Linear Groups of Isometries with Poset Structures

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Abstract

Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$ and $P = \{1, 2, \ldots, n\}$ a poset. We consider on V the poset-metric d_P. In this paper, we give a complete description of groups of linear isometries of the metric space (V, d_P), for any poset-metric d_P. We show that a linear isometry induces an automorphism of order in poset P, and consequently we show the existence of a pair of ordered bases of V relative to which every linear isometry is represented by an $n \times n$ upper triangular matrix.

Key words: Poset codes, poset metrics, linear isometries.

Coding theory takes place in finite dimensional linear spaces over finite fields. One of the main questions of the theory (classical problem) asks to find a k-dimensional subspace in $\mathbb{F}_q^n$, the space of n-tuples over the finite field $\mathbb{F}_q$, with the largest minimum distance possible. There are many possible metrics that can be defined in $\mathbb{F}_q^n$, the most common ones are the Hamming and Lee metrics.

In 1987 Harald Niederreiter generalized the classical problem of coding theory (see [11]). Brualdi, Graves and Lawrence (see [3]) also provided in

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1995 a wider situation for the above problem: using partially ordered sets and defining the concept of poset-codes, they started to study codes with a poset-metric. This has been a fruitful approach, since many new perfect codes have been found with such poset metrics (see [1], [3], [5], [8] and [9]).

We let P be a partially ordered set (abbreviated as *poset*) of cardinality n with order relation denoted, as usual, by $\leq$. An *ideal* of P is a subset $I \subseteq P$ with the property that $x \in I$ and $y \leq x$ implies that $y \in I$. Given $A \subseteq P$, we denote by $\langle A \rangle$ the smallest ideal of P containing A. Without loss of generality, we assume that $P = \{1, 2, \ldots, n\}$ and that the coordinates of vectors in $\mathbb{F}_q^n$ are in one-to-one correspondence with the elements of P.

Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, the *support of* x is the set

$$\operatorname{supp}(x) := \{i \in P : x_i \neq 0\},$$

and we define the *P-weight* of x to be the cardinality of the smallest ideal containing supp(x):

$$w_P(x) = |\langle \operatorname{supp}(x) \rangle|.$$

The function

$$d_P : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{N}$$

defined by $d_P(x, y) = w_P(x - y)$ is a metric in $\mathbb{F}_q^n$ ([3, Lemma 1.1]), called a *poset-metric* or a *P-poset-metric*, when it is important to stress the order taken in consideration. We denote such a metric space by $\left(\mathbb{F}_q^n, d_P\right)$.

An $[n, k, \delta_P]_q$ *poset-code* is a k-dimensional subspace $C \subset \mathbb{F}_q^n$, where $\mathbb{F}_q^n$ is endowed with a poset-metric d_P and

$$\delta_P(C) = \min\{w_P(x) : \mathbf{0} \neq x \in C\}$$

is the *P-minimum distance* of the code C. If P is an *antichain* order, that is, an order with no comparable elements, P-weight, P-poset-metric and P-minimum distance become the Hamming weight, Hamming metric and minimum distance of classical coding theory. Further notice that the Rosenbloom-Tsfasman metric, introduced in [12], can be viewed as a P-poset-metric which corresponds to the poset consisting of finite disjoint union of chains of equal lengths.

A *linear isometry* T of the metric space $\left(\mathbb{F}_q^n, d_P\right)$ is a linear transformation $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ that preserves P-poset-metric,

$$d_P(T(x), T(y)) = d_P(x, y),$$

for every $x, y \in \mathbb{F}_q^n$. Equivalently, a linear transformation T is an isometry if $w_P(T(x)) = w_P(x)$ for every $x \in \mathbb{F}_q^n$. A linear isometry of $\left(\mathbb{F}_q^n, d_P\right)$ is said to be a *P-isometry*. We denote by $GL_P\left(\mathbb{F}_q^n\right)$ the group of linear isometries of $\left(\mathbb{F}_q^n, d_P\right)$. In [4], [6], [10] some authours determined the group of linear isometries of the Rosenbloom-Tsfasman space, generalized Rosenbloom-Tsfasman space and crown space.

In this work, we give a complete description of those groups, for any given poset-metric P. The property of permuting chains of same length, showed in [10], corresponds, in the case of a general poset P, to Theorem 1.1 of the first section, which assures that every linear isometry T induces an automorphism of the poset P. The key-point for these proof is Proposition 1.1, which assures that $\langle \mathrm{supp}\,(T\,(u))\rangle \subseteq \langle \mathrm{supp}\,(T\,(v))\rangle$ if $\langle \mathrm{supp}\,(u)\rangle \subseteq \langle \mathrm{supp}\,(v)\rangle$, $u, v \in \mathbb{F}_q^n$. The characterization of linear isometries is given in Theorem 1.2: there is an ordered base β of $\mathbb{F}_q^n$ relative to which every $T \in GL_P\left(\mathbb{F}_q^n\right)$, is represented by the product $A \cdot U$ of matrices, where U is a monomial matrix corresponding to an isomorphism of the poset P and A is an upper-triangular matrix.

The second section is devoted to some examples, with a complete description of $GL_P\left(\mathbb{F}_q^n\right)$ where we give a detailed description of with some of the most commonly used poset-metrics: when the posets are disjoint union of chains, weak-metric and crown-metric.

1 Linear Isometries for a General Poset Structures

We will present only the concepts of the theory of partially ordered sets that are strictly necessary for this work, refereing the reader to [13] for more details.

A *totally ordered set* (or *linearly ordered set*) is a poset P in which any two elements are comparable. A subset C of a poset P is called a *chain* if C is a totally ordered set when regarded as a subposet of P.

Two posets P and Q are *isomorphic* if there exists an *order-preserving bijection* $\phi : P \to Q$, called of *isomorphism*, whose inverse is order preserving; that is,

$$x \leq y \text{ in } P \text{ if and only if } \phi(x) \leq \phi(y) \text{ in } Q.$$

An isomorphism $\phi : P \to P$ is called an *automorphism*.

Given $x, y \in P$, we say that y *covers* x if $x < y$ and if no element $z \in P$ satisfies $x < z < y$. A chain $x_1 < x_2 < \ldots < x_k$ in a finite poset P is called *saturated* if x_i covers x_{i-1} for $i \in \{1, 2, \ldots, k\}$.

From here on, we denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$.

Given an order automorphism $\phi : P \to P$, we define the *canonical linear P-isometry* T_ϕ *induced by* ϕ as $T_\phi(\sum_{i=1}^n a_i e_i) := \sum_{i=1}^n a_i e_{\phi(i)}$.

We will show that a linear isometry $T \in GL_P\left(\mathbb{F}_q^n\right)$ induces an automorphism of the poset P in the following way: given $i \in \{1, 2, \ldots, n\}$ we consider any saturated chain $i_1 < i_2 < \ldots < i_k$ containing i. Then there are $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$, with j_{s+1} covering j_s for all $s \in \{1, 2, \ldots, k-1\}$, such that $\langle \operatorname{supp}(e_{j_l}) \rangle = \langle \operatorname{supp}(T(e_{i_l})) \rangle$ for any $l \in \{1, 2, \ldots, k\}$. So, if $i = i_l$, we can define the order automorphism ϕ by $\phi(i_l) = j_l$.

The key to prove this is to show that $\langle \operatorname{supp}(T(u)) \rangle \subseteq \langle \operatorname{supp}(T(v)) \rangle$ if $\langle \operatorname{supp}(u) \rangle \subseteq \langle \operatorname{supp}(v) \rangle$, for every $T \in GL_P\left(\mathbb{F}_q^n\right)$.

We will start with some preliminary lemmas.

Lemma 1.1 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $T \in GL_P(\mathbb{F}_q^n)$ and $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$. If $\langle \operatorname{supp}(e_i) \rangle \subseteq \langle \operatorname{supp}(e_j) \rangle$, then*

$$\langle \operatorname{supp}(T(e_i)) \rangle \subseteq \langle \operatorname{supp}(T(e_j)) \rangle .$$

Proof. We observe that, for any vectors $u, v \in \mathbb{F}_q^n$, if $\operatorname{supp}(u) \subseteq \operatorname{supp}(v)$ then $w_P(u) \leq w_P(v)$. Moreover, the inequality is strict if and only if $\langle \operatorname{supp}(u) \rangle \subsetneq \langle \operatorname{supp}(v) \rangle$. We remember that T is a linear isometry, so that $w_P(v) = w_P(T(v))$, for every vector v.

We prove the lemma by contradiction, assuming that $\langle \operatorname{supp}(T(e_i)) \rangle \nsubseteq \langle \operatorname{supp}(T(e_j)) \rangle$.

Suppose $\langle \operatorname{supp}(T(e_i)) \rangle \cap \langle \operatorname{supp}(T(e_j)) \rangle = \varnothing$. Since T is linear,

$$w_P(T(e_i + e_j)) = w_P(T(e_i) + T(e_j))$$

and since the ideals do not intersect, we have that

$$w_P(T(e_i) + T(e_j)) = w_P(T(e_i)) + w_P(T(e_j)) .$$

Since T is an isometry, we find that

$$w_P(T(e_i)) + w_P(T(e_j)) = w_P(e_i) + w_P(e_j) > w_P(e_j)$$
$$w_P(T(e_i + e_j)) = w_P(e_i + e_j) .$$

However, we are assuming that $\langle \operatorname{supp}(e_i) \rangle \subseteq \langle \operatorname{supp}(e_j) \rangle$, so that $w_P(e_i + e_j) = w_P(e_j)$, a contradiction.

Now we can assume that $\langle \operatorname{supp}(T(e_i)) \rangle \cap \langle \operatorname{supp}(T(e_j)) \rangle \neq \varnothing$. If we put $\operatorname{supp}(T(e_i)) \cap \operatorname{supp}(T(e_j)) = \{k_1, \ldots, k_r\}$, we have two cases to consider.

Case 1: $\{k_1, \ldots, k_r\} \neq \varnothing$.

In this case, we can write

$$\operatorname{supp}(T(e_i)) = \{k_1, \ldots, k_r\} \cup \{i_1, \ldots, i_s\}$$

and

$$T(e_i) = \alpha_{k_1} e_{k_1} + \ldots + \alpha_{k_r} e_{k_r} + \beta_{i_1} e_{i_1} + \ldots + \beta_{i_s} e_{i_s}.$$

Let

$$y = e_i - \beta_{i_1} T^{-1}(e_{i_1}) - \ldots - \beta_{i_s} T^{-1}(e_{i_s}).$$

Then

$$w_P(y) \geq w_P(e_i),$$

unless

$$e_i = \beta_{i_1} T^{-1}(e_{i_1}) + \ldots + \beta_{i_s} T^{-1}(e_{i_s}) = T^{-1}(\beta_{i_1} e_{i_1} + ... + \beta_{i_s} e_{i_s}),$$

contradicting the hypothesis that $\{k_1, \ldots, k_r\} \neq \varnothing$. But $T(y) = \alpha_{k_1} e_{k_1} + \ldots + \alpha_{k_r} e_{k_r}$, and since there is $i_l \in \{i_1, \ldots, i_s\} \subseteq \mathrm{supp}(T(e_i))$ such that $i_l \notin \mathrm{supp}(T(e_j))$, we find that $w_P(T(y)) < w_P(T(e_i)) = w_P(e_i)$. So

$$w_P(T(y)) < w_P(y),$$

a contradiction.

Case 2: $\{k_1, \ldots, k_r\} = \varnothing$.

This means that $\mathrm{supp}(T(e_i)) \cap \mathrm{supp}(T(e_j)) = \varnothing$. Put $T(e_i) = \alpha_{i_1} e_{i_1} + \ldots + \alpha_{i_t} e_{i_t}$. Then there is an

$$l \in \langle \mathrm{supp}(T(e_i)) \rangle \backslash \mathrm{supp}(T(e_i)). \tag{1}$$

Let

$$y = e_i - \alpha_{i_1} T^{-1}(e_{i_1}) - \ldots - \alpha_{i_t} T^{-1}(e_{i_t}) + T^{-1}(e_l).$$

Then

$$w_P(y) \geq w_P(e_i),$$

unless $e_i = T^{-1}(e_l)$, and this contradicts (1). But, $T(y) = e_l$ and hence

$$w_P(T(y)) = w_P(e_l) < w_P(e_i) \leq w_P(y),$$

again a contradiction. □

Lemma 1.2 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $T \in GL_P(\mathbb{F}_q^n)$ and $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$. Then,*

$$\bigcup_{i=1}^{s} \langle \mathrm{supp}(T(e_{j_i})) \rangle = \left\langle \mathrm{supp}\left(\sum_{i=1}^{s} T(e_{j_i}) \right) \right\rangle,$$

for every $s \in \{1, 2, \ldots, n\}$ and $j_1, \ldots, j_s \in \{1, \ldots, n\}$.

Proof. If $j \in \langle \operatorname{supp}(\sum_{i=1}^{s} T(e_{j_i})) \rangle$, there is an i such that $j \in \langle \operatorname{supp}(T(e_{j_i})) \rangle$, so that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T(e_{j_i})\right)\right\rangle \subseteq \bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle .$$

We will prove the other inclusion by induction on s. The case $s = 1$ is trivial and we can assume, as the induction hypothesis that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s-1} T(e_{j_i})\right)\right\rangle = \bigcup_{i=1}^{s-1} \langle \operatorname{supp}(T(e_{j_i})) \rangle ,$$

for every subset $\{j_1, \ldots, j_{s-1}\} \subseteq \{1, \ldots, n\}$.

Given $J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n\}$ and $t \in \{1, 2, \ldots, s\}$, we can define

$$\Theta_{J,t} = \langle \operatorname{supp}(T(e_{j_t})) \rangle \setminus \left(\bigcup_{i=1, i \neq t}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle \right).$$

But $\Theta_{J,t} = \varnothing$ means that every $j \in \langle \operatorname{supp}(T(e_{j_t})) \rangle$ we have

$$j \in \bigcup_{i=1, i \neq t}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle$$

so that

$$\bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle = \bigcup_{i=1, i \neq t}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle$$

and by the induction hypothesis we have that

$$\bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle = \left\langle \operatorname{supp}\left(\sum_{i=1, i \neq t}^{s} T(e_{j_i})\right)\right\rangle . \tag{2}$$

Since

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T(e_{j_i})\right)\right\rangle \subseteq \bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle$$

we have that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T(e_{j_i})\right)\right\rangle \subseteq \left\langle \operatorname{supp}\left(\sum_{i=1, i \neq t}^{s} T(e_{j_i})\right)\right\rangle . \tag{3}$$

Since T is a linear isometry, we have that

$$w_P\left(\sum_{i=1}^{s} T\left(e_{j_i}\right)\right) = w_P\left(T\left(\sum_{i=1}^{s} e_{j_i}\right)\right) = w_P\left(\sum_{i=1}^{s} e_{j_i}\right),$$

$$w_P\left(\sum_{i=1, i\neq t}^{s} T\left(e_{j_i}\right)\right) = w_P\left(T\left(\sum_{i=1, i\neq t}^{s} e_{j_i}\right)\right) = w_P\left(\sum_{i=1, i\neq t}^{s} e_{j_i}\right).$$

But

$$w_P\left(\sum_{i=1}^{s} e_{j_i}\right) \geq w_P\left(\sum_{i=1, i\neq t}^{s} e_{j_i}\right) \tag{4}$$

and since by definition, we have that $w_P(v) = |\langle \operatorname{supp}(v)\rangle|$, considering inequality (4) in (3) we find that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T\left(e_{j_i}\right)\right)\right\rangle = \left\langle \operatorname{supp}\left(\sum_{i=1, i\neq t}^{s} T\left(e_{j_i}\right)\right)\right\rangle$$

and from (2) we get that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T\left(e_{j_i}\right)\right)\right\rangle = \bigcup_{i=1}^{s} \left\langle \operatorname{supp}\left(T\left(e_{j_i}\right)\right)\right\rangle,$$

so that the lemma holds if for every $s \geq 2$, there is $J = \{j_1, \ldots, j_s\}$ and $t \in \{1, 2, \ldots, s\}$ such that $\Theta_{J,t} = \varnothing$.

The case of an antichain P is trivial, so we can assume that the poset P is not an antichain order, and hence there are $l_1, l_2 \in \{1, 2, \ldots, n\}$ such that l_2 covers l_1. So, given $s \geq 2$, for every $J = \{l_1, l_2, j_3, \ldots, j_s\}$ we have that $\Theta_{J,l_1} = \varnothing$, since

$$\langle \operatorname{supp}(e_{l_1})\rangle = \langle l_1\rangle \subseteq \langle l_2\rangle = \langle \operatorname{supp}(e_{l_2})\rangle.$$

□

Now we can state and prove the proposition that extends Lemma 1.1 to general vectors.

Proposition 1.1 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $T \in GL_P(\mathbb{F}_q^n)$. Then, for every $u, v \in \mathbb{F}_q^n$,*

$$\langle \operatorname{supp}(T(u))\rangle \subseteq \langle \operatorname{supp}(T(v))\rangle,$$

if $\langle \operatorname{supp}(u)\rangle \subseteq \langle \operatorname{supp}(v)\rangle$.

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$ and express u and v as a linear combination of this base:

$$u = \alpha_1 e_{u_1} + \alpha_2 e_{u_2} + \ldots + \alpha_r e_{u_r}$$
$$v = \beta_1 e_{v_1} + \beta_2 e_{v_2} + \ldots + \beta_s e_{v_s}$$

with $\operatorname{supp}(u) = \{u_1, \ldots, u_r\}$ and $\operatorname{supp}(v) = \{v_1, \ldots, v_s\}$. Since $\langle \operatorname{supp}(u) \rangle \subseteq \langle \operatorname{supp}(v) \rangle$ we have that $\langle \operatorname{supp}(e_{u_i}) \rangle \subseteq \langle \operatorname{supp}(v) \rangle$ for every $i \in \{1, 2, \ldots, r\}$, so there is an $j \in \{1, 2, \ldots, s\}$ such that $\langle \operatorname{supp}(e_{u_i}) \rangle \subseteq \left\langle \operatorname{supp}\left(e_{v_j}\right) \right\rangle$. But Lemma 1.1 assures that $\langle \operatorname{supp}(T(e_{u_i})) \rangle \subseteq \left\langle \operatorname{supp}\left(T\left(e_{v_j}\right)\right) \right\rangle$. It follows that

$$\begin{aligned} \langle \operatorname{supp}(T(u)) \rangle &= \left\langle \operatorname{supp}\left(\sum_{i=1}^{r} T(e_{u_i}) \right) \right\rangle \\ &\subseteq \bigcup_{i=1}^{r} \langle \operatorname{supp}(T(e_{u_i})) \rangle \\ &\subseteq \bigcup_{j=1}^{s} \left\langle \operatorname{supp}\left(T\left(e_{v_j}\right)\right) \right\rangle \end{aligned}$$

and by Lemma 1.2 we have that

$$\begin{aligned} \langle \operatorname{supp}(T(v)) \rangle &= \left\langle \operatorname{supp}\left(\sum_{j=1}^{s} T\left(\beta_j e_{v_j}\right) \right) \right\rangle \\ &= \bigcup_{j=1}^{s} \left\langle \operatorname{supp}\left(T\left(\beta_j e_{v_j}\right)\right) \right\rangle \\ &= \bigcup_{j=1}^{s} \left\langle \operatorname{supp}\left(T\left(e_{v_j}\right)\right) \right\rangle \end{aligned}$$

and we find

$$\langle \operatorname{supp}(T(u)) \rangle \subseteq \langle \operatorname{supp}(T(v)) \rangle .$$

□

An ideal I of a poset P is said to be a *prime ideal* if it contains a unique maximal element.

Lemma 1.3 *Let $P = \{1, 2, ..., n\}$ be a poset, $\beta = \{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$ and $T \in GL_P\left(\mathbb{F}_q^n\right)$. Then, for every $r \in \{1, 2, \ldots, n\}$, we have that $\langle \operatorname{supp}(T(e_r)) \rangle$ is a prime ideal.*

Proof. We want to prove that the ideal $\langle \mathrm{supp}\,(T\,(e_r))\rangle$ is generated by a single greatest element (greater than every other element), or alternatively, it has only one maximal element (no one greater than it). Let $\{j_1, j_2, \ldots, j_k\}$ be a set of maximal elements in $\langle \mathrm{supp}\,(T\,(e_r))\rangle$. Then we have that

$$\begin{aligned}\langle \mathrm{supp}\,(T\,(e_r))\rangle &= \bigcup_{i=1}^{k} \langle j_i \rangle \\ &= \bigcup_{i=1}^{k} \langle \mathrm{supp}\,(e_{j_i})\rangle \\ &= \left\langle \mathrm{supp}\left(\sum_{i=1}^{r} e_{j_i}\right)\right\rangle .\end{aligned}$$

But Proposition 1.1 assures that we can apply T^{-1} to both sides of the equation above preserving the equality, so that

$$\langle \mathrm{supp}\,(e_r)\rangle = \left\langle \mathrm{supp}\left(T^{-1}T\,(e_r)\right)\right\rangle = \left\langle \mathrm{supp}\left(T^{-1}\left(\sum_{i=1}^{r} e_{j_i}\right)\right)\right\rangle . \qquad (5)$$

Since T^{-1} is linear, we have that

$$\left\langle \mathrm{supp}\left(T^{-1}\left(\sum_{i=1}^{r} e_{j_i}\right)\right)\right\rangle = \left\langle \mathrm{supp}\left(\sum_{i=1}^{r} T^{-1}\,(e_{j_i})\right)\right\rangle$$

and by Lemma 1.2, we have that

$$\left\langle \mathrm{supp}\left(\sum_{i=1}^{r} T^{-1}\,(e_{j_i})\right)\right\rangle = \bigcup_{i=1}^{k} \left\langle \mathrm{supp}\left(T^{-1}\,(e_{j_i})\right)\right\rangle . \qquad (6)$$

But looking at equations (5) and (6) we find that $\bigcup_{i=1}^{k} \langle \mathrm{supp}\,(T^{-1}\,(e_{j_i}))\rangle$ is the prime ideal $\langle \mathrm{supp}\,(e_r)\rangle$. Since we are expressing a prime ideal as the union of ideals, one of them, let us say $\langle \mathrm{supp}\,(T^{-1}\,(e_{j_s}))\rangle$ for some $s \in \{1, 2, \ldots, r\}$, must contain the maximal element r and hence $\langle \mathrm{supp}\,(T^{-1}\,(e_{j_s}))\rangle = \langle \mathrm{supp}\,(e_r)\rangle$. Using again Proposition 1.1, we find that

$$\langle \mathrm{supp}\,(e_{j_s})\rangle = \langle \mathrm{supp}\,(T\,(e_r))\rangle$$

so that $\langle \mathrm{supp} T\,(e_r)\rangle$ is a prime ideal and consequently $\{j_1, j_2, \ldots, j_k\} = \{j_s\}$. $\square$

Now we can state and prove the proposition that extends Lemma 1.3 to the general case.

Proposition 1.2 *Let $P = \{1, 2, \ldots, n\}$ be a poset and $T \in GL_P\left(\mathbb{F}_q^n\right)$. Then, for every $v \in \mathbb{F}_q^n$ such that $\langle \text{supp}(v) \rangle$ is a prime ideal, we have that $\langle \text{supp}(T(v)) \rangle$ is also a prime ideal.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$ and $v \in \mathbb{F}_q^n$. Suppose that $v = \alpha_1 e_{i_1} + \ldots + \alpha_s e_{i_s}$. Then

$$\begin{aligned} \langle \text{supp}(v) \rangle &= \langle \text{supp}(\alpha_1 e_{i_1} + \ldots + \alpha_s e_{i_s}) \rangle \\ &= \langle \text{supp}(e_{i_1}) \rangle \cup \ldots \cup \langle \text{supp}(e_{i_s}) \rangle , \end{aligned}$$

and since $\langle \text{supp}(v) \rangle$ is a prime ideal, it follows there is an $k \in \{1, 2, \ldots, s\}$ such that

$$\langle \text{supp}(e_{i_1}) \rangle \cup \ldots \cup \langle \text{supp}(e_{i_s}) \rangle = \langle \text{supp}(e_{i_k}) \rangle$$

so that $\langle \text{supp}(v) \rangle = \langle \text{supp}(e_{i_k}) \rangle$. Lemma 1.1 assures that

$$\langle \text{supp}(T(v)) \rangle = \langle \text{supp}(T(e_{i_k})) \rangle ,$$

and as $\langle \text{supp}(T(e_{i_k})) \rangle$ is a prime ideal (by Lemma 1.3), and we conclude that $\langle \text{supp}(T(v)) \rangle$ is a prime ideal. □

Lemma 1.4 *If k covers i and J is an ideal such that $\langle i \rangle \subseteq J \subseteq \langle k \rangle$, then $J = \langle i \rangle$ or $J = \langle k \rangle$.*

Proof. If $\langle i \rangle = J$, there is nothing to be proved. So, we assume that $\langle i \rangle \subsetneq J \subseteq \langle k \rangle$. Then, there is an $j \in J$ such that $j \gneqq i$. Since $J \subseteq \langle k \rangle$ it follows that $j \leq k$. So $i \lneqq j \leq k$, and since k covers i, we have that $j = k$ and hence $J = \langle k \rangle$. □

Theorem 1.1 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $\{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$ and $T \in GL_P\left(\mathbb{F}_q^n\right)$ linear isometry. Then, for every saturated chain with a minimal element $i_i < i_2 < \ldots < i_r$ there is a unique saturated sequence of prime ideals*

$$\langle \text{supp}(e_{j_1}) \rangle \subset \langle \text{supp}(e_{j_2}) \rangle \subset \ldots \subset \langle \text{supp}(e_{j_r}) \rangle .$$

such that

$$\langle \text{supp}(T(e_{i_k})) \rangle = \langle \text{supp}(e_{j_k}) \rangle$$

for every $k \in \{1, 2, \ldots, r\}$ and

$$\begin{array}{rccl} \phi : & P & \longrightarrow & P \\ & i_k & \longmapsto & \phi(i_k) := j_k \end{array}$$

is a well defined poset automorphism.

Proof. Proposition 1.2 assures us that $\langle \operatorname{supp}(T(e_{i_k}))\rangle$ is a prime for all $k \in \{1, 2, \ldots, r\}$, since $\langle \operatorname{supp}(e_{i_k})\rangle$ is a prime ideal. Then for each $k \in \{1, 2, \ldots, r\}$ there is just one maximal element $j_k \in \langle \operatorname{supp}(T(e_{i_k}))\rangle$. So $\langle \operatorname{supp}(T(e_{i_k}))\rangle = \langle \operatorname{supp}(e_{j_k})\rangle$ for all $k \in \{1, 2, \ldots, r\}$. Since

$$\langle \operatorname{supp}(e_{i_1})\rangle \subset \langle \operatorname{supp}(e_{i_2})\rangle \subset \ldots \subset \langle \operatorname{supp}(e_{i_r})\rangle ,$$

it follows, from Proposition 1.1, that

$$\langle \operatorname{supp}(e_{j_1})\rangle \subset \langle \operatorname{supp}(e_{j_2})\rangle \subset \ldots \subset \langle \operatorname{supp}(e_{j_r})\rangle .$$

We affirm now that the sequence above is saturated. Suppose that for some $k \in \{1, 2, \ldots, r\}$ there is j' such that

$$\langle j_k\rangle \subsetneqq \langle j'\rangle \subsetneqq \langle j_{k+1}\rangle .$$

Since

$$\begin{aligned} \langle j_k\rangle &= \langle \operatorname{supp}(e_{j_k})\rangle = \langle \operatorname{supp}(T(e_{i_k}))\rangle ,\\ \langle j_{k+1}\rangle &= \left\langle \operatorname{supp}\left(e_{j_{k+1}}\right)\right\rangle = \left\langle \operatorname{supp}\left(T\left(e_{i_{k+1}}\right)\right)\right\rangle , \end{aligned}$$

it follows, applying Proposition 1.1) to the linear P-isometry T^{-1}, that

$$\begin{aligned} \langle i_k\rangle &= \left\langle \operatorname{supp}\left(T^{-1}T(e_{i_k})\right)\right\rangle \\ &\subsetneqq \left\langle \operatorname{supp}\left(T^{-1}(e_{j'})\right)\right\rangle \\ &\subsetneqq \left\langle \operatorname{supp}\left(T^{-1}T\left(e_{i_{k+1}}\right)\right)\right\rangle = \langle i_{k+1}\rangle , \end{aligned}$$

what contradicts, by Lemma 1.4, the hypothesis that $i_1 < \ldots < i_r$ is a saturated chain.

Let us now define $\phi : P \to P$ by $\phi(i_l) = j_l$. Since j_l is uniquely defined and does not depends on the choice of the saturated chain containing i_l (but only on $T(e_{i_l})$), we have that ϕ is well defined. Moreover, let us suppose that $x < y$ in P, and let

$$i_1 < \ldots < i_{k-1} < x < i_{k+1} < \ldots < i_{l-1} < y < i_{l+1} < \ldots < i_r$$

be a saturated chain containing x and y. Then there is only one saturated chain

$$j_1 < \ldots < j_{k-1} < j_k < j_{k+1} < \ldots < j_{l-1} < j_l < j_{l+1} < \ldots < j_r$$

such that $\phi(x) = j_k$ and $\phi(y) = j_l$. Since $j_k < j_l$ we get that $\phi(x) < \phi(y)$. Therefore ϕ is an application that preserves the order on P.

Finally, we affirm that ϕ is one-to-one. In fact, suppose that $\phi(x) = \phi(y)$. As $\phi(x) = \max \langle \operatorname{supp}(T(e_x)) \rangle$ and $\phi(y) = \max \langle \operatorname{supp}(T(e_y)) \rangle$ then

$$\langle \operatorname{supp}(T(e_x)) \rangle = \langle \operatorname{supp}(T(e_y)) \rangle ,$$

and from Proposition 1.1 follows that

$$\langle \operatorname{supp}(e_x) \rangle = \left\langle \operatorname{supp}\left(T^{-1}T(e_x)\right) \right\rangle = \left\langle \operatorname{supp}\left(T^{-1}T(e_y)\right) \right\rangle = \langle \operatorname{supp}(e_y) \rangle .$$

As both ideals $\langle \operatorname{supp}(e_x) \rangle$ and $\langle \operatorname{supp}(e_y) \rangle$ are primes, we must have $x = y$. Being ϕ one-to-one and P finite, we find that ϕ is a bijection that preserves the order and we conclude that ϕ is an automorphism of P. □

The *m-th level* $\Gamma^{(m)}(P)$ is the set of elements of P that generates a prime ideal with cardinality m:

$$\Gamma^{(m)}(P) = \{i \in P : |\langle i \rangle| = m\} = \{i \in P : w_P(e_i) = m\} .$$

We now describe the main result of this work:

Theorem 1.2 *Let $P = \{1, 2, \ldots, n\}$ be a poset and $\{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$. Then $T \in GL_P(\mathbb{F}_q^n)$ if and only if*

$$T(e_j) = \sum_{i \in \langle j \rangle} x_{ij} e_{\phi(i)}$$

where $\phi : P \to P$ is an order automorphism and $x_{jj} \neq 0$, for any $j \in \{1, 2, \ldots, n\}$. Moreover, there is a pair of ordered bases β and β' of $\mathbb{F}_q^n$ relative to which every linear isometry $T \in GL_P\left(\mathbb{F}_q^n\right)$ is represented by an $n \times n$ upper triangular matrix $(a_{ij})_{1 \leq i,j \leq n}$ with $a_{ii} \neq 0$ for every $i \in \{1, 2, \ldots, n\}$.

Proof. Since $\langle \operatorname{supp}(e_j) \rangle$ is a prime ideal, it follows from Proposition 1.2 that $\langle \operatorname{supp}(T(e_j)) \rangle$ is also a prime ideal, for every $j \in \{1, 2, \ldots, n\}$. Given $j \in \{1, 2, \ldots, n\}$, let $j' = \phi(j)$ be the unique maximal element of the ideal $\langle \operatorname{supp}(T(e_j)) \rangle$, where $\phi : P \to P$ is the order automorphism induced by the isometry T (see Theorem 1.1). Then

$$\langle \operatorname{supp}(T(e_j)) \rangle = \langle \operatorname{supp}(e_{j'}) \rangle = \left\langle \operatorname{supp}\left(e_{\phi(j)}\right) \right\rangle ,$$

and since ϕ is a automorphism of order we have that

$$\left\langle \operatorname{supp}\left(e_{\phi(j)}\right) \right\rangle = \{\phi(i) : i \in \langle j \rangle\} .$$

Therefore $\langle \mathrm{supp}\,(T\,(e_j))\rangle = \{\phi\,(i) : i \in \langle j\rangle\}$. Being $\phi\,(j) = \max\{\phi\,(i) : i \in \langle j\rangle\}$, we conclude that

$$T(e_j) = \sum_{i\in\langle j\rangle} x_{ij} e_{\phi(i)} \tag{7}$$

with $x_{jj} \neq 0$. It is straightforward to verify that for a given order automorphism $\phi : P \to P$, any linear map defined as in (7) is a P-isometry.

Let $\beta_m = \left\{e_i : i \in \Gamma^{(m)}\,(P)\right\}$ and

$$\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k.$$

be a decomposition of the canonical base of $\mathbb{F}_q^n$ as a disjoint union, where $k = \max\{w_P\,(e_i) : i = 1, 2, \ldots, n\}$. We order this base $\beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ in the following way (and denoted this total order by $\leq_\beta$): if $e_{i_r} \in \beta_{j_r}$ and $e_{i_s} \in \beta_{j_s}$ with $r \neq s$ then, $e_{i_r} \leq_\beta e_{i_s}$ if and only $j_r \leq j_s$. In other words, we begin enumerating the the vectors of β_1 and after exhausting them, we enumerate the vectors of β_2 and so on.

We define another ordered base β' as the base induced by the order automorphism ϕ,

$$\beta' := \left\{e_{\phi(i_1)}, e_{\phi(i_2)}, \ldots, e_{\phi(i_n)}\right\}$$

and let A be the matrix of T relative to the basis β and β':

$$[T]_{\beta,\beta'} = A = (a_{kl})_{1\leq k,l\leq n}\,.$$

We find by the construction of the bases β and β' that $a_{kl} \neq 0$ implies $i_l \in \langle\phi\,(i_k)\rangle$. But $i_l \in \langle\phi\,(i_k)\rangle$ and $\langle i_l\rangle \neq \langle\phi\,(i_k)\rangle$ implies that $l < k$ so that A is upper triangular. Since A is invertible and upper triangular, we must have $\det(A) = \prod_{i=1}^n a_{ii} \neq 0$ so that $a_{ii} \neq 0$, for every $i \in \{1, 2, \ldots, n\}$. □

The upper triangular matrix obtained in the previous theorem is called a *canonical form of* T. We note that the ordered bases chosen in the theorem is unique up to re-ordination within the linearly independent sets β_i, $i = 1, 2, \ldots, k$.

As in [14], a *monomial matrix* is a matrix with exactly one nonzero entry in each row and column. Thus a monomial matrix over $\mathbb{F}_2$ is a permutation matrix, and a monomial matrix over an arbitrary finite field is a permutation matrix times an invertible diagonal matrix.

Corollary 1.1 *Given* $T \in GL_P\left(\mathbb{F}_q^n\right)$ *there is an ordering* $\beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ *of the canonical base such that* $[T]_{\beta,\beta}$ *is given by the product* $A \cdot U$ *where* A

is an invertible upper triangular matrix and U is a monomial matrix obtained from the identity matrix by permutation of the columns, corresponding to the automorphism of order induced by T .

Proof. Let ϕ be the automorphism of order induced by T. Let $T_{\phi^{-1}}$ be the linear isometry defined as $T_{\phi^{-1}}(e_j) = e_{\phi^{-1}(j)}$, for $j \in \{1, 2, \ldots, n\}$. As we saw in Theorem 1.2,

$$T(e_j) = \sum_{i \in \langle j \rangle} x_{ij} e_{\phi(i)}.$$

So,

$$\begin{aligned} T \circ T_{\phi^{-1}}(e_j) &= T\left(e_{\phi^{-1}(j)}\right) \\ &= \sum_{i \in \langle \phi^{-1}(j) \rangle} x_{i\phi^{-1}(j)} e_{\phi(i)} \\ &= x_{i\phi^{-1}(j)} e_j + \sum_{i \in \langle \phi^{-1}(j) \rangle, i \neq \phi^{-1}(j)} x_{i\phi^{-1}(j)} e_{\phi(i)}. \end{aligned}$$

It follows that the automorphism of order induced by $T \circ T_{\phi^{-1}}$ is the identity, so, when taking the base β' as in the Theorem 1.2, we find that $\beta' = \beta$ and the matrix of $T \circ T_{\phi^{-1}}$ relative to this base is an upper triangular matrix $A = [T \circ T_{\phi^{-1}}]_\beta$. But $T_{\phi^{-1}}$ acts on $\mathbb{F}_q^n$ as a permutation of the vectors in β, so that in any ordered base containing those vectors, $U^{-1} = [T_{\phi^{-1}}]$ is obtained from the identity matrix by permutation of the columns. We note that $T_\phi = (T_{\phi^{-1}})^{-1}$ and it follows that

$$\begin{aligned} [T]_\beta &= [T \circ T_{\phi^{-1}} \circ T_\phi]_\beta \\ &= [T \circ T_{\phi^{-1}}]_\beta [T_\phi]_\beta \\ &= A \cdot U. \end{aligned}$$

□

Given a poset $P = \{1, 2, \ldots, n\}$, we denote by $Aut(P)$ the group of the order-automorphisms of P.

Corollary 1.2 *Let $P = \{1, \ldots, n\}$ by a poset and $k = \max\left\{m : \Gamma^{(m)}(P) \neq \varnothing\right\}$. Then*

$$\left|GL_P\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot \left(\prod_{i=1}^{k} q^{(i-1)\left|\Gamma^{(i)}(P)\right|}\right) \cdot |Aut(P)|.$$

Proof. From Corollary 1.1, if $T \in GL_P\left(\mathbb{F}_q^n\right)$ there is an ordered base $\beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ of the canonical base of $\mathbb{F}_q^n$ such that $|\langle i_l \rangle| \leq l$ for all $l \in \{1, 2, \ldots, n\}$ and $[T]_\beta = A \cdot U$, being $A = (a_{kl})_{1 \leq k,l \leq n}$ an upper triangular matrix with $a_{kl} = 0$ if $i_k \notin \langle i_l \rangle$ and $U = [T_\phi]_\beta$ the matrix representing the automorphism ϕ induced by linear isometry T (see Theorem 1.2). Moreover, such base β depends only on ϕ and for every $\phi \in Aut\,(P)$, any matrix A as in the previous Corollary defines a linear P-isometry.

Given $l \in \{1, 2, \ldots, n\}$, there are $(q-1)$ possible different entries for a_{ll} (since $a_{ll} \neq 0$). But A is upper triangular, given $1 \leq i < j \leq n$ we have that $a_{ij} \neq 0$ only if $i \in \langle j \rangle$, so there are at most $|\langle j \rangle| - 1$ possible nonzero indices (i, j) with $1 \leq i < j \leq n$, and for each of those there are q possible different entries. Since there are exactly $\left|\Gamma^{(|\langle j \rangle|)}(P)\right|$ such indices, we find that, up to considering the order automorphism induced by the isometry, there are

$$(q-1)^n \cdot \left(\prod_{i=1}^{k} q^{(i-1)\left|\Gamma^{(i)}(P)\right|} \right)$$

linear P-isometries and we conclude counting the elements of $Aut(P)$. □

Let $M_{n \times n}(\mathbb{F}_q)$ be the set of all $n \times n$ matrices over $\mathbb{F}_q$ and

$$G_P = \left\{ (a_{ij}) \in M_{n \times n}(\mathbb{F}_q) : a_{ij} \in \left| \begin{array}{lc} \mathbb{F}_q & \text{if} \quad i <_P j \\ \mathbb{F}_q^* & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{array} \right. \right\}.$$

As we have seen, this is the set of elements in $GL_P\left(\mathbb{F}_q^n\right)$ that corresponds to isometries that induces the trivial automorphism of order. So, we have the following characterization:

Corollary 1.3 *With the definitions above, the group of isometries of* $\left(\mathbb{F}_q^n, d_P\right)$ *is the semi-direct product* $GL_P\left(\mathbb{F}_q^n\right) \simeq G_P \rtimes Aut\,(P)$.

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be elements in G_P. Since

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \sum_{i \leq_P k \leq_P j} a_{ik} b_{kj}$$

we have that $AB \in G_P$. We note that every element in G_P is an upper triangular matrix with nonzero diagonal entries. Hence, such elements are invertible. Since the inverse of an element in G_P is a polynomial in that element, such an element is in G_P. So, we see that G_P is a subgroup of $GL_P\left(\mathbb{F}_q^n\right)$. Since we already proved that $GL_P\left(\mathbb{F}_q^n\right) = G_P \cdot Aut\left(P\right)$, all is left to show is that G_P is a normal subgroup of $GL_P\left(\mathbb{F}_q^n\right)$. Given $\phi \in S_n$, it acts on $n \times n$ matrices by permuting columns or rows. We denote by A^{ϕ} and ${}^{\phi}A$ respectively the column and row permutation of the matrix A. It is straightforward to show that $\left({}^{\phi}Id\right)^{-1} = Id^{\phi}$([4]). It follows that

$$\left({}^{\phi}Id\right) A \left({}^{\phi}Id\right)^{-1} =^{\phi} A^{\phi}$$

for every $n \times n$ matrix A. If $A = (a_{ij}) \in G_P$, for each $i = 1, 2, ..., n$ we have that

$$\begin{aligned}\left({}^{\phi}Id\right) A \left({}^{\phi}Id\right)^{-1} (e_i) =^{\phi} A^{\phi} (e_i) &= \sum_{k=1}^{n} a_{\phi(k)\phi(i)} e_k \\ &= \sum_{\phi(k) \leq_P \phi(i)} a_{\phi(k)\phi(i)} e_k \\ &= \sum_{k \leq_P i} a_{\phi(k)\phi(i)} e_k\end{aligned}$$

and $a_{\phi(i)\phi(i)} \neq 0$ for every i. Thus, we find that G_P is normal in $GL_P\left(\mathbb{F}_q^n\right)$ and the proposition follows. □

a

Corollary 1.4 *Let P and Q be order posets. Then we have*

1. $G_{P\times Q} = G_P \otimes G_Q$;
2. $G_{P\mathring{\cup}Q} \simeq G_P \times G_Q$;
3. *If Q is a disjoint union of m's posets P on $\{1, 2, ..., n\}$, then we have $Aut\left(Q\right) \simeq Aut\left(P\right) S_n$.*

Proof. All the claims follow straight from the definitions. □

2 Examples

In this section, we illustrate the results of this paper with three examples, the main classes of poset-metrics: the posets that are disjoint union of chains, the weak order and the crown order.

Example 2.1 *Let $D = P_1 \mathring{\cup} P_2 \mathring{\cup} \ldots \mathring{\cup} P_s$ be a poset consisting of a disjoint union of r chains. Denoted by μ_i the cardinality of the i-th chain, $i \in \{1, 2, \ldots, s\}$. For every $j \in \{1, 2, \ldots, n\}$ let $\nu_j = |\{P_i : |P_i| = j\}|$. From Corollary 1.1 follows that there is an ordered base β of $\mathbb{F}_q^n$ relative to which every linear isometry $T \in GL_P(\mathbb{F}_q^n)$ is represented by the product $A \cdot U$ of $n \times n$ matrices, where U is a monomial matrix that acts exchanging coordinate subspaces with isomorphic supports and*

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_s \end{pmatrix},$$

where each A_i is a $\mu_i \times \mu_i$ upper triangular matrix with non zero diagonal entries. If $P = \{1, 2, \ldots, n\}$ be a totally ordered set, then there is an ordered base β of $\mathbb{F}_q^n$ relative to which every linear isometry $T \in GL_P(\mathbb{F}_q^n)$ is represented by the $n \times n$ upper triangular matrix with $x_{ii} \neq 0$ for every $i \in \{1, 2, \ldots, n\}$.

If R consisting of finite disjoint union of chains of equal lengths, then w_R become the Rosenbloom-Tsfasman weight defined on the linear space $M_{n\times m}(\mathbb{F}_q)$ of all $n \times m$ matrices over $\mathbb{F}_q$: if $(a_{ij}) \in M_{n\times m}(\mathbb{F}_q)$, then

$$w_R((a_{ij})) = \sum_{j=1}^{m} |\langle supp(a_{1j}, a_{2j}, \ldots, a_{nj})\rangle| .$$

From Corrollary 1.3 ([10, Theorem 1]) it follows that

$$GL_P(M_{n\times m}(\mathbb{F}_q)) \simeq (T_n)^m \rtimes \mathbf{S}_m,$$

where $(T_n)^m$ denotes the direct product of m copies of the group T_n of all upper triangular matrices of size n over $\mathbb{F}_q$ with nonzero diagonal elements.

Remark 2.1 *For the case of modular rings $\mathbb{Z}_n$, we observed that if $n \neq 2$, there is no partial order $P = \{1, 2, \ldots, m\}$ such that the poset-weight w_P coincide with the Lee weight w_{Lee}: if $x = (\overline{x_1}, \ldots, \overline{x_m}) \in \mathbb{Z}_n^m$ then*

$$w_{Lee}(x) = \sum_{i=1}^{m} \min\{|x_i|, m - |x_i|\},$$

with $0 \leq x_i \leq n$ the representative integer of the class $\overline{x_i}$. If $n = 2$ then $w_{Lee} = w_H$. Therefore, if P is antichain and $n = 2$, then $w_P = w_{Lee}$. Now, if

$n \neq 2$, taking $y = \left(\overline{\left\lfloor \frac{n}{2} \right\rfloor}, \ldots, \overline{\left\lfloor \frac{n}{2} \right\rfloor}\right) \in \mathbb{Z}_n^m$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, follows that $w_P(x) = m$ and $w_{Lee}(x) = m \cdot \left\lfloor \frac{n}{2} \right\rfloor > m$. Hence $w_P(x) \neq w_{Lee}(x)$ ($w_P(x) < w_{Lee}(x)$). In summary: if $n \neq 2$ is a positive integer, then there is no partial order P such that $w_P = w_{Lee}$ over $\mathbb{Z}_n^m$.

Example 2.2 *Let $n_1, \ldots, n_t$ be positive integers with $n_1 + \ldots + n_t = n$. Then $W = n_1\mathbf{1} \oplus \ldots \oplus n_t\mathbf{1}$ will denote the* weak order *given by the ordinal sum of the antichains $n_i\mathbf{1}$ with n_i elements (see [7]). Explicitly, $W = n_1\mathbf{1} \oplus \ldots \oplus n_t\mathbf{1}$ is the poset whose underlying set and order relation are given by*

$$\{1, 2, \ldots, n\} = n_1\mathbf{1} \cup n_2\mathbf{1} \cup \ldots \cup n_t\mathbf{1},$$

$$n_i\mathbf{1} = \{n_1 + \ldots + n_{i-1} + 1, n_1 + \ldots + n_{i-1} + 2, \ldots, n_1 + \ldots + n_{i-1} + n_i\}$$

and

$$x < y \text{ if and only if } x \in n_i\mathbf{1},\ y \in n_j\mathbf{1} \text{ for some } i, j \text{ with } i < j.$$

Notice that if $n_1 = \ldots = n_t = 1$, then $W = 1\mathbf{1} \oplus \ldots \oplus 1\mathbf{1}$ is totally ordered with $1 < 2 < \ldots < t$ and if $t = 1$ then $W = n\mathbf{1}$ is antichain.

For a weak order $W = n_1\mathbf{1} \oplus \ldots \oplus n_t\mathbf{1}$ we have that $\Gamma^{(m)}(W) = n_s\mathbf{1}$ if $m = n_1 + n_2 + \ldots + n_{s-1} + 1$, for any $s \in \{1, 2, \ldots, t\}$ and $\Gamma^{(m)}(W) = \varnothing$ otherwise. The group of the automorphism of order $Aut(W)$ is isomorphic to the cartesian product $\mathbf{S}_{n_1} \times \mathbf{S}_{n_2} \times \ldots \times \mathbf{S}_{n_t}$ ($Aut(W)$ is just the group of the applications ϕ that permutes only the elements of each m-th level). Corollary 1.2 assures us then that

$$\left|GL_W\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot \left(\prod_{i=2}^{t} q^{n_i(n_1+n_2+\ldots+n_{i-1}+1)}\right) \cdot n_1! \cdot n_2! \cdot \ldots \cdot n_t!.$$

From Theorem 1.2 follows that there are bases β and β' of $\mathbb{F}_q^n$ such that the matrix $[T]_{\beta,\beta'}$ is equal

$$\begin{pmatrix} D_{n_1 \times n_1} & * & * & \cdots & * \\ 0 & D_{n_2 \times n_2} & * & \cdots & * \\ 0 & 0 & D_{n_3 \times n_3} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_{n_t \times n_t} \end{pmatrix},$$

where

$$D_{n_s \times n_s} = diag\left(a_{\Sigma n_{s-1}+1, \Sigma n_{s-1}+1}, a_{\Sigma n_{s-1}+2, \Sigma n_{s-1}+2}, \ldots, a_{\Sigma n_{s-1}+n_s, \Sigma n_{s-1}+n_s}\right)$$

is a diagonal matrix for each $s = 1, 2, \ldots, t$, *and* $\Sigma n_{j-1} := n_1 + n_2 + \ldots + n_{j-1}$.

Considering the particular weak order $W = 4\mathbf{1} \oplus 4\mathbf{1} \oplus 4\mathbf{1}$ (*Hasse diagram illustrated in Figure* 1), *the matrix of a linear* P*-isometry* $[T]_{\beta,\beta'}$ *of* $T \in GL_W\left(\mathbb{F}_q^{12}\right)$ *is an upper triangular matrix as bellow:*

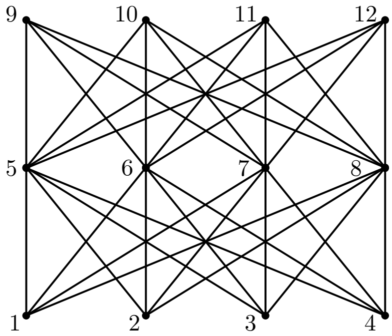

Figure 1: Weak order $W = 4\mathbf{1} \oplus 4\mathbf{1} \oplus 4\mathbf{1}$.

$$\begin{pmatrix}
a_{1,1} & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} & a_{1,11} & a_{1,12} \\
0 & a_{2,2} & 0 & 0 & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} & a_{2,11} & a_{2,12} \\
0 & 0 & a_{3,3} & 0 & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} & a_{3,11} & a_{3,12} \\
0 & 0 & 0 & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & a_{4,9} & a_{4,10} & a_{4,11} & a_{4,12} \\
0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 & 0 & a_{5,9} & a_{5,10} & a_{5,11} & a_{5,12} \\
0 & 0 & 0 & 0 & 0 & a_{6,6} & 0 & 0 & a_{6,9} & a_{6,10} & a_{6,11} & a_{6,12} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{7,7} & 0 & a_{7,9} & a_{7,10} & a_{7,11} & a_{7,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{8,8} & a_{8,9} & a_{8,10} & a_{8,11} & a_{8,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{9,9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10,10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11,11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12,12}
\end{pmatrix}$$

Example 2.3 *The* crown *is a poset with elements* $C = \{1, 2, \ldots, 2n\}$, $n > 1$, *in which* $i < n + i$, $i + 1 < n + i$ *for each* $i \in \{1, 2, \ldots, n - 1\}$, *and* $1 < 2n$, $n < 2n$ *and these are the only strict comparabilities ([1]). The Hasse diagram of crown poset* P *with* $n = 4$ *is illustrated in Figure 2.*

Given a crown $C = \{1, 2, \ldots, 2n\}$, *we have that* $Aut(C)$ *is isomorphic to the dihedral group* D_n, *consisting of the orthogonal transformations which preserve a regular* n*-sided polygon centered at the origin of the euclidian plane. Considering the usual inclusion* $\iota : D_n \to \mathbf{S}_n$, *the action of* D_n *on* C *is defined*

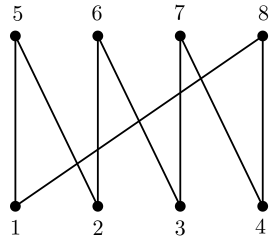

Figure 2: Crown poset $P = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

by

$$g(k) = \begin{cases} \iota \circ g(k) & \text{for } k = 1, 2, \ldots, n \\ \iota \circ g(k-n) & \text{for } k = n+1, \ldots, 2n \end{cases}$$

We note that $\Gamma^{(1)}(C) = \{1, 2, \ldots, n\}$, $\Gamma^{(3)}(C) = \{n+1, \ldots, 2n\}$, *and* $\Gamma^{(k)}(C) = \varnothing$, *for* $k \neq 1, 3$. *So, it follows from Corollary 1.2 that*

$$\left| GL_C \left(\mathbb{F}_q^{2n} \right) \right| = (q-1)^{2n} \cdot q^{2n} \cdot 2n.$$

Theorem 1.2 assures there is a pair of ordered bases β *and* β' *of* $\mathbb{F}_q^n$ *relative to which every linear isometry* $T \in GL_P \left(\mathbb{F}_q^n \right)$ *is represented by the* $[T]_{\beta,\beta'}$ $n \times n$ *upper triangular matrix*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & a_{1,2n} \\ 0 & a_{2,2} & 0 & \cdots & 0 & a_{2,n+1} & a_{2,n+2} & \cdots & 0 & 0 \\ 0 & 0 & a_{3,3} & \cdots & 0 & 0 & a_{3,n+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n,n} & 0 & 0 & \cdots & a_{n,2n-1} & a_{n,2n} \\ 0 & 0 & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{n+2,n+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{2n-1,2n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2n,2n} \end{pmatrix}.$$

In the particular case when $W = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *(see Figure 2), the* *canonical form of a linear P-isometry is*

$$\begin{pmatrix} a_{1,1} & 0 & 0 & 0 & a_{1,5} & 0 & 0 & a_{1,8} \\ 0 & a_{2,2} & 0 & 0 & a_{2,5} & a_{2,6} & 0 & 0 \\ 0 & 0 & a_{3,3} & 0 & 0 & a_{3,6} & a_{3,7} & 0 \\ 0 & 0 & 0 & a_{4,4} & 0 & 0 & a_{4,7} & a_{4,8} \\ 0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{6,6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{7,7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{8,8} \end{pmatrix}.$$

Example 2.4 *The* Boolean *n-cube B^n is the product of n chains of cardinality 2, that is, $B^n = \mathbf{2} \times \mathbf{2} \times \cdots \times \mathbf{2}$ (n times) where $\mathbf{2}$ is a chain of cardinality 2. It is well known ([2]) that $Aut\,(B^n) \simeq S_n$. The Boolean cube may also be described as the Boolean order (defined by the set inclusion order) in the set $\mathcal{P}\,(n)$ of all subsets of $\{1, 2, ..., n\}$. So, we find that the order of subset $\{i_1, i_2, ..., i_k\}$ is 2^k, and there are exactly $\binom{n}{k}$ subsets of cardinality k, that is,*

$$\left|\Gamma^{(m)}\,(P)\right| = \begin{cases} \binom{n}{k} & \text{if } m = 2^k \\ 0 & \text{otherwise} \end{cases}.$$

It follows, from Corollary 1.2 that

$$\left|GL_{B^n}\left(\mathbb{F}_q^n\right)\right| = (q-1)^{2^n} \cdot \left(\prod_{i=0}^{n} q^{(2^i - 1)\binom{n}{i}} \right) n!.$$

From Theorem 1.2, we know we can find ordered bases β and β' of $\mathbb{F}_q^{2^n}$ such the matrix $[T]_{\beta,\beta'}$ is like

$$\begin{pmatrix} D_1 & A_2 & A_3 & A_4 & \cdots & A_n \\ 0 & D_2 & C_{2,3} & C_{2,4} & \cdots & B_2 \\ 0 & 0 & D_3 & C_{3,4} & \cdots & B_3 \\ 0 & 0 & 0 & D_4 & \cdots & B_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D_n \end{pmatrix}$$

where D_i is an $\binom{n}{i} \times \binom{n}{i}$ diagonal matrix with non zero determinant, A_i (B_i) is an $1 \times \binom{n}{i}$ ($\binom{n}{i} \times 1$) matrix, and $C_{i,j}$ is an $\binom{n}{i} \times \binom{n}{j}$ matrix, having (at least) $\binom{n}{j} - \binom{i}{j}$ zero entries in each column and (at least) $\binom{n}{j} - \binom{n-i}{j-i}$ zero entries in each row.

The computations done in all the examples of this work is summarize in the tables bellow. We recall we are denoting by T, D, A, W, C and B total, disjoint union of chains, antichain, weak, crowns and Boolean orders. We recall that ν_j is the number of the components in D with cardinality equal to j (see Exemple 2.1).

Table 1: $Aut\,(P)$ and $|Aut\,(P)|$.

P	$Aut\,(P)$	$\lvert Aut\,(P)\rvert$
T	$\{id\}$	1
D	$\mathbf{S}_{\nu_1} \times \mathbf{S}_{\nu_2} \times \ldots \times \mathbf{S}_{\nu_n}$	$\nu_1! \cdot \nu_2! \cdot \ldots \cdot \nu_t!$
A	$\mathbf{S}_n$	$n!$
W	$\mathbf{S}_{n_1} \times \mathbf{S}_{n_2} \times \ldots \times \mathbf{S}_{n_t}$	$n_1! \cdot n_2! \cdot \ldots \cdot n_t!$
C	D_n	$2n$
B	S_n	$n!$

Table 2: $\Gamma^{(m)}\,(P) \neq \varnothing$ and $\left|\Gamma^{(m)}\,(P)\right|$.

P	$\Gamma^{(m)}\,(P) \neq \varnothing$	$\left\lvert\Gamma^{(m)}\,(P)\right\rvert$
T	$\Gamma^{(m)}\,(T) = \{1, 2, \ldots, m\}$	m
D	$\Gamma^{(m)}\,(D) = \left\{i_m, i_{\Sigma\mu_1+m}, \ldots, i_{\Sigma\mu_{s-1}+m}\right\}$	$\left\lvert\Gamma^{(m)}\,(D)\right\rvert \leq s$
A	$\Gamma^{(1)}\,(A) = A$	n
W	$\Gamma^{(\Sigma n_{s-1}+1)}\,(W) = n_s\mathbf{1}$	n_s
C	$\Gamma^{(1)}\,(C) = \{1, 2, \ldots, n\}$ $\Gamma^{(3)}\,(C) = \{n+1, n+2, \ldots, 2n\}$	n
B	Subsets of cardinality m if $m = 2^k$ $\emptyset$ otherwise	$\binom{n}{k}$ if $m = 2^k$ 0 otherwise

Table 3: $\left|GL_P\left(\mathbb{F}_q^n\right)\right|$.

P	$\left\|GL_P\left(\mathbb{F}_q^n\right)\right\|$
T	$(q-1)^n \cdot \left(\prod_{i=2}^{n} q^{i-1}\right)$
D	$(q-1)^n \cdot \left(\prod_{j=1}^{s} \nu_j!\right) \cdot \left(\prod_{k=1}^{s} q^{\frac{\mu_k(\mu_k-1)}{2}}\right)$
A	$(q-1)^n \cdot n!$
W	$(q-1)^n \cdot \left(\prod_{i=2}^{t} q^{n_i(\Sigma n_{i-1}+1)}\right) \cdot \left(\prod_{j=1}^{t} n_j!\right)$
C	$(q-1)^n \cdot q^n \cdot n$ if n is even
B	$(q-1)^{2^n} \cdot \left(\prod_{i=0}^{n} q^{(2^i-1)\binom{n}{i}}\right) n!$

In the table bellow we explicity compute $\left|GL_P\left(\mathbb{F}_q^n\right)\right|$ for T, D, A, W, C and B with $q=2$ and $n=2,3,\ldots,10$:

Table 4: Numbers of linear isometries of $|GL_P\left(\mathbb{F}_2^n\right)|$.

n	$\|GL_T\left(\mathbb{F}_2^n\right)\|$	$\|GL_A\left(\mathbb{F}_2^n\right)\|$	$\|GL_C\left(\mathbb{F}_2^n\right)\|$	$\left\|GL_B\left(\mathbb{F}_2^{2^n}\right)\right\|$
2	2	2	8	64
3	8	6	$*$	3145 728
4	64	24	64	$\sim 8.8544 \times 10^{20}$
5	1024	120	$*$	$\sim 3.9492 \times 10^{65}$
6	32768	720	384	$\sim 1.1022 \times 10^{203}$
7	2097152	5040	$*$	$\sim 3.3357 \times 10^{623}$
8	268435456	40320	2048	$\sim 3.9778 \times 10^{1902}$
9	$\sim 6.8719 \times 10^{10}$	362880	$*$	$\sim 4.0347 \times 10^{5776}$
10	$\sim 3.5184 \times 10^{13}$	3628800	10240	$\sim 6.6875 \times 10^{17473}$

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