Covering Number for Reflections in Trees

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Abstract

We define a reflection in a tree as an involutive automorphism whose set of fixed points is a geodesic and prove that, for the case of a homogeneous tree of even degree, the group of even automorphisms may be covered by at most 11 reflections.

Reflections are defined as involutive automorphisms having a geodesic as set of fixed points. In a previous work ([6]) we studied the structure of reflections in an homogeneous tree Γ of degree $k \equiv 0 \bmod 4$. We considered the group $\langle \mathcal{R} \rangle$ generated by the set of all reflections $\mathcal{R}$ and the (index two) subgroup $\mathrm{Aut}^+(\Gamma) \subset \mathrm{Aut}(\Gamma)$ consisting of automorphisms with even displacement function and proved that the topological closure of $\langle \mathcal{R} \rangle$ is $\mathrm{Aut}^+(\Gamma)$, that is, given $\varphi \in \mathrm{Aut}^+(\Gamma)$ there is a sequence $(\varphi_n)_{n=1}^{\infty}$ with $\varphi_n \in \langle \mathcal{R} \rangle$ and a sequence of subsets $A_i \subset \Gamma$ with $A_n \subset A_{n+1}$ and $\Gamma = \cup_{n=1}^{\infty} A_n$ such that φ_n coincides with φ on A_n, that is, $\varphi_n|_{A_n} = \varphi|_{A_n}$. The proof given is constructive and actually each φ_n is the product of n reflections, so we could not say that $\mathrm{Aut}^+(\Gamma)$ is finitely generated by $\mathcal{R}$.

In this work we prove with simple arguments that $\mathrm{Aut}^+(\Gamma)$ is finitely generated by $\mathcal{R}$ (Proposition 2) and go further, proving that every $\varphi \in \mathrm{Aut}^+(\Gamma)$ may be expressed as the product of at most eleven reflections (Theorem 12).

1 Basic Concepts

The free monoid X^* of words over the alphabet $X = \{0, ..., n-1\}$ ordered by the prefix relation has a n-regular rooted tree structure in which the empty word is the root and the words of length l constitute the level l in the tree. Denote this n-regular rooted tree by $\mathcal{T}$. If we consider two copies $\mathcal{T}'$ and $\mathcal{T}''$ of the n-regular rooted tree $\mathcal{T}$ and add a single edge, connecting the root of $\mathcal{T}'$ to the root of $\mathcal{T}''$, we get a k-*homogeneous tree* Γ, that is,

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a tree where the number of vertices adjacent to every vertex x is a constant (in this case $k = n + 1$), called the *homogeneity degree of* Γ, denoted $\mathrm{Degree}(\Gamma)$.

Every k-homogeneous tree Γ comes equipped with a natural metric d. For any two vertices x, y of Γ we define $d(x, y)$ to be the minimal number of edges in an edge-path from x to y. If we endow each edge with the metric of the unit interval $[0, 1] \subset \mathbb{R}$, then d naturally extend to a metric on Γ.

We denote by $S(x, R)$ the usual metric sphere of Γ, centered at x with radius R.

By an *automorphism* of Γ we mean an isometry of (Γ, d) which takes vertices to vertices (and therefore edges to edges). We denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ.

An *integer interval* is a subset of $\mathbb{Z}$ of one of the kinds $\mathbb{Z}$, $\mathbb{N}$ or $\mathrm{I}_{a,b} := \{m \in \mathbb{Z} | a \leq m \leq b\}$, with $a, b \in \mathbb{Z}$. A subset $\gamma(\mathrm{I}) \subset \Gamma$ is called a *geodesic, a geodesic ray* or *a geodesic segment* if $\gamma : \mathrm{I} \to \Gamma$ is a map defined on an integer interval respectively of type $\mathbb{Z}, \mathbb{N}$ or $\mathrm{I}_{a,b}$ such that $d(\gamma(n), \gamma(m)) = |n - m|$ for every $n, m \in \mathrm{I}$. We call the map $\gamma : \mathrm{I} \to \Gamma$ a *parametrization* but often make no distinction between the map γ and its image $\gamma(\mathrm{I})$. We denote by $[x, y]$ the geodesic segment joining the vertices $x, y \in \Gamma$.

In this work, we will use a labelling of the tree relative to a geodesic. Since it will be used repeatedly, we will introduce it with some details. Given a geodesic $\gamma \in \Gamma$, we consider (for $n \in \mathbb{N}$) the *n-level around* γ:

$$V_n := V_n(\gamma) = \{x \in \Gamma | d(x, \gamma) = n\}.$$

We note that $V_0 = \gamma$ and

$$\Gamma = \overset{\circ}{\bigcup_{n \geq 0}} V_n.$$

Given $x \in V_n$ $(n > 0)$ there is a unique vertex $x' \in V_{n-1}$ adjacent to x and this defines a surjective map

$$\begin{array}{cccc} p_n : & V_n & \longrightarrow & V_{n-1} \\ & x & \longrightarrow & x' = p_n(x) \end{array}$$

from V_n onto V_{n-1}. If

$$p : \bigcup_{n \geq 1} V_n \to \Gamma = \bigcup_{n \geq 0} V_n$$

be such that $p|_{V_n} = p_n$, we say that x is a *descendent* of x' if $p(x) = x'$.

An automorphism $\mu \in \mathrm{Aut}(\Gamma)$ that leaves the geodesic γ invariant defines a sequence $(\mu^0, \mu^1, \mu^2, \ldots)$ of permutations, where μ^n acts on V_n ($\mu^n(V_n) = V_n$) and satisfies the *invariance condition*:

$$p_n(\mu^n(x)) = \mu^{n-1}(p_n(x)) \;\; \forall n > 0, \forall x \in V_n. \tag{1}$$

Reciprocally, any sequence of permutations $(\mu^0, \mu^1, \mu^2, \ldots)$ with μ^n acting on V_n and satisfying the invariance condition defines an automorphism μ of Γ that leaves γ invariant. We note that γ is fixed by μ iff $\mu^0 = \mathrm{Id}$.

2 Reflections in Trees

There are many possibilities to define a reflection in a tree. The minimal condition for a map $\phi : \Gamma \to \Gamma$ to resemble what is commonly known as a reflection in geometry, is to demand ϕ to be an involutive automorphism, i.e., $\phi^2 = \mathrm{Id}$. Indeed, this is the definition adopted by Moran in [4]. In this work, we adopt a much more restrictive definition:

Definition 1 *A* reflection in a tree Γ *is an automorphism* $\phi : \Gamma \longrightarrow \Gamma$*, satisfying:*

1. *ϕ is an involution.*
2. *The set of fixed points of ϕ is the geodesic γ, i.e., there is a geodesic $\gamma \subset \Gamma$ such that $\phi(x) = x \Leftrightarrow x \in \gamma$.*

In this case we say that ϕ is a *reflection in the geodesic* γ and denote by $\mathcal{R}_\gamma$ the set of all reflections in the geodesic γ.

From here on, we assume that Γ is a k-homogeneous tree of **even** degree, since there are no reflections (as defined before) in case the degree of the tree is odd.

Let $\mathcal{R}$ be the set of all reflections in the tree Γ and $\langle\mathcal{R}\rangle$ the subgroup of $\mathrm{Aut}(\Gamma)$ generated by $\mathcal{R}$. We let

$$\mathrm{Aut}^+(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) \,|\, d_\varphi(x) \equiv 0 \mathrm{mod} 2 \text{ for every } x \in \Gamma\},$$

where $d_\varphi(x) := d(x, \varphi(x))$ is the *displacement function of an automorphism* φ. We call $\mathrm{Aut}^+(\Gamma)$ the *group of even (displacement) automorphisms* and observe it is a normal subgroup of index 2 of $\mathrm{Aut}(\Gamma)$ ([3, Proposition 1]).

Proposition 2 *Let Γ be a k-homogeneous tree and $\langle\mathcal{R}\rangle$ the group generated by reflections. Then $\langle\mathcal{R}\rangle = \mathrm{Aut}^+(\Gamma)$.*

Proof: Given a reflection $\phi \in \mathcal{R}_\gamma \subset \mathcal{R}$ and a vertex $x \in \Gamma$, the (unique) vertex $x_0 \in \gamma$ such that $d(x_0, x) = d(x, \gamma)$ is the middle point of the geodesic segment $[x, \phi(x)]$ and so $\phi \in \mathrm{Aut}^+(\Gamma)$ and hence $\langle\mathcal{R}\rangle \subseteq \mathrm{Aut}^+(\Gamma)$. Given $\varphi \in \mathrm{Aut}(\Gamma)$ and ϕ a reflection in the geodesic γ, we have that $\varphi \circ \phi \circ \varphi^{-1}$ is a reflection in the geodesic $\varphi(\gamma)$ ([6, Proposition 4]). It follows that $\langle\mathcal{R}\rangle$ is normal in $\mathrm{Aut}(\Gamma)$ and hence also in $\mathrm{Aut}^+(\Gamma)$. But $\mathrm{Aut}^+(\Gamma)$ is a simple group ([3, Theorem 7]), and it follows that $\langle\mathcal{R}\rangle = \mathrm{Aut}^+(\Gamma)$. $\square$

3 Covering Number by Reflections

We say that a subset A of a group G *covers* the group if there is a positive integer n such that $G = A^n$ with

$$A^n := \{\phi_1 \circ \phi_2 \circ \cdots \circ \phi_n | \phi_i \in A; i = 1, 2, ..., n\}.$$

In this case, the smallest such integer is called the *covering number of G by A*:

$$\mathrm{cn}(G, A) := \min\{n | A^n = G\}$$

Our main result, that will be proved by the end of this work, is the following:

Theorem 3 *Let Γ be a k-homogeneous tree and $\mathcal{R}$ the set of reflections in geodesics of Γ. Then,* $\mathrm{Aut}^{+}(\Gamma)$ *may be covered by $\mathcal{R}$ and* $\mathrm{cn}\left(\mathrm{Aut}^{+}(\Gamma), \langle\mathcal{R}\rangle\right) \leq 11$.

Given an involution $\varphi \in \mathrm{Aut}(\Gamma)$, if $\varphi(x) = x$ and $\varphi(y) = y$, then $\varphi(z) = z$ for every $z \in [x, y]$ and it follows that $\mathrm{Fix}(\varphi) := \{x \in \Gamma | \varphi(x) = x\}$ is a sub-tree of Γ. We say that a non empty sub-tree $\Gamma' \subset \Gamma$ is an *admissible tree* if there is an involution $\varphi \in \mathrm{Aut}(\Gamma)$ such that $\mathrm{Fix}(\varphi) = \Gamma'$. We say such a tree Γ' is a *trivial admissible tree* if $\Gamma' = \Gamma, \Gamma' = \{x\}$ or $\Gamma = \gamma$ (a geodesic). We start characterizing admissible trees.

Proposition 4 *Let Γ be a k-homogeneous tree and $\Gamma' \subset \Gamma$ a sub-tree. Then, Γ' is an admissible tree iff* $|S(x,1)|_{\Gamma'} \equiv 0 \bmod 2$ *for every $x \in \Gamma'$, where* $|S(x,1)|_{\Gamma'} := |S(x,1) \cap \Gamma'|$.

Proof: Assume that $\Gamma' = \mathrm{Fix}(\varphi)$ for some involution $\varphi \in \mathrm{Aut}(\Gamma)$ and let $x \in \Gamma'$. The action of φ on $S(x,1)$ fixes every vertex $y \in S(x,1) \cap \Gamma'$ and acts as an involution with no fixed points in $S(x,1) \setminus \Gamma'$. It follows that the restriction of φ to $S(x,1) \setminus \Gamma'$ is the product of disjoint transpositions, so that $|S(x,1) \setminus \Gamma'|$ is even. Since k is also even, we find that $|S(x,1)|_{\Gamma'} = |S(x,1) \cap \Gamma'| = k - |S(x,1) \setminus \Gamma'| \equiv 0 \bmod 2$.

Let us assume now that $\Gamma' \subset \Gamma$ is a sub-tree such that $|S(x,1)|_{\Gamma'} \equiv 0 \bmod 2$ for every $x \in \Gamma'$. We label the vertices of Γ' as $\{x_i | i \in I\}$ where I is any set. Since $|S(x_i,1)|_{\Gamma'} \equiv 0 \bmod 2$, we have that $|S(x_i,1) \setminus \Gamma'|$ is also even, and we label those vertices as $x_{i,1}, x_{i,2}, ..., x_{i,2k_i}$,where k_i depends on x_i. Each $x_{i,j} \in \Gamma \setminus \Gamma'$ has exactly $n-1$ adjacent vertex that are descendents of $x_{i,j}$ relatively to Γ', in the sense that $x_{i,j}$ is contained in the geodesic segment joining each $x_{i,j}$ to Γ', so we label them as $x_{i,j,1}, ..., x_{i,j,...,n-1}$. By repeating this process, every vertex in $\Gamma \setminus \Gamma'$ can be labeled as $x_{i,j_1,j_2,...,j_l}$ where $i \in I, j_1 \in \{1, 2, ..., 2k_i\}, j_r \in \{1, 2, ..., n-1\}$ for $r > 1$ and $l = d(x_{i,j_1,j_2,...,j_l}, \Gamma')$. We define a map $\varphi : \Gamma \to \Gamma$ in the following way:

$$\varphi(x_{i,j_1,j_2,...,j_l}) = x_{i,\sigma_i(j_1),j_2,...,j_l}$$

where

$$\sigma_i(j) = \begin{cases} k_i + j & \text{if } 1 \leq j \leq k_i \\ j - k_i & \text{if } k_i + 1 \leq j \leq 2k_i \end{cases}.$$

Since each σ_i is an involution, we find that φ is also an involution. Moreover, it easy to see that $\mathrm{Fix}(\varphi) = \Gamma'$. $\square$

Our next goal is to prove that every involution may be represented as the product of 5 reflections in geodesics. In order to do so, we need to consider two distinct types of admissible trees:

Definition 5 *Let $\Gamma' \subset \Gamma$ be an admissible tree and $x \in \Gamma'$ a vertex. We say that x satisfies the*

- Γ'-even parity condition *(Γ'-EPC) if* $|S(x,1)|_{\Gamma'} \equiv 0 \bmod 4$*;*
- Γ'-odd parity condition *(Γ'-OPC) if* $|S(x,1)|_{\Gamma'} \equiv 2 \bmod 4$.

We also need to define the symmetry condition:

Definition 6 *A sub-tree $\Gamma' \subset \Gamma$ is γ-symmetric if there is a reflection σ in the geodesic $\gamma \subset \Gamma'$ such that $\sigma(\Gamma') = \Gamma'$.*

The next two lemmas will be used to prove our main theorem and could be stated in a unique proposition, but we prefer to separate the two different cases, according to homogeneity degree of Γ.

Lemma 7 *Let Γ be a k-homogeneous tree with $k \equiv 0 \bmod 4$, and $\Gamma' \subset \Gamma$ an admissible sub-tree that is γ-symmetric. If every vertex $x \in \gamma$ satisfies the Γ'-EPC, there are $\mu, \tau \in \mathcal{R}_\gamma$ such that:*

1. $\mathrm{Fix}(\mu) = \mathrm{Fix}(\tau) = \gamma$;
2. $\mathrm{Fix}(\tau \circ \mu) = \Gamma'$.

Proof: The proof consists of making appropriate choices of permutations acting on each $V_n := V_n(\gamma)$ and satisfying the invariance condition (1).

Starting at the initial level $V_0 = \gamma$, we define $\mu^0 = \tau^0 = \mathrm{Id}$, so that $\mu^0(\gamma) = \tau^0(\gamma) = \gamma$ and, since $\gamma \subset \Gamma'$

$$\mathrm{Fix}\left(\tau^0 \circ \mu^0\right) = \Gamma' \cap V_0.$$

Given $x_m \in \gamma\,(m \in \mathbb{Z})$ we have that $p^{-1}(x_m) \subset V_1$ consists of $k-2$ vertices, each of them adjacent to x_m. We label those $k-2$ vertices as $x_{m,1}, x_{m,2}, ..., x_{m,k-2}$. We do so in such a way that $x_{m,j} \in \Gamma' \Leftrightarrow j > (k-2) - l(m)$, where $l(m) := |S(x_m, 1)|_{\Gamma'} - 2$. We define μ_m and τ_m by the product of disjoint involutions as follows:

$$\begin{aligned}\mu_m &:= (x_{m,1}x_{m,3})\,(x_{m,2}x_{m,4}) \ldots \left(x_{m,k-l(m)-5}, x_{m,k-l(m)-3}\right)\left(x_{m,k-l(m)-4}, x_{m,k-l(m)-2}\right)\\ &\quad \left(x_{m,k-l(m)-1}, x_{m,k-l(m)}\right)\left(x_{m,k-l(m)+1}, x_{m,k-l(m)+2}\right) \ldots (x_{m,k-3}, x_{m,k-2})\\ \tau_m &:= (x_{m,1}x_{m,4})\,(x_{m,2}x_{m,3}) \ldots \left(x_{m,k-l(m)-5}, x_{m,k-l(m)-2}\right)\left(x_{m,k-l(m)-4}, x_{m,k-l(m)-3}\right)\\ &\quad \left(x_{m,k-l(m)-1}, x_{m,k-l(m)}\right)\left(x_{m,k-l(m)+1}, x_{m,k-l(m)+2}\right) \ldots (x_{m,k-3}, x_{m,k-2}).\end{aligned}$$

Straightforward calculations show that

$$\tau_m \circ \mu_m = (x_{m,1}x_{m,2})\,(x_{m,3}x_{m,4}) \ldots \left(x_{m,k-l(m)-5}, x_{m,k-l(m)-4}\right)\left(x_{m,k-l(m)-3}, x_{m,k-l(m)-2}\right)$$

so that the only vertices of $p^{-1}(x_m)$ fixed by $\tau_m \circ \mu_m$ are the $l(m)$ vertices $x_{m,j}$ with $j > (k-2) - l(m)$, that is, for every $x_m \in \gamma$ we find that

$$\mathrm{Fix}(\tau_m \circ \mu_m) = \Gamma' \cap p^{-1}(x_m).$$

We define μ^1 and τ^1 by

$$\mu^1(x_{m,i}) := \mu_m(x_{m,i})\ \ , \tau^1(x_{m,i}) := \tau_m(x_{m,i})$$

for $m \in \mathbb{Z}$ and it follows that

$$\mathrm{Fix}\left(\tau^1 \circ \mu^1\right) = \Gamma' \cap V_1$$

Before we get to the next level, we observe that, since $x \in \Gamma' \cap V_2$ implies that $p_2(x) \in \Gamma' \cap V_1$. We are actually interested only on the vertices in V_2 that are descendent of the vertices in

$\Gamma' \cap V_1$. Moreover, the action of μ^1 and τ^1 restricted to Fix $\left(\tau^1 \circ \mu^1\right)$ coincide (it means, they permute the same vertices of Fix $\left(\tau^1 \circ \mu^1\right)$), so that it is enough to show that, restricted to a pair of vertices $x_{m,i}, x_{m,j} \in$ Fix $\left(\tau^1 \circ \mu^1\right)$ such that $\tau^1 \left(x_{m,i}\right) = \mu^1 \left(x_{m,i}\right) = x_{m,j}$, we can define involutions $\mu_{m,i} = \mu_{m,j}$ and $\tau_{m,i} = \tau_{m,j}$ acting on $p^{-1}\left(x_{m,i}\right) \cup p^{-1}\left(x_{m,j}\right) \subset V_2$ such that

$$\text{Fix}\left(\tau_{m,i} \circ \mu_{m,i}\right) = \Gamma' \cap \left(p^{-1}\left(x_{m,i}\right) \cup p^{-1}\left(x_{m,j}\right)\right).$$

We recall we are assuming that Γ' is γ-symmetric, that is, there is $\sigma \in \mathcal{R}_\gamma$ such that $\sigma\left(\Gamma'\right) = \Gamma'$. If $\sigma\left(x_{m,i}\right) = x_{m,j}$ and label the $k-1$ descendents of $x_{m,i}$ and of $x_{m,j}$ as $x_{m,i,1}, x_{m,i,2}, ..., x_{m,i,k-1}$ and $x_{m,j,1}, x_{m,j,2}, ..., x_{m,j,k-1}$ in such a way that $\sigma\left(x_{m,i,q}\right) = x_{m,j,q}$ for every $q \in \{1, 2, ..., k-1\}$ with $x_{m,i,q} \in \Gamma' \Leftrightarrow x_{m,j,q} \in \Gamma'$. We also observe that since Γ' is an admissible tree, Proposition 4 assures that, among the $k-1$ vertices $x_{m,j,1}, x_{m,j,2}, ..., x_{m,j,k-1}$, an odd number must be fixed by the product $\tau_{m,i} \circ \mu_{m,i}$. Let $1 \le l\left(m,i\right) \le k-1$ be the number of vertices in $\Gamma' \cap p^{-1}\left(x_{m,i}\right)$. In the same way as we did in the first level, we may assume without loss of generality that $x_{m,i,q} \in \Gamma' \Leftrightarrow q > (k-1) - l\left(m,i\right)$. We also observe that since $x_{m,i}$ and $x_{m,j}$ are symmetric relatively to γ, $l\left(m,i\right) = l\left(m,j\right)$. We define $\mu_{m,i}$ and $\tau_{m,i}$ as

$$\begin{aligned} \mu_{m,i} &:= \left(x_{m,i,1} x_{m,j,2}\right)\left(x_{m,i,2} x_{m,j,1}\right) ... \left(x_{m,i,k-l(m,i)-2} x_{m,j,k-l(m,i)-1}\right) \\ &\quad \left(x_{m,i,k-l(m,i)-1} x_{m,j,k-l(m,i)-2}\right) \\ \tau_{m,i} &:= \left(x_{m,i,1} x_{m,j,1}\right)\left(x_{m,i,2} x_{m,j,2}\right) ... \left(x_{m,i,k-l(m,i)-2} x_{m,j,k-l(m,i)-2}\right) \\ &\quad \left(x_{m,i,k-l(m,i)-1} x_{m,j,k-l(m,i)-1}\right). \end{aligned}$$

and note that $\mu_{m,i} = \mu_{m,j}$ and $\tau_{m,i} = \tau_{m,j}$. Straightforward calculations show that

$$\begin{aligned} \tau_{m,i} \circ \mu_{m,i} &= \left(x_{m,i,1} x_{m,i,2}\right)\left(x_{m,j,1} x_{m,j,2}\right) ... \left(x_{m,i,k-l(m,i)-2} x_{m,i,k-l(m,i)-1}\right) \\ &\quad \left(x_{m,j,k-l(m,i)-2} x_{m,j,k-l(m,i)-1}\right), \end{aligned}$$

and so there are exactly $2\left(l\left(m,i\right)\right)$ vertices in $p^{-1}\left(x_{m,i}\right) \cup p^{-1}\left(x_{m,j}\right)$ that are fixed by the composition $\tau_{m,i} \circ \mu_{m,i}$, and those are precisely the vertices $x_{m,i,q}$ with $q > (k-1) - l\left(m,i\right)$, that is,

$$\text{Fix}\left(\tau_{m,i} \circ \mu_{m,i}\right) = \Gamma' \cap \left(p^{-1}\left(x_{m,i}\right) \cup p^{-1}\left(x_{m,j}\right)\right)$$

We define

$$\mu^2\left(x_{m,i_1,i_2}\right) := \mu_{m,i_1}\left(x_{m,i_1,i_2}\right), \ \tau^2\left(x_{m,i_1,i_2}\right) := \tau_{m,i_1}\left(x_{m,i_1,i_2}\right)$$

for $m \in \mathbb{Z}$, $i_1 \in \{1, 2, ..., k-2\}$, $i_2 \in \{1, 2, ..., k-1\}$ and it follows that

$$\text{Fix}\left(\tau^2 \circ \mu^2\right) = \Gamma' \cap V_2$$

The definition of μ^r and τ^r, acting at the level V_r is done in a similar way. Suppose $x_{m,i_1,...,i_{r-1}}, x_{m,j_1,...,j_{r-1}} \in$ Fix $\left(\tau^{r-1} \circ \mu^{r-1}\right)$. If σ is the reflection in γ such that $\sigma\left(\Gamma'\right) = \Gamma'$ and $\sigma\left(x_{m,i_1,...,i_{r-1},q}\right) = x_{m,j_1,...,j_{r-1},q}$ for every $q \in \{1, 2, ..., k-1\}$ then, by the invariance condition we have that $x_{m,i_1,...,i_{r-1},q} \in \Gamma' \Leftrightarrow x_{m,j_1,...,j_{r-1},q} \in \Gamma'$. Let $1 \le l\left(m, i_1, ..., i_{r-1}\right) \le k-1$ be the number of vertices in $\Gamma' \cap p^{-1}\left(x_{m,i_1,...,i_{r-1}}\right)$. We may again assume that $x_{m,i_1,...,i_{r-1},q} \in \Gamma' \Leftrightarrow q > (k-1) - l\left(m, i_1, ..., i_{r-1}\right)$. Since $\sigma\left(x_{m,i_1,...,i_{r-1},q}\right) = x_{m,j_1,...,j_{r-1},q}$

we have that $l(m, i_1, ..., i_{r-1}) = l(m, j_1, ..., j_{r-1}) = \Gamma' \cap p^{-1}\left(x_{m,j_1,...,j_{r-1}}\right)$. We define $\mu_{m,i_1,...i_{r-1},q}$ and $\tau_{m,j_1,...,j_{r-1},q}$ as:

$$\begin{aligned}
\mu_{m,i_1,...i_{r-1}} &= \left(x_{m,i_1,...i_{r-1},1} x_{m,j_1,...j_{r-1},2}\right)\left(x_{m,i_1,...i_{r-1},2} x_{m,j_1,...,j_{r-1},1}\right) ... \\
&\left(x_{m,i_1,...i_{r-1},k-l(m,i_1,...,i_{r-1})-2} x_{m,j_1,...,j_{r-1},k-l(m,i_1,...,i_{r-1})-1}\right) \\
&\left(x_{m,i_1,...i_{r-1},k-l(m,i_1,...,i_{r-1})-1} x_{m,j_1,...,j_{r-1},k-l(m,i_1,...,i_{r-1})-2}\right) \\
\tau_{m,i_1,...i_{r-1}} &= \left(x_{m,i_1,...i_{r-1},1} x_{m,j_1,...j_{r-1},1}\right)\left(x_{m,i_1,...i_{r-1},2} x_{m,j_1,...j_{r-1},2}\right) ... \\
&\left(x_{m,i_1,...i_{r-1},k-l(m,i_1,...,i_{r-1})-1} x_{m,j_1,...j_{r-1},k-l(m,i_1,...,i_{r-1})-1}\right)
\end{aligned}$$

so that

$$\begin{aligned}
\tau_{m,i_1,...i_{r-1}} \circ \mu_{m,i_1,...i_{r-1}} &= \left(x_{m,i_1,...i_{r-1},1} x_{m,i_1,...i_{r-1},2}\right)\left(x_{m,j_1,...j_{r-1},1} x_{m,j_1,...j_{r-1},2}\right) ... \\
&\left(x_{m,i_1,...i_{r-1},k-l(m,i_1,...,i_{r-1})-2} x_{m,i_1,...i_{r-1},k-l(m,i_1,...,i_{r-1})-1}\right) \\
&\left(x_{m,j_1,...j_{r-1},k-l(m,i_1,...,i_{r-1})-2} x_{m,j_1,...j_{r-1},k-l(m,i_1,...,i_{r-1})-1}\right).
\end{aligned}$$

We note that $\mu_{m,i_1,...i_{r-1}} = \mu_{m,j_1,...j_{r-1}}$ and $\tau_{m,i_1,...i_{r-1}} = \tau_{m,j_1,...j_{r-1}}$. It follows that the number of vertices in $p^{-1}\left(x_{m,i_1,...,i_{r-1}}\right) \cup p^{-1}\left(x_{m,j_1,...,j_{r-1}}\right)$ fixed by $\tau_{m,i_1,...i_{r-1}} \circ \mu_{m,i_1,...i_{r-1}}$ is $2 \cdot l(m, i_1, ..., i_{r-1})$ and those are precisely the vertices $x_{m,i_1,...,i_{r-1},q}$, $x_{m,j_1,...,j_{r-1},q}$ with $q > (k-1) - l(m, i_1, ..., i_{r-1})$, and as we did before, we define

$$\mu^r\left(x_{m,i_1,...,i_r}\right) := \mu_{m,i_1,...i_{r-1}}\left(x_{m,i_1,...,i_r}\right) \text{ and } \quad \tau^r\left(x_{m,i_1,...,i_r}\right) := \tau_{m,i_1,...i_{r-1}}\left(x_{m,i_1,...,i_r}\right)$$

with $m \in \mathbb{Z}, i_1 \in \{1, 2, ..., k-2\}$ and $i_2, ..., i_r \in \{1, 2, ..., k-1\}$ and we have that

$$\operatorname{Fix}\left(\tau^r \circ \mu^r\right) = \Gamma' \cap V_r.$$

Finally, if we define

$$\begin{aligned}
\mu &:= \left(\mu^0, \mu^1, ..., \mu^r, ...\right) \\
\tau &:= \left(\tau^0, \tau^1, ..., \tau^r, ...\right),
\end{aligned}$$

those are involutions that has γ as set of fixed points, that is, $\mu, \tau \in \mathcal{R}_\gamma$ and by construction

$$\operatorname{Fix}(\tau \circ \mu) = \Gamma'$$

□

The same proposition holds if every vertex $x \in \gamma$ satisfies the Γ'-OPC:

Lemma 8 *Let Γ be a k-homogeneous tree with $k \equiv 2 \bmod 4$, and $\Gamma' \subset \Gamma$ an admissible sub-tree that is γ-symmetric. If every vertex $x \in \gamma$ satisfies the Γ'-OPC, there are $\mu, \tau \in \mathcal{R}_\gamma$ such that:*

1. $\operatorname{Fix}(\mu) = \operatorname{Fix}(\tau) = \gamma$;

2. $\mathrm{Fix}\,(\tau \circ \mu) = \Gamma'$.

Proof: The proof is essentially the same of the preceding lemma, except for the number of vertices in the first level relatively to γ, since it satisfies the Γ'-OPC instead of Γ'-EPC. We adopt the same notations used before and show how to proceed in the firs level. Given $x_m \in \gamma$ we have that $p^{-1}(x_m)$ has $k-2$ vertices, and we label them as $x_{m,1}, x_{m,2}, ..., x_{m,k-2}$.

Let $0 \leq l(m) \leq k-2$ be the number of vertices in $p^{-1}(x_m) \cap \Gamma'$. Since x_m satisfies the Γ'-OPC and two of the vertices adjacent to x_m are in γ, we have that $l(m) \equiv 0 \bmod 4$. We define μ_m and τ_m as follows:

$$\begin{aligned}\mu_m &:= (x_{m,1}x_{m,3})(x_{m,2}x_{m,4}) \ldots \left(x_{m,k-l(m)-5}, x_{m,k-l(m)-3}\right)\left(x_{m,k-l(m)-4}, x_{m,k-l(m)-2}\right) \\ &\quad \left(x_{m,k-l(m)-1}, x_{m,k-l(m)}\right)\left(x_{m,k-l(m)+1}, x_{m,k-l(m)+2}\right) \ldots (x_{m,k-3}, x_{m,k-2}) \\ \tau_m &= (x_{m,1}x_{m,4})(x_{m,2}x_{m,3}) \ldots \left(x_{m,k-l(m)-5}, x_{m,k-l(m)-2}\right)\left(x_{m,k-l(m)-4}, x_{m,k-l(m)-3}\right) \\ &\quad \left(x_{m,k-l(m)-1}, x_{m,k-l(m)}\right)\left(x_{m,k-l(m)+1}, x_{m,k-l(m)+2}\right) \ldots (x_{m,k-3}, x_{m,k-2})\end{aligned}$$

As we did before, we may assume that $x_{m,j} \in \Gamma' \Leftrightarrow j > (k-2) - l_1(m)$. Direct computation shows that

$$\tau_m \circ \mu_m = (x_{m,1}x_{m,2})(x_{m,3}x_{m,4}) \ldots \left(x_{m,k-l(m)-5}, x_{m,k-l(m)-4}\right)\left(x_{m,k-l(m)-3}, x_{m,k-l(m)-2}\right)$$

so that the vertices in $p^{-1}(x_m)$ fixed by the product $\tau_m \circ \mu_m$ are precisely the $l(m)$ vertices $x_{m,j}$ with $j > (k-2) - l_1(m)$. In other words,

$$\mathrm{Fix}\,(\tau_m \circ \mu_m) = \Gamma' \cap p^{-1}(x_m)$$

for every $x_m \in \gamma$. We define

$$\mu^1(x_{m,i}) := \mu_m(x_{m,i}) \text{ and } \tau^1(x_{m,i}) := \tau_m(x_{m,i})$$

and it follows that

$$\mathrm{Fix}\left(\tau^1 \circ \mu^1\right) = \Gamma' \cap V_1.$$

The definition of the permutations acting on subsequent levels is done in exactly the same way it was done in the preceding lemma. □

Theorem 9 *Let Γ be a k-homogeneous tree, $\Gamma' \subset \Gamma$ a non-trivial admissible sub-tree and $\mathcal{R} = \{\varphi \in \mathrm{Aut}(\Gamma) \,|\varphi$ is a reflection$\}$. Then, there are at most five reflections $\mu, \tau, \sigma, \omega, \phi \in \mathcal{R}$ such that Fix $(\phi \circ \omega \circ \sigma \circ \tau \circ \mu) = \Gamma'$ and $(\phi \circ \omega \circ \sigma \circ \tau \circ \mu)^2 = \mathrm{Id}$.*

Proof: Since Γ' is non-trivial, there is a vertex $x_0 \in \Gamma'$ such that at least 2 (and at most $k-2$) vertices adjacent to x_0 are not in Γ'. Let we label those vertices as x_1 and x_{-1} and let η be a geodesic containing both x_1 and x_{-1}. Then we have that $\eta \cap \Gamma' = \{x_0\}$. We label the vertices of η as

$$\eta = \ldots x_{-l}, \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots, x_l \ldots$$

with $d(x_n, x_{n+1}) = 1$ for all $n \in \mathbb{Z}$. Let γ be a geodesic such that $\gamma \cap \eta = x_1$. We consider any reflection in γ that keeps the geodesic η invariant and denote by Γ'' the image of Γ' under such reflection. Since η is invariant under such reflection and it keeps the vertex $x_1 \in \gamma \cap \eta$ fixed, we have that x_0 is carried to x_2, so that Γ'' is a copy of Γ' such that $\Gamma'' \cap \eta = \{x_2\}$. We define a tree

$$\widetilde{\Gamma}^* = \eta \cup \gamma \cup \Gamma' \cup \Gamma''.$$

By construction we have that $\widetilde{\Gamma}^*$ is γ-symmetric and also admissible, since $|S(v,1)|_{\widetilde{\Gamma}^*} \equiv 0 \bmod 2 \; \forall v \in \widetilde{\Gamma}^*$. In order to use one of the preceding lemmas, we must establish the parity condition (even or odd) for every vertex of γ, depending on the parity of $k-2$. First of all we label the geodesic γ as

$$\gamma := ...y_{-j}, ..., y_{-1}, y_0, y_1, ..., y_j, ...$$

with $j \in \mathbb{N}$ and $x_1 = y_0$.

We now consider two different cases:

Case 1 ($\mathbf{k \equiv 0 \bmod 4}$)**:** We consider a family $\lambda_i, i \in \mathbb{Z}, i \neq 0$ of geodesics such that

$$\lambda_i \cap \gamma = \{y_i\} \text{ for every } i \neq 0$$

and define

$$\Gamma^* := \widetilde{\Gamma}^* \bigcup_{\substack{i \in \mathbb{Z} \\ i \neq 0}} \lambda_i.$$

Then we have that Γ^* is an admissible γ-symmetric tree and every vertex of γ satisfies Γ^*-EPC. So, by Lemma 7, there are $\mu, \tau \in \mathcal{R}$ such $\tau \circ \mu$ is an involution with $\mathrm{Fix}(\tau \circ \mu) = \Gamma^*$.

Case 2 ($\mathbf{k \equiv 2 \bmod 4}$)**:** In this case we just consider a geodesic λ such that

$$\lambda \cap \eta = \lambda \cap \gamma = \{y_0\}$$

(this is possible since $k \geq 4$ and $k \equiv 2 \bmod 4$) and define

$$\Gamma^* := \widetilde{\Gamma}^* \cup \lambda.$$

Then we have that Γ^* is an admissible γ-symmetric tree and every vertex $x \in \gamma$ satisfies the Γ^*-OPC. So, by Lemma 8, there are $\mu, \tau \in \mathcal{R}$ such $\tau \circ \mu$ is an involution with $\Gamma' \subset \mathrm{Fix}(\tau \circ \mu) = \Gamma^*$.

We consider as given two reflections $\mu, \tau \in \mathcal{R}$ such that

$$\Gamma' \subset \mathrm{Fix}(\tau \circ \mu) = \Gamma^*. \tag{2}$$

Let $\omega \in \mathrm{Aut}(\Gamma)$ be an automorphism such that $\omega(\eta) = \eta$. We label the vertices of Γ relative to η, so that ω may be described as a sequence $\omega := (\omega^0, \omega^1, \omega^2, ...)$ of permutations where ω^n acts on $V_n := V_n(\eta)$ and $p(\omega^r(x)) = \omega^{r-1}(p(x))$. If $\omega(x_{i_0,i_1,i_2,...,i_r,...}) = x_{j_0,j_1,j_2,...,j_r,...}$ we have that the r-th component of the sequence $(j_0, j_1, j_2...)$ depends on the r first components of the sequence $(i_0, i_1, i_2, ...)$. We state it by using the following notation:

$$\omega(x_{i_0,i_1,i_2,i_3,...}) = x_{\omega^0(i_0),\omega^1_{i_0}(i_1),\omega^2_{i_0,i_1}(i_2),\omega^3_{i_0,i_1,i_2}(i_3),...}$$

where the upper index r in $\omega^r_{i_0,i_1,...,i_r}$ is a redundancy that only helps us (we hope) to track the level where the permutation acts.

Direct computation shows that

$$
\begin{aligned}
\sigma\omega\left(x_{i_0,i_1,i_2,\ldots,i_k,\ldots}\right) &= \sigma\left(x_{\omega^0(i_0),\omega^1_{i_0}(i_1),\omega^2_{i_0,i_1}(i_2),\omega^3_{i_0,i_1,i_2}(i_3),\ldots}\right) \\
&= x_{\sigma^0\omega^0(i_0),\sigma^1_{\omega^0(i_0)}\omega^1_{i_0}(i_1),\sigma^2_{\omega^0(i_0),\omega^1_{i_0}(i_1)}\omega^2_{i_0,i_1}(i_2),\sigma^3_{\omega^0(i_0),\omega^1_{i_0}(i_1),\omega^2_{i_0,i_1}(i_2)}\omega^3_{i_0,i_1,i_2}(i_3),\ldots}
\end{aligned}
$$

where $\omega, \sigma \in \mathrm{Aut}(\Gamma)$ are any automorphism leaving η invariant.

We recall that the reflections μ and τ in (2) keep η invariant and

$$
\begin{aligned}
\mu(x_n) &= \mu^0(x_n) = x_{-n+2} \\
\tau(x_n) &= \tau^0(x_n) = x_{-n+2}
\end{aligned}
$$

for every $x_n \in \eta$. Let β_1 be a geodesic transversal to η at x_1, that is, $\beta_1 \cap \eta = \{x_1\}$. There is $\sigma \in \mathcal{R}_{\beta_1}$ satisfying the following conditions:

$$
\begin{aligned}
\sigma^0(x_n) &= x_{-n+2} \text{ for every } x_n \in \eta \\
\sigma\left(x_{i_0,i_1,i_2,\ldots}\right) &= x_{\sigma^0(i_0),i_1,i_2,\ldots} \text{ for } i_0 \neq 1,
\end{aligned}
\tag{3}
$$

since σ^0 is an involution fixing $\beta_1 \cap \eta$. The action of σ on the vertices having x_1 as prefix (vertices labelled as $x_{1,i_1,i_2,\ldots}$) will be characterized later.

Let β_2 be a geodesic transversal to η at x_2 and let $\omega \in \mathcal{R}_{\beta_2}$ be a reflection satisfying the condition:

$$\omega^0(x_n) = x_{-n-4} \text{ for every } x_n \in \eta.$$

We define now another reflection $\phi \in \mathcal{R}_{\beta_1}$ with $\phi \neq \sigma$, satisfying

$$\phi^0(x_n) = x_{-n-2} \text{ for every } x_n \in \eta.$$

To define the action of $\phi \circ \omega \circ \sigma \in \mathcal{R}$ on the descendents of x_n ($n \in \mathbb{Z}$) we will consider three different cases.

Descendents of $\mathbf{x}_n, \mathbf{n} \neq \mathbf{0}, \mathbf{1}$: We observe that the action of σ, τ and μ are already defined when restricted to those vertices. We want to define the action of the reflections ω and ϕ in such a way that

$$\phi \circ \omega \circ \sigma \circ \tau \circ \mu\left(x_{i_0,i_1,i_2,i_3\ldots}\right) = x_{-i_0,i_1,i_2,i_3} \text{ for } i_0 \neq 0, 1.$$

If we manage to do so, we will find that the composition $\phi \circ \omega \circ \sigma \circ \tau \circ \mu$ will clearly act on those vertices as an involution.

Direct computations shows that

$$
\begin{aligned}
\mu^0(x_n) &= x_{-n+2} \\
\tau^0\mu^0(x_n) &= x_n \\
\sigma^0\tau^0\mu^0(x_n) &= x_{-n+2} \\
\omega^0\sigma^0\tau^0\mu^0(x_n) &= x_{n+2} \\
\phi^0\omega^0\sigma^0\tau^0\mu^0(x_n) &= x_{-n}.
\end{aligned}
$$

If we denote

$$\phi \circ \omega \circ \sigma \circ \tau \circ \mu \left(x_{i_0,i_1,i_2,\ldots,i_r}\right) = x_{j_0,j_1,j_2,\ldots,j_r},$$

since

$$j_0 = \phi^0 \circ \omega^0 \circ \sigma^0 \circ \tau^0 \circ \mu^0 \left(i_0\right)$$

it follows that

$$j_0 = -i_0.$$

From the definition of σ (in 3) we have that

$$\begin{aligned} j_1 &= \phi^1_{\omega^0\sigma^0\tau^0\mu^0(i_0)}\omega^1_{\sigma^0\tau^0\mu^0(i_0)}\sigma^1_{\tau^0\mu^0(i_0)}\tau^1_{\mu^0(i_0)}\mu^1_{i_0}\left(i_1\right) \\ &= \phi^1_{i_0+2}\omega^1_{-i_0+2}\tau^1_{-i_0+2}\mu^1_{i_0}\left(i_1\right) \end{aligned}$$

and we want to impose that $j_1 = i_1$. To get this equality it is enough to define

$$\phi^1_{i_0+2}\omega^1_{-i_0+2} := \left(\tau^1_{-i_0+2}\mu^1_{i_0}\right)^{-1}.$$

We move now to the second level. By definition we have that

$$j_2 = \phi^2_{i_0+2,\omega^1_{-i_0+2}\tau^1_{-i_0+2}\mu^1_{i_0}(i_1)}\omega^2_{-i_0+2,\tau^1_{-i_0+2}\mu^1_{i_0}(i_1)}\tau^2_{-i_0+2,\mu^1_{i_0}(i_1)}\mu^2_{i_0,i_1}\left(i_2\right)$$

and in order to get the equality $j_2 = i_2$, it is enough to define

$$\phi^2_{i_0+2,\omega^1_{-i_0+2}\tau^1_{-i_0+2}\mu^1_{i_0}(i_1)}\omega^2_{-i_0+2,\tau^1_{-i_0+2}\mu^1_{i_0}(i_1)} := \left(\tau^2_{-i_0+2,\mu^1_{i_0}(i_1)}\mu^2_{i_0,i_1}\right)^{-1}.$$

For the third level, we have that

$$\begin{aligned} j_3 = \phi^3_{i_0+2,\omega^1_{-i_0+2}\tau^1_{-i_0+2}\mu^1_{i_0}(i_1),\omega^2_{-i_0+2,\tau^1_{-i_0+2}\mu^1_{i_0}(i_1)}\tau^2_{-i_0+2,\mu^1_{i_0}(i_1)}\mu^2_{i_0,i_1}(i_2)} \\ \omega^3_{-i_0+2,\tau^1_{-i_0+2}\mu^1_{i_0}(i_1),\tau^2_{-i_0+2,\mu^1_{i_0}(i_1)}\mu^2_{i_0,i_1}(i_2)}\tau^3_{-i_0+2,\mu^1_{i_0}(i_1),\mu^2_{i_0,i_1}(i_2)}\mu^3_{i_0,i_1,i_2}\left(i_3\right) \end{aligned}$$

and again, by defining

$$\begin{aligned} &\phi^3_{i_0+2,\omega^1_{-i_0+2}\tau^1_{-i_0+2}\mu^1_{i_0}(i_1),\omega^2_{-i_0+2,\tau^1_{-i_0+2}\mu^1_{i_0}(i_1)}\tau^2_{-i_0+2,\mu^1_{i_0}(i_1)}\mu^2_{i_0,i_1}(i_2)} \\ &\qquad \omega^3_{-i_0+2,\tau^1_{-i_0+2}\mu^1_{i_0}(i_1),\tau^2_{-i_0+2,\mu^1_{i_0}(i_1)}\mu^2_{i_0,i_1}(i_2)} := \left(\tau^3_{-i_0+2,\mu^1_{i_0}(i_1),\mu^2_{i_0,i_1}(i_2)}\mu^3_{i_0,i_1,i_2}\right)^{-1} \end{aligned}$$

we find that $j_3 = i_3$. We proceed in this same manner for the other levels, so that we get five reflections such that the composition $\phi\circ\omega\circ\sigma\circ\tau\circ\mu$ acts on $\Gamma\backslash\left(\{x_{0,i_1,i_2,\ldots}\}\cup\{x_{1,i_1,i_2,\ldots}\}\right)$ as an involution. Moreover, the vertices that are descendents of x_n, for $n \neq 0,1$ are not fixed by the composition of the five reflections.

Descendents of $\mathbf{x}_0$: We remember that μ and τ are given (in 2) and the restriction of σ to descendents of x_0 is defined as

$$\sigma\left(x_{0,i_1,i_2,\ldots}\right) = x_{2,i_1,i_2,\ldots}$$

We still have to define the action of ω and ϕ on the descendents of x_0 in such a way that the composition of the five reflections becomes an involution that fixes only the vertices of the admissible tree Γ'. Given $x_{0,i_1,i_2,i_3,...} \in \Gamma'$ we have that

$$\begin{aligned}
\tau \circ \mu \left(x_{0,i_1,i_2,i_3,...}\right) &= x_{0,i_1,i_2,i_3,...} \\
\sigma \left(x_{0,i_1,i_2,i_3,...}\right) &= x_{2,i_1,i_2 i_3,...} \\
\omega \left(x_{2,i_1,i_2,i_3,...}\right) &= x_{2,\omega_2^1(i_1),\omega_{2,i_1}^2(i_2),\omega_{2,i_1,i_2}^3(i_3),...} \\
\phi \circ \omega \left(x_{2,i_1,i_2,i_3,...}\right) &= x_{0,\phi_2^1\omega_2^1(i_1),\phi^2_{2,\omega_2^1(i_1)}\omega^2_{2,i_1}(i_2),\phi^3_{2,\omega_2^1(i_1),\omega^2_{2,i_1}(i_2)}\omega^3_{2,i_1,i_2}(i_3),...}
\end{aligned}$$

If we define $\phi^i_{...}$ and $\omega^i_{...}$ by the equations

$$\begin{aligned}
\phi_2^1 &:= \left(\omega_2^1\right)^{-1}, \\
\phi^2_{2,\omega_2^1(i_1)} &:= \left(\omega^2_{2,i_1}\right)^{-1}, \\
\phi^3_{2,\omega_2^1(i_1),\omega^2_{2,i_1}(i_2)} &:= \left(\omega^3_{2,i_1,i_2}\right)^{-1}, ...
\end{aligned} \tag{4}$$

we find that

$$\begin{aligned}
\phi_2^1\omega_2^1\left(i_1\right) &= i_1, \\
\phi^2_{2,\omega_2^1(i_1)}\omega^2_{2,i_1}\left(i_2\right) &= i_2, \\
\phi^3_{2,\omega_2^1(i_1),\omega^2_{2,i_1}(i_2)}\omega^3_{2,i_1,i_2}\left(i_3\right) &= i_3, ...
\end{aligned}$$

so that

$$\begin{aligned}
\phi \circ \omega \circ \sigma \circ \tau \circ \mu \left(x_{0,i_1,i_2,i_3,...}\right) &= \phi \circ \omega \left(x_{2,i_1,i_2,i_3,...}\right) \\
&= x_{0,i_1,i_2,i_3,...}
\end{aligned}$$

that is, the vertices of Γ' descending from x_0 are fixed by $\phi \circ \omega \circ \sigma \circ \tau \circ \mu$. Moreover, if $x_{0,i_1,i_2,i_3,...} \notin \Gamma'$, we have that

$$\phi \circ \omega \circ \sigma \circ \tau \circ \mu \left(x_{0,i_1,i_2,i_3...}\right) = x_{0,j_1,j_2,j_3,...}$$

where

$$j_1 = \phi_2^1\omega_2^1\tau_2^1\mu_0^1\left(i_1\right).$$

We observe that if we define $\left(\phi_2^1\omega_2^1\right)$ as in 4, we get that

$$j_1 = \tau_2^1\mu_0^1\left(i_1\right)$$

i.e., j_1 is determined by the action of μ and τ. For j_2 we have that

$$j_2 = \phi^2_{2,\omega_2^1\tau_2^1\mu_0^1(i_1)}\omega^2_{2,\tau_2^1\mu_0^1(i_1)}\tau^2_{2,\mu_0^1(i_1)}\mu^2_{0,i_1}\left(i_2\right)$$

and by defining

$$\phi^2_{2,\omega_2^1\tau_2^1\mu_0^1(i_1)} := \left(\omega^2_{2,\tau_2^1\mu_0^1(i_1)}\right)^{-1}$$

we find that

$$j_2 = \tau^2_{2,\mu^1_0(i_1)}\mu^2_{0,i_1}(i_2)$$

i.e., j_2 is well defined. This is actually the inductive step that should be performed to complete the proof. The same can be done, for example, for the level 3: We have that

$$j_3 = \phi^3_{2,\omega^1_2\tau^1_2\mu^1_0(i_1),\omega^2_{2,\tau^1_2\mu^1_0(i_1)}\tau^2_{2,\mu^1_0(i_1)}\mu^2_{0,i_1}(i_2)}\omega^3_{2,\tau^1_2\mu^1_0(i_1),\tau^2_{2,\mu^1_0(i_1)}\mu^2_{0,i_1}(i_2)}\tau^3_{2,\mu^1_0(i_1),\mu^2_{0,i_1}(i_2)}\mu^3_{0,i_1,i_2}(i_3)$$

So, if we define

$$\phi^3_{2,\omega^1_2\tau^1_2\mu^1_0(i_1),\omega^2_{2,\tau^1_2\mu^1_0(i_1)}\tau^2_{2,\mu^1_0(i_1)}\mu^2_{0,i_1}(i_2)} := \left(\omega^3_{2,\tau^1_2\mu^1_0(i_1),\tau^2_{2,\mu^1_0(i_1)}\mu^2_{0,i_1}(i_2)}\right)^{-1}$$

we find that

$$j_3 = \tau^3_{2,\mu^1_0(i_1),\mu^2_{0,i_1}(i_2)}\mu^3_{0,i_1,i_2}(i_3).$$

Proceeding in this way we will find that $j_n \neq i_n$ for some $n \geq 1$, since $x_{0,i_1,i_2,i_3,\dots} \notin \Gamma'$ and the descendents of x_0 that are fixed by $\tau \circ \mu$ are those ones contained in Γ'.

Descendents of $\mathbf{x}_1$: We proceed in a similar way. We recall that the reflections τ and μ are completely determined and, given $x_n \in \eta$,

$$\begin{aligned} \mu^0(x_n) &= x_{-n+2}, & \tau^0(x_n) &= x_{-n+2} \\ \sigma^0(x_n) &= x_{-n+2}, & \omega^0(x_n) &= x_{-n-4} \\ \phi^0(x_n) &= x_{-n-2}. \end{aligned}$$

We define the action of ω on the descendents of x_1 as

$$\omega(x_{i_0,i_1,i_2,i_3\dots}) = x_{\omega^0(i_0),i_1,i_2,i_3\dots}.$$

If we write

$$\phi \circ \omega \circ \sigma \circ \tau \circ \mu(x_{1,i_1,i_2,i_3\dots}) = x_{-1,j_1,j_2,j_3,\dots}$$

we have that

$$\begin{aligned} j_1 &= \phi^1_3\sigma^1_1\tau^1_1\mu^1_1(i_1) \\ j_2 &= \phi^2_{3,\sigma^1_1\tau^1_1\mu^1_1(i_1)}\sigma^2_{1,\tau^1_1\mu^1_1(i_1)}\tau^2_{1,\mu^1_1(i_1)}\mu^2_{1,i_1}(i_2) \\ j_3 &= \phi^3_{3,\sigma^1_1\tau^1_1\mu^1_1(i_1),\sigma^2_{1,\tau^1_1\mu^1_1(i_1)}\tau^2_{1,\mu^1_1(i_1)}\mu^2_{1,i_1}(i_2)}\sigma^3_{1,\tau^1_1\mu^1_1(i_1),\tau^2_{1,\mu^1_1(i_1)}\mu^2_{1,i_1}(i_2)}\tau^3_{1,\mu^1_1(i_1),\mu^2_{1,i_1}(i_2)}\mu^3_{1,i_1,i_2}(i_3), \dots \end{aligned}$$

We need to define ϕ and σ in such a way that $j_i = i_i$ for every $i \in \mathbb{N}$ and for this it is enough to define the action of ϕ and σ on the descendents of x_1 as follows:

$$\begin{aligned} \phi^1_3\sigma^1_1 &:= \left(\tau^1_1\mu^1_1\right)^{-1} \\ \phi^2_{3,\sigma^1_1\tau^1_1\mu^1_1(i_1)}\sigma^2_{1,\tau^1_1\mu^1_1(i_1)} &:= \left(\tau^2_{1,\mu^1_1(i_1)}\mu^2_{1,i_1}\right)^{-1} \\ \phi^3_{3,\sigma^1_1\tau^1_1\mu^1_1(i_1),\sigma^2_{1,\tau^1_1\mu^1_1(i_1)}\tau^2_{1,\mu^1_1(i_1)}\mu^2_{1,i_1}(i_2)}\sigma^3_{1,\tau^1_1\mu^1_1(i_1),\tau^2_{1,\mu^1_1(i_1)}\mu^2_{1,i_1}(i_2)} &:= \left(\tau^3_{1,\mu^1_1(i_1),\mu^2_{1,i_1}(i_2)}\mu^3_{1,i_1,i_2}\right)^{-1}, \dots \end{aligned}$$

Altogether we have 5 reflections such that the composition $\phi\circ\omega\circ\sigma\circ\tau\circ\mu$ is an involution and $\mathrm{Fix}\,(\phi\circ\omega\circ\sigma\circ\tau\circ\mu) = \Gamma'$ as we wanted. $\square$

By definition, an admissible tree is the set of fixed points of an involution. Since all such involutions are conjugated, if we prove that a single involution is a product of a bounded number of reflections, the same bound will hold for every involution with the same set of fixed points. In the next theorem we will prove that, in case of a trivial admissible tree, such an involution can be attained as the product of 3 reflections.

Let us prove the same Theorem for the case of trivial admissible trees.

Theorem 10 *Let Γ be a k-homogeneous tree, $\Gamma' \subset \Gamma$ a trivial admissible tree and $\mathcal{R} = \{\phi \in \mathrm{Aut}\,(\Gamma)\,|\phi$ is a reflection$\}$. Then, there are at most three reflections $\mu, \tau, \sigma \in \mathcal{R}$ such that $\mathrm{Fix}\,(\sigma\circ\tau\circ\mu) = \Gamma'$ and $(\sigma\circ\tau\circ\mu)^2 = \mathrm{Id}$.*

Proof: If Γ' is a geodesic γ, it is enough to consider μ as any reflection in γ and by definition we have that $\mathrm{Fix}\,(\mu) = \Gamma'$. If $\Gamma' = \Gamma$, we take any reflection $\mu \in \mathcal{R}$ and since a reflection is an involution, we have that $\mathrm{Fix}\,(\mu\circ\mu) = \Gamma = \Gamma'$. So, it is left the case when the trivial admissible tree Γ' consists of a single vertex, that is, $\Gamma' = \{x_0\}$. We shall divide the prove in two cases, according to homogeneity of Γ.

Case 1 ($\mathbf{k \equiv 0 \bmod 4}$): Consider two geodesics γ_1 and γ_2 in Γ such that $\gamma_1 \cap \gamma_2 = \{x_0\}$. We label the vertices adjacent to x_0 as $x_1, x_2, ..., x_{k-1}, x_k$ and assume without loss of generality that $x_1, x_2 \in \gamma_1$ and $x_{k-1}, x_k \in \gamma_2$. Let $\mu \in \mathcal{R}_{\gamma_1}$ and $\tau \in \mathcal{R}_{\gamma_2}$ be reflections such that

$$\begin{aligned}\mu|_{S(x_0,1)} &= (x_3x_4)\,(x_5x_6)\,...\,(x_{k-3}x_{k-2})\,(x_{k-1}x_k)\\ \tau|_{S(x_0,1)} &= (x_1x_2)\,(x_3x_5)\,(x_4x_6)\,...\,(x_{k-5}x_{k-3})\,(x_{k-4}x_{k-2})\,.\end{aligned}$$

The existence of such reflections is assured by the extension property ([6, Proposition 7]). Direct computations shows that

$$\tau\circ\mu|_{S(x_0,1)} = (x_1x_2)\,(x_3x_6)\,(x_4x_5)\,...\,(x_{k-3}x_{k-4})\,(x_{k-1}x_k)$$

so that $\mathrm{Fix}\,(\tau\circ\mu)\,|_{S(x_0,1)} = \emptyset$ and it follows that

$$\mathrm{Fix}\,(\tau\circ\mu) = \{x_0\} = \Gamma'$$

and $(\tau\circ\mu)^2 = \mathrm{Id}$.

Case 2 ($\mathbf{k \equiv 2 \bmod 4}$): Let σ be a reflection fixing the vertex x_0. Let

$$\eta = ...x_{-l}, ..., x_{-2}, x_{-1}, x_0, x_1, x_2, ..., x_l...$$

be a geodesic containing the vertex x_0. Consider now two geodesics γ_1 and γ_2 such that $\gamma_1 \cap \eta = x_1$ and $\gamma_2 \cap \eta = x_2$.

Let $\mu, \sigma \in \mathcal{R}_{\gamma_1}$ and $\tau \in \mathcal{R}_{\gamma_2}$ be reflections satisfying the following conditions

$$\mu^0\,(x_n) = x_{-n+2},\ \ \tau^0\,(x_n) = x_{-n+4} \text{ and } \ \sigma^0\,(x_n) = x_{-n+2}$$

for every $x_n \in \eta$ where the decomposition of the reflections is considered relative to the geodesic η, in the same way we did in the proof of Theorem 9. Direct computations shows that $\sigma^0 \circ \tau^0 \circ \mu^0 (x_n) = x_{-n}$ and it follows that

$$\mathrm{Fix}\left(\sigma^0 \circ \tau^0 \circ \mu^0\right) = \{x_0\}.$$

Since x_0 is the only fixed point of $\sigma \circ \tau \circ \mu$ on γ, in order to have $\mathrm{Fix}\,(\sigma \circ \tau \circ \mu) = \{x_0\}$ it is enough to define the reflections in such a way that it has no fixed points in $S(x_0, 1)$, that is, considering the labeling from γ, we need the condition

$$\sigma \circ \tau \circ \mu (x_{0,i}) \neq x_{0,i}.$$

Considering the decomposition of the three reflections we get that

$$\sigma \circ \tau \circ \mu (x_{0,i}) = \sigma\tau\left(x_{2,\mu_0^1(i)}\right) = \sigma\left(x_{2,\tau_2^1\mu_0^1(i)}\right) = x_{0,\sigma_2^1\tau_2^1\mu_0^1(i)}$$

and for any choice of σ_2^1, τ_2^1 and μ_0^1 such that $\sigma_2^1\tau_2^1\mu_0^1 \neq 1$, we will get that $\mathrm{Fix}\,(\sigma \circ \tau \circ \mu) = \{x_0\}$. If we define, for instance,

$$\mu_0^1(i) = i, \;\; \tau_2^1(i) = k - 1 - i \text{ and } \sigma_2^1(i) = i$$

for every $i = 1, 2, ..., k-2$, we find that $\sigma_2^1\tau_2^1\mu_0^1(i) = k - 1 - i$. Moreover,

$$(\sigma \circ \tau \circ \mu)^2 (x_{0,i}) = (\sigma \circ \tau \circ \mu)(x_{0,k-1-i}) = x_{0,i}$$

so that $(\sigma \circ \tau \circ \mu)^2 = \mathrm{Id}$. □

Before we prove our main result, we need the following lemma:

Lemma 11 *Let Γ be a k-homogeneous tree and $\varphi \in \mathrm{Aut}^+(\Gamma)$. Then φ may be expressed as the product of at most three involutions with fixed points, where at least one of those is a reflection.*

Proof: Suppose there is $x_0 \in \Gamma$ such that $\varphi(x_0) = x_0$. In [4], Moran proved that a rooted tree has the bi-reflection property, that is, every automorphism of the rooted tree may be expressed as the product of two involutions. Since $\varphi(x_0) = x_0$, the automorphism φ may be seen as an automorphism of the tree with root in x_0 and it is the product of two involutions.

Suppose now that φ has no fixed point. Given $x_0 \in \Gamma$, since $0 < d(x_0, \varphi(x_0)) \equiv 0 \bmod 2$ there is a vertex $w \in \Gamma$ that is the middle point of the geodesic segment $[x_0, \varphi(x_0)]$. So there is a reflection ϕ such that $\phi(\varphi(x_0)) = x_0$. Since $\varphi, \phi \in \mathrm{Aut}^+(\Gamma)$ we have that $\phi \circ \varphi \in \mathrm{Aut}^+(\Gamma)$ and since $\phi \circ \varphi(x_0) = x_0$, as we just proved, there are involutions σ_1, σ_2 such that $\phi \circ \varphi = \sigma_1 \circ \sigma_2$ and we have that

$$\varphi = \phi \circ (\phi \circ \varphi) = \phi \circ \sigma_1 \circ \sigma_2.$$

□

Theorem 12 *Let Γ be a k-homogeneous tree and $\psi \in \mathrm{Aut}^+(\Gamma)$. Then, ψ may be expressed as the product of at most 11 reflections in geodesics.*

Proof: In the previous Lemma we proved that $\psi \in \mathrm{Aut}^+(\Gamma)$ may be expressed as the product of (at most) two involutions (say φ' and φ'') and one reflection (let us say ϕ): $\psi = \phi \circ \varphi' \circ \varphi''$. Let Γ' and Γ'' be the trees of fixed points of the involutions φ' and φ''. Theorems 9 and 10 assures there are reflections $\widetilde{\phi}'_1, ..., \widetilde{\phi}'_5$ and $\widetilde{\phi}''_1, ..., \widetilde{\phi}''_5$ (eventually less then 5 are needed) such that

$$\mathrm{Fix}\left(\widetilde{\phi}'_1 \circ ... \circ \widetilde{\phi}'_5\right) = \Gamma' \text{ and } \mathrm{Fix}\left(\widetilde{\phi}''_1 \circ ... \circ \widetilde{\phi}''_5\right) = \Gamma''$$

and $\left(\widetilde{\phi}'_1 \circ ... \circ \widetilde{\phi}'_5\right)^2 = \left(\widetilde{\phi}''_1 \circ ... \circ \widetilde{\phi}''_5\right)^2 = \mathrm{Id}$. It follows that $\widetilde{\phi}'_1 \circ ... \circ \widetilde{\phi}'_5$ and $\widetilde{\phi}''_1 \circ ... \circ \widetilde{\phi}''_5$ are involutions that have the same fixed points as φ' and φ'' respectively. But Lemma 3 in [7] assures that involutions in a homogeneous tree are conjugated if and only if there is an automorphism that maps the set of fixed points of one to the set of fixed points of the other. It follows that

$$\varphi' = \sigma' \left(\widetilde{\phi}'_1 \circ ... \circ \widetilde{\phi}'_5\right) (\sigma')^{-1} = \left(\sigma' \widetilde{\phi}'_1 (\sigma')^{-1}\right) \circ ... \circ \left(\sigma' \widetilde{\phi}'_5 (\sigma')^{-1}\right)$$
$$\varphi'' = \sigma'' \left(\widetilde{\phi}''_1 \circ ... \circ \widetilde{\phi}''_5\right) (\sigma'')^{-1} = \left(\sigma'' \widetilde{\phi}'_1 (\sigma'')^{-1}\right) \circ ... \circ \left(\sigma'' \widetilde{\phi}'_5 (\sigma'')^{-1}\right)$$

and since the conjugate of a reflection is still a reflection, by defining

$$\phi'_i = \sigma' \widetilde{\phi}'_i (\sigma')^{-1}, \quad \phi''_i = \sigma'' \widetilde{\phi}''_i (\sigma'')^{-1}$$

for $i = 1, 2, ..., 5$ we get that

$$\begin{aligned} \psi &= \phi \circ \varphi' \circ \varphi'' \\ &= \phi \circ \phi'_1 \circ ... \circ \phi'_5 \circ \phi''_1 \circ ... \circ \phi''_5. \end{aligned}$$

□

We can restate the previous Theorem in terms of covering number:

Theorem 13 *Let Γ be a k-homogeneous tree and $C_{\mathcal{R}}$ the conjugacy class of reflections in Γ. Then, the covering number satisfies* $\mathrm{cn}\left(\mathrm{Aut}^+(\Gamma), C_{\mathcal{R}}\right) \leq 11$, *that is,* $\mathcal{R}^{11} = \mathrm{Aut}^+(\Gamma)$.

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