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FUZZY GENERALIZED DERIVATIVES WITH APPLICATIONS ON FUZZY DELAY DIFFERENTIAL EQUATIONS

DERIVADAS FUZZY GENERALIZADAS COM APLICAÇÕES EM EQUAÇÕES DIFERENCIAIS FUZZY COM RETARDAMENTO

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Fuzzy Generalized Derivatives with applications on fuzzy delay differential equations

Derivadas Fuzzy Generalizadas com aplicações em equações diferenciais fuzzy com Retardamento

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"No nosso grupo, o mais devagar 'avua' "

Prof. João Meyer, o **Joni**.

Resumo

Este trabalho apresenta resultados obtidos que caracterizam as derivadas generalizadas de Hukuhara em função do comportamento das funções que definem seus α-níveis. Também são apresentados resultados sob pontos de troca da gH-diferenciabilidade de funções a números fuzzy obtidos de resultados recentes da literatura para o caso intervalar. Por último, abordamos as Equações Diferenciais Fuzzy com retardamento, apresentando resultados de existência e unicidade de soluções e resultados de estabilidade via funcionais de Lyapunov.

Palavras-chave: cálculo fuzzy, derivadas generalizadas de Hukuhara, pontos de troca, equações diferenciais fuzzy, equações diferenciais com retardamento.

Abstract

This work present obtained results that characterizes the Hukuhara generalized derivatives related to the behavior of its endpoint functions. Results concerning the concept of switching points of fuzzy functions – obtained from recent results of literature for the interval case– are also presented. Lastly, we deal with delay Fuzzy Differential Equations by presenting existence and uniqueness results of solutions and stability results via Lyapunov functions.

Keywords: fuzzy calculus, Hukuhara generalized derivatives, switching points, fuzzy differential equations, delay differential equations.

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INTRODUCTION

From the arising of fuzzy differential equations (FDE) as a promising field of study of fuzzy mathematics [9], Hukuhara-type derivatives of fuzzy functions are undoubtedly the most common approach encountered in the literature. This success is due to the fact that the fuzzy derivatives based on the Hukuhara difference - and its generalizations - between fuzzy numbers play an important role in both interval and fuzzy analysis.

Historically, the Hukuhara difference between intervals was initially proposed in [23], which inspired the Hukuhara difference (H-difference for short) between fuzzy numbers, as well as the Hukuhara derivative (H-derivative for short) of fuzzy functions in 1983 [35]. The shortcoming of the nondecreasing diameter of solutions to FDEs under the H-derivative, initially studied in [25], inspired the proposal of Strong Generalized Hukuhara difference (SGH-difference for short), and the Strong Generalized Hukuhara derivative (SGH-derivative for short) of fuzzy functions [5, 6], the first one with two different types of differentiability. An FDE under the SGH-derivative may produce solutions, when they exist, with nonincreasing and nondecreasing diameters.

Despite extending the H-derivative, the existence of the SGH-derivative is still very restrictive. For this reason, the generalized Hukuhara difference (gH-difference for short) between intervals and the generalized Hukuhara derivative (gH-derivative for short) of interval-valued functions was proposed in [40]. The fuzzy case was presented in [8] with several new examples of solutions to FDEs. The SGH approach opened a path to the notion of *switching points*, consisting of a point where the type of differentiability of a fuzzy function changes. This notion was inherited by the gH-derivative, and also very studied [11, 37]. The gH-difference and its derivative are still widely used in applications modeled by FDEs, in addition to satisfying important properties in the role of fuzzy calculus [1, 38].

In order to extend the aforementioned proposals to a broader class of fuzzy numbers and fuzzy-valued functions, the generalized difference (g-difference for short) between fuzzy numbers, and the generalized derivative (g-derivative for short) of fuzzy functions were proposed in [8]. The g-difference between fuzzy numbers always exists, that is, $A -_g B$ defines a fuzzy number for all A, B fuzzy numbers given [17]. The path of Hukuhara-type fuzzy differences is enclosed with a levelwise proposal of difference. The LgH-difference between fuzzy numbers is presented in [10], and it is associated with the gH$*$-derivative of fuzzy function.

Some observed nature phenomena may present a timelag between cause and effect. For example, the period between a person's contamination and the identification as infected [33], the interval of response of the organism to any perturbation of the physiological equilibrium (homeostasis) [28], population dynamics, and others. Consequently, the delay differential equations (DDE) have been an important tool for the mathematical modeling, since its main characteristic is to describe systems where the evolutive behavior does not depend only on the present time, but also on past events. For this reason, it was developed an interest on studing how the theory that brings together FDE and DDE operates.

Both similar aproaches to the one of this work and others have been proposed in the literature. For instance, in [24], an HIV model is elaborated considering an scalar delay differential equation and a fuzzy process is obtained by the Zadeh extension of the scalar DDE solution by assuming the delay as a fuzzy number, which is an interesting method when there are uncertainties on the timelag parameter making it hard to obtain a satisfatory value to it.

At first, the main objective was to study delay fuzzy differential equations considering some works on the literature [21, 27] as a starting point. However, a gap in the theory of fuzzy calculus of Hukuhara generalized derivatives was noticed due to the works on interval gH-differentiability [36, 37] and the lack of works characterizing the g-derivative. Such theoretical incompleteness has, as a main consequence to this work, an insufficient understanding on the fuzzy differential equations via gH-, gH$*$- and g-derivatives, since a proper study of the FDE is essential before considering the delay case.

Therefore, our work intends to fulfill this lack in the fuzzy calculus theory bringing results that characterize the gH-, gH$*$-and g-derivatives, results about switching points of the gH-differentiability of fuzzy functions, and lastly the delay FDE are studied.

Some questions that emerged during our research are answered here, while a few open questions and ideas are left for future works. For example, we could find a case when the FDE under g-derivative has only gH$*$-differentiable solutions, while it is still an open question how the function that define the FDE can influence on the possible types of switching points.

This work is organized as follows. Chapter 1 presents some basic notions on fuzzy numbers and fuzzy calculus bringing up important results of differentiation and integration. Inspired by the works from [10, 36], Chapter 2 presents the obtained results that characterizes the possible behaviors of gH-, gH$*$- and g-differentiable fuzzy functions [29] by setting the relation of the fuzzy derivatives with the interval derivatives of the level set functions. In the Chapter 3, we generalize the concepts and results on switching points of interval-valued functions [37] for fuzzy functions. Finally, in the Chapter 4, some Existence and Uniqueness Theorems of delay Fuzzy Differential Equations are presented, such as several stability results obtained based on [21] and the classic theory from [34].

The results from literature are properly referenced in this text, while the results obtained in this work are not.

CHAPTER 1

Mathematical Background

In this chapter, we present the basic theory necessary for the next chapters of this work. The concept of fuzzy numbers and an introduction to fuzzy calculus are presented.

1.1 Fuzzy Numbers

Definition 1.1. [4] Consider a fuzzy subset of the real line $A : \mathbb{R} \to [0, 1]$. Then A is a fuzzy number if it satisfies the following properties:

(i) A is normal, i.e., $\exists x_0 \in \mathbb{R}$ with $A(x_0) = 1$;

(ii) A is fuzzy convex, i,e, $A(tx + (1 - t)x) \geqslant \min\{A(x), A(y)\}, \forall t \in [0, 1], x, y \in \mathbb{R}$;

(iii) A is upper semicontinuous on $\mathbb{R}$, i.e., $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $A(x) - A(x_0) < \varepsilon$ whenever $|x - x_0| < \delta$;

(iv) The set $\text{supp}\,(A) = \{x \in \mathbb{R}_{\mathcal{F}} \mid A(x) > 0\}$ is compact.

The class of all fuzzy numbers is denoted by $\mathbb{R}_{\mathcal{F}}$.

For each $A \in \mathbb{R}_{\mathcal{F}}$ we define the α-levels of A by

$$[A]_\alpha = \begin{cases} \{x \in \mathbb{R} | A(x) \geqslant \alpha\} & \text{if} \quad \alpha \in (0, 1], \\ \overline{\{x \in \mathbb{R} | A(x) > 0\}} & \text{if} \quad \alpha = 0, \end{cases} \tag{1.1}$$

where $\overline{B}$ denotes the closure of $B \subset \mathbb{R}$.

In this work, and in the references, the α-levels are also called α-cuts, α-sets and level sets.

The core of a fuzzy number $A \in \mathbb{R}_{\mathcal{F}}$ is the set of elements $x \in \mathbb{R}$ such that $A(x) = 1$, that is, $\text{core}\,(A) = [A]_1$. If $A \in \mathbb{R}_{\mathcal{F}}$ is a fuzzy number given, then there exist $a^-_\alpha, a^+_\alpha \in \mathbb{R}$ satisfying $a^-_\alpha \leqslant a^+_\alpha$ such that $[A]_\alpha = [a^-_\alpha, a^+_\alpha]$, for all $\alpha \in [0, 1]$ [3].

For all $\alpha \in [0, 1]$, we define the length of $[A]_\alpha$ by $\text{len}\,([A]_\alpha) = a^+_\alpha - a^-_\alpha$, where a^-_α and a^+_α are the left and right endpoints of $[A]_\alpha$, respectively. The diameter of a fuzzy number is given by $\text{diam}\,(A) = \text{len}\,([A]_0) = a^+_0 - a^-_0$ and it can be seen as a measure of the uncertainty of the phenomenon modeled by $A \in \mathbb{R}_{\mathcal{F}}$.

The class of fuzzy numbers with continuous endpoints as functions of α is denoted by $\mathbb{R}^c_{\mathcal{F}}$, that is, $A \in \mathbb{R}^c_{\mathcal{F}}$ if, and only if, a^-_α and a^+_α are continuous functions of $\alpha \in [0, 1]$.

Henceforth, we will denote the set of all compact and convex subsets of $\mathbb{R}$ by $\mathcal{K}_{\mathcal{C}}$. An interval-valued function is an application of the form $F : I \to \mathcal{K}_{\mathcal{C}}$, and a fuzzy number-valued function is an application of the form $F : I \to \mathbb{R}_{\mathcal{F}}$. We denote the α-cuts of a fuzzy-number valued function F by $[F(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$ for all $\alpha \in [0,1]$, and the real functions f_α^- and f_α^+ are called the endpoint functions of F. The length of the level sets of $F(t)$ will be denoted by $\text{len}_\alpha(F)(t) = \text{len}([F(t)]_\alpha)$.

In the following, we recall the notions of gH, LgH, and generalized Hukuhara differences on fuzzy numbers.

Definition 1.2. [39]. Given two fuzzy numbers $A, B \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference (gH-difference, for short) is the fuzzy number C such that

$$A -_{gH} B = C \Leftrightarrow \begin{cases} (i)\ A = B + C \quad \text{or} \\ (ii)\ B = A - C, \end{cases} \tag{1.2}$$

if it exists. Levelwise, we have

$$[A -_{gH} B]_\alpha = \begin{cases} [a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+], \ \ \forall \alpha \in [0,1] \text{ if } A -_{gH} B \text{ exists in case } (i), \\ [a_\alpha^+ - b_\alpha^+, a_\alpha^- - b_\alpha^-], \ \ \forall \alpha \in [0,1] \text{ if } A -_{gH} B \text{ exists in case } (ii). \end{cases}$$

Note that case (i) in Definition 1.2 is the H-difference $A -_H B$.

Definition 1.3. [10]. Given two fuzzy numbers $A, B \in \mathbb{R}_{\mathcal{F}}$, the L-generalized Hukuhara difference (LgH-difference for short) of A and B is the fuzzy number C, if it exists, such that

$$A -_{LgH} B = C \Leftrightarrow [A]_\alpha -_{gH} [B]_\alpha = [C]_\alpha, \ \ \forall \alpha \in [0,1],$$

that is, for each $\alpha \in [0,1]$, either $[A]_\alpha = [B]_\alpha + [C]_\alpha$ or $[B]_\alpha = [A]_\alpha + (-1)[C]_\alpha$.

Levelwise, for all $A, B \in \mathbb{R}_{\mathcal{F}}$, if $A -_{LgH} B$ exists, then we have that

$$\begin{aligned} [A -_{LgH} B]_\alpha &= [A]_\alpha -_{gH} [B]_\alpha \\ &= \Big[\min\{a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+\}, \max\{a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+\}\Big], \end{aligned} \tag{1.3}$$

for all $\alpha \in [0,1]$.

It is noteworthy to point out that the gH-difference, given by Definition 1.2 is a particular case of LgH-difference, given by Definition 1.3. In fact, for $A, B \in \mathbb{R}_{\mathcal{F}}$ given, the set

$$\Big\{[A]_\alpha -_{gH} [B]_\alpha \,\Big|\, \alpha \in [0,1]\Big\}$$

may not define a fuzzy number, that is, it may exist $\alpha, \beta \in [0,1]$, $\alpha < \beta$, such that $[A]_\beta -_{gH} [B]_\beta \not\subset [A]_\alpha -_{gH} [B]_\alpha$. The equality

$$[A -_{gH} B]_\alpha = [A]_\alpha -_{gH} [B]_\alpha,\ \alpha \in [0,1]$$

holds whenever $A -_{gH} B$ exists.

Definition 1.4. [17, 39]. Let $A, B \in \mathbb{R}_\mathcal{F}$ be fuzzy numbers given. The generalized difference (g-difference for short) between A and B is the fuzzy number $A -_g B$ given levelwise by

$$[A -_g B]_\alpha = \overline{\operatorname{conv}\left(\bigcup_{\beta \geqslant \alpha} [A]_\beta -_{gH} [B]_\beta\right)} \tag{1.4}$$

for all $\alpha \in [0,1]$.

In Definition 1.4, $\operatorname{conv}(X)$ denotes the convex hull of a set $X \subset \mathbb{R}$, that is, the smallest convex set that contains X.

Proposition 1.1. [8]. *The g-difference between two fuzzy numbers A and B is givel levelwise by*

$$[A -_g B]_\alpha = \left[\inf_{\beta \geqslant \alpha} \min\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}, \sup_{\beta \geqslant \alpha} \max\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}\right], \tag{1.5}$$

for all $\alpha \in [0,1]$.

In the following, we present some metrics of fuzzy numbers that are used in our work. More properties of fuzzy number metrics can be found in [1, 4].

The Hausdorff-Pompeiu distance between fuzzy numbers is the operator $d_\infty : \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F} \to \mathbb{R}^+$ given by

$$d_\infty(A, B) = \sup_{\alpha \in [0,1]} \max\left\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\right\},$$

for all $A, B \in \mathbb{R}_\mathcal{F}$ given levelwise by $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ and $[B]_\alpha = [b_\alpha^-, b_\alpha^+]$, $\alpha \in [0,1]$. Particularly,

$$d_\infty(A, 0) = \sup_{\alpha \in [0,1]} \max\{|a_\alpha^-|, |a_\alpha^+|\} = \max\{|a_0^-|, |a_0^+|\},$$

and $\|\cdot\| = d_\infty(\cdot, 0)$ may be considered as a norm of the quasivector space $\mathbb{R}_\mathcal{F}$ [31, 42].

Proposition 1.2. [8]. *For any $A, B \in \mathbb{R}_\mathcal{F}$, $d_\infty(A, B) = d_\infty(A -_g B, 0)$.*

Proof. In fact,

$$|a_\alpha^- - b_\alpha^-| \leqslant \sup_{\alpha \in [0,1]} \max\left\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\right\}$$

and

$$|a_\alpha^+ - b_\alpha^+| \leqslant \sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}$$

for all $\alpha \in [0,1]$. Then,

$$\min\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\} \leqslant \sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}$$

and

$$\max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\} \leqslant \sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}$$

for all $\alpha \in [0,1]$. Consequently,

$$\inf_{\alpha\in[0,1]} \min\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\} \leqslant \sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}$$

and

$$\sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\} \leqslant \sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}$$

for all $\alpha \in [0,1]$. Therefore, $d_\infty(A -_g B, 0) \leqslant d_\infty(A, B)$.

On the other hand,

$$\inf_{\beta\in[0,1]} \min\{|a_\beta^- - b_\beta^-|, |a_\beta^+ - b_\beta^+|\} \leqslant a_\alpha^\pm - b_\alpha^\pm \leqslant \sup_{\beta\in[0,1]} \max\{|a_\beta^- - b_\beta^-|, |a_\beta^+ - b_\beta^+|\}$$

for all $\alpha \in [0,1]$. Then,

$$|a_\alpha^\pm - b_\alpha^\pm| \leqslant \max\left\{\inf_{\beta\in[0,1]} \min\{|a_\beta^- - b_\beta^-|, |a_\beta^+ - b_\beta^+|\}, \sup_{\beta\in[0,1]} \max\{|a_\beta^- - b_\beta^-|, |a_\beta^+ - b_\beta^+|\}\right\}$$

for all $\alpha \in [0,1]$. And it implies that

$$\sup_{\alpha\in[0,1]} |a_\alpha^\pm - b_\alpha^\pm| \leqslant \max\left\{\inf_{\beta\in[0,1]} \min\{|a_\beta^- - b_\beta^-|, |a_\beta^+ - b_\beta^+|\}, \sup_{\beta\in[0,1]} \max\{|a_\beta^- - b_\beta^-|, |a_\beta^+ - b_\beta^+|\}\right\}.$$

Therefore, $d_\infty(A, B) \leqslant d_\infty(A -_g B, 0)$, and the equality follows. ■

Some results concerning the metric spaces of fuzzy numbers can be found in [4].

1.2 Introduction to Fuzzy Calculus

In this work, the fuzzy calculus theory studied involves fuzzy number-valued functions of one real variable. In the following, some basic concepts and results are presented. They were obtained from [1, 4, 8, 10, 12, 15]. The limits considered are given with respect to d_∞.

Definition 1.5. [1, 4] Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number-valued function and $t_0 \in I$. We say that $L \in \mathbb{R}_{\mathcal{F}}$ is the limit of F as t goes to t_0, and denote $\lim\limits_{t \to t_0} F(t) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \,|\, x \in I,\, 0 < |t - t_0| < x \Rightarrow d_\infty(F(x), L) < \varepsilon.$$

Also, F is said to be continuous at t_0 if $\lim\limits_{t \to t_0} F(t) = F(t_0)$. Moreover, F is *fuzzy continuous* on I if F is continuous at each $t_0 \in I$.

1.2.1 The Hukuhara generalized differentiabilities

In the following, we recall the notions of generalized Hukuhara, L-generalized Hukuhara, and generalized derivatives of fuzzy number-valued functions. These derivatives are base, respectively, on the gH-, LgH- and g-differences.

Definition 1.6. [4] Let $t_0 \in I \subset \mathbb{R}$ and h be such that $t_0 + h \in I$, then the gH-derivative of a function $F : I \to \mathbb{R}_{\mathcal{F}}$ at t_0 is defined as

$$F'_{gH}(t_0) = \lim_{h \to 0} \frac{1}{h} \cdot \left(F(t_0 + h) -_{gH} F(t_0)\right) . \tag{1.6}$$

If $F'_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at t_0.

Definition 1.7. [8] Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable with f^-_α and f^+_α both differentiable at $t_0 \in (a, b)$. We say that

- f is (i)-gH-differentiable at t_0 if

$$\left[F'_{gH}\left(t_0\right)\right]_\alpha = \left[\left(f^-_\alpha\right)'\left(t_0\right), \left(f^+_\alpha\right)'\left(t_0\right)\right],\ \forall \alpha \in [0, 1];$$

- f is (ii)-gH-differentiable at t_0 if

$$\left[F'_{gH}\left(t_0\right)\right]_\alpha = \left[\left(f^+_\alpha\right)'\left(t_0\right), \left(f^-_\alpha\right)'\left(t_0\right)\right],\ \forall \alpha \in [0, 1].$$

Theorem 1.1. [1]. *Let $F : (, b) \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable at $c \in (a, b)$. Then F is fuzzy continuous at c.*

Theorem 1.2 (Mean Value Theorem for gH-derivative)**.** [1] *Let $F : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy number-valued function. If F has continuous gH-derivative on (a, b), then there exist $c \in (a, b)$ such that*

$$F'_{gH}(c) = \frac{F(b) -_{gH} F(a)}{b - a} .$$

Some derivation rules and other properties, analogous to the classical case, can be found in [1, 4]. As it will be properly discussed in the Chapter 2, the gH-differentiability is not restrict to the cases of Definition 1.7. Also, Chapter 3 will present new results concerning the so called switching points:

Definition 1.8. [8]. Let $F : I \to \mathbb{R}_{\mathcal{F}}$. We say that a point $t_0 \in I$ is a *switching point* for the gH-differentiability of F if in any neighborhood V of t_0 there exist numbers $t_1 < t_0 < t_2$ such that

(type-I) at t_1 F is (i)-gH-differentiable while it is not (ii)-gH-differentiable, and at t_2 F is (ii)-gH-differentiable while it is not (i)-gH-differentiable; or

(type-II) at t_1 F is (ii)-gH-differentiable while it is not (i)-gH-differentiable, and at t_2 F is (i)-gH-differentiable while it is not (ii)-gH-differentiable.

Definition 1.9. [10] Let $t_0 \in I \subset \mathbb{R}$ and h be such that $t_0 + h \in I$, then the gH$*$-derivative of a function $F : I \to \mathbb{R}_{\mathcal{F}}$ at t_0 is defined as

$$F'_{gH*}(t_0) = \lim_{h \to 0} \frac{1}{h} \cdot (F(t_0 + h) -_{LgH} F(t_0)) \ . \tag{1.7}$$

If $F'_{gH*}(t_0) \in \mathbb{R}_{\mathcal{F}}$ exists, we say that F is L-generalized Hukuhara differentiable (gH$*$-differentiable, for short) at t_0. Therefore, in this case, $F'_{*gH}(t_0)$ is the fuzzy number whose levelsets are

$$[F'_{gH*}(t_0)]_\alpha = \lim_{h \to 0} \frac{1}{h} \cdot ([F(t_0 + h)]_\alpha -_{gH} [F(t_0)]_\alpha)$$

for all $\alpha \in [0, 1]$.

Definition 1.10. [4] Let $t_0 \in I \subset \mathbb{R}$ and h be such that $t_0 + h \in I$, then the generalized derivative (g-derivative) of a function $F : I \to \mathbb{R}_{\mathcal{F}}$ at t_0 is defined as

$$F'_g(t_0) = \lim_{h \to 0} \frac{1}{h} \cdot (F(t_0 + h) -_g F(t_0)) \ . \tag{1.8}$$

If $F'_g(t_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (1.8) exists, we say that F is generalized Hukuhara differentiable (gH-differentiable, for short) at t_0.

Theorem 1.3. [8] *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be such that the real-valued functions f^-_α and f^+_α are differentiable at $t_0 \in I$ uniformly in $\alpha \in [0, 1]$. Then F is g-differentiable at t_0 and*

$$[F'_g(t_0)]_\alpha = \left[\inf_{\beta \geqslant \alpha} \min((f^-_\beta)'(t_0), (f^+_\beta)'(t_0)), \sup_{\beta \geqslant \alpha} \max((f^-_\beta)'(t_0), (f^+_\beta)'(t_0)) \right] .$$

1.2.2 Integration of fuzzy number-valued functions

Based on the Aumann's integral of set-valued functions [2], the fuzzy Aumann integral is proposed.

Definition 1.11. [4, 12]

- A mapping $F : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is said to be *strongly measurable* if the interval functions $F_\alpha(t) = [F(t)]_\alpha$ are Borel measurable for all $\alpha \in [0, 1]$.
- A fuzzy number-valued function $F : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is called *integrably bounded* if there exists an integrable function $h : [a, b] \to \mathbb{R}$ such that

$$d_\infty(F(t), 0) \leqslant h(t), \quad \forall t \in [a, b].$$

- A strongly measurable and integrably bounded fuzzy number-valued function is called *integrable*. The *Fuzzy Aumann integral* of $F : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is defined levelwise by

$$\left[(FA)\int_a^b F(t)\,dt\right]_\alpha = (A)\int_a^b [F(t)]_\alpha\,dt, \quad \alpha \in [0, 1].$$

Definition 1.12. [4, 15]. A function $F : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is called *Riemann integrable* on $[a, b]$ if there exists $I \in \mathbb{R}_{\mathcal{F}}$ with the property: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_\infty\left(\sum_{i=0}^{n-1} F(\xi_i)(t_{i+1} - t_i), I\right) < \varepsilon$$

for any division $P = \{a = t_0 < ... < t_n = b\}$ with norm $|P| < \delta$ and any $\xi_i \in [t_i, t_{i+1}]$, $i = 0, ..., n-1$. Then we denote

$$I = (FR)\int_a^b F(t)\,dt$$

and it is called the *fuzzy Riemann integral* of F.

In [41], the *fuzzy Henstock integral* (FH) of a fuzzy number-valued function is introduced, and the fuzzy Riemann integral can be obtained as a particular case. The next result relates the mentioned integrals.

Proposition 1.3. *[4]. A continuous fuzzy number-valued function is fuzzy Aumann integrable, fuzzy Riemann integrable and fuzzy Henstock integrable too. Moreover*

$$(FA)\int_a^b F(t)\,dt = (FR)\int_a^b F(t)\,dt = (FH)\int_a^b F(t)\,dt.$$

From any of the previous integral definitions, we have

$$\left[\int_a^b F(t)\,dt\right]_\alpha = \left[\int_a^b f_\alpha^-(t)dt, \int_a^b f_\alpha^+(t)dt\right].$$

Theorem 1.4. [8]. *If F is gH-differentiable with no switching point in the interval $[a,b]$, then we have*

$$\int_a^b F'_{gH}(t)\,dt = F(b) -_{gH} F(a).$$

Theorem 1.5. [8]. *Let us suppose that function F is gH-differentiable with n switching points at c_i, $i = 1,...,n$, $a = c_0 < c_1 < ... < c_n < c_{n+1} = b$ and exactly at these points. Then we have*

$$F(b) -_{gH} F(a) = \sum_{i=1}^{n}\left[\int_{c_{i-1}}^{c_i} F'_{gH}(t)\,dt -_{gH} (-1)\int_{c_i}^{c_{i+1}} F'_{gH}(t)\,dt\right].$$

Also,

$$\int_a^b F'_{gH}(t)\,dt = \sum_{i=1}^{n+1} F(c_i) -_{gH} F(c_{i-1}).$$

In order to obtain similar results for the gH$*$- and g-derivatives, we have the following.

Theorem 1.6. *Let F be gH$*$-differentiable on $[a,b]$. If $f_\alpha^-(t)$ and $f_\alpha^+(t)$ have integrable detivatives for all $\alpha \in [0,1]$, then*

$$\int_a^b F'_{gH*}(t)\,dt = F(b) -_{LgH} F(a). \tag{1.9}$$

Proof. In fact, since $f_\alpha^-(t)$ and $f_\alpha^+(t)$ have integrable derivatives and

$$[F'_{gH*}(t)]_\alpha = \left[\min((f_\alpha^-)'(t), (f_\alpha^+)'(t)), \max((f_\alpha^-)'(t), (f_\alpha^+)'(t))\right]$$

for all $\alpha \in [0,1]$, we get that

$$\begin{aligned}\left[\int_a^b F'_{gH*}(t)\,dt\right]_\alpha &= \left[\int_a^b \min\left((f_\alpha^-)'(t), (f_\alpha^+)'(t)\right)\,dt, \int_a^b \max\left((f_\alpha^-)'(t), (f_\alpha^+)'(t)\right)\,dt\right]\\ &= \left[\min\left(\int_a^b (f_\alpha^-)'(t)\,dt, \int_a^b (f_\alpha^+)'(t)\,dt\right), \max\left(\int_a^b (f_\alpha^-)'(t)\,dt, \int_a^b (f_\alpha^+)'(t)\,dt\right)\right]\\ &= \left[\min\left(f_\alpha^-(b) - f_\alpha^-(a), f_\alpha^+(b) - f_\alpha^+(a)\right), \max\left(f_\alpha^-(b) - f_\alpha^-(a), f_\alpha^+(b) - f_\alpha^+(a)\right)\right]\end{aligned}$$

for all $\alpha \in [0,1]$. Therefore, equality (1.9) follows. ■

Theorem 1.7. *Let F be g-differentiable on $[a,b]$. If f_α^- and f_α^+ are differentiable on $[a,b]$ with integrable derivatives for all $\alpha \in [0,1]$, then*

$$\int_a^b F'_g(x)dx = F(b) -_g F(a).$$

Proof. Since f_α^- and f_α^+ are differentiable in $[a,b]$ for all $\alpha \in [0,1]$, it follows from Theorem 1.3 that

$$[F_g'(t)]_\alpha = \left[\inf_{\beta \geqslant \alpha} \min((f_\beta^-)'(t),(f_\beta^+)'(t)), \sup_{\beta \geqslant \alpha} \max((f_\beta^-)'(t),(f_\beta^+)'(t))\right]$$

for all $t \in [a,b]$, for all $\alpha \in [0,1]$. Consequently,

$$\begin{aligned}\left[\int_a^b F_g'(t)\,dt\right]_\alpha &= \left[\int_a^b \inf_{\beta \geqslant \alpha} \min\left((f_\beta^-)'(t),(f_\beta^+)'(t)\right)\,dt, \int_a^b \sup_{\beta \geqslant \alpha} \max\left((f_\beta^-)'(t),(f_\beta^+)'(t)\right)\,dt\right]\\ &= \left[\inf_{\beta \geqslant \alpha} \min\left(\int_a^b (f_\beta^-)'(t)\,dt, \int_a^b (f_\beta^+)'(t)\,dt\right), \sup_{\beta \geqslant \alpha} \max\left(\int_a^b (f_\beta^-)'(t)\,dt, \int_a^b (f_\beta^+)'(t)\,dt\right)\right]\\ &= \left[\inf_{\beta \geqslant \alpha} \min\left(f_\beta^-(b)-f_\beta^-(a), f_\beta^+(b)-f_\beta^+(a)\right), \sup_{\beta \geqslant \alpha} \max\left(f_\beta^-(b)-f_\beta^-(a), f_\beta^+(b)-f_\beta^+(a)\right)\right]\\ &= [F(b) -_g F(a)]_\alpha\end{aligned}$$

for all $\alpha \in [0,1]$. Therefore,

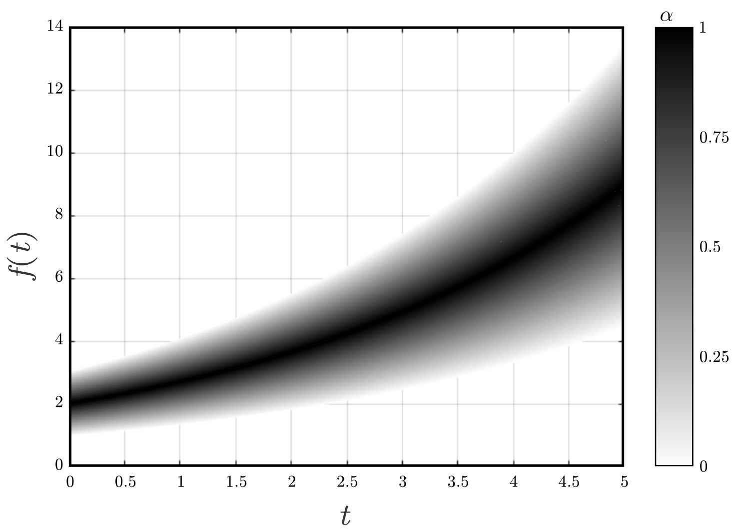
$$\int_a^b F_g'(t)\,dt = F(b) -_g F(a)\,.$$

■

1.2.3 Graphic representation of fuzzy functions

On what concerns the graphic represantation of fuzzy functions, different designs can be found trough the many references on fuzzy calculus. A widely common depiction is given by a grayscale, where the closer the graphic is to black, the closer α is to 1, as it can be seen in Figure 1.1.

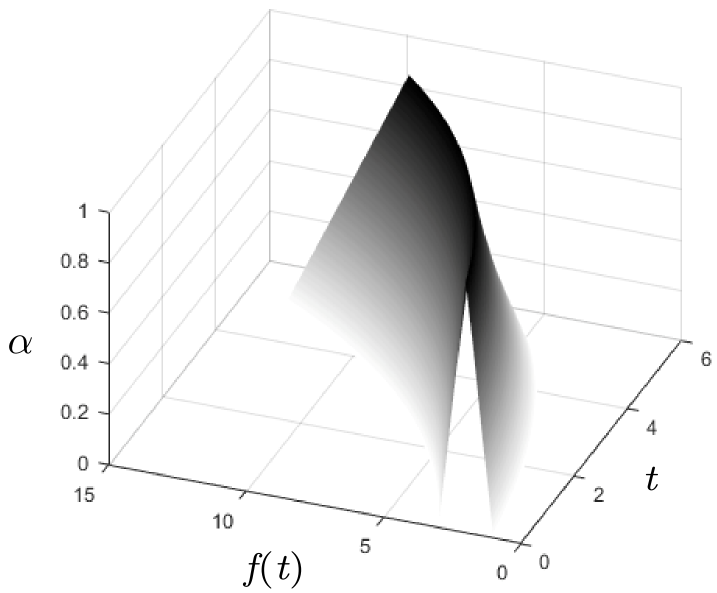
Figure 1.1: Gray-scale representation of a fuzzy function.



Source: Elaborated by the author.

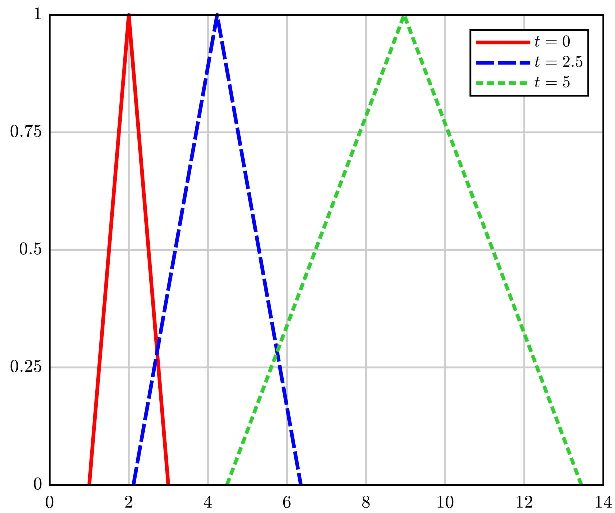
This representation is commonly considered because it is the topview of the three-dimensional graphic in Figure 1.2, which is a good illustration to introduce fuzzy number-valued functions. For a better understanding of a fuzzy function, Figure 1.3 illustrate the membership function for different values of t.

Figure 1.2: Three-dimensional representation of a fuzzy function.



Source: Elaborated by the author.

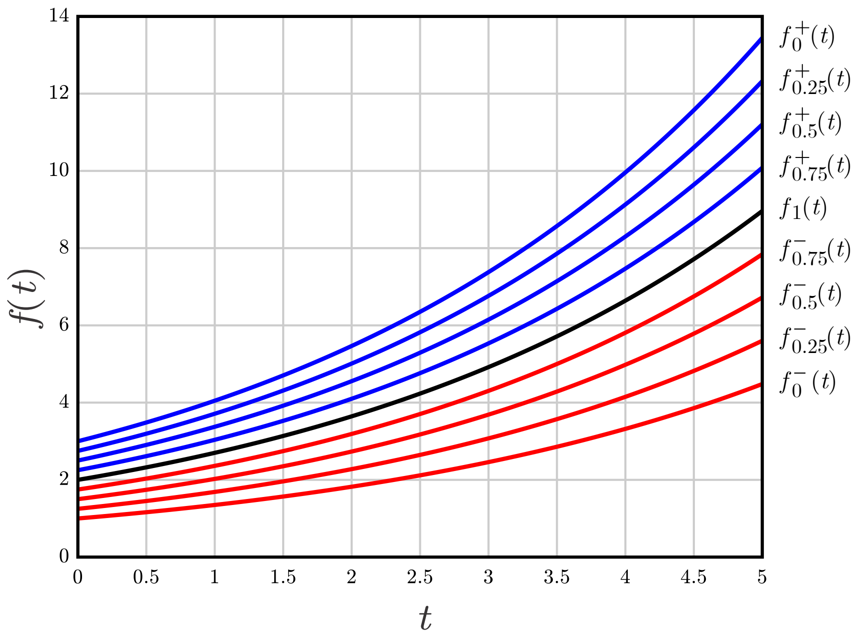
Figure 1.3: Representation of the membership function for different values of t.



Source: Elaborated by the author.

However, the graphic representation that was chosen for this work is given by Figure 1.4 and is also used in [1, 8]. It consists of the levelwise depction of the fuzzy function for some values of α obtained by a convenient discretization of the interval $[0, 1]$. The blue lines represent the upper endpoint functions $F_\alpha^+(t)$ for the values of α in the discretization. Similarly, the red lines represent the lower endpoint functions $F_\alpha^-(t)$ for the same values of α. And the function $F_1^-(t)$ and $F_1^+(t)$, the endpoints of the core of $F(t)$, are represented by black lines. This design is usefull to better understanding the behavior of the α-set of the fuzzy function, even more for the examples of this work with functions with different differentiability in each level set or for the ones with different behavior for rational and irrational numbers.

Figure 1.4: Represantation by a discretization of the interval $[0, 1]$.



Source: Elaborated by the author.

The function represented by these introduction figures is a solution of the fuzzy malthusian model. Note that the diameter of the solution is increasing, and it is due to the type of derivative, as it will be discussed later. Since the malthusian model is linear, the similar behavior of the endpoints of the level sets can be noted. However, most equations present distinct behaviors in each α-cut. Therefore, some cases are illustrated throughout this work.

In the next Chapter we present the obtained results on fuzzy calculus based on generalized Hukuhara derivative and its generalizations.

CHAPTER 2

Fuzzy Calculus for Generalized Differentiable Fuzzy Functions

This chapter presents new properties of the generalized derivatives of fuzzy number-valued functions. The main results were obtained from recent interval calculus papers, namely, [10, 36], and intend to provide a more precise way to work with the derivatives in order to apply to fuzzy differential equations.

2.1 Recent results on the gH-differentiability of interval-valued functions

The following definitions are necessary for characterizing the generalized differentiability of interval-valued functions, as presented in [36].

Definition 2.1. [43]. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers and $c \in \widetilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We say c is a cluster point of $\{x_n\}$ if every neighborhood of c contains infinite many points of $\{x_n\}$.

The set $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is used in the following to consider $-\infty$ and ∞ as possible cluster points of any sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers.

Remark 2.1. $c \in \widetilde{\mathbb{R}}$ is a cluster point of $\{x_n\}$ if, and only if, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{n_k} = c$.

Definition 2.2. [36]. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a real-valued function and $t_0 \in I$. Suppose f is well defined in the interval $(t_0, t_0 + \delta)$ for some positive number δ. For a number $c \in \widetilde{\mathbb{R}}$, if there exists a sequence of positive real numbers $\{h_n\}_{n=1}^{\infty}$ satisfying the following conditions:

(i) $\lim_{n\to\infty} h_n = 0$;

(ii) $\{t_0 + h_n\}_{n=1}^{\infty} \subset (t_0, t_0 + \delta)$;

(iii) c is a cluster point of the sequence $\{f(t_0 + h_n)\}_{n=1}^{\infty}$;

then we say c is a cluster point of f on the right of t_0. The set of cluster points of f on the right of t_0 is denoted by $C_{R(t_0)}(f)$. Analogously, we can define the cluster point of f on the left of t_0 and denote the set of cluster points of f on the left of t_0 by $C_{L(t_0)}(f)$.

Definition 2.3. [36]. Let $f : I \to \mathbb{R}$ be a real-valued function and $t_0 \in I$. Let us define the function $\phi_f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ by

$$\phi_f(h) = \frac{f(t_0 + h) - f(t_0)}{h},$$

for values of h such that $t_0 + h \in I$. The function $\phi_f(h)$ is called the *slope function of secants* at t_0 (*slope function*, for short).

Definition 2.4. [36]. Let f_1 and f_2 be two real-valued functions well-defined in $(t_0, t_0 + \delta)$. If both functions satisfy the following conditions:

(i) $C_{R(t_0)}(f_1) = C_{R(t_0)}(f_2) = \{a^-, a^+\}$, where $a^-, a^+ \in \mathbb{R}$ and $a^- < a^+$;

(ii) $\lim_{h\to 0^+} \min\{f_1(t_0+h), f_2(t_0+h)\} = a^-$ and $\lim_{h\to 0^+} \max\{f_1(t_0+h), f_2(t_0+h)\} = a^+$,

then we say f_1 and f_2 are *right complementary* at t_0.

The case where the real-valued functions f_1 and f_2 are left complementary at t_0 is defined analogously to Definition 2.4.

Henceforth, we denote by $F'(t_0)$ the gH-derivative of an interval-valued function $F : I \to \mathcal{K}_C$ at $t_0 \in I$. Also, the one-side derivatives of a real function $f : I \subset \mathbb{R} \to \mathbb{R}$ at some point t_0 will be denoted by $f'_-(t_0)$ and $f'_+(t_0)$.

The following theorem is the main result of [36].

Theorem 2.1. *[36]. $F : I \to \mathcal{K}_C$ is gH-differentiable at t_0 if, and only if, one of the following cases holds:*

(i) $(f^-)'_+(t_0)$, $(f^+)'_+(t_0)$, $(f^-)'_-(t_0)$ *and* $(f^+)'_-(t_0)$ *exist and*

$$\begin{aligned} F'(t_0) &= \Big[\min\{(f^-)'_+(t_0), (f^+)'_+(t_0)\}, \max\{(f^-)'_+(t_0), (f^+)'_+(t_0)\}\Big] \\ &= \Big[\min\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}, \max\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}\Big]. \end{aligned}$$

(ii) f^- *and* f^+ *are right differentiable at* t_0. ϕ_{f^-} *and* ϕ_{f^+} *are left complementary at 0, i.e.,* $C_{L(0)}(\phi_{f^-}) = C_{L(0)}(\phi_{f^+}) = \{a^-, a^+\}$, *where* $a^-, a^+ \in \mathbb{R}$ *and* $a^- < a^+$. *Moreover,*

$$\begin{aligned} F'(t_0) &= \Big[\min\{(f^-)'_+(t_0), (f^+)'_+(t_0)\}, \max\{(f^-)'_+(t_0), (f^+)'_+(t_0)\}\Big] \\ &= [a^-, a^+]. \end{aligned}$$

(iii) f^- *and* f^+ *are left differentiable at* t_0. ϕ_{f^-} *and* ϕ_{f^+} *are right complementary at 0, i.e.,* $C_{R(0)}(\phi_{f^-}) = C_{R(0)}(\phi_{f^+}) = \{a^-, a^+\}$, *where* $a^-, a^+ \in \mathbb{R}$ *and* $a^- < a^+$. *Moreover,*

$$\begin{aligned} F'(t_0) &= \Big[\min\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}, \max\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}\Big] \\ &= [a^-, a^+]. \end{aligned}$$

(iv) ϕ_{f^-} *and* ϕ_{f^+} *are both left complementary and right complementary at* 0, *i.e.,* $C_{R(0)}(\phi_{f^-}) = C_{R(0)}(\phi_{f^+}) = C_{L(0)}(\phi_{f^-}) = C_{L(0)}(\phi_{f^+}) = \{a^-, a^+\}$. *Moreover,*

$$F'(t_0) = [a^-, a^+].$$

Under the hypothesis of the continuity of $F : I \to \mathcal{K}_C$, the cases stated in Theorem 2.1 boil down to only one case as in the following.

Theorem 2.2. *Let $F : I \to \mathcal{K}_C$ be continuous in $(t_0 - \rho, t_0 + \rho) \subset I$ for some $\rho > 0$. Then F is gH-differentiable at t_0 if, and only if, $(f^-)'_+(t_0)$, $(f^+)'_+(t_0)$, $(f^-)'_-(t_0)$ and $(f^+)'_-(t_0)$ exist, and*

$$\begin{aligned} F'(t_0) &= \Big[\min\{(f^-)'_+(t_0), (f^+)'_+(t_0)\}, \max\{(f^-)'_+(t_0), (f^+)'_+(t_0)\}\Big] \\ &= \Big[\min\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}, \max\{(f^-)'_-(t_0), (f^+)'_-(t_0)\}\Big]. \end{aligned} \tag{2.1}$$

Theorem 2.3 shows that if the interval-valued function F is gH-differentiable in $I \subset \mathbb{R}$, then the gH-derivative of F is characterized by the one-sided differentiability of its endpoint functions.

Theorem 2.3. *If $F : I \to \mathcal{K}_C$ is gH-differentiable in I, then for all $t_0 \in I$, $(f^-)'_+(t_0)$, $(f^+)'_+(t_0)$, $(f^-)'_-(t_0)$ and $(f^+)'_-(t_0)$ exist, and (2.1) holds.*

The next section characterizes the gH-differentiability of a fuzzy number-valued function via the gH-differentiability of its level sets using the results for intrval functions presented.

2.2 Results obtained on gH-differentiability

Let $F : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be given levelwise by $[F(t)]_\alpha = [f^-_\alpha(t), f^+_\alpha(t)]$, and consider the interval-valued functions $F_\alpha : I \to \mathcal{K}_C$ defined by $F_\alpha(t) = [F(t)]_\alpha$ for all $\alpha \in [0,1]$ and $t \in I$. The indexed family $\Big\{F_\alpha\Big\}_{\alpha\in[0,1]}$ is said to be gH-differentiable at $t_0 \in I$ *uniformly in* $\alpha \in [0,1]$ if, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_H\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) < \varepsilon \tag{2.2}$$

for all $0 < |h| < \delta$ and for all $\alpha \in [0,1]$.

The following theorem provides a first characterization of the gH-differentiability of a fuzzy number-valued function in terms of the gH-differentiability of its corresponding family of interval-valued functions.

Theorem 2.4. [29] *The fuzzy number-valued function $F : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at $t_0 \in I$ if and only if the interval-valued functions $F_\alpha(t) = [F(t)]_\alpha$ are gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$, and $F(t_0+h) -_{gH} F(t_0)$ exists for all $|h| < \delta_0$, for some $\delta_0 > 0$ sufficiently small.*

Proof. If F is gH-differentiable in t_0, then exist $\delta_0 > 0$ such that the difference $F(t_0+h) -_{gH} F(t_0)$ exist for $|h| < \delta_0$. Moreover, for a given $\varepsilon > 0$ there exists $0 < \delta < \delta_0$ such that

$$\begin{aligned} 0 < |h| < \delta \Rightarrow d_H &\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, [F'_{gH}(t_0)]_\alpha\right) \\ &= d_H\left(\frac{1}{h}\Big\{[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha\Big\}, [F'_{gH}(t_0)]_\alpha\right) \\ &= d_H\left(\frac{1}{h}\left[F(t_0+h) -_{gH} F(t_0)\right]_\alpha, \left[F'_{gH}(t_0)\right]_\alpha\right) \\ &\leqslant \sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\left[F(t_0+h) -_{gH} F(t_0)\right]_\alpha, \left[F'_{gH}(t_0)\right]_\alpha\right) \end{aligned}$$

$$= d_\infty\left(\frac{1}{h}\Big\{F(t_0+h) -_{gH} F(t_0)\Big\}, F'_{gH}(t_0)\right) < \varepsilon.$$

Therefore, F_α is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$. In addition, $F'_\alpha(t_0) = [F'_{gH}(t_0)]_\alpha$ holds for all $\alpha \in [0,1]$.

Reciprocally, suppose that F_α is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$. From Equation (2.2) and the properties of the supremum, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) \leqslant \varepsilon \tag{2.3}$$

for all $|h|< \delta$. By hypothesis, there exists $\delta_0 > 0$ such that $F(t_0+h) -_{gH} F(t_0)$ exists whenever $|h| < \delta_0$, that is, $[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha$ defines a fuzzy number and it is equal to $[F(t_0+h) -_{gH} F(t_0)]_\alpha$ for all $\alpha \in [0,1]$. Since F_α is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$ and $F(t_0+h) -_{gH} F(t_0)$ exists, then $F'_\alpha(t_0)$ defines a fuzzy number A such that $[A]_\alpha = F'_\alpha(t_0)$. Hence, for a given $\varepsilon > 0$, there exists $0 < \delta < \delta_0$ such that

$$\begin{aligned}
&\sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\left[F(t_0+h) -_{gH} F(t_0)\right]_\alpha, F'_\alpha(t_0)\right)\\
&= \sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\Big\{[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha\Big\}, F'_\alpha(t_0)\right)\\
&= \sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) \leqslant \varepsilon
\end{aligned}$$

for all $0 < |h|< \delta$. Consequently,

$$d_\infty\left(\frac{1}{h}\Big\{F(t_0+h) -_{gH} F(t_0)\Big\}, A\right) < \varepsilon$$

for all $0 < |h|< \delta$. Hence, F is gH-differentiable and $F'_{gH}(t_0) = A$, where $[A]_\alpha = F'_\alpha(t_0)$. ■

It is worth noting the connection between the gH-derivative of a fuzzy number-valued function and the gH-derivative of F_α, $\forall\alpha \in [0,1]$ provided by Theorem 2.4. One can use Theorem 2.1 to obtain the behavior of the α-sets of the gH-derivative of $F : I \subset \mathbb{R} \to \mathbb{R}_\mathcal{F}$.

Theorem 2.5. [29] *The fuzzy number-valued function $F : I \subset \mathbb{R} \to \mathbb{R}_\mathcal{F}$ is gH-differentiable at $t_0 \in I$ if and only if*

- *there exists $\delta_0 > 0$ such that, for each h satisfying $0 < |h|< \delta_0$,*
  - *$f^-_\alpha(t_0+h) - f^-_\alpha(t_0)$ is monotonic increasing and $f^+_\alpha(t_0+h) - f^+_\alpha(t_0)$ is monotonic decreasing as functions of $\alpha \in [0,1]$, and $f^-_1(t_0+h) - f^-_1(t_0) \leqslant f^+_1(t_0+h) - f^+_1(t_0)$; or*
  - *$f^-_\alpha(t_0+h) - f^-_\alpha(t_0)$ is monotonic decreasing and $f^+_\alpha(t_0+h) - f^+_\alpha(t_0)$ is monotonic increasing as functions of $\alpha \in [0,1]$, and $f^-_1(t_0+h) - f^-_1(t_0) \geqslant f^+_1(t_0+h) - f^+_1(t_0)$;*

- *the limits*

$$\lim_{h\to 0^-} \min\Big(\phi_{f_\alpha^-}(h), \phi_{f_\alpha^+}(h)\Big) = a_\alpha^- = \lim_{h\to 0^+} \min\Big(\phi_{f_\alpha^-}(h), \phi_{f_\alpha^+}(h)\Big) \tag{2.4}$$

$$\text{and} \quad \lim_{h\to 0^-} \max\Big(\phi_{f_\alpha^-}(h), \phi_{f_\alpha^+}(h)\Big) = a_\alpha^+ = \lim_{h\to 0^+} \max\Big(\phi_{f_\alpha^-}(h), \phi_{f_\alpha^+}(h)\Big) \tag{2.5}$$

exist uniformly in $\alpha \in [0,1]$.

Moreover, one of the following cases is satisfied for all $\alpha \in [0,1]$*:*

(i) $(f_\alpha^-)'_+(t_0)$, $(f_\alpha^+)'_+(t_0)$, $(f_\alpha^-)'_-(t_0)$ *and* $(f_\alpha^+)'_-(t_0)$ *exist and one of the following equalities holds for all* $\alpha \in [0,1]$*:*

$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\Big] = \Big[(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\Big] = [a_\alpha^-, a_\alpha^+]; \quad \text{or}$$
$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^+)'_+(t_0), (f_\alpha^-)'_+(t_0)\Big] = \Big[(f_\alpha^+)'_-(t_0), (f_\alpha^-)'_-(t_0)\Big] = [a_\alpha^-, a_\alpha^+]; \quad \text{or}$$
$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\Big] = \Big[(f_\alpha^+)'_-(t_0), (f_\alpha^-)'_-(t_0)\Big] = [a_\alpha^-, a_\alpha^+]; \quad \text{or}$$
$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^+)'_+(t_0), (f_\alpha^-)'_+(t_0)\Big] = \Big[(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\Big] = [a_\alpha^-, a_\alpha^+].$$

(ii) f_α^- *and* f_α^+ *are right differentiable at* t_0 *and* $\phi_{f_\alpha^-}$ *and* $\phi_{f_\alpha^+}$ *are left complementary at 0, i.e.,* $C_{L(0)}(\phi_{f_\alpha^-}) = C_{L(0)}(\phi_{f_\alpha^+}) = \{a_\alpha^-, a_\alpha^+\}$*, and one of the following equalities holds for all* $\alpha \in [0,1]$*:*

$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\Big] = [a_\alpha^-, a_\alpha^+]; \quad \text{or}$$
$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^+)'_+(t_0), (f_\alpha^-)'_+(t_0)\Big] = [a_\alpha^-, a_\alpha^+].$$

(iii) f_α^- *and* f_α^+ *are left differentiable at* t_0 *and* $\phi_{f_\alpha^-}$ *and* $\phi_{f_\alpha^+}$ *are right complementary at 0, i.e.,* $C_{R(0)}(\phi_{f^-}) = C_{R(0)}(\phi_{f^+}) = \{a_\alpha^-, a_\alpha^+\}$*, and one of the following equalities holds for all* $\alpha \in [0,1]$*:*

$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\Big] = [a_\alpha^-, a_\alpha^+]; \quad \text{or}$$
$$\big[F'_{gH}(t_0)\big]_\alpha = \Big[(f_\alpha^+)'_-(t_0), (f_\alpha^-)'_-(t_0)\Big] = [a_\alpha^-, a_\alpha^+].$$

(iv) $\phi_{f_\alpha^-}$ *and* $\phi_{f_\alpha^+}$ *are both left complementary and right complementary at* 0*, i.e.,* $C_{R(0)}(\phi_{f_\alpha^-}) = C_{R(0)}(\phi_{f_\alpha^+}) = C_{L(0)}(\phi_{f_\alpha^-}) = C_{L(0)}(\phi_{f_\alpha^+}) = \{a_\alpha^-, a_\alpha^+\}$*, and*

$$[F'_{gH}(t_0)]_\alpha = [a_\alpha^-, a_\alpha^+].$$

Proof. The first condition is equivalent to the existence of $F(t_0 + h) -_{gH} F(t_0)$ and the second is equivalent to the uniform, with respect to α, gH-differentiability of F_α at t_0. Then, the conclusion follows from Theorem 2.1 and 2.4. ■

The following example was adapted from Counterexample 1 of [36], which presents an interval-valued function whose endpoints do not have one-sided derivatives.

Example 2.1. [29] Let $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be given levelwise by

$$[F(t)]_\alpha = \begin{cases} \left[-2+\alpha, t^2+t-\alpha+1\right], & t \in \mathbb{Q} \\ \left[t-2+\alpha, t^2-\alpha+1\right], & t \in \mathbb{R}\backslash\mathbb{Q} \end{cases}, \quad \alpha \in [0,1], \tag{2.6}$$

whose graphical depiction in given by Figure 2.1. Then, for $h \in \mathbb{Q}$,

$$\frac{1}{h}[F(h) -_{gH} F(0)]_\alpha = \left[\frac{f_\alpha^-(h) - f_\alpha^-(0)}{h}, \frac{f_\alpha^+(h) - f_\alpha^+(0)}{h}\right] = [0, h+1],$$

and, for $h \in \mathbb{R}\backslash\mathbb{Q}$,

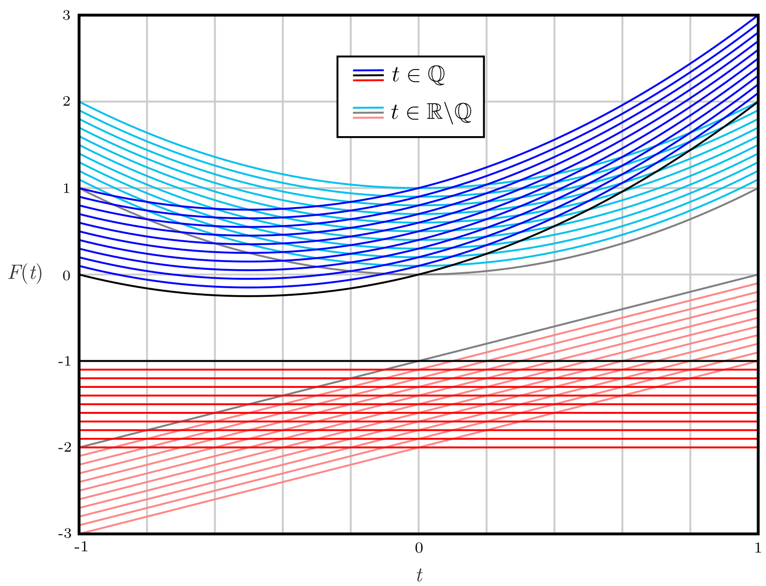
$$\frac{1}{h}[F(h) -_{gH} F(0)]_\alpha = \left[\frac{f_\alpha^+(h) - f_\alpha^+(0)}{h}, \frac{f_\alpha^-(h) - f_\alpha^-(0)}{h}\right] = [h, 1],$$

$\forall\alpha \in [0,1]$. Consequently, the limit

$$F'_{gH}(t_0) = \lim_{h\to 0} \frac{1}{h}\{F(h) -_{gH} F(0)\} = [0,1] \tag{2.7}$$

exists and, therefore, F is gH-differentiable. Thus, $(f_\alpha^-)'_-(0)$, $(f_\alpha^+)'_-(0)$, $(f_\alpha^-)'_+(0)$ and $(f_\alpha^+)'_+(0)$ do not exist, but $F'_{gH}(0)$ does.

Figure 2.1: Graphical representation of the α-cuts of F, given by (2.6), where $\alpha \in [0,1]$.



Source: [29].

Figure 2.1 depicts the discontinuity of the endpoints f_α^- and f_α^+ all over the domain of F. Note This example shows that the gH-differentiability of a fuzzy-number-valued function is not equivalent to the one sided differentiability of its endpoints.

The following theorem is a modified version of Theorem 4 of [10]. It characterizes the gH-differentiability of a fuzzy function in terms of the uniform differentiability of its endpoint functions.

Theorem 2.6. [29] *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy function and $t_0 \in I$. Suppose that $(f_\alpha^-)'_-(t_0)$, $(f_\alpha^+)'_-(t_0)$, $(f_\alpha^-)'_+(t_0)$ and $(f_\alpha^+)'_+(t_0)$ exist uniformly in $\alpha \in [0,1]$. Then F is gH-differentiable at t_0 if, and only if, one of the four cases holds:*

i) *f_α^- and f_α^+ are differentiable at t_0, $(f_\alpha^-)'(t_0)$ is monotonic increasing and $(f_\alpha^+)'(t_0)$ is monotonic decreasing as functions of α and $(f_1^-)'(t_0) \leqslant (f_1^+)'(t_0)$. In this case,*

$$[F'_{gH}(t_0)]_\alpha = \Big[(f_\alpha^-)'(t_0), (f_\alpha^+)'(t_0)\Big],$$

for all $\alpha \in [0,1]$.

ii) *f_α^- and f_α^+ are differentiable at t_0, $(f_\alpha^-)'(t_0)$ is monotonic decreasing and $(f_\alpha^+)'(t_0)$ is monotonic increasing as functions of α and $(f_1^-)'(t_0) \geqslant (f_1^+)'(t_0)$. In this case,*

$$[F'_{gH}(t_0)]_\alpha = \Big[(f_\alpha^+)'(t_0), (f_\alpha^-)'(t_0)\Big],$$

for all $\alpha \in [0,1]$.

iii) *$(f_\alpha^-)'_+(t_0) = (f_\alpha^+)'_-(t_0)$ is monotonic decreasing and $(f_\alpha^-)'_-(t_0) = (f_\alpha^+)'_+(t_0)$ is monotonic increasing as functions of α and $(f_1^+)'_+(t_0) \leqslant (f_1^-)'_+(t_0)$. In this case,*

$$[F'_{gH}(t_0)]_\alpha = [(f_\alpha^+)'_+(t_0), (f_\alpha^-)'_+(t_0)] = [(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)],$$

for all $\alpha \in [0,1]$.

iv) *$(f_\alpha^-)'_+(t_0) = (f_\alpha^+)'_-(t_0)$ is monotonic increasing and $(f_\alpha^-)'_-(t_0) = (f_\alpha^+)'_+(t_0)$ is monotonic decreasing as functions of α and $(f_1^-)'_+(t_0) \leqslant (f_1^+)'_+(t_0)$. In this case,*

$$[F'_{gH}(t_0)]_\alpha = [(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)] = [(f_\alpha^+)'_-(t_0), (f_\alpha^-)'_-(t_0)],$$

for all $\alpha \in [0,1]$.

Additionally, we say that F is (i)-gH-differentiable at t_0 (resp. (ii)-, (iii)-, (iv)-gH- differentiable) if the case (i) (resp. (ii), (iii), (iv)) is satisfied.

Proof. It follows straightforwardly from Theorem 2.5 and item (i) of Theorem 2.1. ∎

Remark 2.2. If $F : I \to \mathbb{R}_{\mathcal{F}}$ is gH-differentiable at t_0 in more than one case, then F is gH-differentiable in all cases, and $F'_{gH}(t_0)$ is a crisp number.

Recall that a given $F : I \to \mathbb{R}_{\mathcal{F}}$ is continuous at $t_0 \in I \subset \mathbb{R}$ if and only if F_α is continuous at t_0 for all α [12]. The next result is a direct consequence of Theorem 2.2.

Theorem 2.7. [29] *Assume $F : I \to \mathbb{R}_{\mathcal{F}}$ continuous in $(t_0 - \rho, t_0 + \rho) \subset I$ for some $\rho > 0$. Then F is gH-differentiable at t_0 if and only if $(f_\alpha^-)'_+(t_0)$, $(f_\alpha^+)'_+(t_0)$, $(f_\alpha^-)'_-(t_0)$ and $(f_\alpha^+)'_-(t_0)$ exist uniformly in $\alpha \in [0,1]$ and one of the items (i)-(iv) of Theorem 2.6 is satisfied.*

Proof. In fact, the continuity of F in $(t_0-\rho, t_0+\rho)$ implies that F_α is continuous in $(t_0-\rho, t_0+\rho)$ for all $\alpha \in [0,1]$. Also, from Theorem 2.4, F is gH-differentiable at t_0 if and only if F_α is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$. Consequently, from Theorem 2.2, F is gH-differentiable at t_0 if and only if $(f_\alpha^-)'_+(t_0)$, $(f_\alpha^+)'_+(t_0)$, $(f_\alpha^-)'_-(t_0)$ and $(f_\alpha^+)'_-(t_0)$ exist uniformly in α, so that at least one of the items (i)-(iv) of Theorem 2.6 is satisfied. ■

From Theorems 2.3, 2.4 and 2.6, we have the following result.

Theorem 2.8. [29] *If $F : I \to \mathbb{R}_\mathcal{F}$ is gH-differentiable in the open set I, then, for each $t_0 \in I$, $(f_\alpha^-)'_+(t_0)$, $(f_\alpha^+)'_+(t_0)$, $(f_\alpha^-)'_-(t_0)$ and $(f_\alpha^+)'_-(t_0)$ exist uniformly in $\alpha \in [0,1]$ and satisfies one of the items (i)-(iv) of Theorem 2.6.*

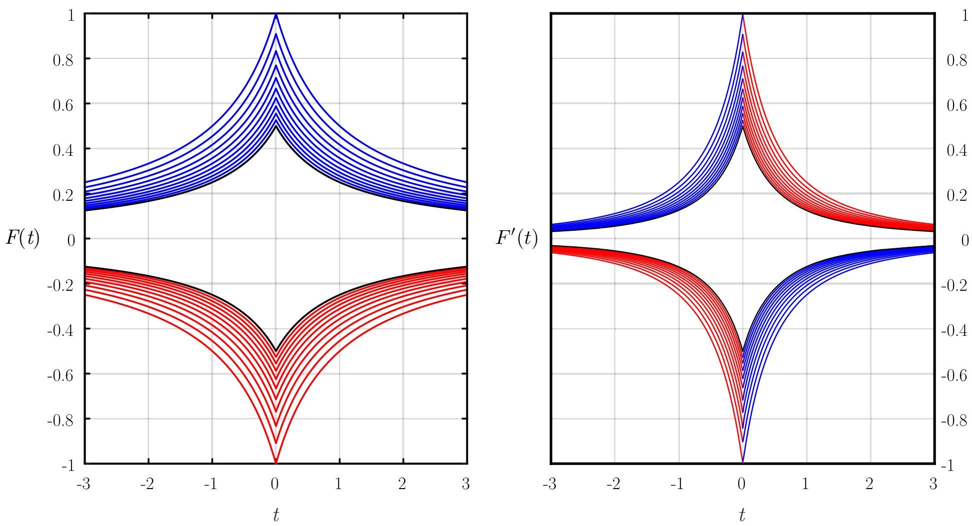
Proof. Theorem 2.4 assures that, if F is gH-differentiable in I, then F_α is gH-differentiable at each point $t_0 \in I$ uniformly in $\alpha \in [0,1]$. Hence, from Theorem 2.3, for each $t_0 \in I$, $(f_\alpha^-)'_+(t_0)$, $(f_\alpha^+)'_+(t_0)$, $(f_\alpha^-)'_-(t_0)$ and $(f_\alpha^+)'_-(t_0)$ exist uniformly in $\alpha \in [0,1]$ and at least one of the items (i)-(iv) of Theorem 2.6 is satisfied. ■

Next, we recall Example 27 of 2.2, which presents a gH-differentiable fuzzy function that satisfies different cases of Theorem 2.6 in its domain. In addition, it illustrates that there are different possibilities to handle the gH-differentiability of fuzzy functions in relation to the switching points.

Example 2.2. [8] Let $F : \mathbb{R} \to \mathbb{R}_\mathcal{F}$ be defined levelwise by

$$[F(t)]_\alpha = \left[\frac{-1}{(1+|t|)(1+\alpha)}, \frac{1}{(1+|t|)(1+\alpha)}\right], \ \alpha \in [0,1]. \tag{2.8}$$

Figure 2.2: From left to right: graphical representation of the endpoints of the α-cuts of F and F'_{gH}, given respectively by Equations (2.8) and (2.9).



Source: Figures 2 and 3 in [8].

Then F is gH-differentiable for all $t \in \mathbb{R}$. More precisely, F (i)-gH-differentiable for $t < 0$, (ii)-gH-differentiable for $t > 0$, and (iii)-gH-differentiable at $t = 0$, while f_α^- and f_α^+ are not differentiable at $t = 0$. Levelwise,

$$\left[F'_{gH}(t)\right]_\alpha = \left[\frac{-1}{(1+|t|)^2(1+\alpha)}, \frac{1}{(1+|t|)^2(1+\alpha)}\right], \quad \alpha \in [0,1], \tag{2.9}$$

for all $t \in \mathbb{R}$. Figure 2.2 presents the depiction of F and F'_{gH} via the endpoints of their levelsets.

Note that the level sets of F has increasing length for $t < 0$ and decreasing length for $t < 0$. Tis situation is related to the change of differentiability of F at $t = 0$. In this situation, $t = 0$ is called a switching point of F (Definition 1.8) and this concept will be discussed on Chapter 3.

2.3 The Hukuhara L-generalized Hukuhara derivative

In this section, the gH∗-differentiability of a fuzzy-number-valued function is characterized as in Section 2.2, which provides similar results for the gH-differentiability.

The following theorem provides a connection between the gH∗-derivative of a fuzzy number-valued function and the gH-derivative of its level-sets.

Theorem 2.9. [29] *The fuzzy number-valued function $F : I \subset \mathbb{R} \to \mathbb{R}_\mathcal{F}$ is gH∗-differentiable at $t_0 \in I$ if and only if the interval-valued functions $F_\alpha(t) = [F(t)]_\alpha$ are gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$, and $F(t_0+h) -_{LgH} F(t_0)$ exists for $0 < |h| < \delta_0$ for some $\delta_0 > 0$ sufficiently small.*

Proof. If F is gH∗-differentiable at t_0, there exists $\delta_0 > 0$ such that the difference $F(t_0+h) -_{LgH} F(t_0)$ exists for $|h| < \delta_0$. Moreover, for a given $\varepsilon > 0$ there exists $0 < \delta < \delta_0$ such that for all $\alpha \in [0,1]$,

$$\begin{aligned}
0 < |h| < \delta \Rightarrow d_H &\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, [F'_{gH*}(t_0)]_\alpha\right) \\
&= d_H\left(\frac{1}{h}\Big\{[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha\Big\}, [F'_{gH*}(t_0)]_\alpha\right) \\
&= d_H\left(\frac{1}{h}\Big\{[F(t_0+h) -_{LgH} F(t_0)]_\alpha\Big\}, [F'_{gH*}(t_0)]_\alpha\right) \\
&\leqslant \sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\Big\{[F(t_0+h) -_{LgH} F(t_0)]_\alpha\Big\}, [F'_{gH*}(t_0)]_\alpha\right) \\
&= d_\infty\left(\frac{1}{h}\Big\{F(t_0+h) -_{LgH} F(t_0)\Big\}, F'_{gH*}(t_0)\right) < \varepsilon
\end{aligned} \tag{2.10}$$

Therefore, F_α is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$. In addition, $F'_\alpha(t_0) = [F'_{gH*}(t_0)]_\alpha$ holds for all $\alpha \in [0,1]$.

On the other hand, suppose that F_α is gH-differentiable at t_0 uniformly in α and $F'_\alpha(t_0) = [A]_\alpha$. By hypothesis, there exists $\delta_0 > 0$ such that $F(t_0+h) -_{LgH} F(t_0)$ exists whenever $|h| < \delta_0$, that is, $[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha$ defines a fuzzy number. Since F_α is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$ and $F(t_0+h) -_{LgH} F(t_0)$ exists, then $F'_\alpha(t_0)$ exist for all $\alpha \in [0,1]$ and define a fuzzy number A such that $[A]_\alpha = F'_\alpha(t_0)$. Also, from (2.2),

$$\sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) < \varepsilon\,. \tag{2.11}$$

Henceforth, for a given $\varepsilon > 0$, there exists $0 < \delta < \delta_0$ such that for all $\alpha \in [0,1]$,

$$\begin{aligned}
0 < |h| < \delta \Rightarrow d_H &\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) \\
&\leqslant \sup_{\alpha\in[0,1]} d_H \left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) \\
&= \sup_{\alpha\in[0,1]} d_H \left(\frac{1}{h}\Big\{[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha\Big\}, F'_\alpha(t_0)\right) \\
&= \sup_{\alpha\in[0,1]} d_H \left(\frac{1}{h}\left[F(t_0+h) -_{LgH} F(t_0)\right]_\alpha, F'_\alpha(t_0)\right) \\
&= d_\infty \left(\frac{1}{h}\Big\{F(t_0+h) -_{LgH} F(t_0)\Big\}, A\right) < \varepsilon\,.
\end{aligned}$$

Therefore, F is gH$*$-differentiable and $F'_{gH*}(t_0) = A$. ■

The next theorem follows straightforwardly from Theorems 2.1 and 2.9.

Theorem 2.10. [29] *The fuzzy number-valued function $F : I \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ is gH$*$-differentiable at $t_0 \in I$ if and only if*

- *there exists $\delta_0 > 0$ such that, for each h satisfying $0 < |h| < \delta_0$, $\min(f^-_\alpha(t_0+h) - f^-_\alpha(t_0), f^+_\alpha(t_0+h) - f^+_\alpha(t_0))$ is monotonic increasing and $\max(f^-_\alpha(t_0+h) - f^-_\alpha(t_0), f^+_\alpha(t_0+h) - f^+_\alpha(t_0))$ is monotonic decreasing as functions of $\alpha \in [0,1]$, and $\min(f^-_1(t_0+h) - f^-_1(t_0), f^+_1(t_0+h) - f^+_1(t_0)) \leqslant \max(f^-_1(t_0+h) - f^-_1(t_0), f^+_1(t_0+h) - f^+_1(t_0))$;*
- *the limits in (2.4) and (2.5) exist uniformly in $\alpha \in [0,1]$.*

Moreover, one of the following cases is satisfied for all $\alpha \in [0,1]$:

(i) $(f^-_\alpha)'_+(t_0)$, $(f^+_\alpha)'_+(t_0)$, $(f^-_\alpha)'_-(t_0)$ *and* $(f^+_\alpha)'_-(t_0)$ *exist and*

$$\begin{aligned}
\left[F'_{gH*}(t_0)\right]_\alpha &= \Big[\min\{(f^-_\alpha)'_+(t_0), (f^+_\alpha)'_+(t_0)\}, \max\{(f^-_\alpha)'_+(t_0), (f^+_\alpha)'_+(t_0)\}\Big] \\
&= \Big[\min\{(f^-_\alpha)'_-(t_0), (f^+_\alpha)'_-(t_0)\}, \max\{(f^-_\alpha)'_-(t_0), (f^+_\alpha)'_-(t_0)\}\Big] \\
&= [a^-_\alpha, a^+_\alpha].
\end{aligned}$$

(ii) f^-_α *and* f^+_α *are right differentiable at* t_0 *and* $\phi_{f^-_\alpha}$ *and* $\phi_{f^+_\alpha}$ *are left complementary at 0, i.e.,* $C_{L(0)}(\phi_{f^-_\alpha}) = C_{L(0)}(\phi_{f^+_\alpha}) = \{a^-_\alpha, a^+_\alpha\}$. *Moreover,*

$$\begin{aligned}
\left[F'_{gH*}(t_0)\right]_\alpha &= \Big[\min\{(f^-_\alpha)'_+(t_0), (f^+_\alpha)'_+(t_0)\}, \max\{(f^-_\alpha)'_+(t_0), (f^+_\alpha)'_+(t_0)\}\Big] \\
&= [a^-_\alpha, a^+_\alpha].
\end{aligned}$$

(iii) f^-_α *and* f^+_α *are left differentiable at* t_0 *and* $\phi_{f^-_\alpha}$ *and* $\phi_{f^+_\alpha}$ *are right complementary at 0, i.e.,* $C_{R(0)}(\phi_{f^-}) = C_{R(0)}(\phi_{f^+}) = \{a^-_\alpha, a^+_\alpha\}$. *Moreover,*

$$\begin{aligned}
\left[F'_{gH*}(t_0)\right]_\alpha &= \Big[\min\{(f^-_\alpha)'_-(t_0), (f^+_\alpha)'_-(t_0)\}, \max\{(f^-_\alpha)'_-(t_0), (f^+_\alpha)'_-(t_0)\}\Big] \\
&= [a^-_\alpha, a^+_\alpha].
\end{aligned}$$

(iv) $\phi_{f_\alpha^-}$ *and* $\phi_{f_\alpha^+}$ *are both left complementary and right complementary at* 0*, i.e.,* $C_{R(0)}(\phi_{f_\alpha^-}) = C_{R(0)}(\phi_{f_\alpha^+}) = C_{L(0)}(\phi_{f_\alpha^-}) = C_{L(0)}(\phi_{f_\alpha^+}) = \{a_\alpha^-, a_\alpha^+\}$*. Moreover,*

$$[F'_{gH*}(t_0)]_\alpha = [a_\alpha^-, a_\alpha^+].$$

Proof. The proof follows from Theorems 2.1 and 2.9. Indeed, items (a) and (b) ensure the existence of the difference $F(t_0+h) -_{LgH} F(t_0)$ for $|h| < \delta_0$; the uniform existence of the limits in (2.4) and (2.5) with respect to $\alpha \in [0,1]$ provides the connection between F'_α and the gH$*$-derivative of F. In addition, items (i)-(iv) are an immediate consequence of Theorem 2.1 applied to each F_α, where $\alpha \in [0,1]$. ■

Remark 2.3. According to [10], in the case where $(f_\alpha^-)'_+(t_0)$, $(f_\alpha^+)'_+(t_0)$, $(f_\alpha^-)'_-(t_0)$ and $(f_\alpha^+)'_-(t_0)$ exist uniformly in $\alpha \in [0,1]$ (case (i) of Theorem 2.10), there are disjoint subjects C_1, C_2, D_1 and D_2 with $C_1 \cup C_2 \cup D_1 \cup D_2 = [0,1]$ such that

- f_α^- and f_α^+ are differentiable at t_0, uniformly in $\alpha \in C_1 \cup C_2$, and

$$[F'_{gH*}(t_0)]_\alpha = \Big[(f_\alpha^-)'(t_0), (f_\alpha^+)'(t_0)\Big], \ \forall \alpha \in C_1,$$

$$[F'_{gH*}(t_0)]_\alpha = \Big[(f_\alpha^+)'(t_0), (f_\alpha^-)'(t_0)\Big], \ \forall \alpha \in C_2;$$

- $(f_\alpha^-)'_+(t_0) = (f_\alpha^+)'_-(t_0)$ and $(f_\alpha^+)'_+(t_0) = (f_\alpha^-)'_-(t_0)$ for all $\alpha \in D_1 \cup D_2$, and

$$[F'_{gH*}(t_0)]_\alpha = \Big[(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\Big] = \Big[(f_\alpha^+)'_-(t_0), (f_\alpha^-)'_-(t_0)\Big], \ \forall \alpha \in D_1,$$

$$[F'_{gH*}(t_0)]_\alpha = \Big[(f_\alpha^+)'_+(t_0), (f_\alpha^-)'_+(t_0)\Big] = \Big[(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\Big], \ \forall \alpha \in D_2.$$

Since the continuity of F in $(\mathbb{R}_\mathcal{F}, d_\infty)$ is equivalent to the continuity of F_α in $(\mathcal{K}_c, d_H)$ [12], the next theorem follows from Theorem 2.1.

Theorem 2.11. [29] *Suppose* $F : I \to \mathbb{R}_\mathcal{F}$ *is continuous in* $(t_0-\rho, t_0+\rho)$ *for some* $\rho > 0$*. Then* F *is gH$*$-differentiable at* t_0 *if and only if* $(f_\alpha^-)'_+(t_0)$*,* $(f_\alpha^+)'_+(t_0)$*,* $(f_\alpha^-)'_-(t_0)$ *and* $(f_\alpha^+)'_-(t_0)$ *exist uniformly in* $\alpha \in [0,1]$ *and*

$$\begin{aligned}\big[F'_{gH*}(t_0)\big]_\alpha &= \Big[\min\{(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\}, \max\{(f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\}\Big] \\ &= \Big[\min\{(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\}, \max\{(f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\}\Big].\end{aligned} \tag{2.12}$$

Proof. The proof is analogous to the proof of Theorem 2.7. ■

The following result follows straightforwardly from Theorems 2.3 and 2.9.

Theorem 2.12. [29] *If* $F : I \to \mathbb{R}_\mathcal{F}$ *is gH$*$-differentiable in* I*, then for each* $t_0 \in I$*,* $(f_\alpha^-)'_+(t_0)$*,* $(f_\alpha^+)'_+(t_0)$*,* $(f_\alpha^-)'_-(t_0)$ *and* $(f_\alpha^+)'_-(t_0)$ *exist and* (2.12) *is satisfied.*

Proof. The proof is analogous to the proof of Theorem 2.8. ■

Example 2.3. Let $F : [-2,2] \to \mathbb{R}_{\mathcal{F}}$ be given levelwise by

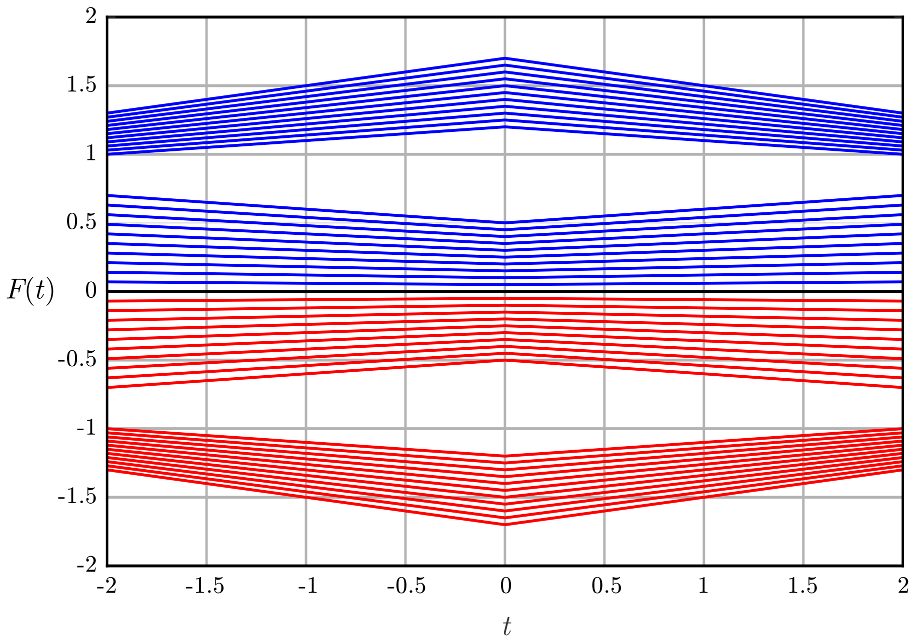
$$[F(t)]_\alpha = \begin{cases} \left[(1-\alpha)\frac{|t|}{5} - 1.7 + \alpha, (\alpha-1)\frac{|t|}{5} + 1.7 - \alpha\right], & \alpha \in \left[0, \frac{1}{2}\right] \\ \left[(\alpha-1)\frac{|t|}{5} - 1 + \alpha, (1-\alpha)\frac{|t|}{5} + 1 - \alpha\right], & \alpha \in \left(\frac{1}{2}, 1\right] \end{cases}, \tag{2.13}$$

for all $t \in [-2,2]$, and whose depiction is given by Figure 2.3. Thereby, F is gH$*$-differentiable and

$$[F'_{gH*}(t)]_\alpha = \left[\frac{\alpha-1}{5}, \frac{1-\alpha}{5}\right], \quad \alpha \in [0,1] \tag{2.14}$$

holds for all $t \in [-2,2]$, that is, $F'_{gH*}(t) = \left(-\frac{1}{5}; 0; \frac{1}{5}\right)$. Note that, F_α is (i)-gH differentiable at $[-2,0)$, (ii)-gH-differentiable at $(0,2]$ and (iii)-gH-differentiable at 0, for all $0 \leqslant \alpha \leqslant 0.5$, and F_α is (ii)-gH differentiable at $[-2,0)$, (i)-gH-differentiable at $(0,2]$ and (iv)-gH-differentiable at 0, for all $0.5 < \alpha \leqslant 1$. The difference of the type of differentiability of the α-sets can be observed in Figure 2.3, where some levels have increasing length and others have decreasing length. Note that this behavior changes at $t = 0$ and, then, it can be considered as a switching point of each F_α.

Figure 2.3: Graphical representation of F, given levelwise by (2.13). The red and blue lines represent, respectively, the left and right endpoints of F, denoted by f_α^- and f_α^+.



Source: [29].

It is worth noting that f_α^- and f_α^+ are discontinuous with respect to $\alpha = 0.5$ but left continuous at the same point. In fact, $f_{0.5}^-(t) = 0.1|t| - 1.2$ while $\lim_{\alpha \to 0.5^+} f_\alpha^-(t) = -0.1|t| - 0.5$, that is,

$$\overline{\bigcup_{\alpha > 0.5} [F(t)]_\alpha} = \lim_{\alpha \to 0.5^+} [f_\alpha^-(t), f_\alpha^+(t)] = [-0.1|t| - 0.5, 0.1|t| + 0.5] \tag{2.15}$$

as illustrated in Figure 2.3.

2.4 The generalized derivative

This is the main section of this paper. It provides several characterization results concerning the generalized differentiability of fuzzy number-valued functions. The next theorem provides a connection between the g-differentiability of a fuzzy function and the uniform gH-differentiability of its endpoint functions.

Theorem 2.13. [29] *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy number valued function and let the family of functions $\{F_\alpha\}_{\alpha\in[0,1]}$ be gH-differentiable at $t_0 \in I$ uniformly in $\alpha \in [0,1]$. Then F is g-differentiable at t_0 and*

$$\left[F'_g(t_0)\right]_\alpha = \overline{\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right)} \tag{2.16}$$

for $\alpha \in [0,1]$.

Proof. Since the family of functions $F_\alpha(t) = [F(t)]_\alpha$ is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\alpha \in [0,1]$,

$$\begin{aligned}
0 < |h| < \delta \Rightarrow d_H &\left(\frac{1}{h}\left[F(t_0+h) -_g F(t_0)\right]_\alpha, \overline{\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right)}\right) \\
&= d_H\left(\overline{\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha}\frac{1}{h}\Big\{[F(t_0+h)]_\beta -_{gH} [F(t_0)]_\beta\Big\}\right)}, \overline{\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right)}\right) \\
&\leqslant d_H\left(\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha}\frac{1}{h}\Big\{[F(t_0+h)]_\beta -_{gH} [F(t_0)]_\beta\Big\}\right), \operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right)\right) \\
&\leqslant d_H\left(\bigcup_{\beta\geqslant\alpha}\frac{1}{h}\Big\{[F(t_0+h)]_\beta -_{gH} [F(t_0)]_\beta\Big\}, \bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right) \\
&\leqslant d_H\left(\frac{1}{h}\Big\{[F(t_0+h)]_\alpha -_{gH} [F(t_0)]_\alpha\Big\}, F'_\alpha(t_0)\right) \\
&\leqslant d_H\left(\frac{1}{h}\Big\{F_\alpha(t_0+h) -_{gH} F_\alpha(t_0)\Big\}, F'_\alpha(t_0)\right) < \varepsilon
\end{aligned}$$

Hence, from the supremum properties, Equation (2.4) yields

$$\begin{aligned}
&\sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\left[F(t_0+h) -_g F(t_0)\right]_\alpha, \overline{\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right)}\right) \\
&= d_\infty\left(\frac{1}{h}\left\{F(t_0+h) -_g F(t_0)\right\}, A\right) \leqslant \varepsilon,
\end{aligned} \tag{2.17}$$

where $A \in \mathbb{R}_{\mathcal{F}}$ is such that $[A]_\alpha = \overline{\operatorname{conv}\left(\bigcup_{\beta\geqslant\alpha} F'_\beta(t_0)\right)}$ for all $\alpha \in [0,1]$.

Therefore, F is g-differentiable at t_0 and

$$F_g'(t_0) = A \Rightarrow [F_g'(t_0)]_\alpha = \overline{\operatorname{conv}\left(\bigcup_{\beta \geqslant \alpha} F_\beta'(t_0)\right)}. \tag{2.18}$$

■

If the gH-differentiability of the interval functions $F_\alpha(t) = [F(t)]_\alpha$ is given by the one-side derivatives of the endpoint functions, the next result follows.

Corollary 2.14. [29] *Let $F : I \to \mathbb{R}_\mathcal{F}$ be a fuzzy function such that $(f_\alpha^-)'_-(t_0)$, $(f_\alpha^+)'_-(t_0)$, $(f_\alpha^-)'_+(t_0)$ and $(f_\alpha^+)'_+(t_0)$ exist uniformly in $\alpha \in [0,1]$ and satisfy*

$$\begin{aligned} &\min\left((f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\right) = \min\left((f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\right) \\ \textit{and} \quad &\max\left((f_\alpha^-)'_-(t_0), (f_\alpha^+)'_-(t_0)\right) = \max\left((f_\alpha^-)'_+(t_0), (f_\alpha^+)'_+(t_0)\right). \end{aligned} \tag{2.19}$$

for some $t_0 \in I$. Then F is g-differentiable at t_0 and

$$\begin{aligned} \left[F_g'(t_0)\right]_\alpha &= \left[\inf_{\beta \geqslant \alpha} \min((f_\beta^-)'_-(t_0), (f_\beta^+)'_-(t_0)), \sup_{\beta \geqslant \alpha} \max((f_\beta^-)'_-(t_0), (f_\beta^+)'_-(t_0))\right] \\ &= \left[\inf_{\beta \geqslant \alpha} \min((f_\beta^-)'_+(t_0), (f_\beta^+)'_+(t_0)), \sup_{\beta \geqslant \alpha} \max((f_\beta^-)'_+(t_0), (f_\beta^+)'_+(t_0))\right] \end{aligned} \tag{2.20}$$

Proof. Since $(f_\alpha^-)'_-(t_0)$, $(f_\alpha^+)'_-(t_0)$, $(f_\alpha^-)'_+(t_0)$ and $(f_\alpha^+)'_+(t_0)$ exist uniformly in $\alpha \in [0,1]$, it follows from (2.19) that the interval functions F_α are gH-differentiable at t_0. Therefore, from Theorem 2.13, F is g-differentiable and

$$\begin{aligned} \left[F_g'(t_0)\right]_\alpha &= \overline{\operatorname{conv}\left(\bigcup_{\beta \geqslant \alpha} F_\beta'(t_0)\right)} \\ &= \left[\inf_{\beta \geqslant \alpha} \min((f_\beta^-)'_-(t_0), (f_\beta^+)'_-(t_0)), \sup_{\beta \geqslant \alpha} \max((f_\beta^-)'_-(t_0), (f_\beta^+)'_-(t_0))\right] \\ &= \left[\inf_{\beta \geqslant \alpha} \min((f_\beta^-)'_+(t_0), (f_\beta^+)'_+(t_0)), \sup_{\beta \geqslant \alpha} \max((f_\beta^-)'_+(t_0), (f_\beta^+)'_+(t_0))\right]. \end{aligned}$$

■

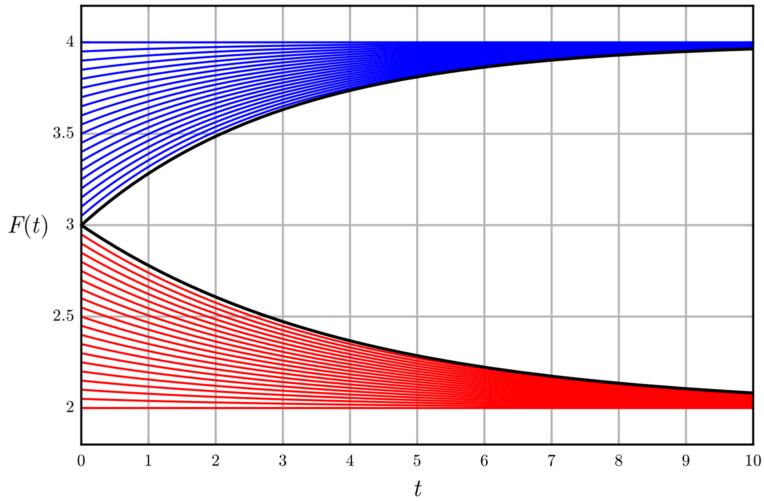
Corollary 2.14 is a generalization of Theorem 34 of [8], which is recalled in Section 1.2.

Example 2.4. [29] Consider $F : \mathbb{R} \to \mathbb{R}_\mathcal{F}$ be given levelwise by

$$[F(t)]_\alpha = \left[2 + \alpha e^{-t/4}, 4 - \alpha e^{-t/3}\right], \quad \forall \alpha \in [0,1], \tag{2.21}$$

whose depiction is given by Figure 2.4. Note that the derivatives of the endpoints functions satisfy $(f_1^-)'(t) \leqslant (f_\alpha^\pm)'(t) \leqslant (f_1^+)'(t)$ for all $\alpha \in [0,1]$ and $t \geqslant 0$. It means that $F_\alpha'(t) \subset F_1'(t)$ for all $\alpha \in [0,1]$ and, then, $F_g'(t) = \left[-\frac{1}{4}e^{-t/4}, \frac{1}{3}e^{-t/3}\right]$, $t \geqslant 0$. Note in Figure 2.4 that the core of F increases its length faster than the other level sets, what illustrate why the gH-derivative of F is given by the derivative of F_1.

Figure 2.4: Graphical representation of the fuzzy function (2.21). The red and blues lines represent the left and right endpoints of F, denoted by f_α^- and f_α^+, respectively, for all $\alpha \in [0, 1]$.



Source: [29].

Theorem 2.13 requires the gH-differentiability of the indexed family of interval-valued functions $\left\{F_\alpha\right\}_{\alpha\in[0,1]}$ as a sufficient condition to the g-differentiability of F. On the other hand, the following result only requires the one-side differentiability of the endpoint functions of F as a sufficient condition to the g-differentiability of F.

Theorem 2.15. [29] *Let $F : I \to \mathbb{R}_\mathcal{F}$ be given. Suppose the one-sided derivatives $(f_\alpha^-)'_-(t_0)$, $(f_\alpha^+)'_-(t_0)$, $(f_\alpha^-)'_+(t_0)$ and $(f_\alpha^+)'_+(t_0)$ exist uniformly in $\alpha \in [0, 1]$ and satisfy*

$$\inf_{\beta\geqslant\alpha} \min\Big((f_\beta^-)'_-(t_0), (f_\beta^+)'_-(t_0)\Big) = \inf_{\beta\geqslant\alpha} \min\Big((f_\beta^-)'_+(t_0), (f_\beta^+)'_+(t_0)\Big) = a_\alpha^- \tag{2.22}$$

and

$$\sup_{\beta\geqslant\alpha} \max\Big((f_\beta^-)'_-(t_0), (f_\beta^+)'_-(t_0)\Big) = \sup_{\beta\geqslant\alpha} \max\Big((f_\beta^-)'_+(t_0), (f_\beta^+)'_+(t_0)\Big) = a_\alpha^+. \tag{2.23}$$

for some $t_0 \in I$. Then F is g-differentiable and $\left[F_g'(t_0)\right]_\alpha = [a_\alpha^-, a_\alpha^+]$.

Proof. Since the one-sided derivatives $(f_\alpha^-)'_-(t_0)$, $(f_\alpha^+)'_-(t_0)$, $(f_\alpha^-)'_+(t_0)$ and $(f_\alpha^+)'_+(t_0)$ exist uniformly in $\alpha \in [0, 1]$, then (2.22) and (2.23) exist uniformly in $\alpha \in [0, 1]$. Consequently, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that, for $0 < |h| < \delta$, we have

$$d_H\left(\left[\inf_{\beta\geqslant\alpha} \min\Big(\phi_{f_\alpha^-}(h), \phi_{f_\alpha^+}(h)\Big), \sup_{\beta\geqslant\alpha} \max\Big(\phi_{f_\alpha^-}(h), \phi_{f_\alpha^+}(h)\Big)\right], [a_\alpha^-, a_\alpha^+]\right) < \varepsilon, \tag{2.24}$$

for all $\alpha \in [0, 1]$, that is,

$$d_H\left(\frac{1}{h}\left[F(t_0 + h) -_g F(t_0)\right], [a_\alpha^-, a_\alpha^+]\right) < \varepsilon, \quad \forall\alpha \in [0, 1] \tag{2.25}$$

which leads us to

$$\begin{aligned}&\sup_{\alpha\in[0,1]} d_H\left(\frac{1}{h}\left[F(t_0+h)-_g F(t_0)\right]_\alpha, [a_\alpha^-, a_\alpha^+]\right)\\&= d_\infty\left(\frac{1}{h}\left\{F(t_0+h)-_g F(t_0)\right\}, A\right) < \varepsilon\,.\end{aligned} \tag{2.26}$$

Therefore, F is g-differentiable at t_0 and, in addition, $[F_g'(t_0)]_\alpha = [A]_\alpha = [a_\alpha^-, a_\alpha^+]$ for all $\alpha \in [0,1]$. ■

Theorems 2.13 and 2.15 present sufficient but not necessary conditions for the g-differentiability of a function. The following example illustrates a case where the fuzzy process is g-differentiable at a point in its domain but does not satisfy the theorems above.

Example 2.5. [29] The function $F:[0,\infty)\to\mathbb{R}_{\mathcal{F}}$, given levelwise by

$$[F(t)]_\alpha = \begin{cases}\Big[-|t|+1-(1-\alpha)e^{-|t|}, |t|+2+(1-\alpha)e^{-|t|}\Big], & t\in\mathbb{Q},\\ \Big[-|t|+1-(1-\alpha), |t|+2+(1-\alpha)\Big], & t\in\mathbb{R}\backslash\mathbb{Q},\end{cases}, \tag{2.27}$$

whose depiction is given by Figure 2.5, is g-differentiable at $t_0 = 0$. However, F_α is gH-differentiable over $[0,\infty)$ iff $\alpha = 1$, since only F_1 is gH-differentiable at 0. In fact, $F_1'(0) = [-1,1]$, but, on the other hand, if $\alpha\in[0,1)$, then consider any sequence $\{h_n\}_{n\in\mathbb{N}}$ with $h_n\to 0$. Hence, we have

$$\lim_{n\to\infty}\frac{1}{h_n}\Big\{F_\alpha(h_n) -_{gH} F_\alpha(0)\Big\} = \begin{cases}[-\alpha,\alpha] & \text{if} \quad \{h_n\}_{n\in\mathbb{N}}\subset\mathbb{Q}\\ [-1,1] & \text{if} \quad \{h_n\}_{n\in\mathbb{N}}\subset\mathbb{R}\backslash\mathbb{Q}\end{cases}, \tag{2.28}$$

that is, F_α is not gH-differentiable at 0 whenever $\alpha\in[0,1)$.

In the meantime, for all $\alpha\in[0,1]$, we have

$$\begin{aligned}\frac{1}{h}[F(h)-_g F(0)]_\alpha &= \Bigg[\inf_{\beta\geqslant\alpha}\min\left(\frac{f_\beta^-(h)-f_\beta^-(0)}{h}, \frac{f_\beta^+(h)-f_\beta^+(0)}{h}\right),\\ &\qquad \sup_{\beta\geqslant\alpha}\left(\frac{f_\beta^-(h)-f_\beta^-(0)}{h}, \frac{f_\beta^+(h)-f_\beta^+(0)}{h}\right)\Bigg]\\ &= \Bigg[\min\left(\frac{f_1^-(h)-f_1^-(0)}{h}, \frac{f_1^+(h)-f_1^+(0)}{h}\right),\\ &\qquad \max\left(\frac{f_1^-(h)-f_1^-(0)}{h}, \frac{f_1^+(h)-f_1^+(0)}{h}\right)\Bigg] = [-1,1].\end{aligned}$$

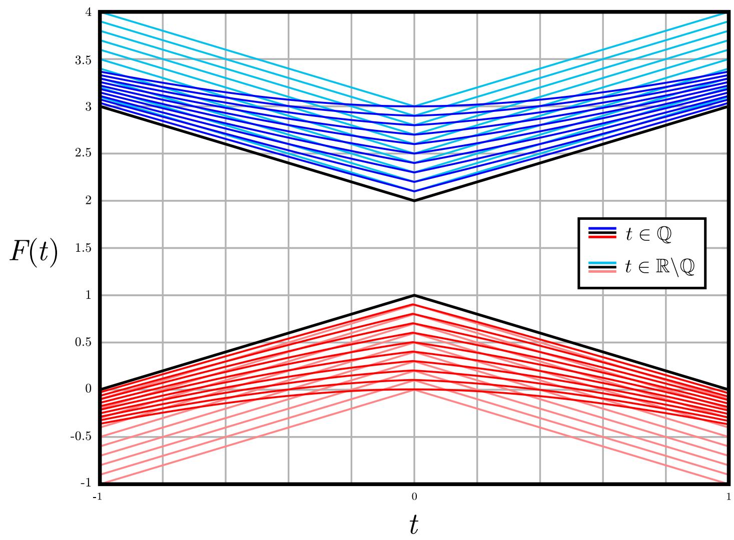
Therefore,

$$\lim_{h\to 0}\frac{1}{h}[F(h)-_g F(0)]_\alpha = [-1,1],\ \forall\alpha\in[0,1], \tag{2.29}$$

that is, F is g-differentiable at 0 and $F_g'(0) = [-1,1]$.

The following lemmas play a fundamental role in structuring a complete characterization of the generalized differentiability of fuzzy number-valued functions. From now on, we shall consider fuzzy functions whose range is contained in $\mathbb{R}_{\mathcal{F}}^c$.

Figure 2.5: Graphical representation of the fuzzy function (2.27).



Source: [29].

Lemma 2.16. [29] *Let* $F : I \to \mathbb{R}^c_{\mathcal{F}}$ *and* $t_0 \in I$ *be given. If*

$$\bigcup_{\beta \in [0,1]} \left(C_{L(0)} \left(\phi_{f^-_\beta} \right) \cup C_{L(0)} \left(\phi_{f^+_\beta} \right) \right)$$

is a lower bounded set, then for each $\alpha \in [0, 1]$ *the limit*

$$\lim_{h \to 0^-} \inf_{\beta \geqslant \alpha} \min \left(\phi_{f^-_\beta}(h), \phi_{f^+_\beta}(h) \right) = a^-_\alpha \tag{2.30}$$

exists if and only if there exists $\gamma \geqslant \alpha$ *such that the limit*

$$\lim_{h \to 0^-} \min \left(\phi_{f^-_\gamma}(h), \phi_{f^+_\gamma}(h) \right) \tag{2.31}$$

exists and satisfies

$$\lim_{h \to 0^-} \min \left(\phi_{f^-_\gamma}(h), \phi_{f^+_\gamma}(h) \right) = a^-_\alpha = \inf \bigcup_{\beta \geqslant \alpha} \left(C_{L(0)} \left(\phi_{f^-_\beta} \right) \cup C_{L(0)} \left(\phi_{f^+_\beta} \right) \right) .$$

Proof. For a given $\alpha \in [0, 1]$, suppose the limit (2.30) exists but, for all $\gamma \geqslant \alpha$, the limit (2.31) does not converges to a^-_α. Thus, for all $\gamma \geqslant \alpha$ there exists $\eta = \eta(\gamma) > 0$ such that, for all $\delta > 0$ there exists $h \in (-\delta, 0)$ satisfying

$$\left| \min \left(\phi_{f^-_\gamma}(h), \phi_{f^+_\gamma}(h) \right) - a^-_\alpha \right| \geqslant \eta \, . \tag{2.32}$$

Meantime, since limit (2.30) exists, given $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$-\delta' < h < 0 \Rightarrow \left| \inf_{\beta \geqslant \alpha} \min \left(\phi_{f^-_\gamma}(h), \phi_{f^+_\gamma}(h) \right) - a^-_\alpha \right| < \varepsilon \, , \tag{2.33}$$

that is, for all $h \in (-\delta', 0)$, there exists $\beta_1 = \beta_1(h) \geqslant \alpha$ such that

$$\left|\min\left(\phi_{f^-_{\beta_1}}(h), \phi_{f^+_{\beta_1}}(h)\right) - a^-_\alpha\right| < \varepsilon\,. \tag{2.34}$$

Thus, for $\varepsilon = \eta(\beta_1)$, there exists $\delta' = \delta'(\eta)$ such that

$$\left|\min\left(\phi_{f^-_{\beta_1}}(h), \phi_{f^+_{\beta_1}}(h)\right) - a^-_\alpha\right| < \varepsilon \tag{2.35}$$

for all $-\delta' < h < 0$ and $\beta_1 = \beta_1(h)$, which leads us to a contradiction with (2.32) by taking $h > -\min\{\delta, \delta'\}$. We conclude that the limit (2.31) exists for some $\gamma \geqslant \alpha$.

Since $\inf\limits_{\beta\geqslant\alpha} \min\left(\phi_{f^-_\beta}(h), \phi_{f^+_\beta}(h)\right) \leqslant \min\left(\phi_{f^-_\mu}(h), \phi_{f^+_\mu}(h)\right)$ for all $\mu \geqslant \alpha$ and for all $h < 0$, a^-_α is a lower bound to the set $\bigcup\limits_{\beta\geqslant\alpha} C_{L(0)}\left(\phi_{f^-_\beta}\right) \cup C_{L(0)}\left(\phi_{f^+_\beta}\right)$. And since

$$\lim_{h\to 0^-} \min\left(\phi_{f^-_\gamma}(h), \phi_{f^+_\gamma}(h)\right) = a^-_\alpha,$$

then

$$a^-_\alpha \in \bigcup_{\beta\geqslant\alpha} C_{L(0)}\left(\phi_{f^-_\beta}\right) \cup C_{L(0)}\left(\phi_{f^+_\beta}\right)\,,$$

consequently

$$a^-_\alpha = \inf \bigcup_{\beta\geqslant\alpha} \left(C_{L(0)}\left(\phi_{f^-_\beta}\right) \cup C_{L(0)}\left(\phi_{f^+_\beta}\right)\right)\,. \tag{2.36}$$

On the other hand, assume that the limit (2.31) exists for some $\gamma \geqslant \alpha$ and

$$a^-_\alpha = \inf \bigcup_{\beta\geqslant\alpha} \left(C_{L(0)}\left(\phi_{f^-_\beta}\right) \cup C_{L(0)}\left(\phi_{f^+_\beta}\right)\right)\,. \tag{2.37}$$

If $c \in \mathbb{R}$ is not in $\bigcup\limits_{\beta\geqslant\alpha} \left(C_{L(0)}\left(\phi_{f^-_\beta}\right) \cup C_{L(0)}\left(\phi_{f^+_\beta}\right)\right)$ then, for any sequence $\{h_n\}_{n=1}^\infty$ of negative numbers going to zero, it follows that $\phi_{f^-_\beta}(h_n)$ and $\phi_{f^+_\beta}(h_n)$ do not converge to c for all $\beta \geqslant \alpha$. In other words, there exists $\varepsilon > 0$ such that, for all $n_0 \in \mathbb{N}$, there is some $n > n_0$ satisfying

$$\left|\phi_{f^-_\beta}(h_n) - c\right| > \varepsilon \quad \text{and} \quad \left|\phi_{f^+_\beta}(h_n) - c\right| > \varepsilon$$

for all $\beta \geqslant \alpha$. And it implies that

$$\left|\min\left(\phi_{f^-_\beta}(h_n), \phi_{f^+_\beta}(h_n)\right) - c\right| > \varepsilon$$

for all $\beta \geqslant \alpha$. Consequently, from the continuity w.r.t. $\alpha \in [0,1]$,

$$\left|\inf_{\beta\geqslant\alpha} \min\left(\phi_{f^-_\beta}(h_n), \phi_{f^+_\beta}(h_n)\right) - c\right| = \left|\min\left(\phi_{f^-_{\beta_n}}(h_n), \phi_{f^+_{\beta_n}}(h_n)\right) - c\right| > \varepsilon\,.$$

Given a sequence $\{h_n\}_{n=1}^{\infty}$ of real negative numbers such that $h_n \to 0$ and

$$\lim_{n\to\infty} \inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h_n), \phi_{f_\beta^+}(h_n)\right) = c \in \widetilde{\mathbb{R}},$$

it follows, from the previous argument, that

$$c \in \bigcup_{\beta\geqslant\alpha} C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)$$

if $c \in \mathbb{R}$ and the idea is similar to $c = \pm\infty$. Then, it can be concluded that $c \geqslant a_\alpha^-$. Assuming $c > a_\alpha^-$ and taking $\varepsilon = (c - a_\alpha^-)/2$, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} n \geqslant n_0 \Rightarrow & \left|\inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h_n), \phi_{f_\beta^+}(h_n)\right) - c\right| < \varepsilon \\ & \Rightarrow \inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h_n), \phi_{f_\beta^+}(h_n)\right) - a_\alpha^- > \varepsilon\,. \end{aligned}$$

At the same time, there exists $n_1 \in \mathbb{N}$ such that

$$n > n_1 \Rightarrow \left|\min\left(\phi_{f_\gamma^-}(h_n), \phi_{f_\gamma^+}(h_n)\right) - a_\alpha^-\right| < \varepsilon\,,$$

which implies that $\inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h_n), \phi_{f_\beta^+}(h_n)\right) > \min\left(\phi_{f_\gamma^-}(h_n), \phi_{f_\gamma^+}(h_n)\right)$ for $n > \max(n_0, n_1)$, which is a contradiction. It means that, independent of the sequence $\{h_n\}$ taken, we must have

$$\lim_{n\to\infty} \inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h_n), \phi_{f_\beta^+}(h_n)\right) = a_\alpha^-.$$

Therefore, the limit (2.30) exists and

$$\lim_{h\to 0^-} \inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h), \phi_{f_\beta^+}(h)\right) = a_\alpha^-.$$

■

Lemma 2.17. [29] *Let $F : I \to \mathbb{R}_{\mathcal{F}}^c$ be given. If If*

$$\bigcup_{\beta\in[0,1]} \left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right)$$

is an upper bounded set, then for each $\alpha \in [0,1]$ the limit

$$\lim_{h\to 0^-} \sup_{\beta\geqslant\alpha} \max\left(\phi_{f_\beta^-}(h), \phi_{f_\beta^+}(h)\right) = a_\alpha^+ \tag{2.38}$$

exists if and only if exists $\gamma \geqslant \alpha$ such that the limit

$$\lim_{h\to 0^-} \max\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) \tag{2.39}$$

exists and satisfies

$$\lim_{h\to 0^-} \max\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) = a_\alpha^+ = \sup \bigcup_{\beta\geqslant\alpha} \left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right).$$

Proof. The proof follows similarly to Lemma 2.16. ■

Lemma 2.18. [29] *Let $F : I \to \mathbb{R}_{\mathcal{F}}^c$ be given. If*

$$\bigcup_{\beta\in[0,1]} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right)$$

is a lower bounded set, then for each $\alpha \in [0,1]$ the limit

$$\lim_{h\to 0^+} \inf_{\beta\geqslant\alpha} \min\left(\phi_{f_\beta^-}(h), \phi_{f_\beta^+}(h)\right) = a_\alpha^- \tag{2.40}$$

exists if and only if there exists $\gamma \geqslant \alpha$ such that the limits

$$\lim_{h\to 0^+} \min\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) \tag{2.41}$$

exist and satisfy

$$\lim_{h\to 0^+} \min\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) = a_\alpha^- = \inf \bigcup_{\beta\geqslant\alpha} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right).$$

Proof. The proof follows similarly to Lemma 2.16. ■

Lemma 2.19. [29] *Let $F : I \to \mathbb{R}_{\mathcal{F}}^c$ be given. If*

$$\bigcup_{\beta\in[0,1]} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right)$$

is an upper bounded set, then for each $\alpha \in [0,1]$ the limit

$$\lim_{h\to 0^+} \sup_{\beta\geqslant\alpha} \max\left(\phi_{f_\beta^-}(h), \phi_{f_\beta^+}(h)\right) = a_\alpha^+ \tag{2.42}$$

exists if and only if there exists $\gamma \geqslant \alpha$ such that

$$\lim_{h\to 0^+} \max\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) \tag{2.43}$$

exists and satisfies

$$\lim_{h\to 0^+} \max\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) = a_\alpha^+ = \sup \bigcup_{\beta\geqslant\alpha} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right).$$

Proof. The proof follows similarly to Lemma 2.16. ■

The following theorem is the main result of this section. It is an immediate consequence of the above lemmas, and it provides a complete characterization of the g-differentiability of fuzzy-number-valued functions.

Theorem 2.20. [29] *Let $F : I \to \mathbb{R}_{\mathcal{F}}^{c}$ with $t_0 \in I \subset \mathbb{R}$, and suppose that*

$$\bigcup_{\beta\in[0,1]} \left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right)$$

and

$$\bigcup_{\beta\in[0,1]} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right)$$

are bounded sets. The function F is g-differentiable at $t_0 \in I$ with $\left[F_g'(t_0)\right]_\alpha = [a_\alpha^-, a_\alpha^+]$ if and only if there exist $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \geqslant \alpha$ such that

$$\begin{aligned} \lim_{h\to 0^-} \min\left(\phi_{f_{\gamma_1}^-}(h), \phi_{f_{\gamma_1}^+}(h)\right) &= \lim_{h\to 0^+} \min\left(\phi_{f_{\gamma_3}^-}(h), \phi_{f_{\gamma_3}^+}(h)\right) = a_\alpha^- , \\ \lim_{h\to 0^-} \max\left(\phi_{f_{\gamma_2}^-}(h), \phi_{f_{\gamma_2}^+}(h)\right) &= \lim_{h\to 0^+} \max\left(\phi_{f_{\gamma_4}^-}(h), \phi_{f_{\gamma_4}^+}(h)\right) = a_\alpha^+ , \end{aligned} \tag{2.44}$$

where

$$\begin{aligned} a_\alpha^- &= \inf \bigcup_{\beta\geqslant\alpha} \left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right) \\ &= \inf \bigcup_{\beta\geqslant\alpha} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right) , \\ a_\alpha^+ &= \sup \bigcup_{\beta\geqslant\alpha} \left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right) \\ &= \sup \bigcup_{\beta\geqslant\alpha} \left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right) , \end{aligned} \tag{2.45}$$

for each $\alpha \in [0,1]$ and, considering γ_1, γ_2, γ_3 and γ_4 as functions of $\alpha \in [0,1]$, then

a) the limit $\lim_{h\to 0^-} \min\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right)$ *exists uniformly in $\gamma \in \gamma_1([0,1])$;*

b) the limit $\lim_{h\to 0^-} \max\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right)$ *exists uniformly in $\gamma \in \gamma_2([0,1])$;*

c) the limit $\lim_{h\to 0^+} \min\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right)$ *exists uniformly in $\gamma \in \gamma_3([0,1])$;*

d) the limit $\lim_{h\to 0^+} \max\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right)$ *exists uniformly in $\gamma \in \gamma_1([0,1])$.*

Proof. The proof follows straightforwardly from Lemmas 2.16, 2.17, 2.18 and 2.19. ■

Theorem 2.20 describes the g-differentiability of a fuzzy process $F : I \to \mathbb{R}_{\mathcal{F}}^{c}$ in terms of the behavior of its endpoint functions. Namely, it assures that under a regular behavior of f_α^- and f_α^+ measured by their slope functions, the g-derivative of F at some $t_0 \in I$ exists. Additionally, it relates the limits of the slope functions to the endpoints of the g-derivative of F at t_0, $F_g'(t_0)$. Also, it provides an accurate depiction of the g-derivative of a given fuzzy function. Nevertheless, it does not represent a practical method to calculate $F_g'(t_0)$. For this reason, some particular cases of its characterization are presented in the following.

Corollary 2.21. *If $F : I \to \mathbb{R}^c_{\mathcal{F}}$ is g-differentiable at $t_0 \in I$, then $F_1(t) = [F(t)]_1$ is gH-differentiable at t_0.*

Proof. The proof follows directly from Theorem 2.20 by considering $\alpha = 1$, and the arguments are analogous to the proof of Theorem 2 in [36]. ■

Theorem 2.22. [29] *Let $F : I \to \mathbb{R}^c_{\mathcal{F}}$ be given and $t_0 \in I$. Denote the subset $D \subset [0,1]$ by*

$$D = \left\{ \gamma \in [0,1] \;\middle|\; F_\gamma(\cdot) = [F(\cdot)]_\gamma \text{ is gH-differentiable at } t_0 \right\}.$$

If $1 \in D$,

$$\begin{aligned} \bigcup_{\beta \geqslant \alpha} \left(C_{L(0)}\left(\phi_{f^-_\beta}\right) \cup C_{L(0)}\left(\phi_{f^+_\beta}\right)\right) &\subset \overline{\operatorname{conv} \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)}, \\ \bigcup_{\beta \geqslant \alpha} \left(C_{R(0)}\left(\phi_{f^-_\beta}\right) \cup C_{R(0)}\left(\phi_{f^+_\beta}\right)\right) &\subset \overline{\operatorname{conv} \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)}, \end{aligned} \tag{2.46}$$

for all $\alpha \in [0,1]$ and F_γ is gH-differentiable at $t_0 \in I$ uniformly in $\gamma \in D$. Then F is g-differentiable at t_0 and

$$[F'_g(t_0)]_\alpha = \overline{\operatorname{conv} \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)} \tag{2.47}$$

for all $\alpha \in [0,1]$.

Proof. Defining $a^-_\alpha = \inf \overline{\operatorname{conv} \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)}$, it follows from the properties of closure and convex hull that $a^-_\alpha = \inf \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)$. Note that a^-_α is either an isolated point or a limit point of $\cup F'_\gamma(t_0)$. The second case will be considered and the first one follows directly.

Thus, for any $\varepsilon > 0$, there exists $x \in \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)$ such that $x - a^-_\alpha < \varepsilon$. Consequently, denoting $F'_\gamma(t_0) = [z^-_\gamma, z^+_\gamma]$, there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} z^-_{\gamma_n} = a^-_\alpha$. In other words, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|z_{\gamma_n} - a^-_\alpha| < \varepsilon/3$ for all $n > n_0$.

Moreover, we have that $\lim\limits_{h\to 0} \min\left(\phi_{f^-_\gamma}(h), \phi_{f^+_\gamma}(h)\right) = z^-_\gamma$ uniformly in $\gamma \in D$. Consequently, given a sequence $\{h_n\}_{n=1}^\infty$ such that $h_n \to 0$, it follows that, for all $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $\left|\min\left(\phi_{f^-_\gamma}(h_n), \phi_{f^+_\gamma}(h_n)\right) - z^-_\gamma\right| < \varepsilon/3$ for all $n > n_1$ and for all $\gamma \in D$.

Furthermore, since $\{\gamma_n\}$ is a bounded sequence, there exists a convergent subsequence $\{\gamma_{n_k}\}_{k=1}^\infty$ that goes to some $\sigma \geqslant \alpha$. And since $\min\left(\phi_{f^-_\alpha}(\cdot), \phi_{f^+_\alpha}(\cdot)\right)$ is continuous with respect to $\alpha \in [0,1]$, it follows that

$$\lim_{k\to\infty} \left(\phi_{f^-_{\gamma_{n_k}}}(h), \phi_{f^+_{\gamma_{n_k}}}(h)\right) = \left(\phi_{f^-_\sigma}(h), \phi_{f^+_\sigma}(h)\right).$$

That is, for all $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ depending on h such that

$$\left|\min\left(\phi_{f^-_{\gamma_{n_k}}}(h), \phi_{f^+_{\gamma_{n_k}}}(h)\right) - \min\left(\phi_{f^-_\sigma}(h), \phi_{f^+_\sigma}(h)\right)\right| < \frac{\varepsilon}{3}$$

for all $k > k_0$.

Consequently, given $\varepsilon > 0$, if $m > n_1$ and $k > \max(n_0, k_0(h_m))$, it follows that

$$\begin{aligned}
&\left|\min\left(\phi_{f_\sigma^-}(h_m), \phi_{f_\sigma^+}(h_m)\right) - a_\alpha^-\right| \\
&\leqslant \left|\min\left(\phi_{f_\sigma^-}(h_m), \phi_{f_\sigma^+}(h_m)\right) - \min\left(\phi_{f_{\gamma_{n_k}}^-}(h_m), \phi_{f_{\gamma_{n_k}}^+}(h_m)\right)\right| \\
&+ \left|\min\left(\phi_{f_{\gamma_{n_k}}^-}(h_m), \phi_{f_{\gamma_{n_k}}^+}(h_m)\right) - z_{\gamma_{n_k}}\right| + |z_{\gamma_{n_k}} - a_\alpha^-| < \varepsilon\,.
\end{aligned}$$

Therefore,

$$a_\alpha^- \in C_{L(0)}\left(\phi_{f_\sigma^-}\right) \cup C_{L(0)}\left(\phi_{f_\sigma^+}\right)$$

and

$$a_\alpha^- \in C_{R(0)}\left(\phi_{f_\sigma^-}\right) \cup C_{R(0)}\left(\phi_{f_\sigma^+}\right)\,.$$

Then

$$\begin{aligned}
a_\alpha^- &\in \bigcup_{\beta \geqslant \alpha}\left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right)\,, \\
a_\alpha^- &\in \bigcup_{\beta \geqslant \alpha}\left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right)\,,
\end{aligned}$$

that is,

$$\begin{aligned}
a_\alpha^- &= \inf \bigcup_{\beta \geqslant \alpha}\left(C_{L(0)}\left(\phi_{f_\beta^-}\right) \cup C_{L(0)}\left(\phi_{f_\beta^+}\right)\right)\,, \\
&= \inf \bigcup_{\beta \geqslant \alpha}\left(C_{R(0)}\left(\phi_{f_\beta^-}\right) \cup C_{R(0)}\left(\phi_{f_\beta^+}\right)\right)\,,
\end{aligned}$$

Consequently, $a_\alpha^- \in \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)$, that is, there exists $\gamma \in D$, $\gamma \geqslant \alpha$, such that

$$\lim_{h \to 0} \min\left(\phi_{f_\gamma^-}(h), \phi_{f_\gamma^+}(h)\right) = a_\alpha^-\,.$$

Some analogous arguments can be used to obtain similar results for

$$a_\alpha^+ = \sup \overline{\text{conv} \bigcup_{\gamma \geqslant \alpha \gamma \in D} F'_\gamma(t_0)}.$$

Lastly, it follows straightforwardly from Theorem 2.20 that F is g-differentiable, and

$$[F'_g(t_0)]_\alpha = [a_\alpha^-, a_\alpha^+] = \overline{\text{conv} \bigcup_{\substack{\gamma \geqslant \alpha \\ \gamma \in D}} F'_\gamma(t_0)}, \quad \forall \alpha \in [0,1]. \tag{2.48}$$

■

Example 2.6. [29] Let the endpoint functions of $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}^c$ be given by

$$\begin{cases} f_\alpha^-(t) = -2\frac{|t|}{1+p(\alpha)} + p(\alpha) - 2 + (\alpha - p(\alpha))e^{-m_1|t|}, \\ f_\alpha^+(t) = 2\frac{|t|}{1+p(\alpha)} - p(\alpha) + 2 - (\alpha - p(\alpha))e^{-m_2|t|}, \end{cases} \tag{2.49}$$

for all $\alpha \in [0,1]$, where m_1 and m_2 are positive real numbers, $m_1 \neq m_2$, and

$$p(\alpha) = \begin{cases} 0.25, & \alpha \in [0, 0.25], \\ 0.5, & \alpha \in (0.25, 0.5], \\ 0.75, & \alpha \in (0.5, 0.75], \\ 1, & \alpha \in (0.75, 1]. \end{cases}$$

Then, the one side derivatives of f_α^- and f_α^+ at $t = 0$ are given by

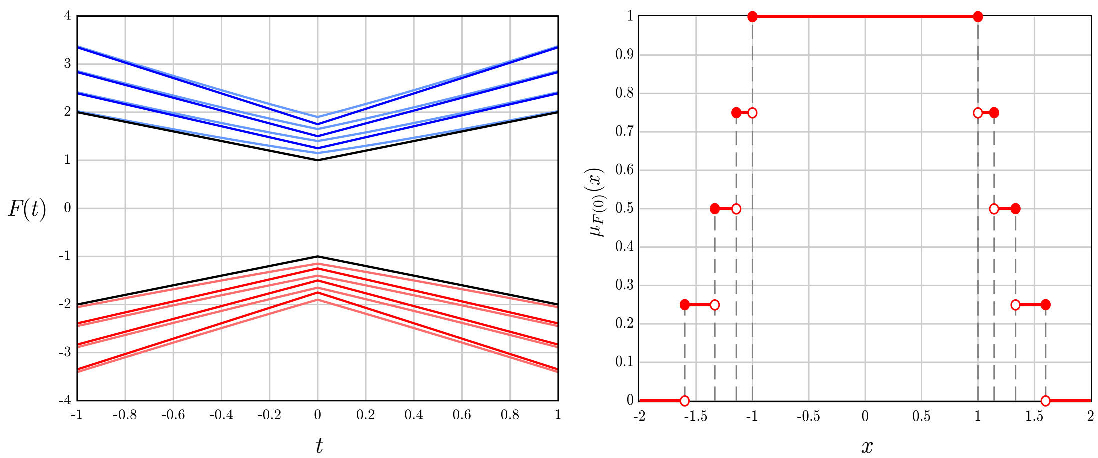
$$\begin{aligned} (f_\alpha^-)'_-(0) &= \frac{2}{1+p(\alpha)} + m_1(\alpha - p(\alpha)) \leqslant \frac{2}{1+p(\alpha)} = (f_{p(\alpha)}^-)'_-(0), \\ (f_\alpha^+)'_-(0) &= \frac{-2}{1+p(\alpha)} - m_2(\alpha - p(\alpha)) \geqslant \frac{-2}{1+p(\alpha)} = (f_{p(\alpha)}^+)'_-(0), \\ (f_\alpha^-)'_+(0) &= \frac{-2}{1+p(\alpha)} - m_1(\alpha - p(\alpha)) \geqslant \frac{-2}{1+p(\alpha)} = (f_{p(\alpha)}^-)'_+(0), \\ (f_\alpha^+)'_+(0) &= \frac{2}{1+p(\alpha)} + m_2(\alpha - p(\alpha)) \leqslant \frac{2}{1+p(\alpha)} = (f_{p(\alpha)}^+)'_-(0). \end{aligned} \tag{2.50}$$

Since m_1 and m_2 are distinct numbers, the interval-valued function F_α is gH-differentiable only for $\alpha \in \{0.25, 0.5, 0.75, 1\}$. Moreover, from (2.50) and Theorem (2.22), F is g-differentiable at $t = 0$ and

$$[F_g'(0)]_\alpha = \left[(f_{p(\alpha)}^+)'_-(0), (f_{p(\alpha)}^-)'_-(0)\right] = \left[(f_{p(\alpha)}^-)'_+(0), (f_{p(\alpha)}^+)'_+(0)\right].$$

Figure 2.6 depicts the endpoints of F, given by (2.49), and $F_g'(0) \in \mathbb{R}_\mathcal{F}$.

Figure 2.6: From left to right: graphical representation of the α-sets of F, given by (2.49), and the graphic representation of the membership function of the fuzzy number $F_g'(0)$.



Source: [29].

The results and examples presented show that, in most cases, the g-derivative does not fully describe the behavior of the fuzzy number-valued function, that is, the lower level-sets in each point can be omitted by higher level-sets of the analyzed function.

The next section provides a study of fuzzy differential equations under the generalized derivatives.

2.5 Consequences on Fuzzy Differential Equations

The main purpose of developing the results that characterize the g-differentiability is to apply them to differential equations and identify the different possibilities provided that can not be obtained with the gH- and gH$*$-differentiabilities.

In this section, we provide a study of the previous properties concerning the gH, gH$*$, and the generalized differentiability of fuzzy functions with applications to the fuzzy Malthusian model. The idea is to illustrate the different behaviors of these derivatives when applied to fuzzy differential equations.

Since any solution of a fuzzy differential equation is differentiable at every point of its domain, it follows from Theorem 2.8 that if $x : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is a solution to the FIVP

$$\begin{cases} x'_{gH}(t) = F\left(t, x(t)\right) \\ \quad x(0) = x_0 \in \mathbb{R}_{\mathcal{F}} \end{cases} \tag{2.51}$$

then the endpoints of x have both left and right derivatives. Moreover, it means that any solution of (2.51) must satisfy the four items of Theorem 2.6. From Theorem 17.9 in [14], the set of points where a real function has both left and right derivatives and they are not equal is countable. Supported by this argument, we shall first consider solutions to FIVP of the form (2.51) given by fuzzy functions with differentiable endpoints.

2.5.1 Malthusian growth and decay models

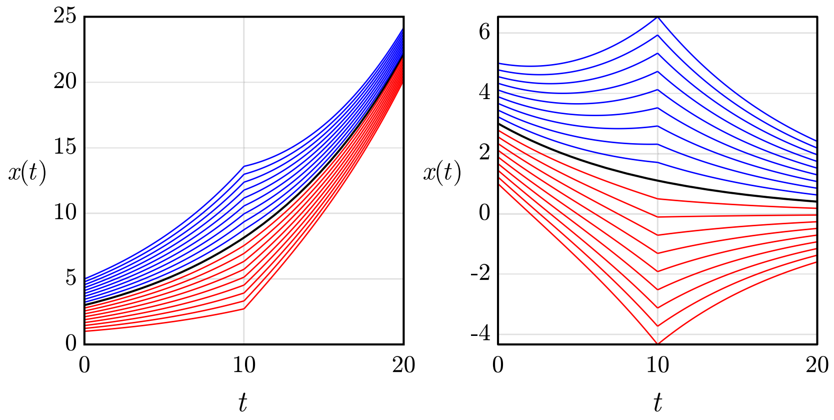
Consider the Malthusian model under the gH-derivative $x'_{gH}(t) = rx(t)$. Then $x(\cdot)$ is the (i)-gH-solution if it satisfies, for all $\alpha \in [0, 1]$, $(x^-_\alpha)'(t) = rx^-_\alpha(t)$ and $(x^+_\alpha)'(t) = rx^+_\alpha(t)$ if $r > 0$, and $(x^-_\alpha)'(t) = rx^+_\alpha(t)$ and $(x^+_\alpha)'(t) = rx^-_\alpha(t)$ if $r < 0$. Similarly, $x(\cdot)$ is the (ii)-gH-solution if $(x^-_\alpha)'(t) = rx^+_\alpha(t)$ and $(x^+_\alpha)'(t) = rx^-_\alpha(t)$ if $r > 0$, and $(x^-_\alpha)'(t) = rx^-_\alpha(t)$ and $(x^+_\alpha)'(t) = rx^+_\alpha(t)$ if $r < 0$.

On the other hand, it is possible to obtain solutions that change the differentiability overtime at the switching points (see Chapter 3). For example, the graphics in Figure 2.7 are solutions of the Malthusian model for $r > 0$ (left) and $r < 0$ (right) with switching point at $t = 10$: they are (i)-gH-differentiable in $[0, 10)$, (ii)-gH-differentiable at $t > 10$ and (iii)-gH-differentiable at $t = 10$.

A solution of an gH-FDE with switching point $T > t_0$ may be obtained by finding the (i)-gH-solution ((ii)-gH-solution) in $[t_0, T]$ and the (ii)-gH-solution ((i)-gH-solution) for $t \geqslant T$.

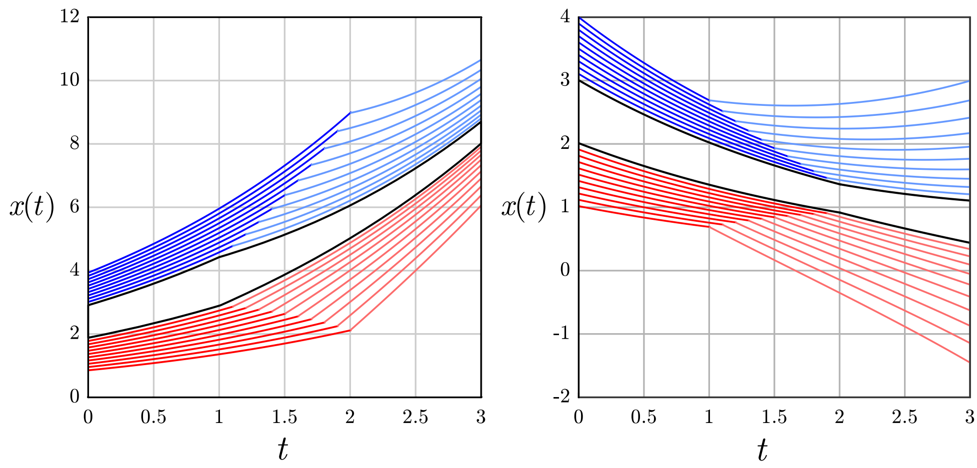
More generally, different solutions can be obtained for the equation $x'_{gH*}(t) = rx(t)$ just by choosing different behaviors for each α-level, what characterizes the main difference when considering the gH$*$-derivative. Figure 2.8 illustrates a possible solution for $r > 0$ (growth model), and another for $r < 0$ (decay model), with different switching points for each $\alpha \in [0, 1]$. To make it easier to understand, we considered a trapezoidal initial condition and diferent shades of blue and red detaching the switching points of each α-set.

Figure 2.7: Graphical representation of the endpoints of gH-differentiable solutions of the malthusian model for $r > 0$ (left) and $r < 0$ (right). Note that the switching point in both cases is given at $t = 10$.



Source: Submitted paper [29].

Figure 2.8: From left to right: graphical representation of the endpoints of gH∗-differentiable solutions to the Malthusian model for $r > 0$ and $r < 0$, respectively. Note that the switching points of the solutions vary on each levelset.



Source: Submitted paper [29].

However, considering the malthusian fuzzy differential equation under g-derivative $x'_g(t) = rx(t)$, it follows that all solutions are gH∗-differentiable. In fact, for $r > 0$, we have from Theorem 1.7 that

$$\inf_{\beta \geqslant \alpha} \min\Big(x^-_\beta(t) - x^-_\beta(t_0), x^+_\beta(t) - x^+_\beta(t_0)\Big) = r\int_{t_0}^{t} x^-_\alpha(s)\,ds, \tag{2.52}$$

$$\sup_{\beta \geqslant \alpha} \max\Big(x^-_\beta(t) - x^-_\beta(t_0), x^+_\beta(t) - x^+_\beta(t_0)\Big) = r\int_{t_0}^{t} x^+_\alpha(s)\,ds, \tag{2.53}$$

for all $\alpha \in [0,1]$ and for all t in a sufficiently small interval on the right of t_0. Given $\alpha \in [0,1)$, we have three possible cases for $x^-_\alpha(t)$:

L1) $x^-_\alpha(t) < x^-_\gamma(t)$ for all $\gamma > \alpha$ and for all $t \in (t_0, t_0 + \delta)$, for some $\delta > 0$;

L2) there exists $\sigma \in (\alpha, 1)$ such that $x^-_\alpha(t) = x^-_\sigma(t)$ and $x^-_\sigma(t) < x^-_\gamma(t)$ for all $\gamma > \sigma$, for all $t \in (t_0, t_0 + \delta)$ for some $\delta > 0$;

L3) $x^-_\alpha(t) = x^-_1(t)$ for all $t \in (t_0, t_0 + \delta)$ for some $\delta > 0$.

If we suppose that two of these cases do not hold, then the remaining one must be valid. Similar cases hold for $x^+_\alpha(t)$:

U1) $x^+_\alpha(t) > x^+_\gamma(t)$ for all $\gamma > \alpha$ and for all $t \in (t_0, t_0 + \delta)$, for some $\delta > 0$;

U2) there exists $\sigma \in (\alpha, 1)$ such that $x^+_\alpha(t) = x^+_\sigma(t)$ and $x^+_\sigma(t) > x^+_\gamma(t)$ for all $\gamma > \sigma$, for all $t \in (t_0, t_0 + \delta)$ for some $\delta > 0$;

U3) $x^+_\alpha(t) = x^+_1(t)$ for all $t \in (t_0, t_0 + \delta)$ for some $\delta > 0$.

In the following reasoning, δ may be taken sufficiently small. The reason for this choice is to avoid that the functions involved do not have different one-side derivatives, satisfying the conditions of Theorem 1.7. If case (L1) is true, we have

$$r \int_{t_0}^{t} x^-_\alpha(s)\, ds < r \int_{t_0}^{t} x^-_\gamma(s)\, ds,$$

for all $\gamma > \alpha$ and for all $t \in (t_0, t_0 + \delta)$. Therefore, from (2.52),

$$\inf_{\beta \geqslant \alpha} \min \left(x^\pm_\beta(t) - x^\pm_\beta(t_0) \right) < \inf_{\beta \geqslant \gamma} \min \left(x^\pm_\beta - (t) - x^\pm_\beta(t_0) \right),$$

for all $\gamma > \alpha$ and for all $t \in (t_0, t_0 + \delta)$. Then,

$$\inf_{\beta \geqslant \alpha} \min \left(x^\pm_\beta(t) - x^\pm_\beta(t_0) \right) = \min \left(x^\pm_\alpha(t) - x^\pm_\alpha(t_0) \right) \tag{2.54}$$

for all $t \in (t_0, t_0 + \delta)$. Analogously, if there exists $\delta > 0$ such that $x^+_\alpha(t) > x^+_\gamma(t)$ for all $\gamma > \alpha$ and for all $t \in (t_0, t_0 + \delta)$, then

$$\sup_{\beta \geqslant \alpha} \max \left(x^\pm_\beta(t) - x^\pm_\beta(t_0) \right) = \max \left(x^\pm_\alpha(t) - x^\pm_\alpha(t_0) \right).$$

Accordingly, if case (L2) holds, it follows that

$$r \int_{t_0}^{t} x^-_\alpha(s)\, ds = r \int_{t_0}^{t} x^-_\sigma(s)\, ds$$

for all $t \in [t_0, t_0 + \delta)$. It means that

$$\inf_{\beta \geqslant \alpha} \min \left(x^\pm_\beta(t) - x^\pm_\beta(t_0) \right) = \inf_{\beta \geqslant \sigma} \min \left(x^\pm_\beta(t) - x^\pm_\beta(t_0) \right).$$

Under the assumptions of σ and (2.54), we have

$$\inf_{\beta \geqslant \sigma} \min \left(x^\pm_\beta(t) - x^\pm_\beta(t_0) \right) = \min \left(x^\pm_\sigma(t) - x^\pm_\alpha(t_0) \right)$$

for all $t \in (t_0, t_0 + \delta)$ so that

$$\begin{aligned} \min\left(x_\alpha^-(t) - x_\alpha^-(t_0), x_\alpha^+(t) - x_\alpha^+(t_0)\right) &= \min\left(x_\sigma^-(t) - x_\alpha^-(t_0), x_\alpha^+(t) - x_\alpha^+(t_0)\right) \\ &= r\int_{t_0}^t x_\sigma^-(s)\,ds = r\int_{t_0}^t x_\alpha^-(s)\,ds\,. \end{aligned}$$

Analogously, if $x_\alpha^+(t) = x_\sigma^+(t)$ for all $t \in [t_0, t_0 + \delta)$ for some $\delta > 0$, where $\sigma > \alpha$ is such that $x_\sigma^+(t) < x_\gamma^+(t)$ for all $\gamma > \sigma$ and for all $t \in (t_0, t_0 + \delta)$, then

$$\min\left(x_\alpha^-(t) - x_\alpha^-(t_0), x_\alpha^+(t) - x_\alpha^+(t_0)\right) = r\int_{t_0}^t x_\alpha^+(s)\,ds.$$

Lastly, we have that

$$\min(x_1^-(t) - x_1^-(t_0), x_1^+(t) - x_1^+(t_0)) = r\int_{t_0}^t x_1^-(s)\,ds$$

for all $t \in [t_0, t_0 + \delta)$, and, if the case (L3) happens, then

$$\begin{aligned} \min(x_\alpha^-(t) - x_\alpha^-(t_0), x_\alpha^+(t) - x_\alpha^+(t_0)) &= \min(x_1^-(t) - x_1^-(t_0), x_1^+(t) - x_1^+(t_0)) \\ &= r\int_{t_0}^t x_1^-(s)\,ds = r\int_{t_0}^t x_\alpha^-(s)\,ds \end{aligned}$$

for all $t \in [t_0, t_0 + \delta)$. The result is analogous for $x_\alpha^+(t)$.

Note that, for any case satisfied by x_α^- and x_α^+, we will have

$$\begin{aligned} \min\left(x_\alpha^-(t) - x_\alpha^-(t_0), x_\alpha^+(t) - x_\alpha^+(t_0)\right) &= r\int_{t_0}^t x_\alpha^-(s)\,ds, \\ \max\left(x_\alpha^-(t) - x_\alpha^-(t_0), x_\alpha^+(t) - x_\alpha^+(t_0)\right) &= r\int_{t_0}^t x_\alpha^+(s)\,ds \end{aligned}$$

for $t > t_0$ sufficiently small and for all $\alpha \in [0, 1]$. And it means that

$$\min\left((x_\alpha^-)'(t), (x_\alpha^+)'(t)\right) = r x_\alpha^-(t) \quad \text{and} \quad \max\left((x_\alpha^-)'(t), (x_\alpha^+)'(t)\right) = r x_\alpha^+(t)$$

since, by the hypothesis of the Theorem 1.7, there is no switching point for the gH-differentiability of $x_\alpha(t) = [x(t)]_\alpha$ for all $\alpha \in [0, 1]$. Consequently, these interval functions satisfy $x'_\alpha(t) = r x_\alpha(t)$, for all $\alpha \in [0, 1]$. Therefore, $x(t)$ is gH$*$-differentiable and it is a solution of $x'_{gH*}(t) = r x(t)$.

Alternatively, consider the fuzzy differential equation given by $x'_g(t) = [a, b] \cdot [x(t)]_1$, where the product considered is the usual product of fuzzy numbers. For the sake of simplicity, we consider that a and b are positive numbers, and the fuzzy initial condition satisfies $x_0^-(t_0) > 0$. Then, one possible solution for this equation that is not gH$*$-differentiable is given by

$$\begin{cases} x_\alpha^-(t) = x_1^-(t_0)e^{a(t-t_0)} + (x_\alpha^-(t_0) - x_1^-(t_0))e^{-\mu(t-t_0)}\,, \\ x_\alpha^+(t) = x_1^+(t_0)e^{b(t-t_0)} + (x_\alpha^+(t_0) - x_1^+(t_0))e^{-\mu(t-t_0)}\,, \end{cases} \tag{2.55}$$

and illustrated in Figure 2.9. After some simple calculations, it is obtained that (2.55) is a non-gH$*$-solution if

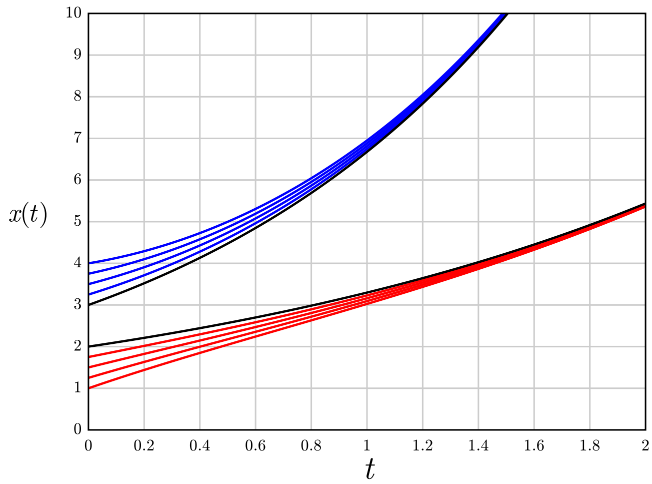
$$0 < \mu \leqslant \frac{bx_1^+(t_0) - ax_1^-(t_0)}{\max(x_1^-(t_0) - x_0^-(t_0), x_0^+(t_0) - x_1^+(t_0))} .$$

Similarly, it is possible to construct solutions of the differential equation $x'(t) = [a,b]x_\gamma(t)$ for some $\gamma \in [0,1)$. From Corollary 2.21, $x_1(t) = [x_1^-(t), x_1^+(t)]$ is gH-differentiable and

$$x_1'(t) = [x_g'(t)]_1 = [a,b]x_\gamma(t) .$$

On the other hand, it may exist $\alpha \in [0,1)$ such that $x_\alpha'(t) \subsetneq x_1'(t)$ and, then, $x_\alpha'(t) \neq [a,b]x_\gamma(t)$.

Figure 2.9: Representation of the α-cuts of the solution (2.55).



Source: Submitted paper [29].

In the following section, we present a result on the non existence of solutions of (2.51) that are not gH$*$-differentiable based on the arguments of the malthusian model.

2.5.2 Generalized result on g-FDE

In this section, based on the assumptions of the malthus model, we obtained a result giving conditions for the non existence of solutions of the g-FDE that are not gH$*$-differentiable.

Considering the Equation (2.51), from Theorem 1.7, any solution must satisfy

$$\inf_{\beta \geqslant \alpha} \min\left(x_\beta^-(t) - x_\beta^-(t_0), x_\beta^+(t) - x_\beta^+(t_0)\right) = \int_{t_0}^t f_\alpha^-(s, x_\alpha^-(s), x_\alpha^+(s))\, ds, \tag{2.56}$$

$$\sup_{\beta \geqslant \alpha} \max\left(x_\beta^-(t) - x_\beta^-(t_0), x_\beta^+(t) - x_\beta^+(t_0)\right) = \int_{t_0}^t f_\alpha^+(s, x_\alpha^-(s), x_\alpha^+(s))\, ds, \tag{2.57}$$

for all $\alpha \in [0,1]$ and for all t in a sufficiently small interval on the right of t_0.

Theorem 2.23. *Let $F : [0,T] \times \mathbb{R}_\mathcal{F} \to \mathbb{R}_\mathcal{F}$ be continuous and lipschitz w.r.t. the second variable, and $T > 0$. For any $t \in [0,T]$ fixed, for all $x \in \mathbb{R}_\mathcal{F}$ and for all $\alpha, \beta \in [0,1]$, $\alpha < \beta$, if $f_\alpha^-(t, x_\alpha^-, x_\alpha^+) <$*

$f^-_\beta(t, x^-_\beta, x^+_\beta)$ and $f^+_\alpha(t, x^-_\alpha, x^+_\alpha) > f^+_\beta(t, x^-_\beta, x^+_\beta)$ whenever $x^-_\alpha < x^-_\beta$ or $x^+_\alpha > x^+_\beta$, then all solutions of (2.51) with differentiable level sets are gH∗-differentiable.

Proof. Let $x(t)$ be any solution of (2.51). Given $\alpha \in [0, 1)$, if $x^-_\alpha(t)$ and $x^+_\alpha(t)$ satisfy (L1) and (U1), respectively, then

$$\int_{t_0}^{t} f^-_\alpha(s, x^-_\alpha(s), x^+_\alpha(s))\, ds < \int_{t_0}^{t} f^-_\gamma(s, x^-_\gamma(s), x^+_\gamma(s))\, ds$$

and

$$\int_{t_0}^{t} f^+_\alpha(s, x^-_\alpha(s), x^+_\alpha(s))\, ds > \int_{t_0}^{t} f^+_\gamma(s, x^-_\gamma(s), x^+_\gamma(s))\, ds$$

for all $\gamma \geqslant \alpha$ and for all $t \in (t_0, t_0 + \delta)$ for some $\delta > 0$. Therefore, from (2.56) and (2.57),

$$\inf_{\beta \geqslant \alpha} \min\left(x^\pm_\beta(t) - x^\pm_\beta(t_0)\right) < \inf_{\beta \geqslant \gamma} \min\left(x^\pm_\beta(t) - x^\pm_\beta(t_0)\right)$$

and

$$\sup_{\beta \geqslant \alpha} \max\left(x^\pm_\beta(t) - x^\pm_\beta(t_0)\right) > \sup_{\beta \geqslant \gamma} \max\left(x^\pm_\beta(t) - x^\pm_\beta(t_0)\right)$$

for all $\gamma > \alpha$ and for all $t \in (t_0, t_0 + \delta)$. Then,

$$\inf_{\beta \geqslant \alpha} \min\left(x^\pm_\beta(t) - x^\pm_\beta(t_0)\right) = \min\left(x^\pm_\alpha(t) - x^\pm_\alpha(t_0)\right)$$

and

$$\sup_{\beta \geqslant \alpha} \max\left(x^\pm_\beta(t) - x^\pm_\beta(t_0)\right) = \max\left(x^\pm_\alpha(t) - x^\pm_\alpha(t_0)\right)$$

for all $t \in (t_0, t_0 + \delta)$. And it means that

$$\int_{t_0}^{t} f^-_\alpha(s, x^-_\alpha(s), x^+_\alpha(s))\, ds = \min\left(x^\pm_\alpha(t) - x^\pm_\alpha(t_0)\right)$$

and

$$\int_{t_0}^{t} f^+_\alpha(s, x^-_\alpha(s), x^+_\alpha(s))\, ds = \max\left(x^\pm_\alpha(t) - x^\pm_\alpha(t_0)\right)$$

for all $t \in (t_0, t_0 + \delta)$. Consequently, $x_\alpha(t) = [x^-_\alpha(t), x^+_\alpha(t)]$ satisfy $x'_\alpha(t) = [F(t, x(t))]_\alpha$ for all $t \in [t_0, t_0 + \delta)$.

Otherwise, suppose that $x^-_\alpha(t)$ satisfies (L1) and $x^+_\alpha(t)$ satisfies (U2), then, from the properties of F, we have that

$$f^-_\alpha(t, x^-_\alpha(t), x^+_\alpha(t)) < f^-_\gamma(t, x^-_\gamma(t), x^+_\gamma(t)),$$
$$f^+_\alpha(t, x^-_\alpha(t), x^+_\alpha(t)) > f^+_\gamma(t, x^-_\gamma(t), x^+_\gamma(t)),$$

for all $\gamma > \alpha$, for all $t \in (t_0, t_0 + \delta)$. Therefore,

$$\int_{t_0}^{t} f^-_\alpha(s, x^-_\alpha(s), x^+_\alpha(s))\, ds < \int_{t_0}^{t} f^-_\gamma(s, x^-_\gamma(s), x^+_\gamma(s)),$$

$$\int_{t_0}^{t} f_\alpha^+(s, x_\alpha^-(s), x_\alpha^+(s))\, ds > \int_{t_0}^{t} f_\gamma^+(s, x_\gamma^-(s), x_\gamma^+(s)),$$

for all $\gamma > \alpha$, for all $t \in (t_0, t_0 + \delta)$. The conclusion follows analogously to the previous case.

Now, if $x_\alpha^-(t)$ satisfy (L2) and $x_\alpha^+(t)$ satisfy (U2), then there exists $\sigma_1, \sigma_2 > \alpha$ such that, for all $t \in (t_0, t_0 + \delta)$, $x_\alpha^-(t) = x_{\sigma_1}^-(t) < x_\gamma^-(t)$ for all $\gamma > \sigma_1$, and $x_\alpha^+(t) = x_{\sigma_2}^+(t) > x_\gamma^- + (t)$ for all $\gamma > \sigma_2$. Consequently,

$$f_\alpha^-(t, x_\alpha^-(t), x_\alpha^+(t)) = f_{\sigma_1}^-(t, x_{\sigma_1}^-(t), x_{\sigma_1}^+(t)) < f_\gamma^-(t, x_\gamma^-(t), x_\gamma^+(t)),$$
$$f_\alpha^+(t, x_\alpha^-(t), x_\alpha^+(t)) = f_{\sigma_2}^+(t, x_{\sigma_2}^-(t), x_{\sigma_2}^+(t)) > f_\gamma^+(t, x_\gamma^-(t), x_\gamma^+(t)),$$

for all $\gamma > \min(\gamma_1, \gamma_2)$, for all $t \in (t_0, t_0 + \delta)$. Once more, the conclusion follows similarly to the first case.

The conclusion for $x_\alpha^-(t)$ and $x_\alpha^+(t)$ satisfying the remaining possible combinations of the cases (L1-L3) and (U1-U3) follows likewise.

We have concluded that every level set $x_\alpha(t) = [x_\alpha^-(t), x_\alpha^+(t)]$ of $x(t)$ satisfy $x'_\alpha(t) = [F(t, x(t))]_\alpha$. Consequently, $x(t)$ is gH$*$-differentiable. ■

Example 2.7. The fuzzy differential equation

$$x'_g(t) = ax(t) + (-b)x(t), \tag{2.58}$$

$a, b > 0$, has only gH$*$-differentiable solutions. Certainly, for any fuzzy number x and for all $\alpha, \beta \in [0, 1]$, $\alpha < \beta$, if $x_\alpha^- < x_\beta^-$ or $x_\alpha^+ > x_\beta^+$, then

$$f_\alpha^-(t, x_\alpha^-, x_\alpha^+) = ax_\alpha^- - bx_\alpha^+ < ax_\beta^- - bx_\beta^+ = f_\beta^-(t, x_\beta^-, x_\beta^+)$$

and

$$f_\alpha^+(t, x_\alpha^-, x_\alpha^+) = ax_\alpha^+ - bx_\alpha^- > ax_\beta^+ - bx_\beta^- = f_\beta^+(t, x_\beta^-, x_\beta^+)\,.$$

The solution illustrated in Figure 2.10 has α-cuts with increasing length for $\alpha \leqslant 0.5$ and decreasing length for $\alpha > 0.5$, what shows that it is gH$*$-differentiable, but not gH-differentiable. The dashed line represents the endpoint functions of $F_{0.51}$ and it was added to illustrate the difference of behavior of the level sets.

Example 2.8. The fuzzy Verhulst model

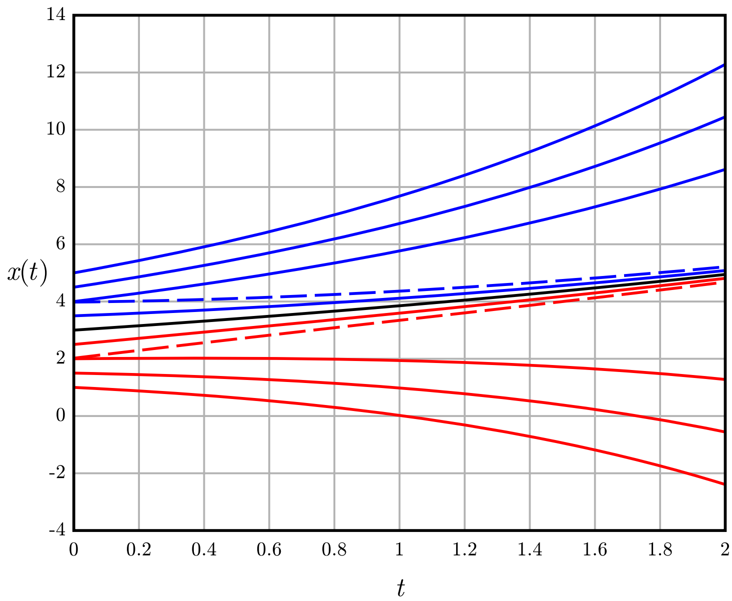
$$x'_g(t) = \frac{r}{K} x(t) \odot (K -_g x(t)), \tag{2.59}$$

$r, K > 0$, does not have non-gH$*$-differentiable solutions, where $\odot$ stands for the cross product of fuzzy numbers [4].

If we consider $x_1^- = x_1^+ = x_1$, then

$$x \odot (K -_g x) = (K - x_1)x + x_1(K -_g x) - x_1(K - x_1)\,.$$

Figure 2.10: Graphical depiction of the endpoint functions, where α varies from 0 to 1, of an gH$*$-solution of (2.58) with $a = 0.45$, $b = 0.2$ and initial condition $[x(t_0)]_\alpha = [1 + 2\alpha, 5 - 2\alpha]$.



Source: Elaborated by the author.

Since $K \in \mathbb{R}$, given $\alpha, \beta \in [0, 1]$, $\alpha < \beta$, we have that

$$K - x_\alpha^+ \leqslant K - x_\beta^+ \leqslant K - x_\beta^- \leqslant K - x_\alpha^- .$$

Therefore, $[K -_g x]_\alpha = [K - x_\alpha^+, K - x_\alpha^-]$. Consequently,

$$\begin{aligned} f_\alpha^-(t, x_\alpha^-, x_\alpha^+) =& \frac{r}{K}\Big[\min\Big((K - x_1)x_\alpha^-, (K - x_1)x_\alpha^+\Big) \\ &+ \min\Big(x_1(K - x_\alpha^+), x_1(K - x_\alpha^-)\Big) \quad - x_1(K - x_1)\Big] \end{aligned}$$

and

$$\begin{aligned} f_\alpha^+(t, x_\alpha^-, x_\alpha^+) =& \frac{r}{K}\Big[\max\Big((K - x_1)x_\alpha^-, (K - x_1)x_\alpha^+\Big) \\ &+ \max\Big(x_1(K - x_\alpha^+), x_1(K - x_\alpha^-)\Big) - x_1(K - x_1)\Big] . \end{aligned}$$

Particularly, if $0 < x_1 < K$, then

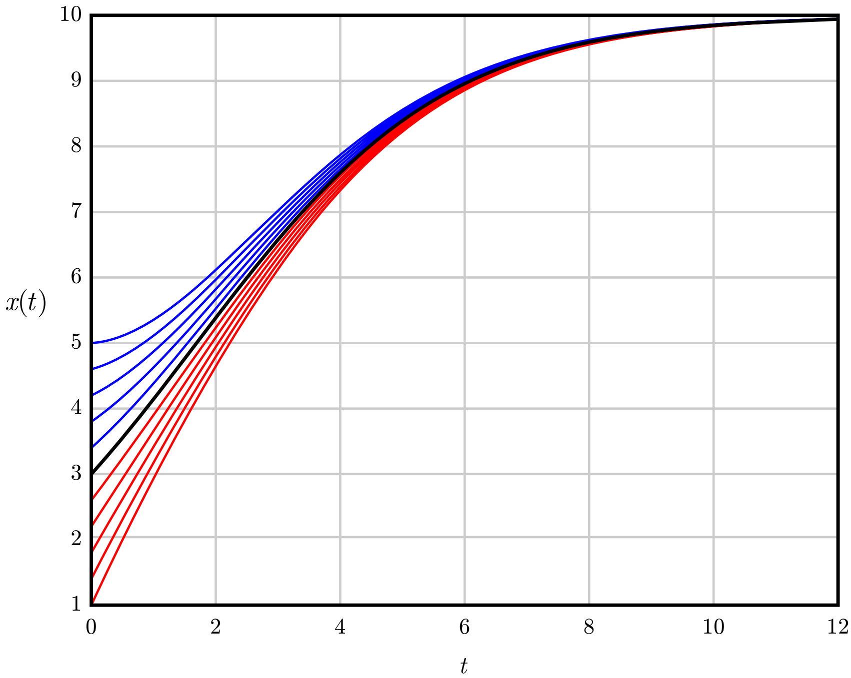
$$\begin{aligned} f_\alpha^-(t, x_\alpha^-, x_\alpha^+) &= \frac{r}{K}\Big[(K - x_1)x_\alpha^- + x_1(K - x_\alpha^+) - x_1(K - x_1)\Big] \\ &< \frac{r}{K}\Big[(K - x_1)x_\beta^- + x_1(K - x_\beta^+) - x_1(K - x_1)\Big] = f_\beta^-(t, x_\beta^-, x_\beta^+), \end{aligned}$$

$$\begin{aligned} f_\alpha^+(t, x_\alpha^-, x_\alpha^+) &= \frac{r}{K}\Big[(K - x_1)x_\alpha^+ + x_1(K - x_\alpha^-) - x_1(K - x_1)\Big] \\ &> \frac{r}{K}\Big[(K - x_1)x_\beta^+ + x_1(K - x_\beta^-) - x_1(K - x_1)\Big] = f_\beta^+(t, x_\beta^-, x_\beta^+), \end{aligned}$$

whenever $x_\alpha^- < x_\beta^-$ or $x_\alpha^+ > x_\beta^+$, for all $\alpha, \beta \in [0, 1]$, $\alpha < \beta$, and for all $t \geqslant t_0$.

Note in Figure 2.11 that, in spite of being a fuzzy version of the logistic model, it does not have the same behavior as the classical problem for all endpoint functions. This can be explained by the

Figure 2.11: Graphical depiction of the endpoint functions, where α varies from 0 to 1, of the (ii)-gH-solution of the fuzzy Verhulst model (2.59) with $r = 0.5$, $K = 10$ and initial condition $[x(t_0)]_\alpha = [1 + 2\alpha, 5 - 2\alpha]$.



Source: Elaborated by the author.

fact that each α-set is obtained by a two equations system, not just a logistic equation, that was obtained by the fuzzy cross-product.

In this chapter we presented results that extend the characterization of the gH-, gH$*$- and g-derivatives. But the generalized Hukuhara differentiability will be even more studied in the next chapter trhough the concept of switching points.

CHAPTER 3

Switching points of gH-differentiable Fuzzy Functions

As pointed out (as examplified) in Chapter 2, the gH-differentiability of a fuzzy function may change at some points in its domain. Therefore, this chapter is dedicated to present the main concepts and results concerning the so called switching points.

In the following, recent results characterizing the switching points of the gH-differentiability of interval-valued functions are presented [37]. Next, these results are used to the generalization for fuzzy number-valued functions.

3.1 Recent results on interval calculus

The study presented in [37] reviews and corrects some previous published results about switching points: Theorems 4.9 and 4.11 in [11], Theorem 28 in [40] and Theorem 41 in [8]. Hence, the new results obtained in [37] are presented below.

A classification of switching points was proposed in [8] (Definition 1.8). However, the next example gives a fuzzy function with a switching point that is both of type I and type II.

Example 3.1. [37] Define $F : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathcal{K}_C$ as follows:

$$F(t) = [f^-(t), f^+(t)] = \begin{cases} \left[-\frac{1}{2}, 0\right], & \text{if } t = 0, \\ \left[-\frac{1}{2}, t^2 \sin\left(\frac{1}{t}\right) + \frac{t}{2}\right], & \text{if } t \neq 0\,. \end{cases}$$

Note that $(f^-)'(t) = 0$ and

$$(f^+)'(t) = \begin{cases} 2t \sin\left(\frac{1}{t}\right) + \frac{1}{2}, & t \neq 0, \\ \frac{1}{2}, & t = 0. \end{cases}$$

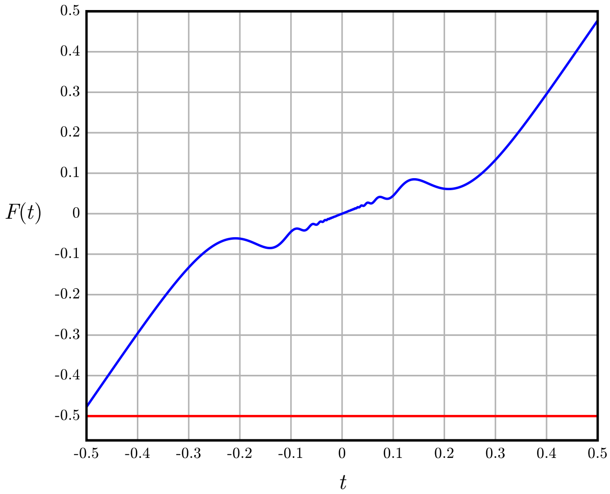
Consequently, F is gH-differentiable in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and

$$F'(t) = \begin{cases} \left[0, \frac{1}{2}\right], & \text{if } t = 0, \\ \left[\min\left\{0, 2t \sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right) + \frac{1}{2}\right\}, \max\left\{0, 2t \sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right) + \frac{1}{2}\right\}\right], & \text{if } t \neq 0, \end{cases}$$

Let $t_n = -\frac{1}{(2n+1)\pi}$ and $t'_n = \frac{1}{2n\pi}$ for all $n \in \mathbb{N}$. We get that $(f^+)'(t_n) = \frac{3}{2}$ and $(f^+)'(t'_n) = -\frac{1}{2}$, and then $F(t_n) = \left[0, \frac{3}{2}\right]$ and $F'(t'_n) = \left[-\frac{1}{2}, 0\right]$ for all $n \in \mathbb{N}$. Consequently, F is (i)-gH-differentiable while it is not (ii)-gH-differentiable at t_n, and F is (ii)-gH-differentiable while it is not (i)-gH-differentiable at t'_n. Thus $t_0 = 0$ is a switching point of type-I. Analogously, defining $s_n = \frac{1}{(2n+1)\pi}$ and $s'_n = -\frac{1}{2n\pi}$ for all $n \in \mathbb{N}$, it follows that $t_0 = 0$ is a switching point of type-II.

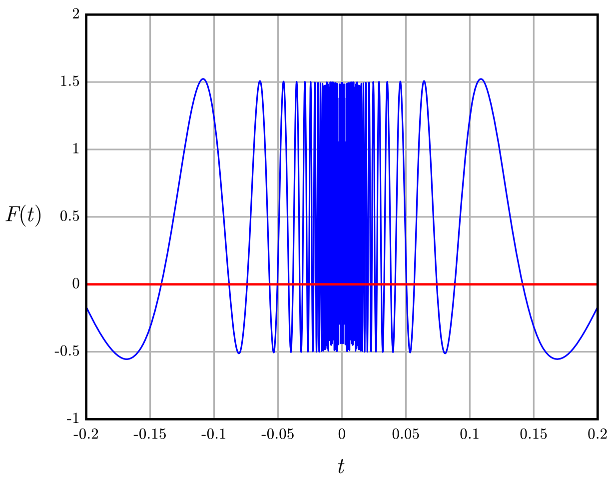
Figures 3.1 and 3.2 illustrate, respectively, the graph of $F(t)$ and $F'(t)$. Note the oscillatory behavior which is directly related to the switching point being of type I and II.

Figure 3.1: Interval-valued function in Example 3.1.



Source: Figure 1 in [37].

Figure 3.2: Derivatives of endpoint functions in Example 3.1.



Source: Figure 2 in [37].

Definition 3.1. [37]. For a gH-differentiable function $F : I \to \mathcal{K}_C$, we say that a point $t_0 \in I$ is

- a switching point of type-I* if it is a switching point of type-I while it is not of type-II,
- a switching point of type-II* if it is a switching point of type-II while it is not of type-I,
- a switching point of mixed-type if it is a switching point of type-I and type-II at the same time.

Lemma 3.1. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function in I and $t_0 \in I$. Then*

- *F is (i)-gH-differentiable at t_0 if and only if $\operatorname{len}(F)(t)$ is differentiable at t_0 and* $\operatorname{len}(F)'(t_0) \geqslant 0$,

- *F is (ii)-gH-differentiable at t_0 if and only if* $\text{len}(F)(t)$ *is differentiable at* t_0 *and* $\text{len}(F)'(t_0) \leqslant 0$*,*

- *F is (iii)-gH-differentiable at t_0 if and only if* $\text{len}(F)'_-(t_0)$ *and* $\text{len}(F)'_+(t_0)$ *exist and satisfy* $\text{len}(F)'_-(t_0) = -\text{len}(F)'_+(t_0) \geqslant 0$*,*

- *F is (iv)-gH-differentiable at t_0 if and only if* $\text{len}(F)'_-(t_0)$ *and* $\text{len}(F)'_+(t_0)$ *exist and satisfy* $\text{len}(F)'_-(t_0) = -\text{len}(F)'_+(t_0) \leqslant 0$*.*

For the switching points of type-I* and type-II*, we have the following result.

Theorem 3.2. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function and $t_0 \in I$. Then*

- *if t_0 is a switching point of type-I*, then there exists a $\delta > 0$ such that F is (i)-gH-differentiable in $(t_0 - \delta, t_0)$ or (ii)-gH-differentiable in $(t_0, t_0 + \delta)$;*

- *if t_0 is a switching point of type-II*, then there exists a $\delta > 0$ such that F is (ii)-gH-differentiable in $(t_0 - \delta, t_0)$ or (i)-gH-differentiable in $(t_0, t_0 + \delta)$.*

The next result about switching points of mixed-type follows straightfowardly from Definitions 1.8 and 3.1.

Theorem 3.3. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function and $t_0 \in I$. Then t_0 is a switching point of mixed-type if and only if for any $\delta > 0$ there exist points $t_1, s_1 \in (t_0 - \delta, t_0)$ and $t_2, s_2 \in (t_0, t_0 + \delta)$ such that*

- *at t_1 F is (i)-gH-differentiable while it is not (ii)-gH-differentiable, and at t_2 F is (ii)-gH-differentiable while it is not (i)-gH-differentiable, and*

- *at s_1 F is (ii)-gH-differentiable while it is not (i)-gH-differentiable, and at s_2 F is (i)-gH-differentiable while it is not (ii)-gH-differentiable.*

The next theorem establishes the relation between the switching point of a gH-differentiable interval-valued function and the type of gH-differentiability at this point.

Theorem 3.4. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function and $t_0 \in I$. If F is only (iii)- or only (iv)-gH-differentiable at t_0, then t_0 is a switching point for the gH-differentiability of F.*

Remark 3.1. At a switching point, the type of gH-differentiability for the interval-valued function does not have to be the (iii)- or (iv)-type, which can be seen in Example 3.1 (F is (i)-gH-differentiable at $t = 0$).

The function F being (iii)-differentiable at t_0 does not mean that t_0 must be a switching point of type-I*. Similarly, if F is (iv)-differentiable at t_0 does not imply that t_0 is a switching point of type-II*.

Example 3.2. [37] Define $F : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathcal{K}_C$, $F(t) = [f^-(t), f^+(t)]$, as follows (see Fig. 3.3)

$$f^-(t) = \begin{cases} -1 & \text{if } t = 0, \\ t^2\left(2 + \sin\left(\frac{1}{t}\right)\right) - 1 - \frac{t}{2} & \text{if } t < 0, \\ t^2\left(2 + \sin\left(\frac{1}{t}\right)\right) - 1 + \frac{t}{2} & \text{if } t > 0, \end{cases}$$

and

$$f^+(t) = \begin{cases} -1 & \text{if } t = 0, \\ -t^2\left(2 + \sin\left(\frac{1}{t}\right)\right) + 1 + \frac{t}{2} & \text{if } t < 0, \\ -t^2\left(2 + \sin\left(\frac{1}{t}\right)\right) + 1 - \frac{t}{2} & \text{if } t > 0. \end{cases}$$

Then we get that $(f^-)'_-(0) = -\frac{1}{2}$, $(f^+)'_-(0) = \frac{1}{2}$, $(f^-)'_+(0) = \frac{1}{2}$ and $(f^+)'_+(0) = -\frac{1}{2}$, which implies that F is (iii)-gH-differentiable at 0 and $F'_{gH}(0) = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Whenever $t \neq 0$, we have (see Fig. 3.4)

$$(f^-)'(t) = \begin{cases} 2t(2 + \sin\left(\frac{1}{t}\right)) - \cos\left(\frac{1}{t}\right) - \frac{1}{2} & \text{if } t < 0, \\ 2t(2 + \sin\left(\frac{1}{t}\right)) - \cos\left(\frac{1}{t}\right) + \frac{1}{2} & \text{if } t > 0, \end{cases}$$

and

$$(f^+)'(t) = \begin{cases} -2x(2 + \sin\left(\frac{1}{x}\right)) + \cos\left(\frac{1}{t}\right) + \frac{1}{2} & \text{if } t < 0, \\ -2x(2 + \sin\left(\frac{1}{x}\right)) + \cos\left(\frac{1}{t}\right) - \frac{1}{2} & \text{if } t > 0. \end{cases}$$

Taking $t_n = -\frac{1}{2n\pi}$ and $t'_n = \frac{1}{(2n+1)\pi}$ for all $n \in \mathbb{N}$, we get

$$F'_{gH}(t_n) = \left[-\frac{3}{2} - \frac{2}{n\pi}, \frac{3}{2} + \frac{2}{n\pi}\right] = [(f^-)'(t_n), (f^+)'(t_n)]$$

and

$$F'_{gH}(t'_n) = \left[-\frac{3}{2} - \frac{4}{(2n+1)\pi}, \frac{3}{2} + \frac{4}{(2n+1)\pi}\right] = [(f^+)'(t_n), (f^-)'(t_n)].$$

Setting $s_n = -\frac{1}{2n\pi}$ and $s'_n = -\frac{1}{(2n+1)\pi}$ for all $n \geqslant 2$, we have

$$F'_{gH}(s_n) = \left[\frac{2}{n\pi} - \frac{1}{2}, \frac{1}{2} - \frac{2}{n\pi}\right] = [(f^-)'(s_n), (f^+)'(s_n)]$$

and

$$F'_{gH}(s'_n) = \left[\frac{4}{(2n+1)\pi} - \frac{1}{2}, \frac{1}{2} - \frac{4}{(2n+1)\pi}\right] = [(f^+)'(s'_n), (f^-)'(s'_n)].$$

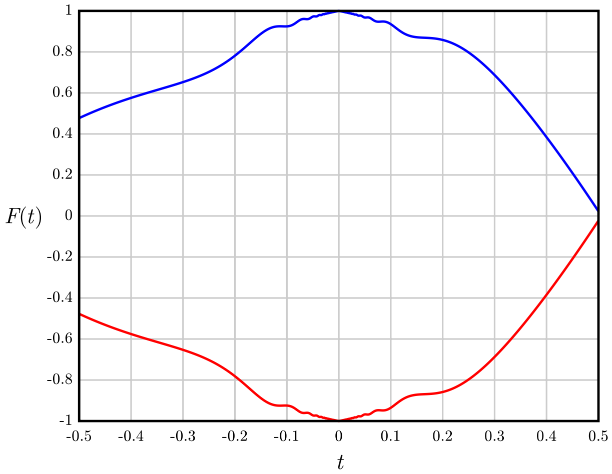
Consequently, by Theorem 3.3, $t_0 = 0$ is a switching point of mixed-type.

Observe that $t_0 = 0$ is a switching point of the function F in Example 3.1, but it is not an extremum point of the length function. The following result relates the switching points of a gH-differentiable interval-valued function and the strict extremum points of its length function.

Theorem 3.5. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function. If t_0 is a strict local stremum point of* $\text{len}(F)$*, then t_0 is a switching point for the gH-differentiability of F.*

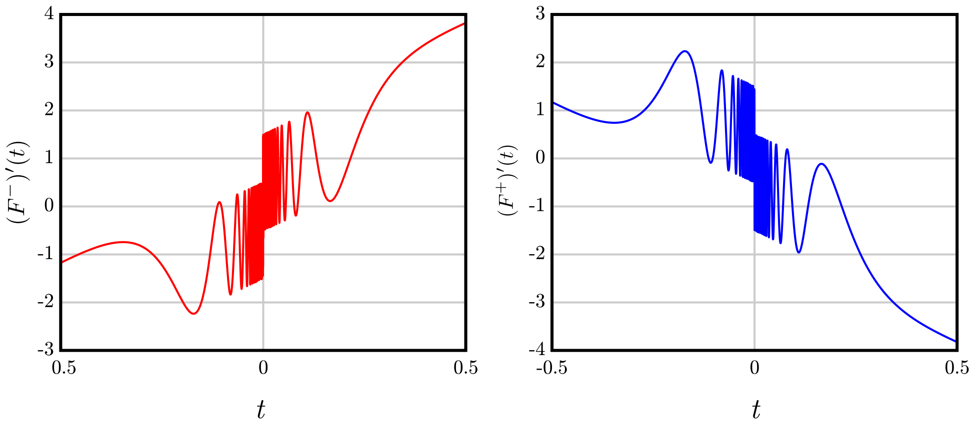
Remark 3.2. A strict local maximum point of the length function is not necessarily a switching point

Figure 3.3: Interval-valued function in Example 3.2.



Source: Figure 6 in [37].

Figure 3.4: Derivatives of lower and upper endpoint functions, respectively, whenever $t \neq 0$ in Example 3.2.



Source: Figures 7 and 8 in [37].

of type-I*. Similarly, a strict local minimum point of $\text{len}(F)$ does not have to be a switching point of type-II*. In Example 3.2, the point $t_0 = 0$ is a switching point of mixed-type for the function F and it is a strict local maximum point of $\text{len}(F)$.

Definition 3.2. [37]. For a gH-differentiable function $F : I \to \mathcal{K}_C$, we say that a point $t_0 \in I$ is

- a switching point of class-I if there exists a $\delta > 0$ such that F is (i)-gH-differentiable in $(t_0 - \delta, t_0)$ while it is not (i)-gH-differentiable in $(t_0, t_0 + \delta)$, and F is (ii)-gH-differentiable in $(t_0, t_0 + \delta)$ while it is not (ii)-gH-differentiable in $(t_0 - \delta, t_0)$;

- a switching point of class-II if there exists a $\delta > 0$ such that F is (ii)-gH-differentiable in $(t_0 - \delta, t_0)$ while it is not (ii)-gH-differentiable in $(t_0, t_0 + \delta)$, and F is (i)-gH-differentiable in $(t_0, t_0 + \delta)$ while it is not (i)-gH-differentiable in $(t_0 - \delta, t_0)$.

Note that the switchings points of class-I and class-II are particular cases of type-I* and type II*, respectively. Then, some results concerning the switching points of class-I and class-II are obtained.

Theorem 3.6. [37]. *Any switching point of class-I or class-II of $F : I \to \mathcal{K}_C$ is the critical point of the length function* $\mathrm{len}(F)$.

Theorem 3.7. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function and $t_0 \in I$. If t_0 is a switching point of class-I, then t_0 is a strict local maximum point of* $\mathrm{len}(F)$.

Theorem 3.8. [37]. *Let $F : I \to \mathcal{K}_C$ be a gH-differentiable interval-valued function and $t_0 \in I$. If t_0 is a switching point of class-II, then t_0 is a strict local minimum point of* $\mathrm{len}(F)$.

3.2 Generalization for fuzzy functions

In the following, based on the previous section, similar results for fuzzy number-valued functions are proved. To do so, it is important to take into account the relation between the gH-differentiability of a fuzzy function and the gH-differentiability of its α-cuts as shown in Chapter 2.

Lemma 3.9. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable, we have that t_0 is a switching point of type-I (type-II) if and only if, for some $\sigma \in (0, 1]$, t_0 is a switching point of type-I (type-II) for the gH-differentiability of $F_\alpha = [F(\cdot)]_\alpha$ for all $\alpha \in [0, \sigma)$ (possibly for $\alpha = \sigma$).*

Proof. The following assertion plays an important role in this proof. Suppose that t_0 is a switching point of type-I for the gH-differentiability of F_β for some $\beta \in (0, 1]$, but it is not for any α in a sufficient small neighborhood of β. Then, for each α in such neighborhood of β, it follows that for any neighborhood V of t_0, there exists $t_1, t_2 \in V$, $t_1 < t_0 < t_2$, such that

- at t_1, F_β is (i)-gH-differentiable while it is not (ii)-gH-differentiable, and F_α is not or not only (i)-gH-differentiable; and
- at t_2, F_β is (ii)-gH-differentiable while it is not (i)-gH-differentiable, and F_α is not or not only (ii)-gH-differentiable.

If $\alpha < \beta$, from Theorem 2.6, it is a contradiction to suppose that F_α is not (i)-gH-differentiable at t_1 or that F_α is not (ii)-gH-differentiable at t_2, since F is gH-differentiable in I and F_β is (i)-gH-differentiable at t_1 and (ii)-gH-differentiable at t_2. Yet, if F_α is not only (i)-gH-differentiable at t_1, it means that F_α is gH-differentiable of all types at t_1 and then $F'_\alpha(t_1) \in \mathbb{R}$, what implies that $F'_\beta(t_1) \in \mathbb{R}$ and, consequently, F_β is (ii)-gH-differentiable at t_1, and this is a contradiction. Analogously, supposing that F_α is not only (ii)-gH-differentiable at t_2 also gets a contradiction. Therefore, F_α must have t_0 as a switching point of type-I for all $\alpha < \beta$. Similarly, if we have $\gamma \in (0, 1]$ such that t_0 is not a switching point of type-I for F_γ, then t_0 is not a switching point of type-I for F_α for all $\alpha \in (\gamma, 1]$. The previous argumentations provide the existence of $\sigma \in (0, 1]$ satisfying the statement of the Lemma.

Now, if t_0 is not a switching point of type-I for the gH-differentiability of F_α for all $\alpha \in [0, 1]$, it is easy to see that t_0 is not a switching point of type-I for the gH-differentiability of F. Then, if t_0 is a switching point of F, t_0 has to be a switching point of type-I of F_β for some $\beta \in (0, 1]$ and the first implication is proved.

On the other hand, if t_0 is a switching point of type-I of F_α, for $\alpha \in [0, \sigma)$, then, for any neighborhoood V of t_0, there are $t_1, t_2 \in V$, $t_1 < t_0 < t_2$, such that

- at t_1, F_α is (i)-gH-differentiable while it is not (ii)-gH-differentiable, for all $\alpha \in [0, \sigma)$; and
- at t_2, F_α is (ii)-gH-differentiable while it is not (i)-gH-differentiable, for all $\alpha \in [0, \sigma)$.

Moreover, for $\alpha \in [\sigma, 1]$, F_α is (i)-gH-differentiable at t_1 and (ii)-gH-differentiable at t_2. Consequently, for any neighborhood V of t_0, there exists $t_1, t_2 \in V$, $t_1 < t_0 < t_2$, such that

- at t_1, F is (i)-gH-differentiable while it is not (ii)-gH-differentiable; and
- at t_2, F is (ii)-gH-differentiable while it is not (i)-gH-differentiable.

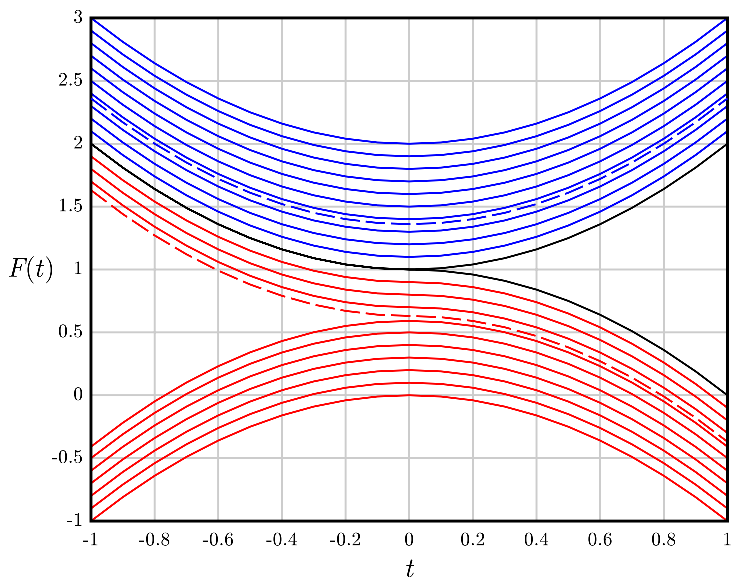
Therefore, t_0 is a switching point of type-I of F. Then the proof is complete. The case for switching point of type-II is analogous. ■

Remark 3.3. Since the level set zero is defined by

$$[F'_{gH}(t_0)]_0 = \overline{\bigcup_{\beta\in[0,1]} [F'_{gH}(t_0)]_\beta},$$

it can not have a different behavior from all other α-cuts, therefore σ must be greater than zero.

Figure 3.5: Graphic representation of the endpoint functions of F in Example 3.3.



Source: Elaborated by the author.

Example 3.3. Let $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be given levelwise by

$$f_\alpha^-(t) = \begin{cases} -t^2 + \alpha, & 0 \leqslant \alpha \leqslant 0.6, \\ -t|t| + \alpha, & 0.6 < \alpha \leqslant 1, \end{cases}$$

and $f_\alpha^+(t) = t^2 - \alpha + 2$ for all $\alpha \in [0, 1]$. Consequently,

$$(f_\alpha^-)'(t) = \begin{cases} -2t, & 0 \leqslant \alpha \leqslant 0.6, \\ -2|t|, & 0.6 < \alpha \leqslant 1, \end{cases}$$

and $(f_\alpha^+)'(t) = 2t$ for all $\alpha \in [0,1]$. Then, F_α is (ii)-gH-differentiable if $0 \leqslant \alpha \leqslant 0.6$ and gH-differentiable of all types if $\alpha > 0.6$ at $t < 0$, and it is (i)-gH-differentiable for all $\alpha \in [0,1]$ at $t > 0$. Therefore, $t_0 = 0$ is a switching point of type-I of F_α for $0 \leqslant \alpha \leqslant 0.6$, but it is not a switching point for $\alpha > 0.6$. In this case, $\sigma = 0.6$ and t_0 is a switching point of the σ-level. Figure 3.5 illustrates and helps understanding the behavior of F. The dashed lines represent the level set $F_{0.61}(t)$ and shows the difference of behavior between the lower and the upper α-cuts of $F(t)$.

Example 3.4. Now, if $F : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ has endpoint functions given by

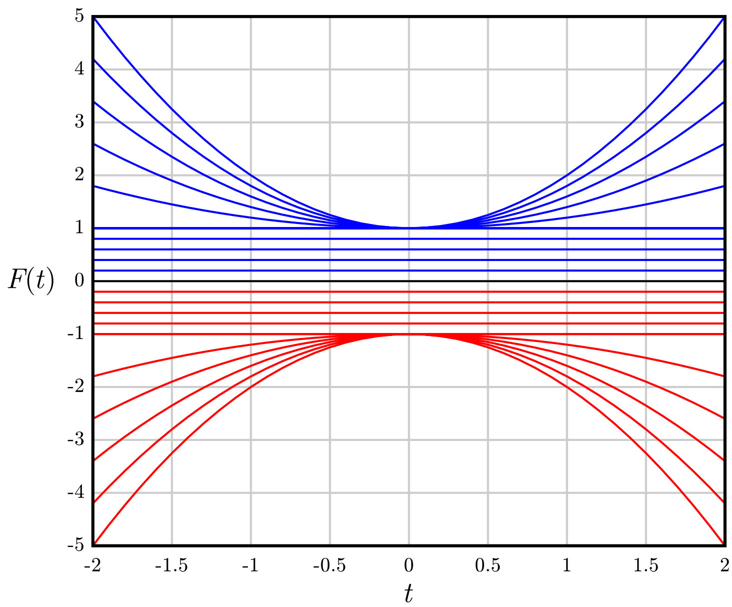
$$f_\alpha^-(t) = \begin{cases} -1 - 2\,(0.5 - \alpha)\,t^2, & 0 \leqslant \alpha \leqslant 0.5 \\ 2\,(-1 + \alpha)\,, & 0.5 < \alpha \leqslant 1, \end{cases}$$

and $f_\alpha^+(t) = -f_\alpha^-(t)$, then

$$(f_\alpha^-)'(t) = \begin{cases} -4\,(0.5 - \alpha)\,t, & 0 \leqslant \alpha \leqslant 0.5 \\ 0, & 0.5 < \alpha \leqslant 1, \end{cases}$$

and $(f_\alpha^+)'(t) = -(f_\alpha^-)'(t)$. Consequently, F_α is (ii)-gH-differentiable if $0 \leqslant \alpha < 0.5$ at $t < 0$, (i)-gH-differentiable if $0 \leqslant \alpha < 0.5$ at $t > 0$, and gH-differentiable of all types if $\alpha \geqslant 0.5$. Accordingly, $t_0 = 0$ is a switching point for F_α if and only if $0 \leqslant \alpha < 0.5$. Here, $\sigma = 0.5$ and $t_0 = 0$ is not a switching point of F_σ. See Figure 3.6.

Figure 3.6: Illustration of the endpoint functions of the fuzzy function F in Example 3.4.



Source: Elaborated by the author.

Similarly to the interval case, the following definition is given.

Definition 3.3. For a gH-differentiable function $F : I \to \mathbb{R}_{\mathcal{F}}$, we say that a point $t_0 \in I$ is

- a switching point of type-I* if it is a switching point of type-I while it is not of type-II,

- a switching point of type-II* if it is a switching point of type-II while it is not of type-I,
- a switching point of mixed-type if it is a switching point of type-I and type-II at the same time.

Lemma 3.10. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be gH-differentiable, it follows that t_0 is a switching point of type-I* (type-II*, mixed-type) if and only if, for some $\sigma \in (0, 1]$, t_0 is a switching point of type-I* (type-II*, mixed-type) for the gH-differentiability of $F_\alpha = [F(\cdot)]_\alpha$ for all $\alpha \in [0, \sigma)$ (possibly for $\alpha = \sigma$).*

Proof. By definition, t_0 is a switching point of type-I* for the gH-differentiability of F if and only if t_0 is type-I while it is not type-II. Then, from Lemma 3.9, t_0 is a switching point of type-I* if and only if, for some $\sigma \in (0, 1]$, t_0 is a switching point of type-I of F_α for all $\alpha \in [0, 1]$, and for all $\sigma' \in (0, 1]$ there exists $\alpha \in [0, \sigma')$ such that t_0 is not a switching point of type-II of F_α. Consequently, from the arguments of the proof of Lemma 3.9, t_0 is not a switching point of type-II for all $\alpha \in [0, 1]$. Therefore, t_0 is a switching point of type-I* if and only if, for some $\sigma \in (0, 1]$, t_0 is a switching point of type-I* of F_α for all $\alpha \in [0, \sigma]$. The proof is analogous for type-II and mixed-type. ■

Lemma 3.11. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function in I and $t_0 \in I$. Then*

- *F is (i)-gH-differentiable at t_0 if and only if $\operatorname{len}_\alpha(F)(t)$ is differentiable at t_0 uniformly in $\alpha \in [0, 1]$ and $\operatorname{len}_\alpha(F)'_\alpha(t_0) \geqslant 0$ for all $\alpha \in [0, 1]$,*
- *F is (ii)-gH-differentiable at t_0 if and only if $\operatorname{len}_\alpha(F)(t)$ is differentiable at t_0 uniformly in $\alpha \in [0, 1]$ and $\operatorname{len}(F)'_\alpha(t_0) \leqslant 0$ for all $\alpha \in [0, 1]$,*
- *F is (iii)-gH-differentiable at t_0 if and only if $\operatorname{len}_\alpha(F)'_-(t_0)$ and $\operatorname{len}_\alpha(F)'_+(t_0)$ exist uniformly in $\alpha \in [0, 1]$ and satisfy $\operatorname{len}_\alpha(F)'_-(t_0) = -\operatorname{len}_\alpha(F)'_+(t_0) \geqslant 0$ for all $\alpha \in [0, 1]$,*
- *F is (iv)-gH-differentiable at t_0 if and only if $\operatorname{len}_\alpha(F)'_-(t_0)$ and $\operatorname{len}_\alpha(F)'_+(t_0)$ exist uniformly in $\alpha \in [0, 1]$ and satisfy $\operatorname{len}_\alpha(F)'_-(t_0) = -\operatorname{len}_\alpha(F)'_+(t_0) \leqslant 0$ for all $\alpha \in [0, 1]$.*

Proof. It follows directly from Theorem 2.8. ■

Theorem 3.12. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function and $t_0 \in I$. Then*

- *if t_0 is a switching point of type-I*, then there exists a $\delta > 0$ such that F is (i)-gH-differentiable in $(t_0 - \delta, t_0)$ or (ii)-gH-differentiable in $(t_0, t_0 + \delta)$;*
- *if t_0 is a switching point of type-II*, then there exists a $\delta > 0$ such that F is (ii)-gH-differentiable in $(t_0 - \delta, t_0)$ or (i)-gH-differentiable in $(t_0, t_0 + \delta)$.*

Proof. From Lemma 3.10, t_0 is a switching point of type-I* of F if and only if, for some $\sigma \in (0, 1]$, t_0 is a switching point of type-I* of F_α for all $\alpha \in [0, \sigma)$. From Theorem 3.2, for each $\alpha \in [0, \sigma)$, there exists $\delta_\alpha > 0$ such that F_α é (i)-gH-differentiable in $(t_0 - \delta_\alpha, t_0)$ or F_α is (ii)-gH-differentiable in $(t_0, t_0 + \delta_\alpha)$. From the gH-differentiability of F, for each $\alpha \in [0, \sigma)$, if F_α is (i)-gH-differentiable in $(t_0 - \delta_\alpha, t_0)$, then F_β is, at least, (i)-gH-differentiable in $(t_0 - \delta_\alpha, t_0)$ for all $\beta \in [0, 1]$. Analagously, if F_α is (ii)-gH-differentiable in $(t_0, t_0 + \delta_\alpha)$, then F_β is, at least, (ii)-gH-differentiable in $(t_0, t_0 + \delta_\alpha)$

for all $\beta \in [0,1]$. Consequently, there exists $\delta > 0$ such that F is (i)-gH-differentiable in $(t_0 - \delta, t_0)$ or (ii)-gH-differentiable in $(t_0, t_0 + \delta)$. The proof is analogous for the second case. ■

Theorem 3.13. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function and $t_0 \in I$. Then t_0 is a switching point of mixed-type if and only if for any $\delta > 0$ there exist points $t_1, s_1 \in (t_0 - \delta, t_0)$ and $t_2, s_2 \in (t_0, t_0 + \delta)$ such that*

- *at t_1 F is (i)-gH-differentiable while it is not (ii)-gH-differentiable, and at t_2 F is (ii)-gH-differentiable while it is not (i)-gH-differentiable, and*
- *at s_1 F is (ii)-gH-differentiable while it is not (i)-gH-differentiable, and at s_2 F is (i)-gH-differentiable while it is not (ii)-gH-differentiable.*

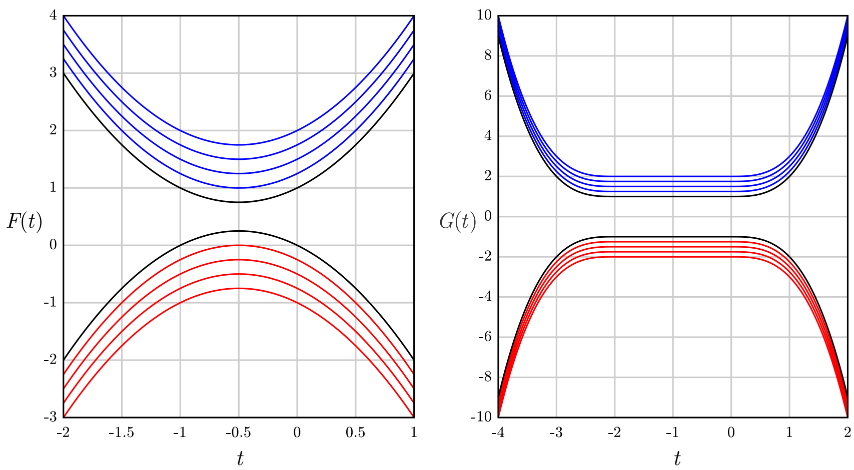
Proof. This result is obtained directly from Definitions 1.8 and 3.3. ■

Theorem 3.14. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function and $t_0 \in I$. If F is only (iii)- or only (iv)-gH-differentiable at t_0, then t_0 is a switching point for the gH-differentiability of F.*

Proof. This proof is obtained from Theorems 3.4 and 2.6. ■

As presented in the next example, we may have a function that is gH-differentiable in all types at t_0 and t_0 is a switching point, and another function that is also gH-differentiable in all types at t_0 but t_0 is not a switching point.

Figure 3.7: Graphic representation of the endpoint functions of the fuzzy-number valued functions F and G in Example 3.5.



Source: Elaborated by the author.

Example 3.5. Let $F, G : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be given by $F(t) = \left[-t^2 - t + \alpha - 1, t^2 + t + 2 - \alpha\right]$, and

$$G(t) = \begin{cases} \left[(2+t)^3 - 2 + \alpha, -(2+t)^3 + 2 - \alpha\right], & \text{if } t \leqslant -2 \\ [\alpha - 2, 2 - \alpha], & \text{if } -2 \leqslant t \leqslant 0 \\ \left[-t^3 - 2 + \alpha, t^3 + 2 - \alpha\right], & \text{if } t \geqslant 0\,. \end{cases}$$

Note that F is gH-differentiable of all types at $t_0 = 0$ and t_0 is a switching point for the gH-differentiability of F. Otherwise, G is gH-differentiable of all types in $[-2, 0]$ and then t_0 is not a switching point of G for any $t_0 \in [-2, 0]$. See Figure 3.7.

Theorem 3.15. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function. If, for some $\sigma \in [0, 1)$, t_0 is a strict local stremum point of $\text{len}_\alpha(F)$ for all $\alpha \in [0, \sigma)$, then t_0 is a switching point for the gH-differentiability of F.*

Proof. Let us suppose that, for some $\beta \in [0, \sigma)$, t_0 is a strict local maximum point of F_β, then there exists $\delta > 0$

$$\begin{cases} \text{len}_\beta(F)'_-(t_0) = -\text{len}_\beta(F)'_+(t_0) \geqslant 0, \\ \text{len}_\beta(F)'(t) > 0, \quad t \in (t_0 - \delta, t_0), \\ \text{len}_\beta(F)'(t) < 0, \quad t \in (t_0, t_0 + \delta). \end{cases}$$

Hence, from Lemma 3.11, for all $\alpha \in [0, 1]$,

$$\begin{cases} \text{len}_\alpha(F)'_-(t_0) = -\text{len}_\alpha(F)'_+(t_0) \geqslant 0, \\ \text{len}_\alpha(F)'(t) > 0, \quad t \in (t_0 - \delta, t_0), \\ \text{len}_\alpha(F)'(t) < 0, \quad t \in (t_0, t_0 + \delta). \end{cases}$$

Therefore, F_α is at least (iii)-gH-differentiable at t_0 for all $\alpha \in [0, 1]$. From Theorem 3.5, t_0 is a switching point for the gH-differentiability of F_α for all $\alpha \in [0, \sigma)$. Consequently, since F_α is at least (iii)-gH-differentiable for $\alpha \in (\sigma, 1]$, t_0 is a switching point of F. The proof is similar if we assume that t_0 is a strict local minimum point of F_β for some $\beta \in [0, \sigma)$. ■

Definition 3.4. For a gH-differentiable function $F : I \to \mathbb{R}_{\mathcal{F}}$, we say that a point $t_0 \in I$ is

- a switching point of class-I if there exists a $\delta > 0$ such that F is (i)-gH-differentiable in $(t_0 - \delta, t_0)$ while it is not (i)-gH-differentiable in $(t_0, t_0 + \delta)$, and F is (ii)-gH-differentiable in $(t_0, t_0 + \delta)$ while it is not (ii)-gH-differentiable in $(t_0 - \delta, t_0)$;

- a switching point of class-II if there exists a $\delta > 0$ such that F is (ii)-gH-differentiable in $(t_0 - \delta, t_0)$ while it is not (ii)-gH-differentiable in $(t_0, t_0 + \delta)$, and F is (i)-gH-differentiable in $(t_0, t_0 + \delta)$ while it is not (i)-gH-differentiable in $(t_0 - \delta, t_0)$.

Theorem 3.16. *Any switching point of class-I or class-II of $F : I \to \mathbb{R}_{\mathcal{F}}$ is the critical point of the length function $\text{len}(F)$.*

Proof. Assume that t_0 is a switching point of class-I. From Definition 3.4 and Lemma 3.11, there exists $\delta > 0$ such that $\text{len}_\alpha(F)'(t) > 0$ if $t \in (t_0 - \delta, t_0)$ and $\text{len}_\alpha(F)'(t) < 0$ if $t \in (t_0, t_0 + \delta)$, for all $\alpha \in [0, 1]$. Therefore, $\text{len}_\alpha(F)'_-(t_0) \geqslant 0$ and $\text{len}_\alpha(F)'_+(t_0) \leqslant 0$ for all $\alpha \in [0, 1]$. Consequently, t_0 is a critical point of $\text{len}_\alpha(F)$ for all $\alpha \in [0, 1]$. The proof is analogous if t_0 is a switching point of class-II. ■

The next Theorems are obtained directly from the proof of Theorem 3.16.

Theorem 3.17. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function and $t_0 \in I$. If t_0 is a switching point of class-I, then t_0 is a strict local maximum point of $\text{len}(F)$.*

Theorem 3.18. *Let $F : I \to \mathbb{R}_{\mathcal{F}}$ be a gH-differentiable fuzzy number-valued function and $t_0 \in I$. If t_0 is a switching point of class-II, then t_0 is a strict local minimum point of $\operatorname{len}(F)$.*

With the ending of this chapter, we give a closure to the fuzzy calculus theory in this work and, in the following, we study the Fuzzy Delay Differential Equations.

CHAPTER 4

gH-Fuzzy Delay Differential Equations

This chapter presents some existence and stability results for delay gH-Fuzzy Differential Equations, that is, delay FDE under gH-derivative. The main references are [27] and [21].

Let $\tau > 0$, $F : [0, \infty) \times \mathcal{C}_\tau \to \mathbb{R}_\mathcal{F}$, $\varphi \in \mathcal{C}_\tau = \mathcal{C}([-\tau, 0]; \mathbb{R}_\mathcal{F})$ and $t_0 \geqslant 0$, we will consider the differential equation

$$\begin{cases} x'_{gH}(t) = F(t, x_t), & t \geqslant t_0, \\ x(t) = \varphi(t - t_0), & t_0 - \tau \leqslant t \leqslant t_0 \, . \end{cases} \tag{4.1}$$

And similarly to the classical case, we denote $x_t(\theta) = x(t + \theta)$, for any continuous function $x : [t_0 - \tau, t_0 + T] \to \mathbb{R}_\mathcal{F}$ and $t \in [t_0, t_0 + T]$.

Definition 4.1. We say that $x(t) = x(t; t_0, \varphi)$ is an (i)-gH-solution of (4.1) if

$$x(t) = x(t_0) + \int_{t_0}^{t} F(s, x_s)\, ds$$

for all $t \in [t_0, T]$ for some $T \leqslant \infty$. Analogously, $x(t) = x(t; t_0, \varphi)$ is a (ii)-gH-solution of (4.1) if

$$x(t) = x(t_0) -_H \int_{t_0}^{t} (-1) F(s, x_s)\, ds$$

for all $t \in [t_0, T]$ for some $T \leqslant \infty$.

For any given interval $I \subset \mathbb{R}$, the metric used for the space $\mathcal{C}(I; \mathbb{R}_\mathcal{F})$ is defined by

$$D(u, v) = \sup_{t \in I} d_\infty(u(t), v(t)) \, , \; \forall u, v \in \mathcal{C}(I; \mathbb{R}_\mathcal{F}) \, .$$

In particular, for all $\varphi, \psi \in \mathcal{C}_\tau$, we have

$$D_\tau(\varphi, \psi) = \sup_{\theta \in [-\tau, 0]} d_\infty(\phi(\theta), \psi(\theta)).$$

4.1 Existence and Uniqueness Theorems

In this section, theorems of existence and unicity are presented for solutions defined in the future and solutions defined for some limited interval. The existence of solutions defined for all $t \geqslant t_0$ is particularly interesting for the stability study that follows.

4.1.1 Solutions defined in the future

For given $a > 0$ and $t_0 \geqslant 0$, E_a is the set of functions $u \in \mathcal{C}([t_0 - \tau, \infty); \mathbb{R}_\mathcal{F})$ such that $u_{t_0} = \varphi$,

$$\sup_{t \geqslant t_0 - \tau} d_\infty(u(t), 0) e^{-at} < +\infty$$

and there exists the Hukuhara difference

$$\varphi(0) -_H \int_{t_0}^{t} (-F(s, u_s))\, ds$$

for all $t \geqslant t_0$. Then, the metric used in E_a is given by

$$D_a(u, v) = \sup_{t \geqslant t_0 - \tau} d_\infty(u(t), v(t)) e^{-at}, \ \forall u, v \in E_a\,.$$

The hypotheses for the existence and unicity theorems presented in [27] are the following:

(h1) there exists $L > 0$ such that

$$d(F(t, \phi), F(t, \psi)) \leqslant L\, D_\tau(\phi, \psi)$$

for all $\phi, \psi \in \mathcal{C}_\tau$ and $t \geqslant 0$;

(h2) $F : [0, \infty) \times \mathcal{C}_\tau \to \mathbb{R}_\mathcal{F}$ is jointly continuous;

(h3) there exist $M, b > 0$ so that

$$d_\infty(F(t, 0), 0) \leqslant M e^{bt}$$

for all $t \geqslant 0$;

(h4) for some $a > \max\{b, L\}$, E_a is non-empty set;

(h5) if $u \in \mathcal{C}([t_0 - \tau, \infty]; \mathbb{R}_\mathcal{F})$ is such that $u_{t_0} = \varphi$ and the Hukuhara difference

$$\varphi(0) -_H \int_{t_0}^{t} (-F(s, u_s))\, ds$$

exists for all $t \geqslant t_0$, then

$$\varphi(0) -_H \int_{t_0}^{t} (-F(s, (Su)_s))\, ds$$

exists for all $t \geqslant t_0$, where, for $t \geqslant t_0$ and $\theta \in [-\tau, 0]$,

$$(Su)_t(\theta) = (Su)(t + \theta) = \begin{cases} \varphi(t + \theta - t_0), & t_0 - \tau \leqslant t + \theta \leqslant t_0, \\ \varphi(0) -_H \displaystyle\int_{t_0}^{t+\theta} (-F(s, u_s))\, ds, & t + \theta \geqslant t_0\,; \end{cases}$$

Remark 4.1. Conditions (h1)-(h3) are the ones imposed in Theorem 4.1 [30] to guarantee the global existence of a unique (i)-gH-solution of (4.1).

The next Lemmas play an important hole on the Existence and Uniqueness Theorem 4.4.

Lemma 4.1. [27]. *If F is jointly continuous, then (E_a, D_a) is a complete metric space.*

Condition (h5) and the choice of the space E_a suggest the definition of a mapping $S : D \subset \mathcal{C}([-\tau, \infty); \mathbb{R}_\mathcal{F}) \to \mathcal{C}([-\tau, \infty); \mathbb{R}_\mathcal{F})$ as

$$(Su)(t) = \begin{cases} \varphi(t - t_0), & t_0 - \tau \leqslant t \leqslant t_0, \\ \varphi(0) -_H \int_{t_0}^{t} (-F(s, u_s))\, ds, & t \geqslant t_0\,, \end{cases} \tag{4.2}$$

provided these H-differences exist.

Lemma 4.2. [27] *If $F : [0, \infty) \times \mathcal{C}_\tau \to \mathbb{R}_\mathcal{F}$ satisfies assumptions (h1)-(h5), then $S(E_a) \subseteq E_a$.*

Lemma 4.3. [27] *If $F : [0, \infty) \times \mathcal{C}_\tau \to \mathbb{R}_\mathcal{F}$ satisfies assumptions (h1)-(h5), then S is a contraction mapping on E_a.*

Theorem 4.4. *[27]. Suppose that the functions $F : [0, \infty) \times \mathcal{C}_\tau \to \mathbb{R}_\mathcal{F}$ and $\varphi \in \mathcal{C}_\tau$ satisfy assumptions (h1)-(h5). Then the fuzzy delay differential equation (4.1) has exactly one (i)-gH-differentiable solution and exactly one (ii)-gH-differentiable solution defined globally on $[t_0 - \tau, \infty)$.*

The following lemma has the purpose to provide tools for an Existence and Uniqueness Theorem based on Lemma 2.2 from [7] and Theorem 9.10 from [4].

Lemma 4.5. *Let $x \in \mathbb{R}_\mathcal{F}$, $[x]_\alpha = [x^-_\alpha, x^+_\alpha]$, be such that x^-_α and x^+_α are differentiable with respect to $\alpha \in [0, 1]$, with x^- strictly increasing and x^+ strictly decreasing on $[0, 1]$ such that there exist constants $c_2 < 0 < c_1$ satisfying $(x^-_\alpha)' \geqslant c_1$ and $(x^+_\alpha)' \leqslant c_2$ for all $\alpha \in [0, 1]$. Let $F : [a, \infty) \to \mathbb{R}_\mathcal{F}$, $[F(t)]_\alpha = [f^-_\alpha(t), f^+_\alpha(t)]$, be continuous with respect to t and have endpoint functions $f^-_\alpha(t)$ and $f^+_\alpha(t)$ with bounded partial derivatives $\dfrac{\partial f^-_\alpha(t)}{\partial \alpha}$ and $\dfrac{\partial f^+_\alpha(t)}{\partial \alpha}$ satisfying*

$$0 \leqslant \int_a^t \frac{\partial f^-_\alpha(s)}{\partial \alpha}\, ds \leqslant c_1 \qquad \textit{and} \qquad 0 \geqslant \int_a^t \frac{\partial f^+_\alpha(s)}{\partial \alpha}\, ds \geqslant c_2\,,$$

for all $\alpha \in [0, 1]$ and every $t \geqslant a$. If

a) $x^-_1 < x^+_1$ and $\displaystyle\int_a^t \mathrm{len}([F(s)]_1)\, ds \leqslant \mathrm{len}([x]_1)$ for all $t \geqslant a$,

or if

b) $x^-_1 = x^+_1$ and $[F(t)]_1$ consists of exactly one element for any $t \geqslant a$,

then the H-difference $x -_H \displaystyle\int_a^t F(s)\, ds$ exists for all $t \geqslant a$.

Proof. To prove the existence of $X -_H \displaystyle\int_a^t F(s)\, ds$, we must get that, for any $t \geqslant a$,

$$\int_a^t f^+_1(s)\, ds - \int_a^t f^-_1(s)\, ds \leqslant x^+_1 - x^-_1 = \mathrm{len}([x]_1),$$

$x^-_\alpha - \int_a^t f^-_\alpha(s)\, ds$ is non-decreasing with respect to $\alpha \in [0, 1]$, and $x^+_\alpha - \int_a^t f^+_\alpha(s)\, ds$ is non-increasing with respect to $\alpha \in [0, 1]$. And this is equivalent to show that, for any $t \geqslant a$,

$$\int_a^t \mathrm{len}([F(s)]_1)\, ds \leqslant \mathrm{len}([x]_1)\,, \tag{4.3}$$

$$(x_\alpha^-)' - \int_a^t \frac{\partial f_\alpha^-(s)}{\partial \alpha}\, ds \geqslant 0\,,\ \forall \alpha \in [0,1]\,, \tag{4.4}$$

and

$$(x_\alpha^+)' - \int_a^t \frac{\partial f_\alpha^+(s)}{\partial \alpha}\, ds \leqslant 0\,,\ \forall \alpha \in [0,1]\,. \tag{4.5}$$

Observe that (4.3) follows straightfowardly from assumptions (a) and (b). Besides, (4.4) e (4.5) can be quickly obtained because

$$\int_a^t \frac{\partial f_\alpha^-(s)}{\partial \alpha}\, ds \leqslant c_1 \leqslant (x_\alpha^-)'\,,\ \forall \alpha \in [0,1],$$

and

$$-\int_a^t \frac{\partial f_\alpha^+(s)}{\partial \alpha}\, ds \leqslant -c_2 \leqslant -(x_\alpha)'\,,\ \forall \alpha \in [0,1].$$

for all $t \geqslant a$. ■

Theorem 4.6. *Suppose that the function $F : [0,\infty) \times \mathcal{C}_\tau \to \mathbb{R}_\mathcal{F}$, $[F(t,\phi)]_\alpha = [f_\alpha^-(t,\phi), f_\alpha^+(t,\phi)]$, satisfy the assumptions (h1)-(h3) and*

- *i) $\varphi_\alpha^-(\theta)$ and $\varphi_\alpha^+(\theta)$ are differentiable with respect to $\alpha \in [0,1]$ for all $\theta \in [-\tau, 0]$, $\varphi_\alpha^-(0)$ is strictly increasing and $\varphi_\alpha^+(0)$ is strictly decreasing on $[0,1]$, such that there exist constants $c_2 < 0 < c_1$ satisfying $\frac{\partial \varphi_\alpha^-(0)}{\partial \alpha} \geqslant c_1$ and $\frac{\partial \varphi_\alpha^+(0)}{\partial \alpha} \leqslant c_2$ for all $\alpha \in [0,1]$;*
- *ii) there exists $u \in \mathcal{C}([t_0-\tau,\infty);\mathbb{R}_\mathcal{F})$ such that $u_{t_0} = \varphi$, $\sup_{t \geqslant t_0 - \tau} d(u(t),0)e^{-at} < \infty$ for some $a > \max\{L,b\}$, $f_\alpha^-, f_\alpha^+ : [0,\infty) \times \mathcal{C}_\tau \to \mathbb{R}$ have bounded partial derivatives, with respect to α, satisfying*

$$c_2 \leqslant -\int_{t_0}^t \frac{\partial f_\alpha^-(s,u_s)}{\partial \alpha}\, ds \leqslant 0 \leqslant -\int_{t_0}^t \frac{\partial f_\alpha^+(s,u_s)}{\partial \alpha}\, ds \leqslant c_1\,,$$

  for all $t \geqslant t_0$, and one of the following situations holds:

  - *a) $\varphi_1^-(0) < \varphi_1^+(0)$ and $\int_{t_0}^t \mathrm{len}([F(s,u_s)]_1)\, ds \leqslant \mathrm{len}([\varphi(0)]_1) = \varphi_1^+(0) - \varphi_1^-(0)$ for all $t \geqslant t_0$, or*
  - *b) $\varphi_1^-(\theta) = \varphi_1^+(\theta)$, for all $\theta \in [-\tau,0]$, and $[F(t,u_t)]_1$ consists of exacly one element for all $t \geqslant t_0$ whenever $[u_t(\theta)]_1$ consists of one element for all $\theta \in [-\tau,0]$;*
- *iii) if $u \in \mathcal{C}([t_0-\tau,\infty);\mathbb{R}_\mathcal{F})$ satisfies (ii), then Su also does, where S is given by (4.2).*

Therefore, the fuzzy delay differential equation (4.1) has exactly one (i)-gH-solution and exactly one (ii)-gH-differentiable solution defined on $[t_0-\tau,\infty)$.

Proof. As mentioned before, the existence and uniqueness of the (i)-gH-solution follows from the hypothesis (h1)-(h3). Now, from itens (i) and (ii) and Lemma 4.5, there exist $a > \max\{L,b\}$ such that E_a is not empty and, then, hypothesis (h4) is satisfied. At last, assumption (h5) follows directly from (iii). Consequently, the conclusion follows from 4.4. ■

4.1.2 Solutions defined on compact sets

For the results that follows, given $T > 0$, we define the set

$$E = \left\{ u \in \mathcal{C}([t_0 - \tau, T]; \mathbb{R}_{\mathcal{F}}) \;\middle|\; u_{t_0} = \varphi,\ \exists\, \varphi(0) -_H \int_{t_0}^{t} (-F(s, u_s))\, ds\ \forall\, t \in [t_0, T] \right\},$$

and consider the following assumptions:

($\overline{\text{h1}}$) hypothesis (h1) is satisfied for $t \in [t_0, T]$;

($\overline{\text{h2}}$) the continuity of F is assumed in $[t_0, T] \times \mathcal{C}_\tau$;

($\overline{\text{h4}}$) there exists $u \in \mathcal{C}([t_0 - \tau, T]; \mathbb{R}_{\mathcal{F}})$ such that $u_{t_0} = \varphi$ and the H-difference

$$\varphi(0) -_H \int_{t_0}^{t} (-F(s, u_s))\, ds$$

exists for all $t \in [t_0, T]$;

($\overline{\text{h5}}$) if $u \in \mathcal{C}([t_0 - \tau, T]; \mathbb{R}_{\mathcal{F}})$ is such that $u_{t_0} = \varphi$ and the H-difference

$$\varphi(0) -_H \int_{t_0}^{t} (-F(s, u_s))\, ds$$

exists for all $t \in [t_0, T]$, then the H-difference

$$\varphi(0) -_H \int_{t_0}^{t} (-F(s, (Su)_s))\, ds$$

exists for all $t \in [t_0, T]$.

In this case, we consider the mapping S defined similarly to (4.2) but only for functions with domain $[-\tau, T]$.

Lemma 4.7. [27] *If $F : [0, T] \times \mathcal{C}_\tau \to \mathbb{R}_{\mathcal{F}}$ satisfies assumptions ($\overline{h2}$), ($\overline{h4}$) and ($\overline{h5}$), then $S(E) \subseteq E$.*

Lemma 4.8. [27] *If $F : [0, T] \times \mathcal{C}_\tau \to \mathbb{R}_{\mathcal{F}}$ satisfies assumptions ($\overline{h1}$), ($\overline{h2}$), ($\overline{h4}$) and ($\overline{h5}$), and $a > 0$ is chosen in such a way that $L < a/(1 - e^{-at})$ (where L is the Lipschitz constant in (h1)), then S is a contraction mapping on (E, D_a).*

Theorem 4.9. [27]. *Suppose that the functions $F : [t_0, T] \times \mathcal{C}_\tau \to \mathbb{R}_{\mathcal{F}}$ and φ satisfy assumptions ($\overline{h1}$), ($\overline{h2}$), ($\overline{h4}$) e ($\overline{h5}$). Then the fuzzy delay differential equation* (4.1) *has exactly one (i)-gH-solution and exactly one (ii)-gH-solution on $[t_0 - \tau, T]$.*

Similarly to the previous case, a new Existence and Uniqueness Theorem for solutions defined on compact sets is proposed based on the results from [4, 7].

Lemma 4.10. [7] *Let $x \in \mathbb{R}_{\mathcal{F}}$, $[x]_\alpha = [x_\alpha^-, x_\alpha^+]$, be such that x_α^- and x_α^+ are differentiable with respect to $\alpha \in [0, 1]$, with x^- strictly increasing and x^+ strictly decreasing on $[0, 1]$ such that there exist* *constants* $c_2 < 0 < c_1$ *satisfying* $(x_\alpha^-)' \geqslant c_1$ *and* $(x_\alpha^+)' \leqslant c_2$ *for all* $\alpha \in [0,1]$. *Let* $F : [a,b] \to \mathbb{R}_{\mathcal{F}}$, $[F(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$, *be continuous with respect to t and have endpoint functions* $f_\alpha^-(t)$ *and* $f_\alpha^+(t)$ *with bounded partial derivatives* $\dfrac{\partial f_\alpha^-(t)}{\partial \alpha}$ *and* $\dfrac{\partial f_\alpha^+(t)}{\partial \alpha}$. *If*

a) $x_1^- < x_1^+$, *or*

b) $x_1^- = x_1^+$ *and* $[F(t)]_1$ *consists of exactly one element for all* $t \geqslant a$.

Then there exists $h > a$ *such that the* $x -_H \int_a^t F(s)\,ds$ *exists for all* $t \in [a,h]$.

Theorem 4.11. *Suppose the function* $F : [t_0,T] \times \mathcal{C}_\tau \to \mathbb{R}_{\mathcal{F}}$, $[F(t,\phi)]_\alpha = [f_\alpha^-(t,\phi), f_\alpha^+(t,\phi)]$, *satisfies the assumptions* ($\overline{h1}$) *and* ($\overline{h2}$), *and*

i) $\varphi_\alpha^-(\theta)$ *and* $\varphi_\alpha^+(\theta)$ *are differentiable with respect to* $\alpha \in [0,1]$ *for all* $\theta \in [-\tau, 0]$, $\varphi_\alpha^-(0)$ *is strictly increasing and* $\varphi_\alpha^+(0)$ *is strictly decreasing on* $[0,1]$ *such that there exist constants* $c_2 < 0 < c_1$ *satisfying* $\frac{\partial \varphi_\alpha^-(0)}{\partial \alpha} \geqslant c_1$ *and* $\frac{\partial \varphi_\alpha^+(0)}{\partial \alpha} \leqslant c_2$ *for all* $\alpha \in [0,1]$;

ii) there exists $u \in \mathcal{C}([t_0 - \tau, T]; \mathbb{R}_{\mathcal{F}})$ *such that* $u_{t_0} = \varphi$, $f_\alpha^-, f_\alpha^+ : [t_0,T] \times \mathcal{C}_\tau \to \mathbb{R}$ *have bounded partial derivatives with respect to* $\alpha \in [0,1]$ *satisfying*

$$\frac{c_2}{T - t_0} \leqslant -\frac{\partial f_\alpha^-(t,u_t)}{\partial \alpha} \leqslant 0 \leqslant -\frac{\partial f_\alpha^+(t,u_t)}{\partial \alpha} \leqslant \frac{c_1}{T - t_0} \tag{4.6}$$

for all $\alpha \in [0,1]$ *and for all* $t \in [t_0,T]$, *and one of the following situations holds:*

a) $\varphi_1^-(0) < \varphi_1^+(0)$ *and* $\mathrm{len}([F(t,u_s)]_1) \leqslant \dfrac{\mathrm{len}([\varphi(0)]_1)}{T - t_0}$ *for all* $t \in [t_0,T]$;

b) $\varphi_1^-(0) = \varphi_1^-(0)$ *and* $[F(t,u_s)]_1$ *consists exactly of one element whenever* $[u_s(\theta)]_1$ *consists of exactly one element for all* $\theta \in [-\tau, 0]$;

iii) if $u \in \mathcal{C}([t_0 - \tau, \infty); \mathbb{R}_{\mathcal{F}})$ *satisfy (ii), then* Su *also does.*

Therefore, the fuzzy delay differential equation (4.1) *has exactly one (i)-gH-solution and exactly one (ii)-gH-solution defined on* $[t_0,T]$.

Proof. This proof is similar to the proof of Theorem 4.6. The existence and unicity of an (i)-gH-solution is obtained from assumptions ($\overline{\text{h1}}$) and ($\overline{\text{h2}}$). Hypothesis ($\overline{\text{h4}}$) follows from items (i) and (ii), and (h5) is provided by item (iii). Consquently, the existence and uniqueness of a (ii)-gH-solutions is obtained. ■

Remark 4.2. The inequalities from (4.6) and item (a) were chosen based on the proof of Lemma 4.10 so that we could get $h = T$.

Subsequently, some pratical observations are made concerning the existence of a (ii)-gH-solution.

4.1.3 Examples

In most cases, it is not easy to find a function $u \in \mathcal{C}([t_0-\tau,\infty);\mathbb{R}_{\mathcal{F}})$ satisfying assumption (h5) or item (iii) from Theorem 4.6 for all $t \geqslant t_0$. Then, to ensure the existence of an (ii)-gH-solution of (4.1), we may analize the parameterized system

$$\begin{cases} (x_\alpha^-)'(t) = f_\alpha^+(t,(x_\alpha^-)_t,(x_\alpha^+)_t)\,, & \\ (x_\alpha^+)'(t) = f_\alpha^-(t,(x_\alpha^-)_t,(x_\alpha^+)_t)\,, & t \geqslant t_0, \\ x_\alpha^-(t) = \varphi_\alpha^-(t-t_0)\,, & \\ x_\alpha^+(t) = \varphi_\alpha^+(t-t_0)\,, & t_0-\tau \leqslant t \leqslant t_0. \end{cases} \tag{4.7}$$

It can be done by proving the existence of $x_\alpha^-(t)$ e $x_\alpha^+(t)$ for each $\alpha \in [0,1]$, and verifing that $[x_\alpha^-(t),x_\alpha^+(t)]$ and $[(x_\alpha^+)'(t),(x_\alpha^-)'(t)]$ are α-sets defining a fuzzy number. Then we obtain the (ii)-gH-solution given levelwise by

$$[x(t)]_\alpha = [x_\alpha^-(t),x_\alpha^+(t)]\,,$$

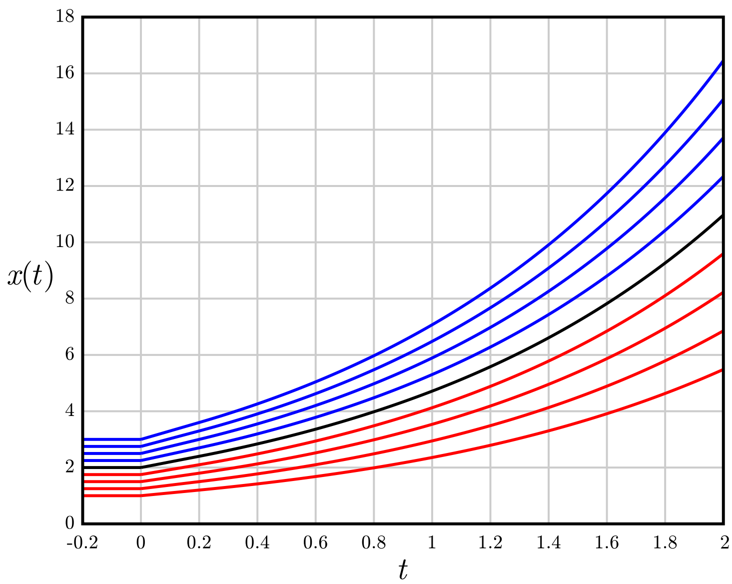
for all $\alpha \in [0,1]$ and $t \geqslant t_0$.

Example 4.1 (Fuzzy malthusian model with delay)**.** Consider the system

$$\begin{cases} x'_{gH}(t) = rx(t-\tau), & t \geqslant t_0, \\ x(t) = \varphi(t-t_0), & t_0-\tau \leqslant t \leqslant t_0\,. \end{cases} \tag{4.8}$$

The existence and uniqueness of the (i)-gH-solution defined in the future is simple to be obtained because $F(t,x_t) = rx_t(-\tau)$ satisfies the hypothesis (h1)-(h3). This solution is illustrated in Figure 4.1.

Figure 4.1: Graphic representation of the (i)-gH-solution of Equation (4.8) with $r=1$, $\tau=0.2$, $t_0=0$ and initial condition $[\varphi(\theta)]_\alpha = [1+\alpha,3-\alpha]$, $\alpha \in [0,1]$, $\theta \in [-\tau,0]$.



Source: Elaborated by the author.

For the (ii)-gH-solution, considering $r > 0$, we have the system

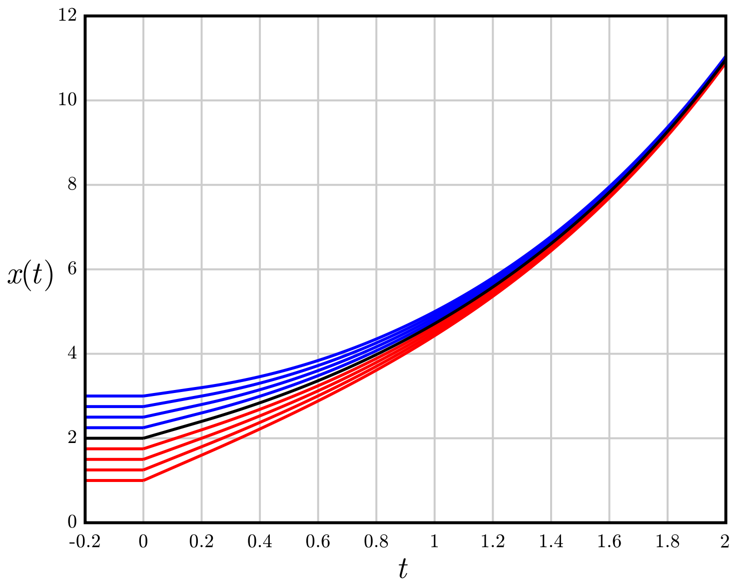
$$\begin{cases} (x_\alpha^-)'(t) = r\, x_\alpha^+(t-\tau)\,, \\ (x_\alpha^+)'(t) = r\, x_\alpha^-(t-\tau)\,, \quad t \geqslant t_0, \\ x_\alpha^-(t) = \varphi_\alpha^-(t-t_0)\,, \\ x_\alpha^+(t) = \varphi_\alpha^+(t-t_0)\,, \qquad t_0-\tau \leqslant t \leqslant t_0, \end{cases} \tag{4.9}$$

for the growth model (Figure 4.2) and

$$\begin{cases} (x_\alpha^-)'(t) = -r\, x_\alpha^-(t-\tau)\,, \\ (x_\alpha^+)'(t) = -r\, x_\alpha^+(t-\tau)\,, \quad t \geqslant t_0, \\ x_\alpha^-(t) = \varphi_\alpha^-(t-t_0)\,, \\ x_\alpha^+(t) = \varphi_\alpha^+(t-t_0)\,, \qquad t_0-\tau \leqslant t \leqslant t_0, \end{cases} \tag{4.10}$$

for the malthusian decay (Figure 4.3). The stacking of the intervals $[x_\alpha^-(t), x_\alpha^+(t)]$ will define a fuzzy number if $x_\beta^-(t) - x_\alpha^-(t) \geqslant 0$ and $x_\alpha^+(t) - x_\beta^+(t) \geqslant 0$ for all $0 \leqslant \alpha < \beta \leqslant 1$, and $x_1^+(t) - x_1^-(t) \geqslant 0$. Since we have a linear function, ensuring these inequalities for systems (4.9) and (4.10) is the same as guaranteeing the existence of positive solutions of the *crisp* equation $y'(t) = -ry(t-\tau)$. Consequently, from the assumptions of Section A.1, there exist a (ii)-gH-solution if $\tau < 1/re$.

Figure 4.2: Graphic representation of the (ii)-gH-solution of Equation (4.9) with $r = 1$, $\tau = 0.2$, $t_0 = 0$ and initial condition $[\varphi(\theta)]_\alpha = [1+\alpha, 3-\alpha]$, $\alpha \in [0,1]$, $\theta \in [-\tau, 0]$.



Source: Elaborated by the author.

In particular, from Section A.1, for $\tau < 1/4r$, the (ii)-gH-solution exists and is well defined in the future if

$$\varphi_\beta^-(0) - \varphi_\alpha^-(0) > \frac{1-\sqrt{1-4r\tau}}{2\tau} \int_{-\tau}^{0} \left(\varphi_\beta^-(\theta) - \varphi_\alpha^-(\theta)\right) d\theta,$$

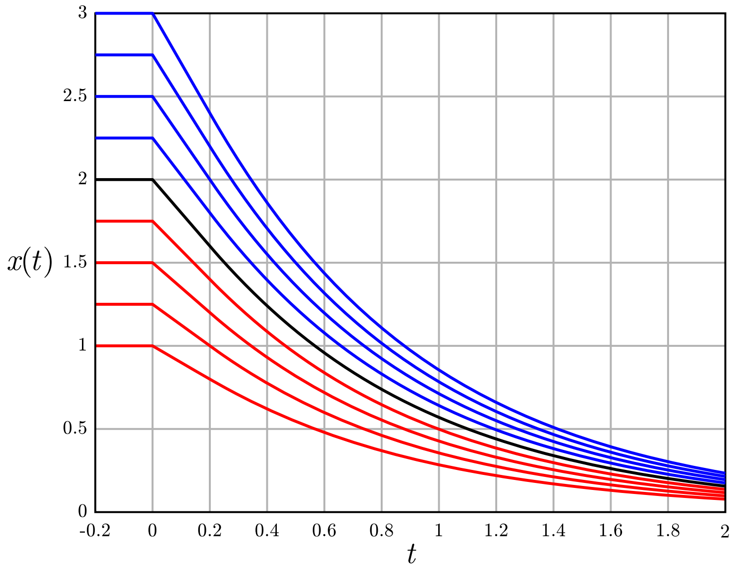
$$\varphi_\alpha^+(0) - \varphi_\beta^+(0) > \frac{1-\sqrt{1-4r\tau}}{2\tau} \int_{-\tau}^{0} \left(\varphi_\alpha^+(\theta) - \varphi_\beta^-(\theta)\right) d\theta$$

for any $\alpha, \beta \in [0,1]$, $\alpha < \beta$, and

$$\varphi_1^+(0) - \varphi_1^-(0) > \frac{1-\sqrt{1-4r\tau}}{2\tau} \int_{-\tau}^{0} \left(\varphi_1^+(\theta) - \varphi_1^-(\theta)\right)\, d\theta\,.$$

Also from Section A.1, if the initial condition is constant, then the (ii)-gH-solution exists and is well defined in the future if $\tau < 1/re$.

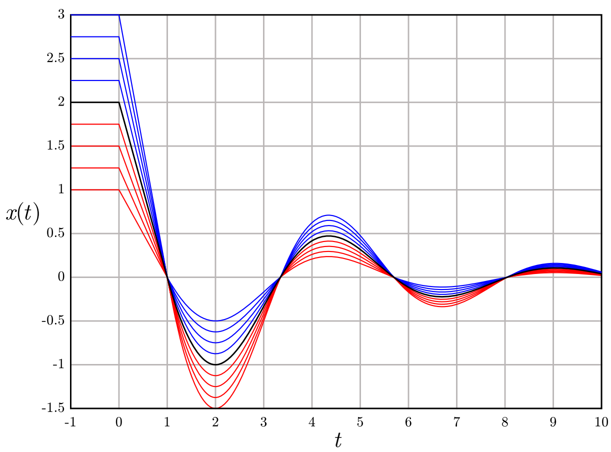
Figure 4.3: Graphic representation of the (ii)-gH-solution of Equation (4.10) with $r = 1$, $\tau = 0.2$, $t_0 = 0$ and initial condition $[\varphi(\theta)]_\alpha = [1+\alpha, 3-\alpha]$, $\alpha \in [0,1]$, $\theta \in [-\tau, 0]$.



Source: Elaborated by the author.

4.1.4 Solutions with switching points

Figure 4.4: Solution with switching points of the malthusian model with $r = 1$, $\tau = 1$, $t_0 = 0$ and initial condition $[\varphi(\theta)]_\alpha = [1+\alpha, 3-\alpha]$, $\alpha \in [0,1]$, $\theta \in [-\tau, 0]$.



Source: Elaborated by the author.

As pointed out in Section A.1, if we have two different solutions of the crisp model $y'(t) =$

$-ry(t-\tau)$ both with constant initial conditions, then the solutions has the same zeros. From this, consider for each $\alpha \in [0,1]$ the system

$$\begin{cases} (u^-)'(t) = -r\,u^-(t-\tau)\,, \\ (u^+)'(t) = -r\,u^+(t-\tau)\,, \quad t \geqslant t_0, \\ u^-(t) = (u^-_\alpha)_0\,, \\ u^+(t) = (u^+_\alpha)_0\,, \qquad\qquad t_0-\tau \leqslant t \leqslant t_0, \end{cases} \tag{4.11}$$

with $r > 0$ and $\tau > 1/re$, and define $x^-_\alpha(t) = \min\left(u^-_\alpha(t), u^-_\alpha(t)\right)$ and $x^+_\alpha(t) = \max\left(u^-_\alpha(t), u^-_\alpha(t)\right)$ where $u^-_\alpha(t)$ and $u^+_\alpha(t)$ are the solutions of (4.11). Then, it follows that $[x(t)]_\alpha = [x^-_\alpha(t), x^+_\alpha(t)]$ is a solution with switching points of the malthusian fuzzy model $x'(t) = rx(t-\tau)$. This solution is illustrated in Figure 4.4. From the stability theory of DDE [19, 20, 34], the solutions oscillate if $\tau > 1/re$, with $x \equiv 0$ assimptotically stable if $\tau < \pi/2r$ and unstable otherwise.

4.2 Stability via Lyapunov-type functions

In this section, we present some stability results from [21] using a Lyapunov type function with three variables. After that, we obtained some stability results based on classical theory of delay differential equations as in [19, 20, 34]. For the following, we assume that $F(t,0) = 0$, that is, equation (4.1) has the solution $x \equiv 0$.

Definition 4.2. The trivial solution $x \equiv 0$ is

i) **stable** if, given $\varepsilon > 0$ and $t_0 \geqslant 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that, if $D_\tau(\varphi, 0) < \delta$, then $d_\infty(x(t;t_0,\varphi),0) < \varepsilon$ for all $t \geqslant t_0$;

ii) **uniformly stable** if, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $D_\tau(\varphi, 0) < \delta$, then $d_\infty(x(t;t_0,\varphi),0) < \varepsilon$ for all $t \geqslant t_0$;

iii) **assimptotically stable** when it is stable and, given $t_0 \geqslant 0$, there exists $\rho = \rho(t_0) > 0$ such that $\lim_{t\to\infty} d_\infty(x(t;t_0;\varphi),0) = 0$ whenever $D_\tau(\varphi,0) < \rho$;

iv) **equiassimptotically stable** if, given $\varepsilon > 0$ and $t_0 \geqslant 0$, there exist $\rho = \rho(t_0) > 0$ and $T = T(\varepsilon) > 0$ such that $d_\infty(x(t;t_0,\varphi),0) < \varepsilon$ if $D_\tau(\varphi,0) < \rho$ e $t \geqslant t_0 + T$;

v) **uniformly assimptotically stable** if it is uniformly stable and there is $\rho > 0$ such that $D_\tau(\varphi,0) < \rho$ implies that, for all $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ satisfying $d_\infty(x(t;t_0,\varphi),0) < \varepsilon$ for all $t \geqslant t_0 + T$;

vi) **exponentially stable** is there exist positive numbers ρ, α and M such that

$$d_\infty(x(t;t_0,\varphi),0) \leqslant Me^{-\alpha(t-t_0)} D_\tau(\varphi,0)$$

para todo $t \geqslant t_0$ whenever $D_\tau(\varphi,0) < \rho$;

vii) **globally assimptotically stable** if condition (iii) is satisfied for $\rho(t_0) = \infty$.

viii) **globally uniformly assimptotically stable** if it is uniformly stable and, given $\rho > 0$ e $\varepsilon > 0$, there exist $T = T(\varepsilon, \rho) > 0$ such that $d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$ if $D_\tau(\varphi, 0) < \rho$ and $t \geqslant t_0 + T$.

It is important to note that, since $x \equiv 0$ is a crisp number and the (i)-gH-solutions have increasing diameter, some concepts of Definition 4.2 are applicable only with respect to the (ii)-gH-solutions. However, if we have a system $y'(t) = G(t, y_t)$ such that $G(t, K) = 0$ for all $t \geqslant 0$ and for some $K \in \mathbb{R}_\mathcal{F}$, that is, $y \equiv K$ is a constant solution, then we can change variables by defining $x_t(\theta) = y_t(\theta) -_{gH} K$ and, then, $F(t, x_t) = G(t, y_t)$ with $F(t, 0) = G(t, K) = 0$ for all $t \geqslant 0$. Consequently, the study of the stability of $y \equiv K$ with respect to any (ii)-gH-solution and (i)-gH-solutions $y(t; t_0, \psi)$ with initial condition satisfying $[\psi(\theta)]_\alpha \subset [K]_\alpha$ may be done by studying the stability of $x \equiv 0$ of $x'(t) = F(t, x_t)$ with respect to the (ii)-gH-solutions. In fact, each (i)-gH-solution $y(t; t_0, \psi)$ of $y'(t) = G(t, x_t)$ with initial condition satisfying $[\psi(\theta)]_\alpha \subset [K]_\alpha$ correspond to a (ii)-gH-solution of $x'(t) = F(t, x_t)$ through the change of variables proposed.

The following example is presented in [21].

Example 4.2. The zero solution of the differential equation

$$x'_{gH}(t) = \frac{r}{(1+t)^2} x(t-1) \tag{4.12}$$

is *uniformly stable*. Indeed, for any (i)- or (ii)-gH-solution $x(t) = x(t; t_0, \varphi)$, it follows that

$$d(x(t), 0) \leqslant d(\varphi(0), 0) + |r| \int_{t_0}^{t} \frac{d(x(s-1), 0)}{(1+s)^2} \, ds$$

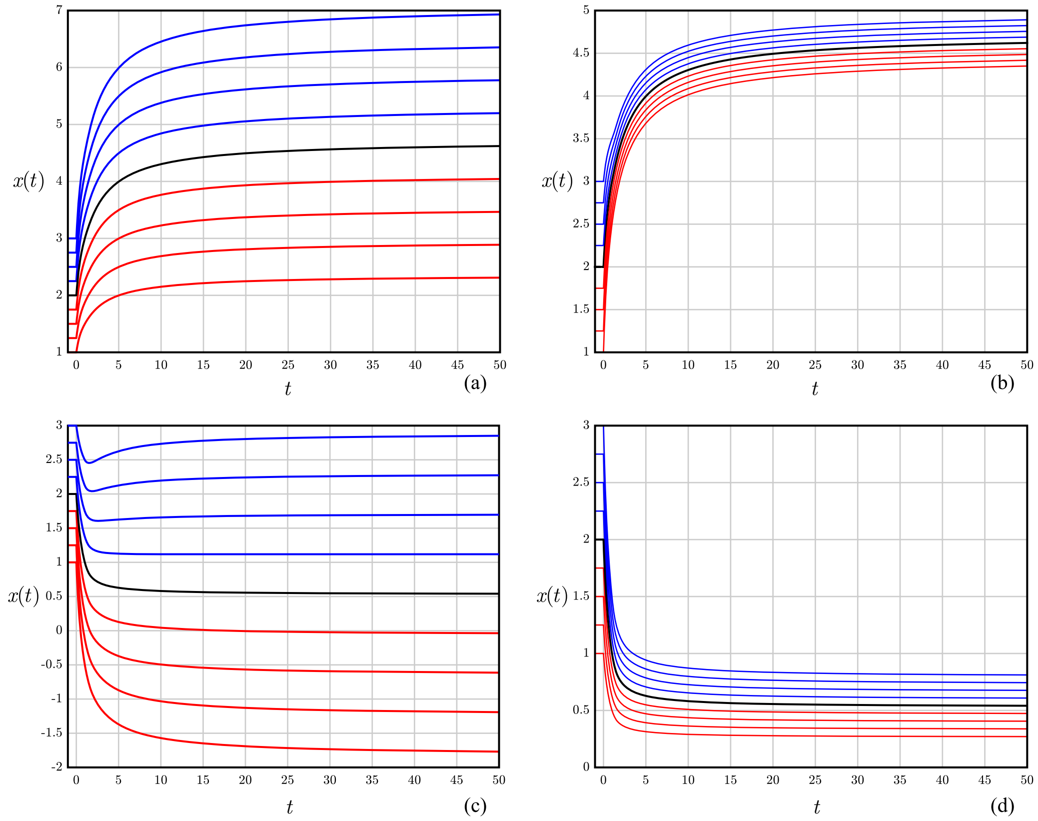
for all $t \geqslant t_0$. Then, let $D_\tau(\varphi, 0) < \delta$ for some $\delta > 0$, then

$$\begin{aligned} d_\infty(x(t), 0) &\leqslant d_\infty(\varphi(0), 0) + |r| \int_{t_0}^{t} \frac{d_\infty(x(s-1), 0)}{(1+s)^2} \, ds \\ &= d_\infty(\varphi(0), 0) + |r| \int_{-1}^{t-t_0-1} \frac{d_\infty(\varphi(\theta), 0)}{(\theta+2+t_0)^2} \, d\theta \\ &< \delta + |r|\delta \frac{t-t_0}{(1+t_0)^2} < \delta + |r|\delta \frac{1}{(1+t_0)^2} < \delta(1+|r|) = \delta a_1 \end{aligned}$$

for all $t \in [t_0 + 1, t_0]$. Now, for $t \in [t_0 + 1, t_0 + 2]$,

$$\begin{aligned} d_\infty(x(t), 0) &\leqslant d_\infty(x(t_0+1), 0) + |r| \int_{t_0+1}^{t} \frac{d_\infty(x(s-1), 0)}{(1+s)^2} \, ds \\ &< \delta a_1 + |r|\delta a_1 \frac{(t-t_0-1)}{(t_0+2)^2} \leqslant \delta a_1 + |r|\delta a_1 \frac{1}{(t_0+2)^2} \\ &< \delta \left(1 + \frac{|r|}{4}\right) a_1 = \delta a_2 \,. \end{aligned}$$

Figure 4.5: Graphic representation of the solutions of (4.12):(a) (i)-gH-solution and $r > 0$; (b) (ii)-gH-solution and $r > 0$; (c) (i)-gH-solution and $r < 0$; and (d) (ii)-gH-solution and $r < 0$.



Source: Elaborated by the author.

Repeating this process successively, we have

$$d_\infty(x(t), 0) < \delta \left(1 + \frac{|r|}{(n+1)^2}\right) a_n = \delta a_{n+1}$$

for all $t \in [t_0 + n, t_0 + n + 1]$. Then, we obtain the sequence

$$a_{n+1} = \left(1 + \frac{|r|}{(n+1)^2}\right) a_n > a_n$$

whose solution satisfies

$$a_n = \prod_{k=1}^{n} \left(1 + \frac{|r|}{k^2}\right).$$

We claim that the sequence (a_n) is convergent. In fact, from Taylor's Theorem (with Lagrage remainder), $f(x) = \ln(1+x) \approx x + \frac{f''(\xi)}{2} x^2$, for some $\xi \in [0, 1]$. Consequently, denoting $a_n = \prod(1 + b_n)$, we have

$$\begin{aligned} \ln(a_n) &= \ln\left(\prod_{k=1}^{n} (1 + b_k)\right) = \sum_{k=1}^{n} \ln(1 + b_k) \\ &= \sum_{k=1}^{n} \left[b_k - \frac{\eta_k}{2} b_k^2\right] = \sum_{k=1}^{n} b_k - \frac{1}{2} \sum_{k=1}^{n} \eta_k b_k^2. \end{aligned}$$

where η_k is such that $\ln(1+b_k) = b_k - \frac{\eta_k}{2} b_k^2$ and, then, $0 < \eta_k \leqslant 1$. Henceforth, the convergence of a_n is obtained from the convergence of $\sum b_k$ and $\sum b_k^2$, and the fact that η_k is a bounded sequence because $0 < \eta_k < 1$. Since $b_n = 1 + |r|/n^2$, the convergence of a_n follows. Let $\lim_{n\to\infty} a_n = c$, from the monotonicy of (a_n), it follows that $a_n < c$ for all $n \in \mathbb{N}$. For this reason, we have that $d_\infty(x(t), 0) < c\delta$. Accordingly, given $\varepsilon > 0$, we can take $\delta = \varepsilon/c$ and, then, $d(x(t; t_0, \varphi), 0) < \varepsilon$ whenever $D_\tau(\varphi, 0) < \delta$, for any $t_0 \geqslant 0$. The uniform stability of $x \equiv 0$ follows.

The uniformly stability of the solution $x \equiv 0$ of (4.12), as it can be seen in the previous calculus, results from the decreasing monotocity and boudedness of the term $1/(1+t)^2$. On the other hand, from the numerical solutions of Figure 4.5, we may note that the trivial solution is not assimptotically stable, but the solutions have assimptotical behavior in all cases illustrated.

Given $B > 0$, consider the sets $S(B) = \{\phi \in \mathcal{C}_\tau \,|\, D_\tau(\phi, 0) \leqslant B\}$ and $\Omega(B) = \{x \in \mathbb{R}_\mathcal{F} \,|\, d(x, 0) \leqslant B\}$. In the definitions of local stability (i)-(vi), we have $\varepsilon, \delta, \rho < B$. Otherwise, for the global stability definitions, $B = \infty$ and ε, δ and ρ may be taken as big as necessary.

Definition 4.3. A function $a(w)$ is said to be of class $\mathcal{K}$ if $a \in \mathcal{C}([0, B), \mathbb{R}^+)$, $a(0) = 0$ and $a(w)$ is strictly increasing with respect to w.

In the following sections, some auxiliary results are necessary to obtain the stability ones. These results follow from the Theorems A.9 and A.10 from the Appendix.

4.2.1 Lyapunov-type functional with three-variables

Definition 4.4. Let $V : [0, \infty) \times \Omega(B) \times S(B) \to \mathbb{R}^+$ be continuous, we define

$$D^+V(t, y, \phi) := \limsup_{h\to 0^+} \frac{1}{h}\Big[V(t+h, y+hf(t, x_t), x_{t+h}) - V(t, y, x_t)\Big], \tag{4.13}$$

as the upper right derivative of $V(t, y, \phi)$ along the solution $x(\cdot; t, \phi)$ of (4.1). Note that, then, $x_t = \phi$ in (4.13).

In the following, we present some comparision results necessary for the stability theorems.

Lemma 4.12. [21] *Assume that*

i) for all $(t, y, \phi), (t, z, \phi) \in \mathbb{R}^+ \times \Omega(B) \times S(B)$, there exists $L > 0$ such that

$$|V(t, y, \phi) - V(t, z, \phi)| \leqslant Ld(y, z)\,;$$

ii) $g \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ is locally Lipschitz in the second variable and satisfies

$$D^+V(t, y, \phi) \leqslant g(t, V(t, y, \phi))\,.$$

If $x(t; t_0, \varphi)$ is any (i)-gH-solution of (4.1) existing on $[t_0, \infty)$ such that $V(t_0, \varphi(0), \varphi) \leqslant u_0$, then

$$V(t, x(t; t_0, \varphi), x_t(t_0, \varphi)) \leqslant r(t; t_0, u_0),\ t \geqslant t_0,$$

where $r(t) = r(t; t_0, u_0)$ is the maximal solution of the scalar differential equation

$$u'(t) = g(t, u(t)), \quad u(t_0) = u_0 \geqslant 0$$

existing on $[t_0, \infty)$.

Proof. Let $m(t) = V(t, x(t), x_t)$, where $x(t) = x(t; t_0, \varphi)$, then

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, x(t+h), x_{t+h}) - V(t, x(t), x_t) \\ &= V(t+h, x(t+h), x_{t+h}) - V(t+h, x(t) + hf(t, x_t), x_{t+h}) \\ &+ V(t+h, x(t) + hf(t, x_t), x_{t+h}) - V(t, x(t), x_t) \\ &\leqslant L d_\infty(x(t+h), x(t) + hf(t, x_t)) \\ &+ V(t+h, x(t) + hf(t, x_t), x_{t+h}) - V(t, x(t), x_t)\,. \end{aligned}$$

Consequently,

$$\begin{aligned} D^+m(t) &\leqslant L \limsup_{h\to 0^+} \frac{1}{h} d(x(t+h), x(t) + hf(t, x_t)) + D^+V(t, x(t), x_t) \\ &\leqslant L \limsup_{h\to 0^+} d_\infty\left(\frac{x(t+h) -_H x(t)}{h}, f(t, x_t)\right) + D^+V(t, x(t), x_t) \\ &= D^+V(t, x(t), x_t) \leqslant g(t, V(t, x(t), x_t)) = g(t, m(t))\,. \end{aligned}$$

Therefore, it follows from Theorem A.9 that $m(t) \leqslant r(t; t_0, u_0)$ for all $t \geqslant t_0$. ■

It is important to note that Corollary 4.1 from [21] does not assume g to be locally Lipschitz with respect to the second variable, but this condition was necessary to the conclusion of Theorem A.9 and it was added in Lemma 4.12.

Corollary 4.13. [21] *Let $V(t, y, \phi)$ be continuous and locally Lipschitz in y. Assume that, for $t > t_0$, $D^+V(t, y, \phi) \leqslant 0$. Let $x(t; t_0, \varphi)$ be any solution of* (4.1) *such that $x(t; t_0, \varphi) \in \Omega(B)$ for $t \in [t_0, T]$. Then*

$$V(t, x(t; t_0, \varphi), x_t(t_0, \varphi)) \leqslant u_0,$$

for all $t \in [t_0, T]$, $u_0 \geqslant 0$, whenever $V(t_0, \varphi(0), \varphi) \leqslant u_0$. Particularly,

$$V(t, x(t; t_0, \varphi), x_t(t_0, \varphi)) \leqslant V(t_0, \varphi(0), \varphi),$$

for all $t \in [t_0, T]$.

Proof. Similarly to the proof of Lemma 4.12, we have $D^+m(t) \leqslant 0$, $t \in [t_0, T]$. Then, it follows from Theorem A.10 that $m(t) \leqslant u_0$ for all $t \in [t_0, T]$. ■

From these comparision results, some stability results follows.

Theorem 4.14. [21] *Assume that*

i) for $(t, y, \phi), (t, z, \phi) \in \mathbb{R}^+ \times \Omega(B) \times S(B)$*, there exists* $L > 0$ *such that*

$$|V(t, y, \phi) - V(t, z, \phi)| \leqslant L d_\infty(y, z)\,;$$

ii) if $(t, y, \phi) \in [0, \infty) \times \Omega(B) \times S(B)$*, then*

$$D^+ V(t, y, \phi) \leqslant 0\,;$$

iii) for $(t, y, \phi) \in [0, \infty) \times \Omega(B) \times S(B)$*, we have that*

$$b(d_\infty(y, 0)) \leqslant V(t, y, \phi) \leqslant a(D_\tau(\phi, 0))\,,$$

where $b, a \in \mathcal{K}$*.*

Then, the trivial solution of (4.1) *is* uniformly stable.

Proof. Let $0 < \varepsilon < B$ and $t_0 \geqslant 0$ be given. Choose $\delta(\varepsilon) > 0$ such that $a(\delta) < b(\varepsilon)$. We claim that, if $D_\tau(\varphi, 0) < \delta$, then $d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$, $t \geqslant t_0$. Suppose that this is not true. Then, there exists a solution $x(t; t_0, \varphi)$ of (4.1) with $D_\tau(\varphi, 0) \leqslant \delta$ such that

$$d_\infty(x(t_1; t_0, \varphi), 0) = \varepsilon$$

and

$$d_\infty(x(t; t_0, \varphi), 0) < \varepsilon, \quad t \in [t_0, t_1].$$

Consequently, from item (iii),

$$V(t_1, x(t_1; t_0, \varphi), x_{t_1}(t_0, \varphi)) \geqslant b(\varepsilon)\,.$$

Furthermore, this means that $x(t; t_0, \varphi) \in \Omega(B)$ for $t \in [t_0, t_1]$. Since

$$V(t_0, \varphi(0), \varphi) \leqslant a(D_\tau(\varphi, 0))\,,$$

from item (iii), and $D^+ V(t, y, \phi) \leqslant 0$, it follows from Corollary 4.13 that

$$V(t, x(t; t_0, \varphi), x_t(t_0, \varphi)) \leqslant a(D_\tau(\varphi, 0)), \quad t \in [t_0, t_1]\,.$$

Hence,

$$b(\varepsilon) \leqslant V(t_1, x(t_1; t_0, \varphi), x_{t_1}(t_0, \varphi)) \leqslant a(D_\tau(\varphi, 0)) < a(\delta)\,,$$

which is a contradiction with the assumption that $a(\delta) < b(\varepsilon)$. Therefore, the trivial solution of (4.1) is uniformly stable. ■

Corollary 4.15. [21] *Let the assumptions of Theorem 4.14 hold except that the condition (iii) is replaced* *by*

$$b(d(y,0)) \leqslant V(t,y,\phi) \leqslant a(t, D_\tau(\phi,0))\,,$$

for $(t,y,\phi) \in [t_0,\infty) \times \Omega(B) \times S(B)$*, where* $b, a(t,\cdot) \in \mathcal{K}$*. Then, the trvial solution of* (4.1) *is* stable.

Proof. This result is obtained similarly to the proof of Theorem 4.14, except that, now, we take $\delta > 0$ such that $a(t_0,\delta) < b(\varepsilon)$. This condition is related to the dependence of δ with respect to t_0. ■

Theorem 4.16. [21] *Let the assumptions of Theorem 4.14 hold except that the condition (ii) is replaced by*

$$D^+V(t,y,\phi) \leqslant -d(D_\tau(\phi,0))\,,$$

with $d \in \mathcal{K}$*. Then the trivial solution of* (4.1) *is* uniformly assimptotically stable.

Proof. The uniform stability follos from Theorem 4.14. Then, from definition 4.2-(ii), for $\varepsilon = B$, we take $\rho = \delta(B)$ and, for any $0 < \varepsilon < B$, $\delta = \delta(\varepsilon)$. Hence, take $T(\varepsilon) = 1 + \dfrac{a(\rho)}{d(\delta)} > 0$ and consider $D_\tau(\varphi,0) < \rho$. Suppose that there exists a solution $x(t) = x(t;t_0,\varphi)$ of (4.1) such that $D_\tau(x_t,0) \geqslant \delta$ for all $t \in [t_0, t_0+T]$, then $d(D_\tau(x_t,0)) \geqslant d(\delta)$. Since $D^+V(t,x(t),x_t) \leqslant -d(D_\tau(x_t,0))$, $t \geqslant t_0$, it follows that $D^+V(t,x(t),x_t) \leqslant -d(\delta)$, $t_0 \leqslant t \leqslant t_0+T$. Consequently,

$$\begin{aligned} V(t,x(t),x_t) &\leqslant V(t_0,\varphi(0),\varphi) - d(\delta)(t-t_0) \\ &\leqslant a(D_\tau(\varphi,0)) - (\delta)(t-t_0) \\ &\leqslant a(\rho) - d(\delta)(t-t_0), \quad t_0 \leqslant t \leqslant t_0+T\,. \end{aligned}$$

Therefore,

$$V(t_0+T, x(t_0+T), x_{t_0+T}) \leqslant a(\rho) - d(\delta)T = -d(\delta) < 0\,,$$

which is a contradiction to the fact that V is positive. It means that there exists $\bar{t} \in [t_0, t_0+T]$ such that $D_\tau(x_{\bar{t}},0) < \delta$ and, from the uniform convergence, $d_\infty(x(t),0) < \varepsilon$ for all $t \geqslant \bar{t}$, that is, $d_\infty(x(t),0) < \varepsilon$ for all $t \geqslant t_0+T$, and the proof is complete. ■

4.2.2 Lyapunov-type functional with two-variables

In the following, we present some new stability results obtained following the ideas for the classical case from [34].

Definition 4.5. Let $B > 0$ and $V : [0,\infty) \times S(B) \to \mathbb{R}^+$ be continuous with $V(t,0) = 0$, we define

$$D^+V(t,\phi) := \limsup_{h\to 0^+} \frac{1}{h}\Big[V(t+h,x_{t+h}) - V(t,x_t)\Big]\,, \tag{4.14}$$

as the upper right derivative of $V(t,\phi)$ along the solution $x(\cdot;t,\phi)$ of (4.1). Note that $x_t = \phi$ in Equation (4.14).

In the following results, V is assumed to be continuous.

Lemma 4.17. *Suppose that, for* $(t, \phi) \in [0, \infty) \times S(B)$*,*

$$D^+V(t, \phi) \leqslant g(t, V(t, \phi))\,,$$

where $g \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ *is locally Lipschitz with respect to the second variable. If* $x(t; t_0, \varphi)$ *any solution of* (4.1) *existing in* $[t_0, \infty)$ *such that* $V(t_0, \varphi) \leqslant u_0$*, then*

$$V(t, x_t(t_0, \varphi)) \leqslant r(t; t_0, u_0), \quad t \geqslant t_0,$$

where $r(t) = r(t; t_0, u_0)$ *is the maximal solution of the scalar differential equation*

$$u'(t) = g(t, u(t)), \quad u(t_0) = u_0 \geqslant 0$$

existing on $[t_0, \infty)$*.*

Proof. Defining $m(t) = V(t, x_t)$, $x(t) = x(t; t_0, \varphi)$, we get

$$m(t + h) - m(t) = V(t + h, x_{t+h}) - V(t, x_t)$$

and, then, $D^+m(t) = D^+V(t, x_t) \leqslant g(t, V(t, x_t)) = g(t, m(t))$. the conclusiom follows from Theorem A.9. ■

Corollary 4.18. *Let* $V(t, \phi)$ *be such that, for* $t > t_0$*,* $D^+V(t, \phi) \leqslant 0$*. Let* $x(t; t_0, \varphi)$ *be any solution of* (4.1) *such that* $x_t(t_0, \varphi) \in S(B)$ *for* $t \in [t_0, T]$*. Then*

$$V(t, x_t(t_0.\varphi)) \leqslant u_0,$$

for all $t \in [t_0, T]$*,* $u_0 \geqslant 0$*, whenever* $V(t_0, \varphi) \leqslant u_0$*. Particularly,*

$$V(t, x_t(t_0, \varphi)) \leqslant V(t_0, \varphi),$$

for all $t \in [t_0, T]$*.*

Proof. Similarly to the proof of 4.17, we have that $D^+m(t) \leqslant 0$, where $m(t) = V(t, x_t(t_0, \varphi))$, and the result follows from Theorem A.10. ■

Now, the stability results follow.

Theorem 4.19. *Suppose that* $D^+V(t, \phi) \leqslant 0$ *and there exists* $b \in \mathcal{K}$ *such that* $b(d_\infty(\phi(0), 0)) \leqslant V(t, \phi)$ *for any* $t \geqslant 0$ *and* $\phi \in S(B)$*. then the trivial solution* $x \equiv 0$ *is* stable.

Proof. From the continuity of V at 0, it follows that, given $t_0 \geqslant 0$ and $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that $V(t_0, \varphi) < b(\varepsilon)$ whenever $D_\tau(\varphi, 0) < \delta$. We claim that

$$D_\tau(\varphi, 0) < \delta \Rightarrow d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$$

for all $t \geqslant t_0$. Indeed, if the opposite holds, there exists $t_1 > t_0$ such that $d_\infty(x(t_1; t_0, \varphi), 0) = \varepsilon$ an, then,

$$V(t_1, x_{t_1}(t_0, \varphi)) \geqslant b(d_\infty(x(t_1), 0)) = b(\varepsilon) > V(t_0, \varphi)\,.$$

However, from Corollary 4.18, it follows that $V(t, x_t) \leqslant V(t_0, \varphi)$ for all $t \in [t_0, t_1]$. Hence, we have a contradiction by supposing the existence of t_1. The proof is complete. ■

Theorem 4.20. *Assume that $D^+V(t, \phi) \leqslant 0$ and that there exist $a, b \in \mathcal{K}$ such that $b(d_\infty(\phi(0), 0)) \leqslant V(t, \phi) \leqslant a(D_\tau(\phi, 0))$ for all $t \geqslant 0$ and $\phi \in S(B)$. Then the trivial solution $x \equiv 0$ is* uniformly stable.

Proof. The proof is similar to Theorem 4.14. ■

Theorem 4.21. *Suppose that there exist $a, b, c \in \mathcal{K}$ such that*

$$b(d_\infty(\phi(0), 0)) \leqslant V(t, \phi) \leqslant a(D_\tau(\phi, 0))$$

and

$$D^+V(t, \phi) \leqslant -c(d_\infty(\phi(0), 0))$$

for all $t \geqslant 0$ and $\phi \in S(B)$. Then the solution $x \equiv 0$ is uniformly assimptotically stable.

Proof. It follows similarly to Theorem 4.16. ■

Theorem 4.22. *Let $B = \infty$ ($S(B) = \mathcal{C}_\tau$). If there exist $a, b, c \in \mathcal{K}$ such that*

$$b(d_\infty(\phi(0), 0)) \leqslant V(t, \phi) \leqslant a(D_\tau(\phi, 0))$$

and

$$D^+V(t, \phi) \leqslant -c(d_\infty(\phi(0), 0))$$

for all $t \geqslant 0$ and $\phi \in \mathcal{C}_\tau$. If $\lim_{r\to\infty} b(r) = \infty$, then the solution $x \equiv 0$ is globally uniformly assimptotically stable.

Remark 4.3. If there were $M > 0$ such that $b(r) < M$ for all $r \geqslant 0$, and $R < \infty$ such that $a(R) = M$, then $a(r) > M$ if $r > R$. Then, given $\varepsilon > 0$, we can not have $\delta > R$ for theuniform stability. It means that, if $D_\tau(\varphi, 0) > R$, we may not have $d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$ for all $t \geqslant t_0$. Hence, we can not have $\rho > R$ for the uniform assimptotic stability, implicating that it can not be global.

Accordingly, the condition $\lim_{r\to\infty} b(r) = \infty$ implicates that $b([0, \infty)) = a([0, \infty)) = [0, \infty)$. Hence, defining $\delta(\varepsilon) = \sup\{r \,|\, a(r) < b(\varepsilon)\}$, we have that $\delta \to \infty$ when $\varepsilon \to \infty$. From this fact, the proof of Theorem 4.22 is analogous to Theorem 4.16 for any $\rho > 0$.

If we assumed that $\lim_{r\to\infty} b(r) = M = \lim_{r\to\infty} a(r)$, that is, $b([0, \infty)) = a([0, \infty)) = [0, M)$, then it would follow that $\delta(\varepsilon) \to \infty$ when $\varepsilon \to \infty$. Therefore, we could still obtain the desired.

Theorem 4.23. *Suppose that $b(d_\infty(\phi(0), 0)) \leqslant V(t, \phi) \leqslant W(\phi)$ and $D^+V(t, \phi) \leqslant 0$ for all $t \geqslant 0$ and $\varphi \in S(B)$, where $b \in \mathcal{K}$ and $W : S(B) \to \mathbb{R}$ is a continuous function with $W(0) = 0$. Then the solution $x \equiv 0$ of Equation (4.1) is* uniformly stable.

Proof. Given $0 < \varepsilon < B$, since W is continuousr, we can choose $\delta = \delta(\varepsilon) > 0$ such that $W(\phi) < b(\varepsilon)$ when $D_\tau(\phi, 0)$. Hence, for $t_0 \geqslant 0$ and $D_\tau(\varphi, 0) < \delta$,

$$V(t_0, \varphi) \leqslant W(\varphi) < b(\varepsilon).$$

Accordingly, from Corollary 4.18, it follws that

$$V(t, x_t(t_0, \varphi)) \leqslant V(t_0, \varphi) < b(\varepsilon), \quad \forall t \geqslant t_0.$$

We claim that $d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$ for all $t \geqslant t_0$. If it is not true, then there exist $t_1 > t_0$ such that $d_\infty(x(t; t_0, \varphi), 0) = \varepsilon$ and, then,

$$V(t_1, x_{t_1}(t_0, \varphi)) \geqslant b(d_\infty(x(t_1; t_0, \varphi), 0)) = b(\varepsilon),$$

getting into contradiction. Therefore, $D_\tau(\varphi, 0) < \delta$ implicates that $d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$ for all $t \geqslant t_0$, independently from t_0. Consequently, $x \equiv 0$ is uniformly stable. ■

Theorem 4.24. *Suppose that* $b(d(\phi(0), 0)) \leqslant V(t, \phi) \leqslant W(\phi)$ *for all* $t \geqslant 0$ *and* $\varphi \in S(B)$*, where* $b \in \mathcal{K}$ *and* $W : S(B) \to \mathbb{R}$ *is a continuous functional with* $W(0) = 0$*. Moreover, suppose that there is* $c \in \mathcal{K}$ *such that* $D^+V(t, x_t) \leqslant -c(d_\infty(x(t), 0))$ *for all solutions* $x(t)$ *of* (4.1)*, and suppose that* $F(t, \phi)$ *is limited on* $[0, \infty) \times S(B)$*. Then the solution* $x \equiv 0$ *of Equation* (4.1) *is* assimptotically stable.

Proof. Given $0 < \varepsilon < B$, let $0 < \rho < \varepsilon$ be chosen so that $W(\phi) < c(\varepsilon)$ whenever $D_\tau(\phi, 0) < \rho$. just like in the proof of Theorem 4.23, if $D_\tau(\varphi, 0) < \rho$, then $d_\infty(x(t; t_0, \varphi), 0) < \varepsilon$, for all $t \geqslant t_0$. We claim that

$$t_0 \geqslant 0,\ D_\tau(\varphi, 0) < \rho \Rightarrow \lim_{t\to\infty} d_\infty(x(t; t_0, \varphi), 0) = 0\,.$$

Then, suppose that there is $t_0 \geqslant 0$ and $\varphi \in S(B)$ so that $d_\infty(x(t), 0)$ does not go to zero as $t \to \infty$, where $x(t) = x(t; t_0, \varphi)$. Then, there exist an increasing sequence (t_m) and $\eta > 0$ such that $t_{m+1} - t_m > 2$ and $d_\infty(x(t_m), 0) > \eta$ for all $m \in \mathbb{N}$. Let $\sigma > 0$ be such that $d_\infty(F(t, \phi), 0) \leqslant \sigma$ for all $t \geqslant 0$ and $\phi \in S(B)$. From Theorem 1.2, for each $t \geqslant t_0$, there is s between t and t_m satisfying

$$x'(s) = \frac{x(t) -_{gH} x(t_m)}{t - t_m}\,.$$

Hence,

$$\begin{aligned}\left|\frac{d_\infty(x(t), 0) - d_\infty(x(t_m), 0)}{t - t_m}\right| &\leqslant \frac{d_\infty(x(t), x(t_m))}{|t - t_m|} = \frac{d_\infty(x(t) -_{gH} x(t_m), 0)}{|t - t_m|} \\ &= d_\infty\left(\frac{x(t) -_{gH} x(t_m)}{t - t_m}, 0\right) = d_\infty(x'_{gH}(s), 0) \\ &= d_\infty(f(s, x_s), 0) \leqslant \sigma\,.\end{aligned}$$

Consequently

$$\frac{d_\infty(x(t), 0) - d_\infty(x(t_m), 0)}{|t - t_m|} \geqslant -\sigma$$

$$\Rightarrow d_\infty(x(t),0) \geqslant d_\infty(x(t_m),0) - \sigma|t-t_m| \geqslant \eta - \sigma|t-t_m|\,.$$

Therefore, $d_\infty(x(t),0) \geqslant \frac{\eta}{2}$ se $|t-t_m| \leqslant \frac{\eta}{2\sigma}$. Conveniently, let $k = \min\left\{\frac{\eta}{2\sigma}, 1\right\}$, then $\frac{\eta}{2} \leqslant d_\infty(x(t),0) \leqslant \varepsilon$ for $|t-t_m| \leqslant k$. Accordingly, for $|t-t_m| \leqslant k$,

$$\begin{aligned} D^+V(t,x_t) &\leqslant -c(d_\infty(x(t),0)) \leqslant \sup_{|s-t_m|\leqslant k} -c(d_\infty(x(s),0)) \\ &= \sup_{\frac{\eta}{2}\leqslant r\leqslant\varepsilon} -c(r) = -\inf_{\frac{\eta}{2}\leqslant r\leqslant\varepsilon} c(r) = -c\left(\frac{\eta}{2}\right) < 0\,. \end{aligned}$$

Since $k \leqslant 1$ and $t_{m+1} - t_m > 2$, it follows that, for all $N \in \mathbb{N}$,

$$\begin{aligned} V(t_N+k, x_{t_N+k}) - V(t_1-k, x_{t_1-k}) &\leqslant \sum_{m=1}^{N} \left[V(t_m+k, x_{t_m+k}) - V(t_m-k, x_{t_m-k})\right] \\ &\leqslant \sum_{m=1}^{N} -2kc\left(\frac{\eta}{2}\right) = -2kc\left(\frac{\eta}{2}\right)N, \end{aligned}$$

that is, $V(t_N+k, x_{t_N+k}) = V(t_1-k, x_{t_1-k}) - 2kc\left(\frac{\eta}{2}\right)N$, what implicates that

$$\lim_{N\to\infty} V(t_N+k, x_{t_N+k}) = -\infty,$$

and it is a contradiction with the fact that $V(t,\phi) \geqslant 0$ for all $t \geqslant 0$ and $\phi \in S(B)$. Therefore, $\lim_{t\to\infty} d_\infty(x(t;t_0,\varphi),0) = 0$ for all $t_0 \geqslant 0$ and $\varphi \in \mathcal{C}_\tau$ such that $D_\tau(\varphi,0) < \rho$. ■

The next results is a consequence of Theorem 4.24 and can be proved by the same arguments of Theorem 4.22.

Corollary 4.25. *Suppose that $b(d_\infty(\phi(0),0)) \leqslant V(t,\phi) \leqslant W(\phi)$ for all $t \geqslant 0$ and $\phi \in \mathcal{C}_\tau$, where $b \in \mathcal{K}$, $\lim_{r\to+\infty} b(r) = +\infty$, and $W : \mathcal{C}_\tau \to \mathbb{R}$ is continuous and $W(0) = 0$. Moreover, suppose that there is $c \in \mathcal{K}$ such that $D^+V(t,x_t) \leqslant -c(d_\infty(x(t),0))$ for all solutions $x(t)$ of (4.1), and suppose that $F(t,\phi)$ is bounded on $[0,\infty) \times S(B)$ for all $B > 0$. Then the solution $x \equiv 0$ of Equation (4.1) is* globally assimptotically stable.

It can be noted that the classical idea of a Lyapunov-type function with two variables provides more wide hypothesis than the three variables function proposed by [21].

Example 4.3. Let $q(t)$ be a scalar continuous function for all $t \geqslant 0$, and λ and σ be positive constants such that $\lambda \geqslant |q(t)| + \sigma$ for all $t \geqslant 0$. Then, the zero solution $x \equiv 0$ of

$$x'_{gH}(t) = -\lambda x(t) + q(t)x(t-\tau) \tag{4.15}$$

is globally assimptotically stable.

In fact, defining

$$V(\phi) = \phi_0^-(0)^2 + \phi_0^+(0)^2 + \lambda \int_{-\tau}^{0} \phi_0^-(\theta)^2 + \phi_0^+(\theta)^2\, d\theta, \tag{4.16}$$

we have

$$\begin{aligned} V(\phi) &= \phi_0^-(0)^2 + \phi_0^+(0)^2 + \lambda \int_{-\tau}^{0} \phi_0^-(\theta)^2 + \phi_0^+(\theta)^2 \, d\theta \\ &\geqslant d_\infty(\phi(0), 0)^2 + \lambda \int_{-\tau}^{0} d_\infty(\phi(\theta), 0)^2 \, d\theta \geqslant d_\infty(\phi(0), 0)^2, \end{aligned}$$

that is, $b(d_\infty(\phi(0), 0)) \leqslant V(\phi) = W(\phi)$ for all $t \geqslant 0$ and $\phi \in \mathcal{C}_\tau$, where $b(r) = r^2$. Moreover, given any (ii)-gH-solution $x(t)$, it follows that

$$\begin{aligned} D^+V(x_t) &= \limsup_{h\to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h} \\ &= \limsup_{h\to 0^+} \frac{1}{h} \bigg[(x_0^-)_{t+h}(0)^2 + (x_0^+)_{t+h}(0)^2 + \lambda \int_{-\tau}^{0} (x_0^-)_{t+h}(\theta)^2 + (x_0^+)_{t+h}(\theta)^2 \, d\theta \\ &- (x_0^-)_t(0)^2 - (x_0^+)_t(0)^2 - \lambda \int_{-\tau}^{0} (x_0^-)_t(\theta)^2 + (x_0^+)_t(\theta)^2 \, d\theta \bigg] \\ &= \limsup_{h\to 0^+} \frac{1}{h} \bigg[x_0^-(t+h)^2 + x_0^+(t+h)^2 + \lambda \int_{-\tau}^{0} x_0^-(t+h+\theta)^2 + x_0^+(t+h+\theta)^2 \, d\theta \\ &- x_0^-(t)^2 - x_0^+(t)^2 - \lambda \int_{-\tau}^{0} x_0^-(t+\theta)^2 + x_0^+(t+\theta)^2 \, d\theta \bigg] \\ &= 2x_0^-(t)(x_0^-)'(t) + 2x_0^+(t)(x_0^+)'(t) + \lambda \Big[x_0^-(t)^2 + x_0^+(t)^2 - x_0^-(t-\tau)^2 - x_0^+(t-\tau)^2 \Big] \\ &= 2x_0^-(t) \Big[-\lambda x_0^-(t) + \max(q(t)x_0^-(t-\tau), q(t)x_0^+(t-\tau)) \Big] \\ &+ 2x_0^+(t) \Big[-\lambda x_0^+(t) + \min(q(t)x_0^-(t-\tau), q(t)x_0^+(t-\tau)) \Big] \\ &+ \lambda \Big[x_0^-(t)^2 + x_0^+(t)^2 - x_0^-(t-\tau)^2 - x_0^+(t-\tau)^2 \Big]. \end{aligned}$$

If $q(t) > 0$, then

$$\begin{aligned} D^+V(x_t) &= 2x_0^-(t) \Big[-\lambda x_0^-(t) + q(t)x_0^+(t-\tau)) \Big] \\ &+ 2x_0^+(t) \Big[-\lambda x_0^+(t) + q(t)x_0^-(t-\tau) \Big] \\ &+ \lambda \Big[x_0^-(t)^2 + x_0^+(t)^2 - x_0^-(t-\tau)^2 - x_0^+(t-\tau)^2 \Big] \\ &= -\lambda \Big[x_0^-(t)^2 + x_0^+(t)^2 \Big] - \lambda \Big[x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2 \Big] \\ &+ q(t) \Big[2x_0^-(t)x_0^+(t-\tau) + 2x_0^+(t)x_0^-(t-\tau) \Big] \\ &\leqslant -\lambda \Big[x_0^-(t)^2 + x_0^+(t)^2 \Big] - \lambda \Big[x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2 \Big] \\ &+ q(t) \Big[x_0^-(t)^2 + x_0^+(t-\tau)^2 + x_0^+(t)^2 + x_0^-(t-\tau)^2 \Big] \\ &= (-\lambda + q(t)) \Big[x_0^-(t)^2 + x_0^+(t)^2 + x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2 \Big] \\ &\leqslant -\sigma \Big[x_0^-(t)^2 + x_0^+(t)^2 + x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2 \Big] \leqslant -\sigma \Big[x_0^-(t)^2 + x_0^+(t)^2 \Big] \\ &= \leqslant -\sigma d_\infty(x(t), 0)^2 = -c(r), \end{aligned}$$

where $c(r) = \sigma r^2$. And if $q(t) < 0$, then

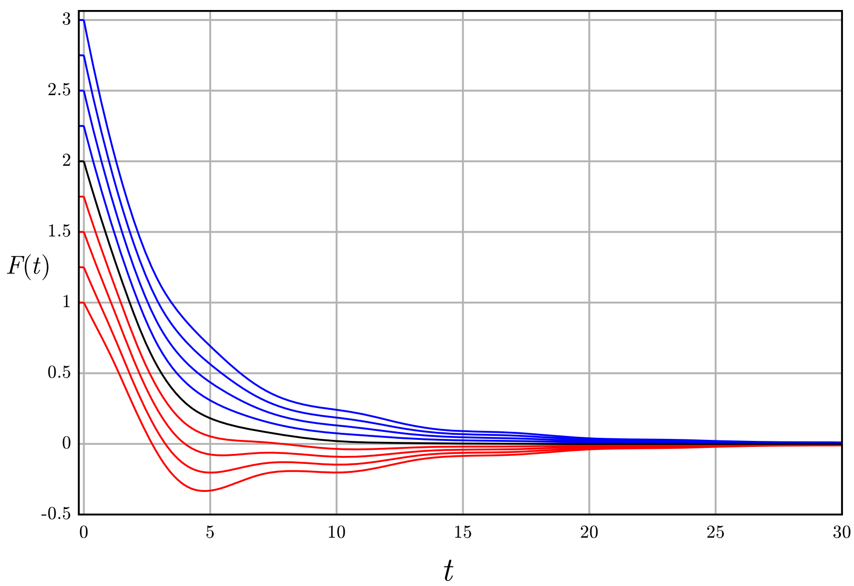
$$\begin{aligned}
D^+V(x_t) &= 2x_0^-(t)\Big[-\lambda x_0^-(t) + q(t)x_0^-(t-\tau))\Big] \\
&+ 2x_0^+(t)\Big[-\lambda x_0^+(t) + q(t)x_0^+(t-\tau)\Big] \\
&+ \lambda\Big[x_0^-(t)^2 + x_0^+(t)^2 - x_0^-(t-\tau)^2 - x_0^+(t-\tau)^2\Big] \\
&= -\lambda\Big[x_0^-(t)^2 + x_0^+(t)^2\Big] - \lambda\Big[x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2\Big] \\
&+ q(t)\Big[2x_0^-(t)x_0^-(t-\tau) + 2x_0^+(t)x_0^+(t-\tau)\Big] \\
&\leqslant -\lambda\Big[x_0^-(t)^2 + x_0^+(t)^2\Big] - \lambda\Big[x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2\Big] \\
&+ |q(t)|\Big[x_0^-(t)^2 + x_0^+(t-\tau)^2 + x_0^+(t)^2 + x_0^-(t-\tau)^2\Big] \\
&= (-\lambda + |q(t)|)\Big[x_0^-(t)^2 + x_0^+(t)^2 + x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2\Big] \\
&\leqslant -\sigma\Big[x_0^-(t)^2 + x_0^+(t)^2 + x_0^-(t-\tau)^2 + x_0^+(t-\tau)^2\Big] \leqslant -\sigma\Big[x_0^-(t)^2 + x_0^+(t)^2\Big] \\
&\leqslant -\sigma d_\infty(x(t), 0)^2 = -c(r).
\end{aligned}$$

Finally, for any $B > 0$, if $(t, \phi) \in [0, \infty) \times S(B)$, then

$$\begin{aligned}
d_\infty(F(t,\phi), 0) &= d_\infty(-\lambda\phi(0) + q(t)\phi(-\tau), 0) \leqslant \lambda d_\infty(\phi(0), 0) + |q(t)| d_\infty(\phi(-\tau), 0) \\
&\leqslant \lambda B + |q(t)|B \leqslant \lambda B + (\lambda - \sigma)B = (2\lambda - \sigma)B.
\end{aligned}$$

Therefore, all conditions of Corollary 4.25 are fulfilled and the conclusion follows.

Figure 4.6: Graphic representation of the endpoint functions of the (ii)-gH-solution of Equation (4.15) with $\lambda = 0.3$, $q(t) = 0.12\cos(t) - 0.14$, $\tau = 0.2$ and initial condition $[\varphi(\theta)]_\alpha = [1 + \alpha, 3 - \alpha]$ for all $\alpha \in [0, 1]$ and $\theta \in [-\tau, 0]$.



Source: Elaborated by the author.

In Figure 4.6, we have a numerical (ii)-gH-solution of Equation 4.15 with triangular initial condition. Note that the oscillatory behavior caused by $q(t) = 0.12\cos(t) - 0.14$ does not avoid the assymptotical behavior of the solution since the condition $\lambda \geqslant |q(t)| + \sigma$ is satisfied for $\sigma = 0.04$.

Note that we obtained the global assymptotic stability of $x \equiv 0$, but we can not guarantee to be uniform due to the function $q(t)$.

The theory presented in this chapter is just a starting point for future works on Fuzzy Delay Differential Equations and its application on mathematical modeling. In the next, some considerations are made about our work, what we obtained and some questions to be answered in the future.

FINAL CONSIDERATIONS

Observing the obtained g-derivative characterization results of Chapter 2, it can be immediately noted that some results are very abstract and difficult to be applied on FDE. However, they make part of the porpose to fulfill the understanding of the g-derivative.

As mentioned before, while studying the possibilities of FDE for each one of these derivatives, a question emerged: when there exist solutions of a g-FDE that are not gH$*$-differentiable? The answer to this question is not quite complete, but Section 2.5.2 present a result and some examples showing very common equations whose solutions are, at most, gH$*$-differentiable.

In Theorem 2.23, we consider only the solutions whose level sets are differentiable, that is, given any solution $x(t)$, $x_\alpha(t) = [x(t)]_\alpha$ is an interval gH-differentiable function. From the mathematical modeling point of view, differentiable processes are generally considered and, under the considered assumptions, we have endpoint functions $x_\alpha^-(t)$ and $x_\alpha^+(t)$ that are differentiable in the whole domain, except for a countable set where they have one-sided derivatives.

Summarily, in this chapter, we generalized some interval calculus results for gH- and gH$*$-derivatives of fuzzy functions, obtained characterization results for the generalized derivative, and pointed the difference of these three derivarives when applied on FDE.

From the studies presented in Chapter 3, we may note that the notions of switching points from Definitions 1.8 and 3.3 may envolve more complex situations while the cases from Definition 3.4 are easier to be studied on solutions of FDE.

The function G on Example 3.5, as it was mentioned before, is (ii)-gH-differentiable at $t < -2$ and (i)-gH-differentiable at $t > 0$, but has no switching point. For this kind of situation, the concept of transitional region is presented in [8].

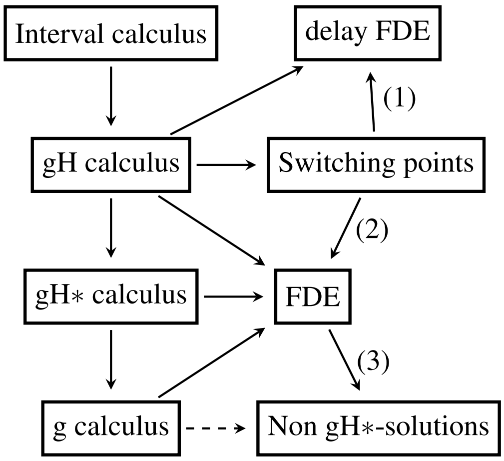
On what concerns the FDE, a question that arised is the behavior of switching points of the solutions. How can the function F that describes the equation determinate the type of switching points that the solution may have? This is an investigation to be done in the future, but it is an undoubtedly interesting subject in the study of gH-fuzzy differential equations.

We generalized the results about the switching points for the gH-differentiability of fuzzy functions with the espectation of bringing new properties on the solutions of FDE.

Chapter 4 is the closure of this work and presentes the combined theories of fuzzy differential equations and delay differential equations. Under assumptions (h1)-(h5), Theorem 4.4 was proposed in [27] and provides the existence of one unique (i)-gH-solution and one unique (ii)-gH-solution. Motivated by the Existence and Uniqueness Theorems from [4], we obtained Theorems 4.6 and 4.11 with the previous idea of providing a more practical theorem. However, as commented before, it is not simple to work with the conditions of the theorems. Therefore, an analysis of the α-parametrized system (4.7) can be an easier way to guarantee the existence of the solutions.

The existence of solutions defined in the future is a first step to the study of the stability of a FDE, then this is the main conection between the two subjects of the last chapter. The α-parametrized system is always a good alternative to analyze a fuzzy model. However, the stability results presented shows how simply the results from DDE can be extended to FDE.

Figure 4.7: Diagram of the studied subjects of this work.



Source: Elaborated by the author.

In the diagram of Figure 4.7, we intend to illustrate the connection of the subjects studied in our work. Connections (1) and (2) represent the open questions on how the different classifications of switching points are present on FDE and delay FDE. And (3) is the continuity on the studying of the solutions of FDE under g-derivative. Moreover, there are more concepts to study on delay FDE such as: assymptotic behavior, continuity on the initial conditions and parameters, periodic solutions and others.

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APPENDIX

AUXILIARY RESULTS

The following sections present auxiliary concepts and results necessary to the study of the Fuzzy Delay Differential Equations.

A.1 Delay differential equations

Since Chapter 4 consists of FDE with delay, this section intends to introduce the classical theory of Delay Differential Equations (DDE).

In the DDE theory, the derivative of an unknown function at each instant t depends on the function itself in previous instants such as $t-\tau$, for some $\tau > 0$, the so called delay. For instance, the equation $x'(t) = x(t-\tau)$ is a DDE.

If $x : [t_0 - \tau, t_0 + T) \to \mathbb{R}^n$ is a continuous function with $t_0 \geqslant 0$, $0 < \tau < \infty$ and $0 < t \leqslant \infty$, we define, for each $t \in [t_0, t_0 + T)$, a function $x_t : [-\tau, 0] \to \mathbb{R}^n$ given by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$.

Accordingly, the DDE are given by $x'(t) = f(t, x_t)$, where $f : \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R} \times \mathcal{C}_\tau$ and $\mathcal{C}_\tau = \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ is the Banach space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the supremum norm.

One important difference to the ODE is that, because of the delay, the initil condition is not assumed only in the initial instant, but in the whole interval $[t_0 - \tau, t_0]$, that is, the initial condition is a continuous function $\varphi : [-\tau, 0] \to \mathbb{R}^n$ such that $x(t) = \phi(t - t_0)$ for $t \in [t_0 - \tau, t_0]$. The notions of stability for DDE and ODE are similar, except that the stability may depend on the initial time t_0 in a DDE. From this, we have the concept of uniform stability, when the stability does not depend on t_0.

The theory, examples and applications of DDE can be found in [19, 20, 34].

A.1.1 Positive Solutions of DDE

Let us consider the linear system with constant coefficients

$$y'(t) = -By(t-\tau), \tag{A.1}$$

with $\tau > 0$ and $B = (b_{ij})_{n\times n}$, $b_{ij} \in \mathbb{R}$. From the same idea of ODEs, we want to find the values of λ so that $y(t) = e^{\lambda t}v$ is a solution of Equation (A.1) for some $v \in \mathbb{R}^n$, $v \neq 0$. Since $y'(t) = \lambda e^{\lambda t} v$, then

$$y'(t) = -By(t-\tau) \Leftrightarrow \lambda e^{\lambda t} v = -e^{\lambda(t-\tau)} Bv \Leftrightarrow \lambda v = -e^{-\lambda\tau} Bv,$$

that is, λ must be an eigenvalue of $-Be^{-\lambda\tau}$ and, then, satisfy the characteristic equation $\det(-Be^{-\lambda\tau} - \lambda I) = 0$.

This type of equation have been widely studied through out the decades and it is well known that all zeros of this characteristic equation remains in a left semi-plane of complex numbers. Also, the biggest real solution has the biggest real part among all complex roots.

In the following, some positive solutions existence results from [13] are presented for the system (A.1) with $b_{ij} > 0$ for all $i, j = 1, ..., n$.

Theorem A.1. [13]. *Let there exist positive constants k_i, $i = 1, ..., n$, such that*

$$e\tau \left(b_{ii} + \frac{1}{k_i} \sum_{j=1//j\neq i}^{n} k_j b_{ij} \right) \leqslant 1 \tag{A.2}$$

for each $i = 1, ..., n$. Then system (A.1) has a positive solution on interval $[t^* - r, \infty)$ for some $t^* > t_0 + r$.

Particularly, if $k_i = 1$ for all $i = 1, ..., n$, the condition (A.2) turns into condition $e\tau b_M \leqslant 1$, where $b_M = \max\limits_{1\leqslant i\leqslant n} \sum\limits_{j=1}^{n} b_{ij}$.

Theorem A.2. [13]. *Let us suppose that the matrix B is indecomposable. Then the system (A.1) has a positive solution on interval $[t^* - r, \infty)$ if $e\tau\rho(B) < 1$, where $\rho(B)$ is the spectral radius of the matrix B.*

In addition to the these theorems, others can be found in [13] for other cases of delay differential equations.

Delay malthusian model

Let us consider

$$\begin{cases} u'(t) = -ru(t-\tau), & t \geqslant t_0, \\ u(t) = \psi(t - t_0), & t_0 - \tau \geqslant t \geqslant t_0\,, \end{cases} \tag{A.3}$$

with $t_0 \geqslant 0$, $r > 0$, $\tau > 0$ and $\psi \in \mathcal{C}_\tau$ is positive. From Theorem (A.2), $r\tau < e^{-1}$ is an enough condition for the existence of positive solutions of (A.3). Then, the main objective in the following is to find conditions on the initial

Note that, for $n \in \mathbb{N}$, if $u(t) > 0$ for all $t \in [t_0 + (n-1)\tau, t_0 + n\tau]$, then

$$u(t) = u(t_0 + n\tau) - r\int_{t_0+(n-1)\tau}^{t-\tau} u(s)\,ds \geqslant u(t_0 + n\tau) - r\int_{t_0+(n-1)\tau}^{t_0+n\tau} u(s)\,ds = u(t_0 + (n+1)\tau)$$

for all $t \in (t_0 + n\tau, t_0 + (n+1)\tau]$. Hence, it is enough to obtain the positivity of $u(t_0 + (n+1)\tau)$. Therefore, observe that $u(t_0 + \tau) > 0$ if and only if

$$\psi(0) > r\int_{-\tau}^{0} \psi(\theta)\,d\theta$$

since

$$\begin{aligned} u(t_0 + \tau) &= u(t_0) - r\int_{t_0}^{t_0+\tau} u(s-\tau)\,ds = u(t_0) - r\int_{t_0-\tau}^{t_0} u(s)\,ds \\ &= \psi(0) - r\int_{-\tau}^{0} \psi(\theta)\,d\theta. \end{aligned}$$

Similarly,

$$\psi(0) > \frac{r}{1-r\tau}\int_{-\tau}^{0}\psi(\theta)\,d\theta \Rightarrow u(t_0+2\tau) > 0,$$

$$\psi(0) > \left(\frac{1-r\tau}{1-2r\tau}\right) r\int_{-\tau}^{0}\psi(\theta)\,d\theta \Rightarrow u(t_0+3\tau) > 0$$

and

$$\psi(0) > \left(\frac{1-2r\tau}{1-3r\tau+r\tau^2}\right) r\int_{-\tau}^{0}\psi(\theta)\,d\theta \Rightarrow u(t_0+4\tau) > 0\,.$$

Consequently, from successive calculations, we obtain a sequence $(p_n)_{n\in\mathbb{N}}$ satisfying

$$p_{n+1} = \frac{r}{1-\tau p_n},\quad p_1 = r\,.$$

Then, if $r\tau < 1/4$, the sequence is increasing and convergent with

$$\lim_{n\to\infty} p_n = c = \frac{1-\sqrt{1-4r\tau}}{2\tau}\,.$$

Therefore, if $\psi(0) > c\int_{-\tau}^{0}\psi(\theta)\,d\theta$, we have

$$\psi(0) > p_n\int_{-\tau}^{0}\psi(\theta)\,d\theta$$

for all $n\in\mathbb{N}$, that is, the solution with initial condition ψ is positive for all $t \geqslant t_0$.

Now, consider the system

$$\begin{cases} u'(t) = -rv(t-\tau), \\ v'(t) = -ru(t-\tau), \end{cases} t \geqslant t_0, \quad \begin{cases} u(t) = \psi(t-t_0), \\ v(t) = \phi(t-t_0), \end{cases} t_0-\tau \leqslant t \leqslant t_0\,. \tag{A.4}$$

with $t_0 \geqslant 0$, $r > 0$, $\tau > 0$, $\psi,\phi \in \mathcal{C}_\tau$ are positive and $r\tau < e^{-1}$. From Equation (A.4), we have the characteristic equation $(\lambda - r\exp(-\lambda\tau))(\lambda + r\exp(-\lambda\tau)) = 0$ and, then, there exist a positive real eigenvalue. Consequently, solution $(u,v)\equiv(0,0)$ unstable and both functions u and v diverges, one goes to $+\infty$ and the other goes to $-\infty$. Accordingly, we may not assure positive solutions u and v in a general sense. However, if we consider $\psi\equiv\phi$, then $u\equiv v$ and we get back to the one dimension case argumented before.

On the other hand, we can obtain by the method of steps [20, 34] and prove by induction that, if the initial condition is constant and positive, $\varphi\equiv x_0$, then the solution of (A.3) can be given by

$$x(t;0,x_0) = x_0\sum_{k=0}^{N}\frac{(-1)^k r^k}{k!}\Big(t-(k-1)\tau\Big)^k$$

if $t\in[N\tau,(N+1)\tau]$ for each $N\in\mathbb{N}$. And two facts follow from this expression:

- if $\tau < 1/re$, then $x(t;0,x_0) > 0$ for all $t \geqslant t_0$, that is, solutions with positive constant initial condition are positive;

- if $x(t) = x(t; 0, x_0)$ and $y(t) = y(t; 0, y_0)$ are solutions, then $x(t)$ and $y(t)$ have the same roots.

A.2 Upper Dini derivative

The main purpose of this section is to provide comparision results the upper Dini derivative for the Lyapunov stability analysis of Section 4.2.

Definition A.1. [26]. Let $f : [a, b] \to \mathbb{R}$ be a real-valued function and $t_0 \in [a, b)$, we define the upper Dini derivative (upper right-hand derivative) of f at t_0 by

$$D^+ f(t_0) = \limsup_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h}\,, \tag{A.5}$$

where it can be finite or assume the values $\pm\infty$.

Remark A.1. If the limit (A.5) is finite, then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < h < \delta \Rightarrow 0 < \sup_{0 < s \leqslant h} \frac{f(x_0 + s) - f(x_0)}{h} - D^+ f(x_0) < \varepsilon\,.$$

Remark A.2. The upper limit of a function in an acumulation point is finite if the function is bounded on a neighborhood of this point. Accordingly, denoting $q_t(h) = \frac{f(t+h) - f(t)}{h}$, we have that $\limsup_{h \to 0^+} q_t(h)$ is finite if, and only if, there exists $\delta_0 > 0$ such that $0 < h < \delta_0$ implies $|q_t(h)| \leqslant K$, for some $K > 0$. In other words, $D^+ f(t)$ is finete if, and only if, there exist $\delta_0, K > 0$ such that

$$|f(t + h) - f(t)| \leqslant Kh\,,$$

for all $h \in (0, \delta_0)$.

Lemma A.3. [32, 44]. *Let f and g be real functions defined and finite on a neighborhood of $t_0 \in \mathbb{R}$. Then,*

$$D^+[f + g](t_0) \leqslant D^+ f(t_0) + D^+ g(t_0)$$

and

$$D^+[f + g](t_0) \geqslant D^+ f(t_0) + D_+ g(t_0)\,,$$

where

$$D_+ f(t_0) = \liminf_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h}\,.$$

Particularly, if $g'(t_0)$ exists,

$$D^+[f + g](t_0) = D^+ f(t_0) + g'(t_0)\,.$$

Proof. From the properties of supremum and infimum, it follows that

$$D^+[f + g](t_0) = \limsup_{h \to 0^+} \frac{1}{h}\Big[f(t_0 + h) + g(t_0 + h) - f(t_0) - g(t_0)\Big] \tag{A.6}$$

$$
\begin{aligned}
&= \limsup_{h\to 0^+}\left[\frac{f(t_0+h)-f(t_0)}{h}+\frac{g(t_0+h)-g(t_0)}{h}\right] && \text{(A.7)}\\
&\leqslant \limsup_{h\to 0^+}\left[\frac{f(t_0+h)-f(t_0)}{h}\right]+\limsup_{h\to 0^+}\left[\frac{g(t_0+h)-g(t_0)}{h}\right] && \text{(A.8)}\\
&= D^+f(t_0)+D^+g(t_0)\,, && \text{(A.9)}
\end{aligned}
$$

and also

$$
\begin{aligned}
D^+[f+g](t_0) &\geqslant \limsup_{h\to 0^+}\left[\frac{f(t_0+h)-f(t_0)}{h}\right]+\liminf_{h\to 0^+}\left[\frac{g(t_0+h)-g(t_0)}{h}\right]\\
&= D^+f(t_0)+D_+g(t_0)\,.
\end{aligned}
$$

If $g'(t_0)$ exists, the equality is obvious since $g'(t_0)=D^+(t_0)=D_+(t_0)$. ■

In Lemma A.3, we could just assume the existence of the right-hand derivative $g'_+(t_0)$ to obtain the equality.

Lemma A.4. [16] *Let $f:(a,b)\to\mathbb{R}$ be continuous on the right and $g:(a,b)\to\mathbb{R}$ differentiable at $t_0\in(a,b)$. If $g(t_0)\geqslant 0$, then*

$$
D^+[fg](t_0)=f(t_0)g'(t_0)+g(t_0)D^+f(t_0)\,.
$$

Proof.

$$
\begin{aligned}
D^+[fg](t_0) &= \limsup_{h\to 0^+}\frac{1}{h}\Big[f(t_0+h)g(t_0+h)-f(t_0)g(t_0)\Big]\\
&= \limsup_{h\to 0^+}\frac{1}{h}\Big[f(t_0+h)g(t_0+h)-f(t_0)g(t_0)+f(t_0+h)g(t_0)-f(t_0+h)g(t_0)\Big]\\
&= \limsup_{h\to 0^+}\left[g(t_0)\frac{f(t_0+h)-f(t_0)}{h}+f(t_0+h)\frac{g(t_0+h)-g(t_0)}{h}\right]\\
&= \limsup_{h\to 0^+}\left[g(t_0)\frac{f(t_0+h)-f(t_0)}{h}\right]+\limsup_{h\to 0^+}\left[f(t_0+h)\frac{g(t_0+h)-g(t_0)}{h}\right]\\
&= g(t_0)\limsup_{h\to 0^+}\frac{f(t_0+h)-f(t_0)}{h}+\lim_{h\to 0^+}\left[f(t_0+h)\frac{g(t_0+h)-g(t_0)}{h}\right]\\
&= g(t_0)D^+f(t_0)+f(t_0)g'(t_0)\,.
\end{aligned}
$$

■

Lemma A.5. [22] *Let f be finite and continuous on the interval $[a,b]$. If $D^+f(t)=0$ for all $t\in[a,b)$, then f is constant on $[a,b]$.*

Proof. Suppose that f is not constant in $[a,b]$, then there exists $c\in(a,b]$ such that $f(c)-f(a)=p\neq 0$. Let us assume that $p>0$. Defining $g_k(t)=f(t)-f(a)-k(t-a)$, $k>0$, it follows that $g_k(a)=0$ and $g_k(c)=p-k(c-a)$. Fix q such that $0<q<p$ and take $k<(p-q)/(c-a)$, then $g_k(c)>q$. If $A=\{t\in(a,c)\,|\,g_k(t)\leqslant q\}$ and $\xi=\sup A$, then $\xi<c$ and $g_k(\xi)=q$. Thus, if $0<h\leqslant c-\xi$, then $g_k(\xi+h)>q$ and

$$
\frac{g_k(\xi+h)-g_k(\xi)}{h}>0\,.
$$

Therefore, $D^+g_k(\xi) \geqslant 0$, contradicting the assumption that

$$D^+g_k(t) = D^+f(t) - k = -k < 0$$

for all $t \in [a, b)$. Analogously, for $p < 0$, the conclusion follows for $g_k(t) = f(t) - f(a) + k(t - a)$. ■

Lemma A.6. [18] *If f is continuous on $[a, b]$ and $D^+f(x) \geqslant 0$ at every point of (a, b), then f is non decreasing on (a, b).*

Proof. Let us suppose that there are $c, d \in (a, b)$ such that $c < d$ and $f(t) < f(s)$ for all $t, s \in [c, d]$ with $s < t$. Then,

$$\frac{f(t+h) - f(t)}{h} \leqslant 0, \quad 0 < h < d - t$$

for all $t \in [c, d)$, that is, $D^+f(t) \leqslant 0$ on $[c, d)$. Since $D^+f(t) \geqslant 0$ for all $t \in (a, b)$, it follows that $D^+f(t) = 0$ on $[c, d)$. Consequently, f is constant on $[c, d)$, but, by the continuity of f, it follows that $f(d) = f(c)$, contradicting the assumptiong that $f(d) < f(c)$. ■

Lemma A.7. [18] *Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $D^+f(t)$ finite for all $t \in [a, b]$, then*

$$\underline{\int_a^b} D^+f(t)\, dt \leqslant f(b) - f(a) \leqslant \overline{\int_a^b} D^+f(t)\, dt\,.$$

Proof. Let $P = \{a = t_0 < t_1 < ... < t_n = b\}$ be a partition of $[a, b]$. If $\ell_i = \inf\{D^+f(t) \,|\, t_{i-1} \leqslant t \leqslant t_i\}$ and $L_i = \sup\{D^+f(t) \,|\, t_{i-1} \leqslant t \leqslant t_i\}$, then $\ell_i \leqslant D^+f(t) \leqslant L_i$ for all $t \in [t_{i-1}, t_i]$. From Lemma A.6 applied to the functions $g_1(t) = f(t) - \ell$ e $g_2(t) = L - f(t)$, it follows that

$$\ell_i(t_i - t_{i-1}) \leqslant f(t_i) - f(t_{i-1}) \leqslant L_i(t_i - t_{i-1}),$$

what means that

$$s(f; P) \leqslant f(b) - f(a) \leqslant S(f; P)\,.$$

Consequently, since P is an arbitrary partition of $[a, b]$,

$$\underline{\int_a^b} D^+f(t)\, dt \leqslant f(b) - f(a) \leqslant \overline{\int_a^b} D^+f(t)\, dt\,.$$

■

The following Lemma plays an important role in the proof of the Comparision Theorems A.9 and A.10.

Lemma A.8. *Let $w : [a, b] \to \mathbb{R}$ be continuous with $D^+w(t)$ finite for all $t \in [a, b)$. If $D^+w(t) \leqslant Kw(t)$, $K > 0$, then $w(b) \leqslant w(a)e^{K(b-a)}$.*

Proof. In fact, for all $t \in [a, b)$,

$$D^+w(t) \leqslant Kw(t) \Rightarrow D^+w(t)\, e^{-K(t-a)} \leqslant Ke^{-K(t-a)}w(t)$$

$$\Rightarrow D^+ w(t)\, e^{-K(t-a)} - K e^{-K(t-a)} w(t) \leqslant 0$$
$$\Rightarrow D^+ \Big(w(t) e^{-K(t-a)} \Big) \leqslant 0\,.$$

Therefore, from Lemma A.7,

$$w(b)e^{-K(b-a)} - w(a) \leqslant \overline{\int_a^b} D^+ \Big(w(t) e^{-K(t-a)} \Big)\, dt \leqslant 0 \Rightarrow w(b) \leqslant w(a) e^{K(b-a)}\,.$$

■

Theorems A.9 and A.10 are adaptations of classic comparision results for the upper right-hand derivative of real functions and are necessary for the stability results presented.

Theorem A.9. *Suppose that the functions* $u(t)$ *and* $v(t)$ *are continuous on the interval* $[a, \infty)$ *and that they have upper right-hand derivative on* $[a, \infty)$*. Also, suppose that* $f(t, y)$ *is continuous and locally lipschitz with respect to y,* $u(a) \leqslant v(a)$ *and*

$$D^+ u(t) - f(t, u(t)) \leqslant D^+ v(t) - f(t, v(t)) \tag{A.10}$$

on $[a, \infty)$*. Then* $u(t) \leqslant v(t)$ *for all* $t \geqslant a$*.*

Proof. Suppose that $u(s) > v(s)$ at some point $s \in (a, \infty)$. Then, since $u - v$ is continuous and $u(a) - v(a) \leqslant 0$, there exist $c \in [a, \infty)$ such that $u(c) = v(c)$ and $u(t) > v(t)$ on $(c, s]$. Consequently, defining $w(t) = u(t) - v(t)$, it follows that

$$D^+ w(t) \leqslant D^+ u(t) - D^+ v(t) \leqslant f(t, u(t)) - f(t, v(t)) \leqslant K|u(t) - v(t)| \leqslant K w(t)$$

for $t \in [c, s]$. Therefore, from Lemma A.8, $w(s) \leqslant w(c) e^{K(s-c)} = 0$, contradicting the assumption that $w(s) > 0$. ■

Theorem A.10. *Suppose that the functions* $u(t)$ *and* $v(t)$ *are continuous on the interval* $[a, b]$ *and that they have upper right-hand derivative on* $[a, b)$*. Also, suppose that* $f(t, y)$ *is continuous and locally lipschitz with respect to y,* $u(a) \leqslant v(a)$ *and*

$$D^+ u(t) - f(t, u(t)) \leqslant D^+ v(t) - f(t, v(t)) \tag{A.11}$$

on $[a, b)$*. Then* $u(t) \leqslant v(t)$ *for all* $t \in [a, b]$*.*

Proof. Analogous to the previous proof by supposing that $u(s) > v(s)$ at some point $s \in (a, b]$. ■