A dynamical surface interacting with rarefied walls

L.R.G. Fontes*,[1]   M. Vachkovskaia†,[2]   A. Yambartsev‡,[1]

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[1]Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Rua do Matão 1010, CEP 05508–090, São Paulo SP, Brazil.
E-mails: lrenato@ime.usp.br, yambar@ime.usp.br
[2]Department of Statistics, Institute of Mathematics, Statistics and Scientific Computation, University of Campinas, Caixa Postal 6065, CEP 13083–970, Campinas SP, Brazil.
E-mail: marinav@ime.unicamp.br

Abstract

We consider the motion of a discrete random surface interacting by exclusion with a rarefied wall. The dynamics is given by the serial harness process. We prove that the process delocalizes iff the mean number of visits to the set of sites where the wall is present by a random walk is infinite. When the surface delocalizes, bounds on its average speed are obtained.

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1 Introduction

In [1], the serial harness process interacting by exclusion with a solid wall located at the origin was studied with the aim of ascertaining the existence or not of a (substantial) effect of repulsion on the surface due to the presence of the wall.

In Theorem 1.1 of that reference, it is shown that in all dimensions the average height of the surface (say, at the origin) diverges to $+\infty$ as time increases. This should be compared to the average absolute height of the surface at the origin in the *free* case (where there is no wall). In the latter case, that quantity is bounded in dimensions 3 and higher [2, 1]. An effect of repulsion on the wall is thus established in those dimensions.

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In the present work, we ask ourselves if and how much of the repulsion effect remains in those dimensions when part of the wall is removed. We place solid walls in a subset of the space only, and examine the asymptotic behavior of the average height of the surface at the origin as time diverges.

In Theorem 3.1 below we relate the occurrence of repulsion (in all dimensions) with the mean number of visits of the wall set $\mathcal{W}$ by a random walk. It readily follows that repulsion occurs in the case where $\mathcal{W}$ is chosen according to a product of Bernoullis with *fixed* positive density. Furthermore, it follows from the lower bound of (3.2) below (see also Lemma 3.1 of [3]) that in dimensions 3 and higher and for a class of noise distributions (including the Gaussian one), the average height of the surface at the origin in the latter case has the *same* asymptotic behavior to leading order (apart from a constant) as time diverges as in the case with walls everywhere.

We are thus led to consider the case where $\mathcal{W}$ is not homogeneously distributed in space, with the probability of wall presence in a site vanishing as the distance from the site to the origin diverges.

This setting is also inspired in the distribution of traps in [4, 5]. There is a remarkable similarity of the transition delocalization/localization discussed below in the context of Corollary 3.1, and the transition recurrence/transience established in Theorem 3.1 of [4] (both transitions take place at the same point).

In the next section we define precisely the relevant models. In the following and final section we present our results, some of which were announced in [6].

2 The model

Denote by $|i-j|$ the number of edges in a minimal path connecting i and j (we will use this definition not only for $\mathbb{Z}^d$, but also for other graphs). Let $\mathcal{P} = \{p(i,j)\}_{i,j\in\mathbb{Z}^d}$ be a symmetric stochastic matrix which satisfies $p(i,j) = p(0,j-i) =: p(j-i) = p(i-j)$ (homogeneity) and $p(j) = 0$ for all $|j| > v$ for some v (finiteness). Assume also that $\mathcal{P}$ is truly d-dimensional: $\{j \in \mathbb{Z}^d : p(j) \neq 0\}$ generates $\mathbb{Z}^d$. The weights $p(i,j)$ can be interpreted as transition probabilities of a random walk on $\mathbb{Z}^d$; denote by $\mathcal{P}$ its transition matrix and by $p_m(i,j)$ its m-step transition probabilities. By homogeneity, $p_m(i,j) = p_m(0,j-i) =: p_m(j-i)$.

Let $\mathcal{E} := \{\varepsilon, \varepsilon_n(i),\, i \in \mathbb{Z}^d, n \in \mathbb{Z}\}$ be a family of i.i.d. integrable symmetric random variables with unbounded support.

Let us place a piece of flat wall located at height 0 at some sites of $\mathbb{Z}^d$, thus defining the environment

$$\mathcal{W} = \{i \in \mathbb{Z}^d : \text{ there is a wall in } i\}.$$

The set $\mathcal{W}$ can be either random or deterministic. Then, with the environment fixed, we start the serial harness which is not allowed to go below the wall (in the sites where the wall is present). So, the wall process $X_n^{\mathcal{W}}(i)$, $i \in \mathbb{Z}^d$, $n \geq 0$

is defined by

$$X_n^{\mathcal{W}}(i) = \begin{cases} 0, & \text{if} \quad n = 0, \\ \left(\sum_{j \in \mathbb{Z}^d} p(i,j) X_{n-1}^{\mathcal{W}}(j) + \varepsilon_n(i) \right)^+, & \text{if} \quad n \geq 1 \text{ and } i \in \mathcal{W}, \\ \sum_{j \in \mathbb{Z}^d} p(i,j) X_{n-1}^{\mathcal{W}}(j) + \varepsilon_n(i), & \text{if} \quad n \geq 1 \text{ and } i \notin \mathcal{W}, \end{cases} \tag{2.1}$$

where $(\cdot)^+ = (\cdot) \vee 0 = \max\{(\cdot), 0\}$.

The case where $\mathcal{W} = \emptyset$, in which we denote $X^{\mathcal{W}}$ by Y, was introduced by Hammersley in [2] as the *serial harness*. In that paper it is shown that $\mathbb{E}Y_n^2(0)$ is of order 1 in $d \geq 3$. (Notice that under our assumptions on $X_0^{\mathcal{W}}$ and ε, $\mathbb{E}Y_n(0) \equiv 0$.)

The case where $\mathcal{W} = \mathbb{Z}^d$, in which we denote $X^{\mathcal{W}}$ by Z, was studied in [1]. We now quote some of the results of that paper, which will be relevant here.

Theorem 2.1 *(Part of Theorem 1.2 from [1]). Let F be the law of ε, $\bar{F}(x) = \mathbb{P}(\varepsilon > x)$ and define*

$$\mathcal{L}_\alpha^- := \{F : \bar{F}(x) \leq ce^{-c'x^\alpha}, x > 0, \text{ for some positive } c,\, c'\}, \tag{2.2}$$

$$\mathcal{L}_\alpha^+ := \{F : \bar{F}(x) \geq ce^{-c'x^\alpha}, x > 0, \text{ for some positive } c,\, c'\}, \tag{2.3}$$

and

$$\mathcal{L}_\alpha := \mathcal{L}_\alpha^- \cap \mathcal{L}_\alpha^+. \tag{2.4}$$

For $d \geq 3$, there exist constants c and C that may depend on the dimension such that

(i) if $F \in \mathcal{L}_\alpha$ for some $1 \leq \alpha \neq 1 + d/2$, then

$$c(\log n)^{\frac{1}{\alpha}} \leq \mathbb{E}\, Z_n(0) \leq C(\log n)^{\frac{1}{\alpha} \vee \frac{2}{2+d}}; \tag{2.5}$$

(ii) if $F \in \mathcal{L}_{1+d/2}$, then

$$c(\log n)^{\frac{2}{2+d}} \leq \mathbb{E}\, Z_n(0) \leq C(\log n)^{\frac{2}{2+d}} (\log \log n)^{\frac{d}{2+d}}. \tag{2.6}$$

For a discussion and references on related surface models, dynamical and in equilibrium, see [1] and [3].

3 Results

Denote by ξ_n^i the position at time n of a random walk with transition matrix $\mathcal{P}$ starting from i, and independent of $\mathcal{E}$.

Suppose that the set $\mathcal{W}$ is such that either for all $i \in \mathbb{Z}^d$ the mean number of visits to the set $\mathcal{W}$ by $\{\xi_n^i\}_{n \in \mathbb{N}}$ is infinite, or there exists a constant C such that for all $i \in \mathbb{Z}^d$ the mean number of visits to the set $\mathcal{W}$ by $\{\xi_n^i\}_{n \in \mathbb{N}}$ is less than C. (So, the following situation is impossible: there exists a sequence $i_1, i_2, \ldots \in \mathbb{Z}^d$

such that the mean number of visits to the set $\mathcal{W}$ by $\{\xi_n^{i_k}\}_{n\in\mathbb{N}}$ is equal to M_k and $M_k \to \infty$ as $k \to \infty$.) This assumption is not very restrictive and it holds e.g. in the particular case considered in Theorem 3.2.

For fixed $\mathcal{W}$, we say that the process *localizes* (and that there occurs *localization*), if $\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) \le const$ for all n and i, and *delocalizes* (there occurring *delocalization*) if there exists at least one site $i \in \mathbb{Z}^d$ such that $\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W})$ is unbounded as $n \to \infty$ (the next theorem shows that those two alternatives cover in fact all the possible cases for which $\mathcal{W}$ satisfies the assumption on the latter paragraph). Our first result is quite general.

Theorem 3.1 *Under the above assumption on $\mathcal{W}$, the process delocalizes iff the mean number of visits to the set $\mathcal{W}$ by $\{\xi_n^i\}_{n\in\mathbb{N}}$ is infinite.*

Remark 3.1 *This results actually holds not only for $\mathbb{Z}^d$, but for any homogeneous graph $\mathcal{G}$, if $\mathcal{G}$ and $\mathcal{P}$ are such that (besides $\mathcal{P}$ being symmetric, homogeneous and of finite range) either*

$$\sum_k \sum_{i\in\mathcal{G}} p_k(0,i)^2 < \infty, \tag{3.1}$$

or $\{\xi_n^i\}_{n\in\mathbb{N}}$ is recurrent. Condition (3.1) is important for localization, as it guarantees the localization of the free process, used in (3.9). For example, Theorem 3.1 is valid for a simple random walk on a regular d-ary tree (which obviously satisfies (3.1)).

Remark 3.2 *Note that, as the localization implies that the number of visits to the set $\mathcal{W}$ by $\{\xi_n^i\}_{n\in\mathbb{N}}$ is bounded, the assumption on $\mathcal{W}$ is satisfied in this case. In the opposite direction, if the assumption on $\mathcal{W}$ is not satisfied, it can happen that the process neither localizes, nor delocalizes (accordingly to our definition), that is, $\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W})$ is bounded in n for any fixed i, but the set $\{\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) : i \in \mathbb{R}^d, n = 0,1,\dots\}$ is unbounded.*

We study in more detail the case when the wall is placed at each site $i \in \mathbb{Z}^d$ with probability q_i, independently of all other sites, where $q_i = q(|i|)$, $q : \mathbb{R}_+ \to [0,1]$. If $q_i = 1$ for all i, we get the process considered in [1]. It can be easily shown that all upper bounds obtained in [1] are valid for this modification also. In this case the fact that the mean number of visits to the set $\mathcal{W}$ is finite (a.s. with respect to $\mathcal{W}$) is equivalent to a.s. transience of $\mathcal{W}$ (see [4, 5]).

In the case when the process delocalizes, we are able to obtain lower bounds on $\mathbb{E}X_n^{\mathcal{W}}(i)$ (here the expectation is taken with respect to $\mathcal{W}$ also), which depend on the tail of the distribution of ε and on $q(x)$, under a technical restriction on the transition matrix $\mathcal{P}$.

Theorem 3.2 *Suppose that $p(0,0) = \delta > \frac{1}{2d+1}$, $p(0,i) = \frac{1-\delta}{2d}$, if $|i| = 1$, and q is nonincreasing. Let $d \ge 3$ and $F_\varepsilon \in \mathcal{L}_\alpha^+$ for some $\alpha > 0$. Then, there exist constants c_1 and c_2 such that for all i, $n \ge 1$*

$$c_1(i)\left(\log \mathbb{E}L_n^{\mathcal{W}}\right)^{\frac{1}{\alpha}} \le \mathbb{E}X_n^{\mathcal{W}}(i) \le c_2 \sum_{k=1}^n kq(k), \tag{3.2}$$

where $L_n^{\mathcal{W}} = \sum_{k=0}^{n-1} 1\{\xi_k^0 \in \mathcal{W}\}$ *is the number of visits of* ξ^0 *to* $\mathcal{W}$ *from time* 0 *to time* $n-1$.

See at the end of the proof of this theorem the remark about what happens on graphs other than $\mathbb{Z}^d$.

Remark 3.3 *The size* 1 *of the range of* $\mathcal{P}$ *is adopted for convenience. The restrictions on* $p(0,0)$ *and* q *are on the other hand substantial, but technical.*

Remark 3.4 *The expected value on the left hand side of (3.2) is with respect to the probability distributions of both* $\mathcal{W}$ *and* ξ^0. *It can also be expressed as*

$$\sum_{k=1}^{n} \mathbb{E}q(\xi_k^0), \tag{3.3}$$

where the expected value is with respect to ξ^0.

Remark 3.5 *In [1] other types of interaction between the surface and the wall were mentioned (see (1.19), (1.20) of [1]), namely, when the noise would push the surface below the wall, then do nothing, or take convex combination without noise. All lower bounds obtained in the present paper are valid also for those types of interactions. As of upper bounds, we do not have specific results for those types of interactions; we may always of course use the upper bounds for the solid wall case of [1], which are valid for those interactions as well; see last paragraph of Section 1 of that reference.*

Also, as all the proofs use the fact that the free process localizes (which is true only for $d \geq 3$*) the question about bounds in dimensions 1 and 2 remains open.*

To compare the model with rarefied wall to the solid wall model, we consider two specific functions $q(x)$. The results below follow readily from the lower bound in (3.2), or alternatively (3.3).

Corollary 3.1 *Under the assumptions of Theorem 3.2, let*

$$q(x) = \lambda(x+1)^{-\beta},\ \beta \geq 0,\ 0 < \lambda \leq 1.$$

(i) If $\beta < 2$, *then for any* i *there exists* $c = c(i,\beta)$ *such that for all* n

$$\mathbb{E}X_n^{\mathcal{W}}(i) \geq c(\log n)^{\frac{1}{\alpha}}. \tag{3.4}$$

(ii) If $\beta = 2$, *then for any* i *there exists* $c = c(i)$ *such that for all* n

$$\mathbb{E}X_n^{\mathcal{W}}(i) \geq c(\log\log n)^{\frac{1}{\alpha}}. \tag{3.5}$$

In both cases above, q is convex, so the results follow from the following lower bound to (3.3). By Jensen

$$\sum_{k=1}^{n} q(\mathbb{E}|\xi_k^0|) \geq \sum_{k=1}^{n} q(\mathrm{const}\sqrt{k}). \tag{3.6}$$

Note that in the case (i) the lower bound is of the same order as in the process with $q_i \equiv 1$. If $\alpha < 1 + d/2$, then Theorems 2.1 and Corollary 3.1 give upper and lower bounds of the same order. So, the case when $q(x) \geq x^{-\beta}$, $\beta < 2$, is similar to $q_i \equiv 1$ from the point of view of the repulsion speed. The case $\beta = 0$ is that of a wall with constant positive density (less than one) of holes.

Another remark is that under the assumption of Corollary 3.1, if $\beta > 2$, then Theorem 3.1 implies localization, which together with the results of Corollary 3.1, establish a transition delocalization/localization at $\beta = 2$.

A further point is that again under the assumption of Corollary 3.1, our better upper bound for case (ii) is by using the same bound as for the case $q_i \equiv 1$ of Theorem 2.1. Thus we do not resolve the issue of whether or not there is a difference in repulsion speed (in leading order) between the cases treated in Theorems 2.1 and in Corollary 3.1(ii). We can nevertheless establish a difference between the case of Theorem 2.1 and some of the ones treated in the following result.

Corollary 3.2 *Under the assumptions of Theorem 3.2, let*

$$q(x) = (x^2 \log^\gamma (x+1))^{-1}, \gamma > 0.$$

(i) If $\gamma < 1$, then

$$c_1(i,\gamma)(\log\log n)^{\frac{1}{\alpha}} \leq \mathbb{E}X_n^{\mathcal{W}}(i) \leq c_2(\log n)^{1-\gamma}. \tag{3.7}$$

(ii) If $\gamma = 1$, then

$$c_1(i)(\log\log\log n)^{\frac{1}{\alpha}} \leq \mathbb{E}X_n^{\mathcal{W}}(i) \leq c_2 \log\log n. \tag{3.8}$$

From Theorem 2.1 and Corollary 3.2, when $\alpha \geq 1$ and $\gamma > (\alpha-1)/\alpha$, we can differentiate between the repulsion speeds of the two cases.

Note that under the hypotheses of Corollary 3.2, there is a delocalization/localization transition at $\gamma = 1$, the localization upper bounds for the case $\gamma > 1$ follows from Theorem 3.1.

Proof of Theorem 3.1. First, we prove that if the mean number of visit to $\mathcal{W}$ by $\{\xi_n^i\}_{n\in\mathbb{N}}$ is bounded uniformly in i, then there is localization. So, consider our process in $\mathbb{Z}^d$, $d \geq 3$.

Note that if $a > c$, then $(a+b)^+ - (c+b) \leq a - c + b^- + c^-$. Let us couple $X_n^{\mathcal{W}}$ and the free process Y_n using the same $\varepsilon_k(i)$, so that $X_n^{\mathcal{W}}(i) \geq Y_n(i)$ for all n and i. We have for $i \in \mathcal{W}$

$$\begin{aligned}
&\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) = \mathbb{E}(X_n^{\mathcal{W}}(i) - Y_n(i) \mid \mathcal{W}) \\
= \;& \mathbb{E}\Big(\Big(\sum_{j\in\mathbb{Z}^d}(p(i,j)X_{n-1}^{\mathcal{W}}(j) + \varepsilon_n(j))\Big)^+ - \sum_{j\in\mathbb{Z}^d}(p(i,j)Y_{n-1}(j) + \varepsilon_n(j))\Big|\mathcal{W}\Big) \\
\leq \;& \mathbb{E}\Big(\sum_{j\in\mathbb{Z}^d} p(i,j)(X_{n-1}^{\mathcal{W}}(j) - Y_{n-1}(j))\Big|\mathcal{W}\Big) + \mathbb{E}(\varepsilon_n(i)^-) + \mathbb{E}((\mathcal{P}Y_n(i))^-)
\end{aligned}$$

$$\leq \sum_{j\in\mathbb{Z}^d} p(i,j)\mathbb{E}(X_{n-1}^{\mathcal{W}}(j) - Y_{n-1}(j) \mid \mathcal{W}) + C, \tag{3.9}$$

where C is a positive constant; the latter inequality following from the boundedness of $\mathbb{E}(\varepsilon_n(i)^-)$, $\mathbb{E}((\mathcal{P}Y_n(i))^-)$ $(d \geq 3)$.

For $i \notin \mathcal{W}$, we get

$$\begin{aligned}
&\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) = \mathbb{E}(X_n^{\mathcal{W}}(i) - Y_n(i) \mid \mathcal{W}) \\
= \;& \mathbb{E}\Big(\Big(\sum_{j\in\mathbb{Z}^d}(p(i,j)X_{n-1}^{\mathcal{W}}(j) + \varepsilon_n(j))\Big) - \Big(\sum_{j\in\mathbb{Z}^d}(p(i,j)Y_{n-1}(j) + \varepsilon_n(j))\Big)\Big|\mathcal{W}\Big) \\
= \;& \sum_{j\in\mathbb{Z}^d} p(i,j)\mathbb{E}(X_{n-1}^{\mathcal{W}}(j) - Y_{n-1}(j) \mid \mathcal{W}).
\end{aligned} \tag{3.10}$$

So in general

$$\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) \leq \sum_{j\in\mathbb{Z}^d} p(i,j)\mathbb{E}(X_{n-1}^{\mathcal{W}}(j) - Y_{n-1}(j) \mid \mathcal{W}) + C1\{i \in \mathcal{W}\}. \tag{3.11}$$

Iterating (3.11), we obtain

$$\mathbb{E}(X_n^{\mathcal{W}}(0) \mid \mathcal{W}) \leq C \sum_{j\in\mathcal{W}}\sum_{k=0}^{n} p_k(0,j) \leq C \sum_{j\in\mathcal{W}}\sum_{k=0}^{\infty} p_k(0,j) \tag{3.12}$$

and the latter double sum represents the mean number of visit to $\mathcal{W}$ by $\{\xi_n^i\}_{n\in\mathbb{N}}$, thus establishing necessity.

For sufficiency, suppose that the process localizes, and thus $\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) \leq$ *const* for all n and i. This implies that there exists $K > 0$ such that $\mathbb{P}[\mathcal{P}X_n^{\mathcal{W}}(i) > K \mid \mathcal{W}] < 1/2$ for all n and i. As ε has infinite support, $\mathbb{P}[\varepsilon < -K-1] > \delta$ for some $\delta > 0$. Those two facts imply $\mathbb{P}[\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i) \leq -1 \mid \mathcal{W}] \geq \delta/2$. Note that if $\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i) \leq -1$, then $(\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i))^- \geq 1$. We can couple $X_n^{\mathcal{W}}$ and the free process Y_n using the same $\varepsilon_k(i)$, so that $X_n^{\mathcal{W}}(i) \geq Y_n(i)$ for all n and i. Thus,

$$\begin{aligned}
&\mathbb{E}[(\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i))^+ - (\mathcal{P}Y_{n-1}(i) + \varepsilon_n(i)) \mid \mathcal{W}] \\
&\quad - \mathbb{E}[\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i) - (\mathcal{P}Y_{n-1}(i) + \varepsilon_n(i)) \mid \mathcal{W}] \\
= \;& \mathbb{E}[(\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i))^+ - (\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i)) \mid \mathcal{W}] \\
= \;& \mathbb{E}[(\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i))^- \mid \mathcal{W}] \\
\geq \;& \mathbb{E}[(\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i))^- \mid \mathcal{W},\ \mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i) \leq -1] \\
&\quad \times \mathbb{P}[\mathcal{P}X_{n-1}^{\mathcal{W}}(i) + \varepsilon_n(i) \leq -1 \mid \mathcal{W}] \\
\geq \;& \delta/2.
\end{aligned}$$

So, if $i \in \mathcal{W}$, we have

$$\mathbb{E}(X_n^{\mathcal{W}}(i) \mid \mathcal{W}) = \mathbb{E}(X_n^{\mathcal{W}}(i) - Y_n(i) \mid \mathcal{W})$$

$$
\begin{aligned}
&= \mathbb{E}[(\mathcal{P}X^{\mathcal{W}}_{n-1}(i)+\varepsilon_n(i))^+ - (\mathcal{P}Y_{n-1}(i)+\varepsilon_n(i)) \mid \mathcal{W}] \\
&\geq \mathbb{E}[\mathcal{P}X^{\mathcal{W}}_{n-1}(i)+\varepsilon_n(i) - (\mathcal{P}Y_{n-1}(i)+\varepsilon_n(i)) \mid \mathcal{W}] + \delta/2 \\
&= \mathbb{E}[\mathcal{P}(X^{\mathcal{W}}_{n-1}(i) - Y_{n-1}(i)) \mid \mathcal{W}] + \delta/2. \qquad (3.13)
\end{aligned}
$$

If $i \notin \mathcal{W}$, then

$$
\begin{aligned}
\mathbb{E}(X^{\mathcal{W}}_n(i) \mid \mathcal{W}) &= \mathbb{E}(X^{\mathcal{W}}_n(i) - Y_n(i) \mid \mathcal{W}) \\
&= \mathbb{E}[\mathcal{P}(X^{\mathcal{W}}_{n-1}(i) - Y_{n-1}(i)) \mid \mathcal{W}] \\
&= \mathcal{P}\mathbb{E}[X^{\mathcal{W}}_{n-1}(i) - Y_{n-1}(i) \mid \mathcal{W}]. \qquad (3.14)
\end{aligned}
$$

So, in general,

$$
\mathbb{E}(X^{\mathcal{W}}_n(i) \mid \mathcal{W}) \leq \mathcal{P}\mathbb{E}[X^{\mathcal{W}}_{n-1}(i) - Y_{n-1}(i) \mid \mathcal{W}] + 1\{i \in \mathcal{W}\}\,\delta/2. \qquad (3.15)
$$

Iterating (3.15), we get

$$
\mathbb{E}(X^{\mathcal{W}}_n(i) \mid \mathcal{W}) \geq \frac{\delta}{2}\sum_{k=0}^{n}\sum_{j\in\mathcal{W}} p_k(i,j). \qquad (3.16)
$$

So, if $\mathbb{E}(X^{\mathcal{W}}_n(i) \mid \mathcal{W})$ is bounded, the mean number of visits to $\mathcal{W}$ by the random walk with transition matrix $\mathcal{P}$ is also bounded. Theorem 3.1 is proved. ■

Proof of Theorem 3.2. As in the proof of Lemma 3.1 of [3], using the symmetry of $\varepsilon_n(i)$, we have

$$
\mathbb{E}X^{\mathcal{W}}_n(i) \geq \mathcal{P}\mathbb{E}X^{\mathcal{W}}_{n-1}(i) + G(\mathcal{P}\mathbb{E}X^{\mathcal{W}^i}_{n-1}(i))q_i \qquad (3.17)
$$

and

$$
\mathbb{E}X^{\mathcal{W}^i}_n(j) - \mathbb{E}X^{\mathcal{W}}_n(j) \leq C_0. \qquad (3.18)
$$

We now define an auxiliary family of variables $\nu_n(i)$ by the recursive relation:

$$
\nu_n(i) = \mathcal{P}\nu_{n-1}(i) + \tilde{G}(\mathcal{P}\nu_{n-1}(i))q_i, \qquad (3.19)
$$

where $\tilde{G}(x) = G(x + C_0)$ and $\nu_0(i) = 0$ for all i. To proceed, we need the following

Lemma 3.1 *We have*

(i) $\mathbb{E}X^{\mathcal{W}}_n(i) \geq \nu_n(i)$ *for all* $i \in \mathbb{Z}^d$ *and for all* n*;*

(ii) for all $i \in \mathbb{Z}^d$, $\nu_n(i)$ *is non-decreasing in* n*;*

(iii) for all n, $\nu_n(i)$ *non-increasing in* $|i|$.

Proof. Note that $\tilde{G}(x)$ is decreasing, but $x + \tilde{G}(x)$ is increasing (and thus, $H(x) := x + \tilde{G}(x)q_i$ is also increasing). This fact and (3.17) imply *(i)* and *(ii)*.

To prove *(iii)*, suppose $\nu_{n-1}(i) \le \nu_{n-1}(j)$, if $|i| \ge |j|$. If $|j-i| > 1$, it is immediate that $\mathcal{P}\nu_n(i) \le \mathcal{P}\nu_n(j)$. If $|j-i| = 1$, then, as $\delta > \frac{1}{2d+1}$,

$$\begin{aligned}
\mathcal{P}\nu_{n-1}(i) &= \delta\nu_{n-1}(i) + \sum_{l:|l-i|=1} \frac{1-\delta}{2d}\nu_{n-1}(l) \\
&= \delta\nu_{n-1}(i) + \frac{1-\delta}{2d}\nu_{n-1}(j) + \sum_{l\neq j,|l-i|=1} \frac{1-\delta}{2d}\nu_{n-1}(l) \\
&\le \delta\nu_{n-1}(j) + \frac{1-\delta}{2d}\nu_{n-1}(i) + \sum_{m\neq i,|m-j|=1} \frac{1-\delta}{2d}\nu_{n-1}(m) \\
&= \mathcal{P}\nu_{n-1}(j).
\end{aligned}$$

So,

$$\begin{aligned}
\nu_n(i) &= \mathcal{P}\nu_{n-1}(i) + \tilde{G}(\mathcal{P}\nu_{n-1}(i))q_i \\
&\le \mathcal{P}\nu_{n-1}(j) + \tilde{G}(\mathcal{P}\nu_{n-1}(j))q_i \\
&\le \mathcal{P}\nu_{n-1}(j) + \tilde{G}(\mathcal{P}\nu_{n-1}(j))q_j \\
&= \nu_n(j),
\end{aligned}$$

as q is nonincreasing. Lemma 3.1 is proved. ■

We continue the proof of Theorem 3.2. For $j = 0$, we have

$$\begin{aligned}
\nu_n(0) &= \mathcal{P}\nu_{n-1}(0) + \tilde{G}(\mathcal{P}\nu_{n-1}(0))q_0 \\
&\ge \mathcal{P}\nu_{n-1}(0) + \tilde{G}(\nu_n(0))q_0 \\
&\ge \tilde{G}(\nu_n(0)) \sum_{k=1}^{n} \mathcal{P}^k q_0 \\
&= \tilde{G}(\nu_n(0)) \sum_{k=1}^{n} \mathbb{E}q(\xi_k^0), \qquad (3.20)
\end{aligned}$$

where the inequalities follow from the monotonicity properties of $\nu_n(i)$ in n and i. (We recall that ξ_k^0 is a random walk with transition matrix $\mathcal{P}$ starting from site i.)

Below, c and c' are positive constants which may be different in different appearances. If $\mathbb{P}[\varepsilon > x] \ge ce^{-c'x^\alpha}$, $\alpha > 0$, then also $G(y) = \mathbb{E}(\varepsilon - y)^+ \ge ce^{-c'y^\alpha}$. So,

$$\nu_n(0) \ge ce^{-(\nu_n(0)+C_0)^\alpha} \sum_{k=1}^{n} \mathbb{E}q(|\xi_k^0|). \qquad (3.21)$$

Using the fact that $\log \nu_n(0) \le c\nu_n^\alpha(0)$, where c does not depend on $\nu_n(0)$, we get

$$\nu_n(0) \ge c\Big(\log \sum_{k=1}^{n} \mathbb{E}q(|\xi_k^0|)\Big)^{\frac{1}{\alpha}}. \qquad (3.22)$$

For general j, we have for $n \geq |j|$

$$\nu_n(j) \geq \mathcal{P}^{|j|}\nu_{n-|j|}(j) \geq \left(\frac{1-\delta}{2d}\right)^{|j|} \nu_{n-|j|}(0), \tag{3.23}$$

and the left hand side of (3.2) follows.

As for the upper bound, using (3.12) we have (note that C in (3.12) is independent of $\mathcal{W}$)

$$\begin{aligned} \mathbb{E}X_n^{\mathcal{W}}(0) &\leq c\,\mathbb{E}\sum_{j\in\mathcal{W}}\sum_{k=0}^{n} p_k(0,j) &(3.24)\\ &\leq c\,\mathbb{E}\sum_{\substack{j\in\mathcal{W}\\|j|\leq n}} G(0,j) \;\leq\; c\,\mathbb{E}\sum_{\substack{j\in\mathcal{W}\\|j|\leq n}} \frac{1}{|j|^{d-2}} &(3.25)\\ &\leq c\sum_{k=1}^{n} kq(k), &(3.26)\end{aligned}$$

where $G(0,j) = \sum_{k=0}^{\infty} p_k(0,j)$ is the Green's function and (3.25) is based on the well known behavior of G under our assumptions on $\mathcal{P}$. Theorem 3.2 is proved. ■

Remark 3.6 *Equation (3.24) holds for any homogeneous graph $\mathcal{G}$ with*

$$\sum_k \sum_{i\in\mathcal{G}} p_k(0,i)^2 < \infty, \tag{3.27}$$

and equation (3.21) holds if $\mathcal{P}$ satisfies (3.27) and if also $a(i) \leq a(j)$, $|i| \geq |j|$, implies $\mathcal{P}a(i) \leq \mathcal{P}a(j)$, $i,j \in \mathcal{G}$. Equations (3.21) and (3.24) then provide bounds on $\mathbb{E}X_n^{\mathcal{W}}(0)$. For example, for regular d-ary tree with $p(0,0) = \delta > \frac{1}{d+1}$, $p(0,i) = \frac{1-\delta}{d}$, if $|i| = 1$, q convex and nonincreasing, and $F_\varepsilon \in \mathcal{L}_\alpha^+$, one can obtain

$$c_1(i)\Big(\log\sum_{k=1}^{n} q(k)\Big)^{\frac{1}{\alpha}} \leq \mathbb{E}\; X_n^{\mathcal{W}}(i) \leq c_2\sum_{k=1}^{n} q(k). \tag{3.28}$$

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