

$$\{\nabla g_0(x)\}, \{\nabla g_1(x)\}, \{\nabla g_0(x), \nabla g_1(x)\}$$

have constant rank equal to 1 for all x near x^*. This shows that the definition of CRCQ from [11] is not a constraint qualification, as this property is characterized by the fact that the Karush-Kuhn-Tucker conditions hold at any local minimizer.

We next briefly point out the possible mistake in the approach followed in [11]. It is based on the proof of RCRCQ from [10], which is also similar to [1]. It is shown therein that $\mathscr{L}(x^*) \subseteq \mathscr{T}(x^*)$, for apropriate definitions of the linearized cone $\mathscr{L}(x^*)$ and tangent cone $\mathscr{T}(x^*)$ for second-order cone programming, by means of applying an implicit function-type theorem (Lyusternik's theorem [8]). This theorem allows constructing a suitable tangent curve and can be applied provided the constant rank assumption holds true. However, in the nonlinear programming context, when constraint $g_j(x^*) = 0$ is analyzed, direction $d \in \mathscr{L}(x^*)$ must be orthogonal to the gradient $\nabla g_j(x^*)$ in order to ensure the existence of a tangent curve to $\{x \mid g_j(x) = 0\}$ along the direction d. This seems to be ignored in [11].

Instead of applying the implicit function approach, constant rank constraint qualifications may be defined using the approach of sequential optimality conditions [2]. See, for instance, [4, 5, 6]. For this, one would need a proper extension of the so-called Carathéodory Lemma (see, e.g., [5]), which permits rewriting a linear combination $y := \sum_{i=1}^m \lambda_i v_i$ with $\lambda_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^n$ for all i in the following way: $y = \sum_{i \in I} \tilde{\lambda}_i v_i$ with $I \subseteq \{1, \ldots, m\}$, $\{v_i\}_{i \in I}$ linearly independent, and $\tilde{\lambda}_i$ with the same sign of λ_i for each i. In the case of second-order cones, for which the vector of scalars $(\alpha_i)_{i=1}^m$ belongs to the second-order cone K_m, one would want to rewrite the same vector y by only using a linearly independent subset of $\{v_i\}_{i=1}^m$ and such that the new scalars still belong to the cone. However, this is not possible in general as the following examples show.

Example 1. *Take* $y := \beta_0 v_0 + \beta_1 v_1 + \beta_2 v_2$*, with* $(\beta_0, \beta_1, \beta_2) := (\sqrt{2}, 1, 1) \in K_3$*,* $v_0 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$*,* $v_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$*,* $v_2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$*. There is no way of rewriting y using new scalars* $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) \in K_3$ *such that* $\hat{\beta}_i = 0$ *for some* $i = 0, 1, 2$.

In the case of more than one block of constraints ($\ell > 1$), even assuming more regularity for each block, a conic variant of Carathéodory's Lemma seems not possible to obtain.

Example 2. *Take* $y := \beta_0 v_0 + \beta_1 v_1 + \gamma_0 w_0 + \gamma_1 w_1$ *with* $(\beta_0, \beta_1) := (1, 1) \in K_2$*,* $(\gamma_0, \gamma_1) := (1, 1) \in K_2$*, and vectors*

$$v_0 := \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \; v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; w_0 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } w_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is not possible to rewrite y with new scalars $(\hat{\beta}_0, \hat{\beta}_1) \in K_2$*,* $(\hat{\gamma}_0, \hat{\gamma}_1) \in K_2$ *in such a way that at least one component vanishes. Note that both* $\{v_0, v_1\}$ *and* $\{w_0, w_1\}$ *are linearly independent sets, but the necessity of dealing with the product of two second-order cones makes it impossible to fulfill the desired property.*

We end this erratum with the following observation. Since it is well-known that linear second-order cone programs may possess duality gap, a definition of CRCQ could not be automatically satisfied by linear problems at the vertex. In [3], a naive proposition of CRCQ is presented where the "multi-dimensionally" active constraints are treated similarly to Robinson's CQ while the remaining constraints are treated similarly to CRCQ for nonlinear programming.

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