A DISCUSSION ON ELECTRICITY PRICES, OR THE TWO SIDES OF THE COIN

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Abstract. We examine to which extent different pricing frameworks are suitable for dealing with nonconvex features typical of day-ahead energy markets. For the system operator, requirements of minimum generation, ramps, and start-up or shut-down costs, translate into feasibility issues that need to be resolved with minimal cost. Classical pricing systems, based on marginal costs and Lagrange multipliers, fail to capture a signal that is suitable for the agents. We discuss pricing systems that, combined with compensations, provide a compromise between the needs of the agents and those of the system operator. For some of those approaches the compensation, determined after fixing the price, may not be enough and still entail a loss for efficient generators. In order to guarantee revenue-adequacy to all the agents, we propose a new mechanism that, once the dispatch is known, computes simultaneously prices and compensations. Simple, yet representative examples illustrate the pros and cons of the considered methodologies.

1. Price signals for power systems

Setting prices at appropriate levels is a key driver for success in any business. This is particularly true nowadays for electricity, with energy markets transitioning to a power mix dominated by distributed and renewable sources. As pointed out in the comprehensive guide of electricity markets [1], the generation of adequate price signals becomes more and more challenging in such setting. In this work we examine the issue under the light of the following through line:

Pricing mechanisms in energy markets
are like coins, and have two sides

Signals for energy prices have the Independent System Operator (ISO) on the heads, and generation agents on the tail side. The latter are in charge of providing energy, the former is responsible for dispatching the generators in a manner that is both reliable and sufficient, so that the electricity flows through the network to attend the demand.

Assessing a signal as a mechanism that provides a "good" price will naturally depend on from which side of the coin the appraisal is being made. Generators are concerned about making their business profitable, and expect the price to be high enough to cover the cost of all the "ingredients" necessary to produce electricity, including fixed costs for starting-up and shutting-down a production unit. The ISO main interest, on the other hand, is to ensure the generated electricity is carried through the network and reaches the end consumers, who care for low prices. Having a global view, rather than focusing on the individual cost of the generated energy,

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for the ISO the signal of price is related to the effort of having one more unit of energy delivered through the power system.

These concerns of *feasibility* and *profit*, respectively of the ISO and the generators, define the two sides of our coin. Three prototypical examples are examined throughout to show the challenges that appear with nonconvex configurations. Marginal costs of operation, representing the incremental costs incurred when producing an additional unit of energy, break down when there are discontinuities induced by fixed costs. Furthermore, for optimization problems with 0-1 variables, as in the considered setting, there are no Lagrange multipliers associated with the demand constraint. In what follows we discuss how different pricing mechanisms give preference to the heads or to the tails of the coin. To formalize those two views in mathematical terms, we shall consider the optimization problem solved by the ISO both in its primal and dual formulations, respectively in Sections 2 and Section 3 below. Specifically, in Section 2, after describing the unit-commitment primal problem of interest for the ISO, concepts of relevance for the agents, such as perceived loss and profit, are introduced. Section 3 presents the dual point of view, associated with generators who are in the business for profit. As explained in Subsection 33.2, the primal problem that corresponds to the generators preference can be interpreted as a relaxed version of the ISO problem, with randomized decisions. Section 4 starts with the proposal in [2], that determines prices and compensations as multipliers of a dispatch problem defined with the output of the unit-commitment problem. The discussion then focuses on a new rule, presented in Subsection 44.2, that computes prices and compensations as the solution of an optimization problem, of economic nature, and is also defined using the unit-commitment output. Contrary to the rule [2], the new approach is always revenue-adequate. We study their respective merits for some stylized examples, including a dynamic one, using a demand profile corresponding to two days in the Brazilian summer.

2. Tossing the coin for Unit Commitment problems

When the ISO determines how to dispatch in the short term the energy produced by the generators, the optimization problem said of *unit commitment* (UC) comes into play. Solving the UC problem provides the ISO with a dispatch that allows for an optimal operation of the power system while attending the demand. Associated with this output, of primal nature from the optimization point of view, there is an important indicator, the system's marginal cost of operation (MCO). Whether the power system is run under market premises, like in Europe or the US, or is centralized like in Brazil, the MCO acts as a price signal, akin to the well-known shadow prices in Linear Programming.

We now illustrate the main issues that arise when setting prices in a day-ahead setting, using an idealized UC model, that we named UNITOY. We refer to [3] for more details, noting that the thermal power plant modeling shares some features with the model [4], that deals with the whole hydro-thermal power system in Brazil.

2.1. **Formulation of the UC problem.**

The optimization horizon covers T time steps for a system with m generation units. The UC variables are the energy $p_i^t \in \mathbb{R}$ generated by i at time t, and the commitment $u_i^t \in \{0,1\}$ that indicates whether unit i at time t is on ($u_i^t = 1$) or it is off ($u_i^t = 0$). The overall generation and commitment of the ith unit are the vectors $p_i = (p_i^1, \dots, p_i^T)$ and $u_i = (u_i^1, \dots, u_i^T)$.

Technological constraints are written abstractly as $(p_i, u_i) \in \mathcal{P}_i \subset \mathbb{R}^T$. Typical relations in this set are the capacity and ramp constraints, given below:

$$(1) \qquad (p_i, u_i) \in \mathcal{P}_i \quad \text{contains} \quad \left\{ \begin{array}{ll} p^t_{i,\min} u^t_i \leq p^t_i \leq p^t_{i,\max} u^t_i & t = 1, \dots, T \\ |p^t_i - p^{t-1}_i| \leq \Delta p_i & t = 1, \dots, T\,, \end{array} \right.$$

for p^0_i the initial generation level for the ith unit.

Each unit has variable generation cost $C_i(p^t_i)$ as well as fixed operational costs F^+_i and F^-_i, the latter being incurred whenever the unit is turned on and off, respectively. The total operational cost for unit i is given by

$$\mathrm{GCost}_i(p_i, u_i) = \sum_{t=1}^{T} C_i(p^t_i) + F^+_i[u^t_i - u^{t-1}_i]^+ + F^-_i[u^{t-1}_i - u^t_i]^+\,,$$

where $[\cdot]^+ = \max(\cdot, 0)$ denotes the positive-part function.

Feasibility is often ensured by an artificial unit, with very large capacity $p_{\mathtt{a}\max}$ and cost, derived from the cost of shedding the load. In that case, the generation of this unit represents the system deficit of energy. Here the slack unit is used to eliminate border effects that pollute the price determination if the demand is small. For this reason the slack variable has low variable cost; see Table 1, with the information for the system that is used as a basis for our illustrations.

TABLE 1. A very simple power system with all generating units off at departure and $m = 2\,, T = 1$ in (2)

unit	C^t_i	$F^+_i = F^-_i$	$p^t_{i,\min}$	$p^t_{i,\max}$	u^0_i in (1)
1	5	0.0	0.0	150	0
2	12	0.0	0.0	150	0
$\mathtt{a}$	4	0.0	5000	5000	1

Given a system demand $D = (D^1, \dots, D^T) \in \mathbb{R}^T$, the *unit-commitment* problem is

$$(2) \qquad \left\{ \begin{array}{lll} \displaystyle\min_{(u_i,p_i)_{i=1}^m, (p^t_{\mathtt{a}})_{t=1}^T} & \displaystyle\sum_{i=1}^{m} \mathrm{GCost}_i(p_i, u_i) + \sum_{t=1}^{T} C_a(p^t_{\mathtt{a}}) & \\ \text{s.t.} & & \\ & \displaystyle\sum_{i=1}^{m} p^t_i + p^t_{\mathtt{a}} = D^t & t = 1, \dots, T \\ & (p_i, u_i) \in \mathcal{P}_i, & i = 1, \dots, m \\ & u_i \in \{0,1\}^T, & i = 1, \dots, m \end{array} \right.$$

The goal of the UC model is to minimize the total generation cost while satisfying the demand D, respectively represented by the objective function and the first constraint in (2). This goal must be achieved respecting the second constraint, with technological rules for the particular features of each source of energy. The inclusion in the second constraint of (2) states conditions to be satisfied for one unit i, independently of the behavior of other units in the system. This setting, that represents purely thermal power systems, needs to be slightly modified for hydro-thermal systems. Namely, for cascade hydro-power plants, constraints on units that are uphill have an impact on the units that are downhill. The corresponding technological set is therefore defined for the ith cascade as a whole, with the commitment u_i^t becoming a vector; its components indicate the status of the units that are distributed along the sequence of reservoirs.

The final constraint in (2) models the commitment decision, to define which units are to be turned on to actually generate energy. We shall see that the binary nature of the commitment variables is one of the main sources of difficulties when designing sound pricing mechanisms.

2.2. **Preferred profit and perceived loss. The value function.** The ISO is responsible for two main tasks. The first one is to determine the commitment and levels of dispatch for each unit, by solving the UC problem (2). The second one is to set a price at which each unit will be remunerated for the generated energy. This second task is not simple and so far there is no satisfactory answer in the literature, due to the two sides of the coin mentioned in the introduction. The ISO objectives conflict with those of the generators: the former looks for the cheapest way of satisfying the demand while the latter are in the market for profit and care for higher prices. The definition of the price signal needs to take into account those two sides, and find a compromise.

Let (p^*, u^*) denote a generation plan set by the ISO after solving (2). Given a price $\pi = (\pi_1, \ldots, \pi_t) \in \mathbb{R}^T_+$, the generator's revenue from unit i is given by

$$\mathrm{Prof}_i(\pi, p^*, u^*) = \pi^\top p_i^* - \mathrm{GCost}_i(p_i^*, u_i^*)\,,$$

where $x^\top y = \sum_i x_i y_i$ stands for the inner product of two column vectors x and y.

In this definition it is important to notice that, since both the generation plan and the price are imposed by the ISO, the value of $\mathrm{Prof}_i(\pi, p^*, u_i^*)$ may not represent an actual profit for the generator: its value might even be negative if the price π is too low and insufficient to cover the generation cost. This remark naturally leads to the following concept, of *preferred profit*, quantifying the gain that the ith unit would get, if able to determine its optimal generation level in response to the price π paid by the ISO:

$$\mathrm{PProf}_i(\pi) = \begin{cases} \max\limits_{(p_i,u_i)} & \pi^\top p_i - \mathrm{GCost}_i(p_i, u_i) \\ \quad \text{s.t.} & (p_i, u_i) \in \mathcal{P}_i \\ & u_i^t \in \{0,1\}\,, t = 1, \ldots, T\,. \end{cases} \tag{3}$$

In fact, these two concepts provide an estimation of the thickness of our figurative coin. Since the plan (p^*, u^*) is feasible for this maximization problem, the relation

$$\mathrm{PProf}_i(\pi) \geq \mathrm{Prof}_i(\pi, p^*, u^*)\,,$$

always holds, and the difference, called *uplift* in [2], is always nonnegative.

The preferred profit indicates to which degree the price clears the operational expenses. As such, it represents the view of the generator, while the actual profit the view of the ISO. When positive, the difference between these two measures is considered a lost opportunity by the generator. In order to bring closer these two notions of profit, many markets put in place a system of *compensations.* If e_i^t denotes the lump sum the ith generator receives as time t, say to compensate fixed costs, then

$$\mathrm{PercL}_i(\pi, e) = \mathrm{PProf}_i(\pi) - \mathrm{Prof}_i(\pi, p^*, u^*) - \sum_{t=1}^{T} e_i^t\,, \tag{4}$$

represents the *perceived loss*, that we shall use as a measure of satisfaction of the ith generator with the pricing system (a negative value of the perceived loss means the generator sees a profit opportunity).

A price π that zeroes the perceived losses for all the units represents the best possible scenario, with the expectation of the generators coinciding with the ISO's view. When the price and dispatch decided by the ISO make all units completely satisfied by their profit, the price π is said to *support* the generation plan (p^*, u^*).

A fundamental quantity in Economics for determining price signals is the *Marginal Cost* which, roughly speaking, represents the cost variation that results from an increase in production. In the context of energy generation, the *Marginal Cost of Operation (MCO)* quantifies the rate of change of the operational cost of the whole system when the demand varies in, say, 1KW/h.

In order to measure the MCO, we need to compute the derivatives of certain value function. To define this function, it is convenient to handle the slack unit as one more generator and set

$$u_{\mathtt{a}}^t = 1 \text{ for } t = 1, \ldots, T, \quad \mathrm{GCost}_{\mathtt{a}}(p_{\mathtt{a}}, u_{\mathtt{a}}) = \sum_{t=1}^{T} C_{\mathtt{a}}(p_{\mathtt{a}}^t), \quad \text{and} \quad \mathcal{P}_{\mathtt{a}} = [0, p_{\mathtt{a}\max}]^T ,$$

so that $\mathcal{M}_{\mathtt{a}} := \{\mathtt{a}\} \cup \{1, \ldots, m\}$ groups all the units. To shorten the expressions, we also include the binary relations on the commitment in the technological sets, and let

$$\mathcal{Q}_i := \left\{(u_i, p_i) \in \mathcal{P}_i | u_i \in \{0,1\}^T\right\} \text{ and } \mathcal{Q}_{\mathtt{a}} := \left\{(u_{\mathtt{a}}, p_{\mathtt{a}}) \in \mathcal{P}_{\mathtt{a}} | u_{\mathtt{a}}^t = 1\,, t = 1, \ldots, T\right\}$$

In the new notation, the feasible set in (2) is expressed as

$$\left\{(u_i, p_i) \in \mathcal{Q}_i\,, i \in \mathcal{M}_{\mathtt{a}} \mid \sum_{i \in \mathcal{M}_{\mathtt{a}}} p_i = D\right\},$$

and the *value function* $v : \mathbb{R}^T \to (-\infty, +\infty]$ associated with the UC problem (2) is

$$v(D) = \left\{ \begin{array}{lll} \min\limits_{(p_i,u_i)\in\mathcal{Q}_i\,, i\in\mathcal{M}_{\mathtt{a}}} & \sum\limits_{i\in\mathcal{M}_{\mathtt{a}}} \mathrm{GCost}_i(p_i, u_i) & \\ \text{s.t.} & \sum\limits_{i\in\mathcal{M}_{\mathtt{a}}} p_i^t = D^t\,, & t = 1, \ldots, T\,. \end{array} \right. \tag{5}$$

Since the feasible set is bounded, the function values are finite unless the problem is unfeasible, in which case $v(D) = +\infty$ (for example if D is a negative).

The data in Table 1 for the slack unit $\mathtt{a}$ was chosen to eliminate possible pathological effects on the price that appear when the demand is too low. This is why in our plots the abscissa parses the demand in an interval that is beyond the maximum capacity of the artificial unit ($D \geq 5000$). In Figure 1, the full line represents the value function and the shaded areas the generation level of units 1 and 2 for the system in Table 1.

For convex cases as the example in Figure 1, the following important property holds.

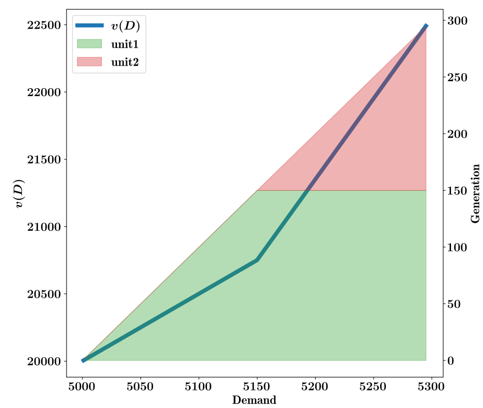

FIGURE 1. Convex value function for a system as in Table 1 and $T = 1$ in (5). The change of slope reflects the more expensive variable cost of unit 2, that enters into operation when the capacity of the other units is insufficient to attend the demand ($D = 5150$).

Theorem 2.1 (Subgradients support generation plans)**.** *Suppose the value function v is convex, as in Figure 1, and let $\partial v(D)$ denote the subdifferential of v at D:*

$$\partial v(D) = \left\{\pi \in \mathbb{R}^T \mid v(D') \geq v(D) + \pi^\top (D' - D) \text{ for all } D' \in \mathbb{R}^T\right\} .$$

Then any subgradient $\pi \in \partial v(D)$ is a price that supports the generation plan for the demand D.

Proof. Given a demand D and a subgradient $\pi \in \partial v(D)$, the function $W(D') = v(D') - \pi^\top D'$ satisfies $0 \in \partial W(D)$, which means that D is a global minimizer of W and, hence,

$$v(D) - \pi^\top D = W(D) = \min_{D'} W(D') = \min_{D'} \Big(v(D') - \pi^\top D' \Big) .$$

Since, in addition, $v(D) = \sum_{i \in \mathcal{M}_\mathsf{a}} \mathrm{GCost}_i(\bar{p}_i, \bar{u}_i)$, for $\{(\bar{p}_i, \bar{u}_i)\}_{i \in \mathcal{M}_\mathsf{a}}$ solving (5), expanding the inner product gives the following chain of equalities

$$\begin{aligned} v(D) - \pi^\top D &= \min_{D'} \min_{\substack{(p_i,u_i) \in \mathcal{Q}_i, i \in \mathcal{M}_\mathsf{a} \\ \sum_{i \in \mathcal{M}_\mathsf{a}} p_i^t = D'^t, 1 \leq t \leq T}} \Big(\sum_{i \in \mathcal{M}_\mathsf{a}} \mathrm{GCost}_i(p_i, u_i) - \pi^\top D' \Big) \\ &= \min_{(p_i,u_i) \in \mathcal{Q}_i, i \in \mathcal{M}_\mathsf{a}} \Big(\sum_{i \in \mathcal{M}_\mathsf{a}} \mathrm{GCost}_i(p_i, u_i) - \sum_{t=1}^T \pi^t \sum_{i \in \mathcal{M}_\mathsf{a}} p_i^t \Big) \\ &= \min_{(p_i,u_i) \in \mathcal{Q}_i \, i \in \mathcal{M}_\mathsf{a}} \sum_{i \in \mathcal{M}_\mathsf{a}} \Big(\mathrm{GCost}_i(p_i, u_i) - \sum_{t=1}^T \pi^t p_i^t \Big) \\ &= \sum_{i \in \mathcal{M}_\mathsf{a}} \min_{(p_i,u_i) \in \mathcal{Q}_i} \Big(\mathrm{GCost}_i(p_i, u_i) - \pi^\top p_i \Big) \end{aligned}$$

Therefore, $(\bar{p}_i, \bar{u}_i)$ solves (3), and π supports the generation plan $\{(\bar{p}_i, \bar{u}_i)\}_{i \in \mathcal{M}_\mathsf{a}}$, as stated. $\square$

The interest of this result is twofold. First, the statement itself ensuring that for convex value functions it is always possible for the ISO to find prices that will satisfy completely all the generators. Second, the realization that convex value functions are associated with many real-life applications, for instance when they are modeled as linear programs. In this case, the Lagrange multipliers associated with (5) (cf. Section 3 below) yield the subdifferential $\partial v(D)$, which gives ground for considering that they are genuine *shadow prices*.

When the value function is differentiable, its derivative gives the MCO and, if in addition the value function is convex, its subdifferential set is just the singleton derivative. In this case, the MCO is equal to the shadow price and supports the generation plan. When the value function is not differentiable but still convex, the concept of MCO becomes ambiguous: it is not well defined because now π can take any value in the set of subgradients. Yet, according to Theorem 2.1, any element in the subdifferential $\partial v(D)$ will support the generation plan.

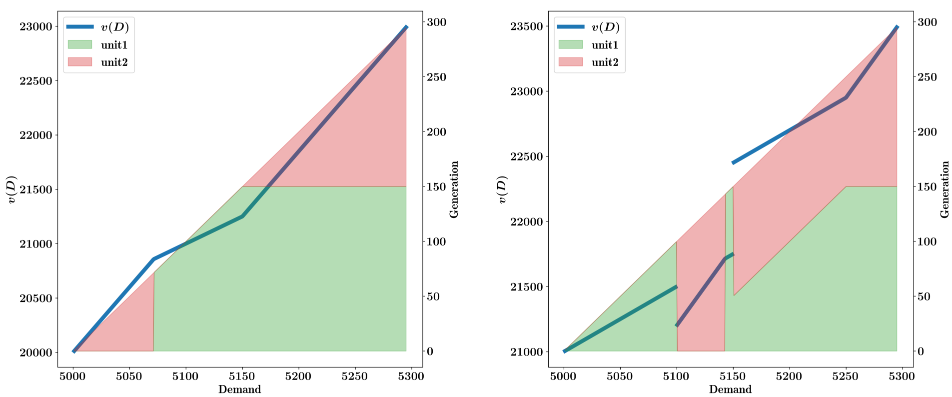

FIGURE 2. Positive values for fixed costs and minimal generation introduce nonconvexities and discontinuities in the value function. The plots were obtained by changing the data from Table 1 to $F_1^+ = F_1^- = 500$ (left) and to $F_1^+ = F_1^- = 1000$, $p_{2,\min} = 500$ (right). On the left, approximately in the range $D \in (5050, 5150)$, the change in the slopes indicates a reduction in the MCO, when unit 1 is generating instead of unit 2 (unit 1 has cheaper variable cost but high fixed cost). This drop occurs in spite of an increase in demand, a behavior that goes against the market expectation and could harm the effectiveness of demand-response programs. On the right, near $D = 5100$, unit 2 (with minimal generation but no fixed cost) starts generating instead of unit 1. This creates a discontinuity in the value function that translates into a MCO taking any value in a set that is unbounded. Such a behavior can produce an infinite signal that translates into unduly large prices in practice.

The challenge in defining a price mechanism for UC problems as (5) comes from the fact that the value function v might be nonconvex or even discontinuous, if there are fixed costs and/or non-null minimum generation constraints in (1).

The phenomenon is illustrated by the left and right plots in Figure 2, leading to situations that go against the expected behavior of prices in the energy business (and require compensations for generators in the market not to incur into losses). For the convex case considered in Figure 1, the perceived losses are all null, but the cases in Figure 2 result in positive values. Having positive start-up and shut-down costs and minimum generation requirements is not uncommon in power systems. In the sequel we analyze a dual pricing mechanism that gives a consistent MCO in those settings.

3. Tossing twice the coin: bi-dual considerations

Generators are interested in a problem that from the optimization point of view is *dual* to (5). Starting from this UC problem, the ISO's side of the coin, we now derive certain *dual function*. The dual function is separable, and its computation amounts to computing the individual preferred profit (3) for all the generators. These issues are discussed below, as well as examining what happens if the game is played in the opposite direction, flipping the coin from the generators' side, to obtain certain *bi-dual* problem for the ISO.

3.1. **Generators have a dual point of view.**

In the UC problem (5), requiring satisfaction of demand couples the decisions of all the generators. Associating a multiplier $\lambda \in \mathbb{R}^T$ to the demand constraint, the Lagrangian function

$$\begin{aligned} L(p,u,\lambda) &:= \lambda^\top D + \sum_{i\in\mathcal{M}_\mathsf{a}} \Big(\mathrm{GCost}_i(p_i,u_i) - \lambda^\top p_i\Big) \\ &= \lambda^\top D + \sum_{i\in\mathcal{M}_\mathsf{a}} L_i(p_i,u_i,\lambda) \end{aligned}$$

is separable into partial Lagrangians $L_i(p_i,u_i,\lambda) := \mathrm{GCost}_i(p_i) - \lambda^\top p_i$, depending only on the variables of the ith generator. A problem dual to (5) is given by maximinimizing the Lagrangian. One of the reasons that make it particularly interesting is that it inherits the separable structure present in the Lagrangian. Specifically, the dual problem is defined as

$$\max\left\{\theta(\lambda) : \lambda \in \mathbb{R}^T\right\}\,, \tag{6}$$

where

$$\begin{aligned} \theta(\lambda) &:= \min\left\{L(p,u,\lambda) : (p_i,u_i) \in \mathcal{Q}_i\,, i \in \mathcal{M}_\mathsf{a}\right\} \\ &= \lambda^\top D + \sum_{i\in\mathcal{M}_\mathsf{a}} \min_{(p_i,u_i)\in\mathcal{Q}_i} L_i(p_i,u_i,\lambda)\,. \end{aligned} \tag{7}$$

Each term corresponds to a partial dual function of the form $\theta_i(\lambda) := \min_{(p_i,u_i)\in\mathcal{Q}_i} L_i(p_i,u_i,\lambda)\,.$ These functions are the negative of the profit maximization problems (3), written with $\pi = \lambda$:

$$-\theta_i(\pi) = \max_{(p_i,u_i)\in\mathcal{Q}_i} -L_i(p_i,u_i,\pi) = \left\{\begin{array}{ll} \displaystyle\max_{(p_i,u_i)} & \pi^\top p_i - \mathrm{GCost}_i(p_i,u_i) \\ \text{s.t.} & (p_i,u_i) \in \mathcal{Q}_i \end{array}\right. = \mathrm{PProf}_i(\pi)\,.$$

As a result, solving the dual problem (6) amounts to solving

$$\max\left\{\lambda^\top D - \sum_{i\in\mathcal{M}_\mathsf{a}} \mathrm{PProf}_i(\lambda) : \lambda \in \mathbb{R}^T\right\}.$$

The prices for our three examples, solving the dual problem (6), are reported in Figure 3.

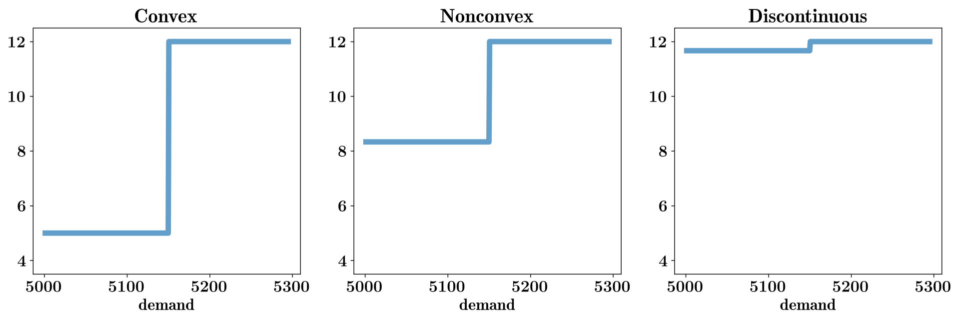

FIGURE 3. Dual price π^* obtained with (6) for the three situations in Section 22.2. The dual price increases together with the demand. In a manner that is consistent with the intuition, the inclusion of start-up and shut-down costs (middle) and of minimal generation and fixed costs (right) yielded higher prices than for the case without nonconvexities (left). Notice that at $D = 5150$ there is a set of possible prices, for instance on the convex case (left), any value in $[5, 12]$ is a legitimate price.

When the primal problem is convex, the dual price supports the dispatch. This can be seen relating the dual function and the value function defined in (5). Namely, by the various definitions, the convexity assumption implies that

$$v(D) = \min_{(p,u)} \max_{\lambda} L(p, u, \lambda) = \max_{\lambda} \min_{(p,u)} L(p, u, \lambda) = \theta(\pi^*)\,.$$

By linearity of the Lagrangian with respect to D, this implies, for any $D' \in \mathbb{R}^T$:

$$v(D') = \max_{\lambda} \{\theta(\lambda) + \lambda^\top(D' - D)\} \geq \theta(\pi^*) + \pi^{*\top}(D' - D) = v(D) + \pi^{*\top}(D' - D)\,.$$

Hence, $\pi^* \in \partial v(D)$, and as such it supports the dispatch, by Theorem 2.1.

For systems with many heterogeneous units, the primal problem (5) may be too difficult, if not impossible, to solve. The ISO must decide every day the dispatch for the next day, calculations need to be done in relatively short times and in a reliable manner. In this setting, the separable structure of the dual function in (6) can be exploited by an iterative procedure based on decomposition, that maximizes the dual function. Because the dual function is concave but nonsmooth, special techniques must be put in place. Since a maximizer of the dual problem provides a price signal, accuracy is of foremost importance too. The family of bundle algorithms for nonsmooth minimization are the methods of choice in this case, [5, Part II]. We do not enter into further details here. For success stories of decomposition methods applied to energy optimization, we refer to [6] and the many references therein.

3.2. **What primal problem corresponds to the generators preference?** We have just seen that solving the dual problem (6) gives a price satisfying the generators point of view. Nothing guarantees that the generation associated with that price will satisfy the demand (the dual function was defined by relaxing the demand constraint).

The ISO, on the other side of the coin, is interested in attending the demand and solving the primal problem (5). Both views, primal and dual, are equivalent only when the problem is sufficiently convex. This is certainly not the case in (5), because of the fixed costs and the binary relations that model the commitment of the units. We now examine the properties of the the primal pairs $(\bar{p}, \bar{u})$ that are available when solving the dual, both in terms of feasibility and optimality with respect to (5).

As explained in the thorough geometric study [7], such pairs always solve a problem that is dual to the dual, or the *bi-dual*. When the ISO problem has no 0-1 variables and the objective and constraints functions are convex, the bi-dual coincides with the original primal problem (the optimality conditions are necessary and sufficient to characterize a minimizer). In other words, the problem dual to (6) is precisely our initial UC problem (5): any side of the coin is dual to the other one. In the optimization jargon, it is said that *there is no duality gap*. In terms of perceived losses, the generation computed by the ISO is already optimal for the generators, and in the parlance of [2] and [8], the minimal uplifts are all null.

When there are 0-1 variables, the duality gap may be positive and the bi-dual may differ from the ISO problem (5). Notwithstanding, an interesting interpretation for the bi-dual is given in [9]. More precisely, when a bundle method [5] is employed to solve the dual problem (6), together with an optimal dual price π^*, the method outputs a very special set of primal points and simplicial coefficients, that we denote respectively by

$$\left\{\Big(\bar{p}(k), \bar{u}(k)\Big) \text{ and } \bar{\alpha}_k \mid k = 1, \ldots, K\right\}, \text{ noting that } K \leq T+1.$$

Each pair of dispatch and commitment $(\bar{p}(k), \bar{u}(k))$ was computed at some iteration k of the bundle method, so that it satisfies the technological constraints including the binary relations:

$$\Big(\bar{p}_i(k), \bar{u}_i(k)\Big) \in \mathcal{P}_i \text{ and } \bar{u}_i(k) \in \{0,1\}^T \text{ for } i \in \mathcal{M}_{\mathtt{a}}\,,$$

As for the coefficients $\bar{\alpha}$, they are in the unit simplex:

$$\bar{\alpha} \in \Delta^K := \left\{\alpha \in \mathbb{R}^K : 0 \leq \alpha_k \leq 1\,, k = 1, \ldots, K \text{ and } \sum_{k=1}^{K} \alpha_k = 1\right\}.$$

With the bundle output, it is possible to define the following convex combinations

$$\hat{p}_i(\bar{\alpha}, \bar{p}) := \sum_{k=1}^{K} \bar{\alpha}_k \bar{p}_i(k)\,, \quad \hat{u}_i(\bar{\alpha}, \bar{u}) := \sum_{k=1}^{K} \bar{\alpha}_k \bar{u}_i(k)\,,$$

called the pseudo-planning in [10]. Regarding feasibility for the UC problem (2), notice that the commitment component is such that $\hat{u}_i \in [0,1]^T$, as in Table 2 below for $D = 5149$ and 5151. This is typical from bi-dual solutions, as the dual of the dual "convexifies" the original problem.

As a manner of preserving the 0-1 nature present in the bundle output, we can rely on a nice interpretation from [11] that arises when considering α_k as the *probability* of the ISO taking the decision $(p(k), u(k))$. More precisely, in [9, Section 6] it is shown that the bundle elements solve the following bi-dual, a relaxed version of the original UC problem, in which the demand constraint is satisfied by the generation component of the pseudo-planning:

$$
\left\{
\begin{array}{llll}
\displaystyle\min_{\alpha,(u,p)} & \displaystyle\sum_{k=1}^{K}\alpha_k \sum_{i\in\mathcal{M}_{\mathtt{a}}} \mathrm{GCost}_i(p_i(k),u_i(k)) & & \\
\text{s.t.} & \alpha\in\Delta^K & & \\
 & (p_i(k),u_i(k))\in\mathcal{P}_i, & i\in\mathcal{M}_{\mathtt{a}}\,, & k=1,\ldots,K \\
 & u_i(k)\in\{0,1\}^T, & i=1,\ldots,m\,, & k=1,\ldots,K \\
 & u^t_{\mathtt{a}}(k)=1, & t=1,\ldots,T\,, & k=1,\ldots,K \\
 & \displaystyle\sum_{i\in\mathcal{M}_{\mathtt{a}}}\sum_{k=1}^{K}\alpha_k p^t_i(k)=D^t\,, & t=1,\ldots,T\,, & k=1,\ldots,K\,.
\end{array}
\right.
$$

With respect (5), notice that the bi-dual feasible set was enlarged to include all randomized decisions; see also [12, Section 3.4]. If the simplicial coefficients are thought of as being probabilities, the objective value in the bi-dual problem represents the expected value of the operational cost. Being a solution to the bi-dual, the bundle output $(\bar\alpha,\bar p,\bar u)$ minimizes the expected-value formulation of the ISO problem given above.

As an illustration, we applied the bundle method in [13] to solve the dual problem of the system yielding the discontinuous value function on the right in Figure 2. We considered three different demand instances, $D\in\{5149,5150,5151\}$, that capture the behavior in one of the regions where the value function jumps. The respective prices are $\pi=\{11.67,11.84,12\}$, the duality gap is 0.03%, 0.00%, and 3.00%. Pseudo-plannings are reported in Table 2.

TABLE 2. Solution to UC problem (2) and output of the bundle method when solving (6) for the system with fixed costs and minimal generation

	$D=5149$		$D=5150$		$D=5151$	
(p^*,u^*) solving (2)	unit 1	(149,1)	unit 1	(150,1)	unit 1	(51,1)
	unit 2	(0,0)	unit 2	(0,0)	unit 2	(100,1)
pseudo-planning $(\hat p,\hat u)$	unit 1	(148.9,1)	unit 1	(150,1)	unit 1	(150,1)
	unit 2	$(0,\frac{2}{300})$	unit 2	(0,0)	unit 2	$(1,\frac{2}{300})$

The largest mismatch between the pseudo-planning and the UC solution (p^*,u^*) is observed with $D=5151$, when unit 2 enters into operation with its minimal generation ($p_{2\min}=100$). For these runs, the pseudo-planning results from a convex combination of the same two pairs,

$$\{(\bar p_i(k),\bar u_i(k))\,,k=1,2\}=(150,1)\text{ and }(0,1)\,,$$

using the simplicial coefficients (probabilities)

$$
\bar\alpha=\left\{
\begin{array}{cl}
(1-\frac{2}{300},\frac{2}{300}) & \text{if}\quad D=5149\\
(1,0) & \text{if}\quad D=5150\\
(\frac{2}{300},1-\frac{2}{300}) & \text{if}\quad D=5151\,.
\end{array}
\right.
$$

The points with highest probability can be used as starting point in a heuristic of primal recovery. Finally, regarding Lagrange multipliers, it should be noted that the dual function of the bi-dual problem is still (7) (the dual of the dual of the dual is equal to the dual, because the dual problem is always convex, and therefore coincides with its bi-dual).

4. Comparison of pricing mechanisms

Defining reasonable prices is a challenging problem of practical relevance. As seen in Section 2, if the value function v is convex, any subgradient in $\partial v(D)$ is a price that supports the dispatch and yields a genuine MCO. The nonconvex configuration is not as straightforward, as the concept of MCO is no longer well defined. Similarly, the notion of Lagrange multiplier becomes vacuous for problems like (5), with 0-1 variables. The comprehensive review [14] considers many proposals for computing price signals in the presence of nonconvexities. We now discuss the mechanism from [2] and a new proposal, with limited compensation of fixed costs, that is revenue-adequate.

Revenue adequacy is an important property for a pricing mechanism, as it allows generators not only to cover operating expenses, but also to get a reasonable return on capital, a feature that retains and attracts agents to the business.

4.1. **Multipliers for prices and compensations.** The rule in [2] compensates positive perceived losses, in the sense of (4), by means of lump payments. The idea is to determine the rate of change not only of the demand constraint but also of the integrality nature associated with the commitment. This is achieved in two steps. First, the original UC problem (5) is solved, for instance using an algorithm for mixed-integer problems. The corresponding optimal dispatch and commitment are p^* and u^*. In a second step, the integer variables are fixed at the values of the computed optimal solution, relaxing the integrality constraints. This yields the following dispatch-like problem

$$
\left\{\begin{array}{lll}
\displaystyle\min_{(p_i,u_i)\in\mathcal{P}_i\,,i\in\mathcal{M}_{\mathsf{a}}} & \displaystyle\sum_{i\in\mathcal{M}_{\mathsf{a}}} \mathrm{GCost}_i(p_i,u_i) & \\
\text{s.t.} & u_i = u_i^*\,, & i\in\mathcal{M}_{\mathsf{a}}\\
 & u_i\in[0,1]^T\,, & i\in\mathcal{M}_{\mathsf{a}}\\
 & \displaystyle\sum_{i\in\mathcal{M}_{\mathsf{a}}} p_i^t = D^t\,, & t=1,\dots,T\,.
\end{array}\right. \tag{8}
$$

Having only continuous variables, this optimization problem does have Lagrange multipliers in particular, the one associated with the new constraint measures the impact of moving away from the original 0-1 decisions ($u = u^*$).

In (8), the multipliers associated with the demand constraint measure how much the optimal cost would change if the respective demand was increased (or decreased) while keeping the 0-1 decisions fixed. This gives a MCO for a frozen configuration of the power system (no new unit is turned on or off, but generation can change). Figure 4 shows the value of the price for our three systems, convex, with fixed costs, and with both fixed costs and minimal generation, respectively on the left, middle and right graphs. Notice that, contrary to the dual scheme in Figure 3, the prices in Figure 4 do not accompany all the increases in demand. As mentioned, this can make less effective demand-response programs. Also, the prices in Figure 4 are lower than those in Figure 3, which do follow the demand. This phenomenon was observed in [8]: the dual pricing scheme (6) achieves the minimum necessary uplifts that compensate generators for being dispatched with the centralized solution, instead of self-scheduling.

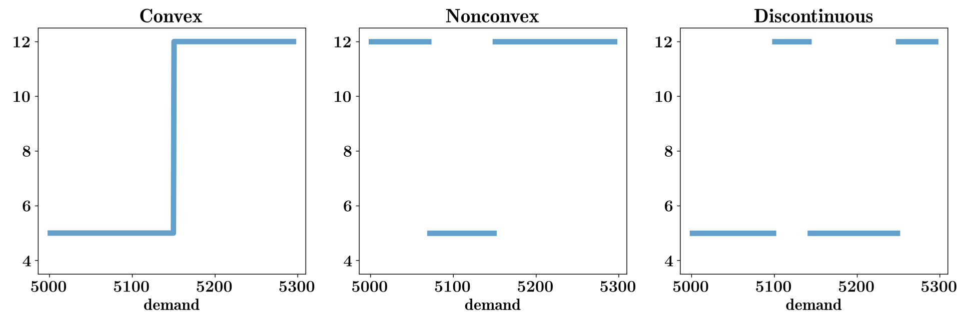

FIGURE 4. Price π^* given by a multiplier in (8) for the three situations in Section 22.2, and dispatch the output of the UC problem (2). For the nonconvex case (middle), the decrease in the price at some regions, even if the demand increases, is explained by the cheaper variable cost of the unit 1, not dispatched for values of D below 5050 because of its high start-up cost. A similar phenomenon is observed in the discontinuous case (right), with the minimal generation requirement for unit 1 inducing another change.

The multiplier associated with the new constraints ($u = u^*$) measures the impact of changing the unit status from 0 to 1 or visceversa. This value signals the payment e_i^t in (4), compensating generators for fixed start-up or shut-down costs. In an idealized market, surplus is avoided by zeroing the profit of an infinite number of energy suppliers. But a real market has only a finite number of generators and the compensation computed from the multiplier can be negative. Since this penalizes efficiency, a fix suggested by [2] is to discard negative compensations, so that

$$e_i^t \quad \text{is replaced by} \quad [e_i^t]^+, \quad i = 1, \ldots, m\,, \tag{9}$$

allowing efficient generators to make a positive profit. The perceived losses of generators 1 and 2 with the prices from Figure 4 and the corrected compensation (9), are shown in Figure 5.

According to [2], this is the pricing mechanism employed by the New York Independent System Operator and the Pennsylvania-New Jersey-Maryland Interconnection market. A similar rule is planned to be put in practice in Brazil, starting in 2021; see [4]. It is important to be aware that, being defined as a multiplier of the constraints that relax the integrality, the value of the compensation will depend on the solver that is used to solve (8) (different solvers can give different compensations if the multiplier is not unique). Also, the sign of the compensation depends on how the constraint was written ($u - u^* = 0$ or $u^* - u = 0$). Furthermore, different formulations of the technological sets $\mathcal{P}_i$, even if equivalent on paper, do change the multiplier set and, therefore, also produce a different compensation. These are delicate issues that can have a significant impact on the business, if not suitably taken into consideration.

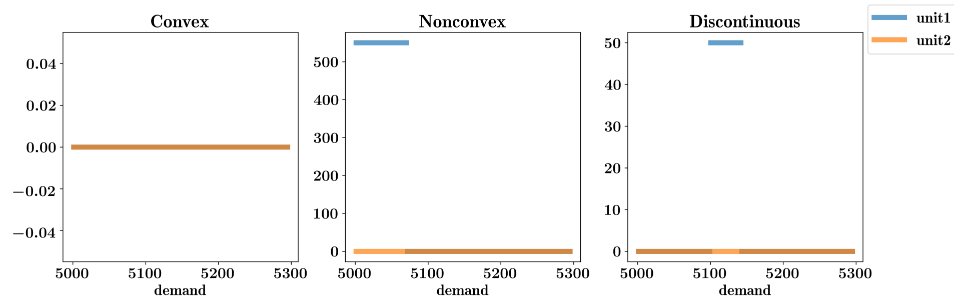

FIGURE 5. Perceived losses for prices and compensations determined by multipliers in (8), using the correction (9). Generator 2 is satisfied in all cases. Generator 1, with cheaper variable cost, perceives losses at demand ranges for which it is not dispatched, due to the high start-up cost ($D \in [5000, 5050]$ in the middle graph, and $D \in [5100, 5150]$ on the right.

4.2. **A revenue-adequate pricing with limited compensations.** It is stated in [14] that prices computed from optimal Lagrange multipliers of (8) suffer from volatility and can incur into excessively large compensations. This feature has the effect of changing the incentives for the generators, bringing the pricing system closer to a pay as bid one (an infra-marginal unit may recover a high startup cost from the high price itself). In this case the lump payment may be unnecessarily compensating the generator, rewarding the high start-up cost, even if this high startup is an inefficiency of the generator. The work also comments on the importance of guaranteeing that all the dispatched units clear their expenses.

We now present a new revenue-adequate pricing scheme that is somewhat reminiscent of a natural interpretation of duality. Namely, given a linear production problem under inventory constraints, the dual problem can be interpreted as searching prices for the items in the inventory in such a way that they compensate any possible production level. This goal is achievable with Linear Programming models because there is no duality gap and any real production level is allowed. With the UC problem (5), however, the situation is different, especially if the duality gap is large due to the integrality constraints, the minimal generation levels, and significant fixed costs. Indeed, when the value function is discontinuous and exhibits a jump, as in the right plot in Figure 2, only an "infinite" price could capture the instantaneous change in the optimal cost. In this setting, the price must be accompanied by some compensation.

Prices and compensations in Section 44.1 are given by multipliers of the dispatch problem (8), a relaxed UC problem that belongs to the ISO side of the coin. Our proposal is to "flip the coin" and consider instead an *economic* problem that represents the interest of the generators. This is consistent with the concerns of feasibility and profit, respectively of the ISO and the generators, mentioned in the introduction.

To define the new approach, we introduce the overall fixed cost of unit i

$$\mathrm{FCost}_i(u_i^*) = \sum_{t=1}^{T} F_i^+[(u_i^*)^t - (u_i^*)^{t-1}]^+ + F_i^-[(u_i^*)^{t-1} - (u_i^*)^t]^+ ,$$

where a solution (p^*, u^*) to the UC problem (2) is given. The economic problem whose minimization gives simultaneously prices and compensations depends on scalars $a, b \geq 0$ and a parameter $\beta \in (0,1)$ to limit the level of compensations:
(10)

$$\left\{\begin{array}{lll} \min\limits_{(\pi^t, e_i^t, s_i)} & \frac{a}{2}\|\pi - \pi^*\|_2^2 + b\|\pi\|_1 & \\ \text{s.t.} & & \\ & E_i = \sum_{t=1}^T e_i^t, & i = 1, \ldots, m \\ & E_i \leq s_i M, & i = 1, \ldots, m \\ & E_i^t \leq \sum_{t=1}^T \mathrm{FCost}_i(u_i^*), & i = 1, \ldots, m \\ & \sum_{t=1}^T \pi^t p_i^{*t} + E_i \geq \mathrm{GCost}_i(p_i^*, u_i^*), & i = 1, \ldots, m \\ & \sum_{t=1}^T \pi^t p_i^{*t} + E_i \leq \mathrm{GCost}_i(p_i^*, u_i^*) + (1 - s_i)M, & i = 1, \ldots, m \\ & \sum_{i=1}^m E_i \leq \beta \sum_{t=1}^T \pi^t \sum_{i=1}^m p_i^{*t} & \\ & \pi \geq 0, \quad e \geq 0\,, \quad s \in \{0,1\}^m\,. & \end{array}\right.$$

In this problem, the target price π^* is chosen for having some desirable properties (for example the dual price from Section 3). The objective function can be anyone that is meaningful, as long as the optimization problem remains tractable. We use the squared Euclidean distance to the target for simplicity, and in addition, if $b > 0$, an ℓ_1-term that induces smaller prices. The binary variable s_i models if a compensation is needed for generator i to avoid a loss (the large constant $M > 0$ can be computed from the market configuration). Regarding the first three constraints, they prevent generators from making a profit when being compensated for a fixed cost. The forth and fifth constraints ensure revenue adequacy, so that generators do not incur into losses. In the next constraint, the parameter β times overall expenditure bounds compensations from above. The level of β is typically determined by the regulating agency. The final constraint states that prices and compensations can not be negative and that variables s_i are all binary.

The prices computed by solving (10) are reported in Figure 6. All the pricing systems yield some profit for generator 1, the most efficient one. With the new rule prices are higher than those in Figure 4, and compensations are smaller.

In the static setting considered so far ($T = 1$), the ith generator receives the compensation $E_i = e_i^t$. For UC problems (2) with longer time horizon, as the one considered next, the total amount is more significant that its specific distribution along time (the ith generator is concerned about the sum E_i and not the individual values e_i^t).

4.3. **Dynamics of pricing mechanisms.** The demand typically varies along the day between "base", "average", and "peak" levels. This is illustrated by the top line in Figure 7, with a typical demand hourly distribution for two weekdays of

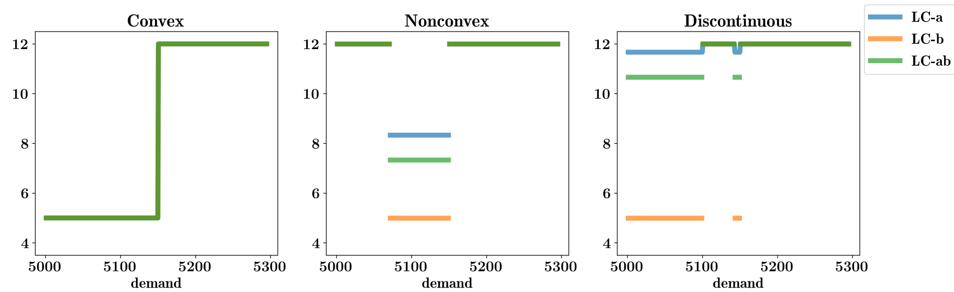

FIGURE 6. Price π^* given by the economic problem (10) with $\beta = 0.05$, for the three situations in Section 22.2, using as dispatch the output of the UC problem (2). The variants `LC-a, b`, `ab` correspond to taking $(a, b) \in \{(1, 0), (0, 1), (1, 1)\}$, respectively. Differently from the prices in Figure 4, the discontinuous setting reported on the right does not alternate between 12 and 5, in a bang-bang fashion. In the middle graph the prices from Figure 4 coincides with those one computed by the variant `LC-b`. The variants with $a > 0$, in the same region ($D \in [5050, 5150]$), bring the price closer to the target $\pi^* = 8$ (see the middle graph in Figure 3), making the drop in prices less abrupt than in Figure 4.

summer in Rio de Janeiro and São Paulo states in Brazil (source www.ons.org). The different pricing rules along the two days are shown in Figure 7.

The power system is composed of three ideal units, representing the base average and peak generation, with increasingly faster ramping dynamics and decreasing capacity (and a slack artificial unit with very high cost, to ensure feasibility). All units have start-up and shut-down costs and minimal generation requirements. The parameters in the limited compensation approach were $\beta = 0.05$ and two combinations $(a, b) = (1, 0)$, $(0.5, 0.1)$, denoted by `LC-a` and `LC-ab`, respectively. In the figures, the rule from [2] is referred to as `IP`.

The generators' perception is shown in Figure 8, for two of the units.

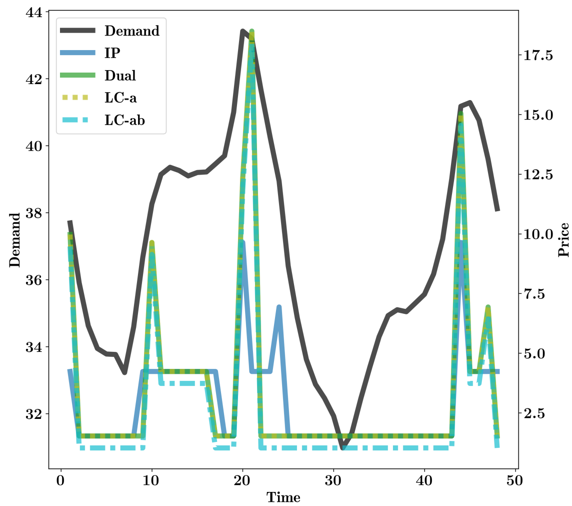

FIGURE 7. Different pricing schemes for a power system over an horizon with $T = 48$ hours. All the prices accompany the demand, with the rule `IP` paying lower prices in general, most notably at peak times (near hours 10, 20, and 40). The effect of using a positive b-term in (10) is noticeable by the rule `LC-ab` yielding lower prices than `LC-a`. The dual price is informed as a reminder of the level of prices that is needed in the market to ensure minimal uplifts to generators.

5. Conclusion

There is no definite answer on how to suitably remunerate the generators in the presence of nonconvexities. As shown, this yields to a positive duality gap, and this gap in fact measures the discrepancy between the ISO and the generators' expectations of how the system should be operated for them to make some profit. Depending on the system configuration, some approaches may be preferred. One needs to assess the behavior of the pricing system from different angles, including determining if prices follow the demand, to ensure that demand-response programs are effective. Also, compensations should ensure that no generator is at a loss.

The simple model derived from Table 1, when particularized in its three instances, illustrates well issues that need to be resolved. The different pricing mechanisms explored have all pros and cons. The proposal in Section 44.1 combines variable prices with compensations, so that no losses take place for the dispatched agents. Since the rule is designed to zero the surplus, some agents may end up without profit [14]. The new approach introduced in Subsection 44.2 presents some differential edge, as it makes explicit the generators' search for revenue-adequacy. Additionally, the economic nature of problem (10) makes it possible for the regulator to include price caps to correct market distortions, if needed.

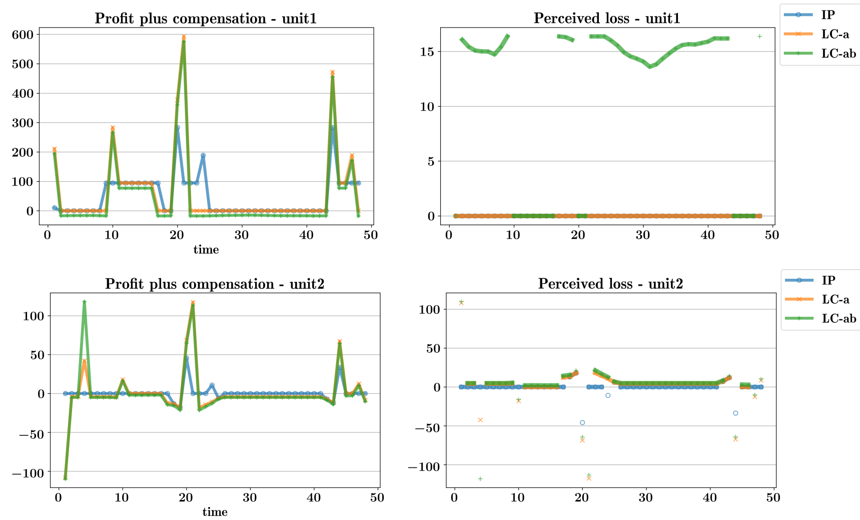

FIGURE 8. Profit, including compensations, and perceived losses for units 1 and 2. The agents profit mostly near peak times. At "valley" times (hours 0 to 10 and approximately 30 to 40), the lower prices paid by LC-ab, induced by taking $b > 0$ in (10), make unit 1 incur into a (small) loss; the rule remains revenue-adequate over the time horizon, by construction. With the pricing system IP, from [2], the more efficient unit 1 makes a profit and perceives no losses. Similarly for LC-a, only that with this rule the profit is always higher than IP, particularly at peak times. Neither rule paid a compensation to agent 1. With all the pricing systems, the less efficient unit 2 perceives losses only occasionally, also at the peak hour. The total compensation received by unit 2 along the period was IP=216.72, LC-a=46.37, and LC-ab=61.71.

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