Soliton like solutions of a Lorentz invariant equation in dimension 3*

Vieri Benci†, Donato Fortunato‡ & Lorenzo Pisani‡
†Dipart. Matematica Appl. "U. Dini",
Università degli Studi di Pisa, Via Bonanno, 25/b
50126 PISA (Italy)
‡Dipartimento di Matematica
Università degli Studi di Bari, Via Orabona, 4
70125 BARI (Italy)

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1. Introduction

A soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect solitons have a particle-like behavior and they occur in many questions of mathematical physics, such as classical and quantum field theory, non linear optics, fluid mechanics, plasma phisics (see [7], [12], [16], [19]).

Probably, the simplest equation which has soliton-like solutions is the sin-Gordon equation which is a semilinear hyperbolic equation defined in a two dimensional space-time.

The goal of this paper is to determine an equation defined in a four dimensional space-time which preserves the main peculiarities of the sin-Gordon equation, namely:

- it has soliton solutions;
- it is Lorentz invariant;
- the linearized equation at $u = 0$ is the Klein-Gordon equation

$$u_{tt} - c^2 \Delta u + \omega_0^2 u = 0$$

- the mass of a soliton which is the matrix m_{ij} defined by the formula

$$p_i = m_{ij} v_j$$

  is in fact a scalar, namely $m_{ij} = m\delta_{ij}$ (here p_i is the momentum of the field and v_i is the velocity of the soliton).

- the solitons satisfy the celebrated Einstein equation

$$E = mc^2 \tag{1.1}$$

The paper is organized as follows.

In Section 2 we introduce our model equation and we state the existence result.

In Section 3 we give the topological classification of static solutions by means of the topological charge.

In Section 4 we prove the existence of static solutions with non trivial charge. To prove this result we need to study the behaviour of sequences with bounded energy, in the spirit of the concentration-compactness principle.

In the last Section we define the mass of soliton and prove that it is a scalar which equals the energy.

2. Statement of the Problem

2.1. Nonlinear Wave Equations and Solitons

We introduce some notation.

If n, k are positive integer, $\mathbb{R}^{n+1}$ and $\mathbb{R}^k$ will denote respectively the *physical space-time* (tipically $n = 3$) and the *internal parameters space*. A point in $\mathbb{R}^{n+1}$ will be denoted by (x, t), with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

The fields we are interested in are maps

$$u\colon \mathbb{R}^{n+1} \to \mathbb{R}^k, \qquad u = (u^1, ..., u^k)$$

Since we require the Lorentz invariance, we shall consider Lagrangian densities of the form

$$\mathcal{L} = \mathcal{L}(u, \rho)$$

where

$$\rho = c^2 \left|\nabla u\right|^2 - \left|u_t\right|^2,$$

c being the light velocity, ∇u and u_t denote respectively the Jacobian with respect to x and the derivative with respect to t.

We shall consider

$$\mathcal{L}(u, \rho) = -\frac{1}{2}\alpha(\rho) - V(u), \tag{2.1}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ and the function V is defined in an open subset $\Omega \subset \mathbb{R}^k$.

The action functional related to 2.1 is

$$\mathcal{S}(u) = \int_{\mathbb{R}^{3+1}} \mathcal{L}(u, \rho)\, dxdt = \int_{\mathbb{R}^{3+1}} \left(-\frac{1}{2}\alpha(\rho) - V(u)\right)\, dxdt.$$

So the Euler-Lagrange equation is

$$\frac{\partial}{\partial t}\left(\alpha'(\rho)\, u_t\right) - c^2 \nabla\left(\alpha'(\rho)\, \nabla u\right) + V'(u) = 0 \tag{2.2}$$

where $\nabla\left(\alpha'(\rho)\, \nabla u\right)$ denotes the vector whose j-th component is given by

$$\operatorname{div}\left(\alpha'(\rho)\, \nabla u^j\right)$$

and V' denotes the gradient of V.

When $\alpha(\rho) = \rho$, the equation 2.2 reduces to the semilinear wave equation

$$\Box u + V'(u) = 0$$

where

$$\begin{aligned} \Box u &= u_{tt} - c^2 \Delta u, \\ \Delta u &= \frac{\partial^2 u}{\partial x_1^2} + ... + \frac{\partial^2 u}{\partial x_n^2}. \end{aligned}$$

Under suitable assumptions on α and V every static solution of 2.2

$$u(x,t) = \varphi(x),$$

i.e. a solution of

$$-c^2 \nabla \left(\alpha'(\rho) \nabla u \right) + V'(u) = 0 \tag{2.3}$$

has energy localized in a compact region.

Let $\varphi = \varphi(x_1, x_2, x_3)$ be a solution of the (2.3) and consider a vector $\mathbf{v}$ with $|\mathbf{v}| < c$. For simplicity we take $\mathbf{v} = (v, 0, 0)$. Then it is easy to verify that

$$u_{\mathbf{v}}(x_1, x_2, x_3, t) = \varphi \left(\frac{x_1 - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, x_2, x_3 \right) \tag{2.4}$$

is a solution of the equation 2.2.

Notice that the function u experiences a contraction of a factor

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

in the direction of the motion; this is a consequence of the fact that 2.2 is Lorentz invariant.

Moreover, it is clear that the 2.4 has some kind of stability if the static solution of 2.2 is a local minimum of the energy functional

$$E(\varphi) = \int_{\mathbb{R}^n} \left[\frac{1}{2} \alpha \left(c^2 \left| \nabla \varphi \right|^2 \right) + V(\varphi) \right] dx.$$

These observations lead to the following definition.

Definition 2.1. *We call soliton a solution of the equation 2.2 having the form 2.4, where φ is a local minimum of the energy functional.*

Before making our choice of the functions α and V, we make some remarks about the difference between the one dimensional case ($n = 1$) and the three dimensional case ($n = 3$).

If $n = 1$, a classical non linear equation having soliton solutions is the sin-Gordon equation

$$\Box u + \sin u = 0. \tag{2.5}$$

Observe that 2.5 can be obtained from 2.2 choosing

$$k = 1, \qquad \alpha(\rho) = \rho, \qquad V(\xi) = 1 - \cos\xi.$$

The static solutions of 2.5 are critical point of the energy funtional

$$E(\varphi) = \int_{\mathbb{R}} \frac{c^2}{2} \left(\dot{\varphi}(x)\right)^2 + \left(1 - \cos\varphi(x)\right) dx$$

over the space H of the smooth real functions φ satisfying the asymptotic conditions

$$\varphi(-\infty) = 2k_1\pi, \qquad \varphi(+\infty) = 2k_2\pi, \tag{2.6}$$

with $k_1, k_2 \in \mathbb{Z}$; observe that the asymptotic values of φ are minima of the potential V, so that the energy $E(\varphi)$ can be finite.

Explicit calculations shows that these solutions possess a solitonic behaviour (see [7], [19]). Furthermore, since the space H can be divided in infinitely many connected components according to the asymptotics conditions 2.6, the solitons can be topologically classified (more precisely, since $E(\varphi) = E(\varphi + 2k_1\pi)$, it is sufficient to fix k_1 and consider the difference $k_2 - k_1$).

Now consider the more realistic (from a physical point of view) case $n = 3$. Consider, as before, purely scalar fields (*i.e.* $k = 1$) and take $\alpha(\xi) = \xi$. Then the energy functional for the static solutions is

$$E(\varphi) = \int_{\mathbb{R}^3} \frac{c^2}{2} |\nabla\varphi|^2 + V(\varphi)\, dx. \tag{2.7}$$

If the potential V is positive, then a simple scaling argument shows that any function φ minimizing 2.7 is necessarily trivial, *i.e.* it takes a constant value which is a minimum point of V; this fact can be easily seen by imposing that $\frac{d}{d\lambda} E(\varphi(\lambda x))_{|\lambda=1} = 0$ (see [7]). So if we want to consider purely scalar fields and $\alpha(\xi) = \xi$, we cannot take $V(\xi) \geqslant 0$.

On the other hand, if we consider non positive potential, we are forced to seek saddle points, instead of minima, and for these static solutions we have lack of stability. As an example we recall that, if we take

$$V(\xi) = \frac{1}{2}\xi^2 - \frac{1}{4}\xi^4,$$

critical points of the energy functional

$$E(\varphi) = \int_{\mathbb{R}^3} \frac{c^2}{2} |\nabla \varphi|^2 + \frac{1}{2}\varphi^2 - \frac{1}{4}\varphi^4 \, dx.$$

have been found in [5] and [14] and for more general potentials in [4], [18]; but in [1] and [3] it has been proved that these static solutions are not stable.

2.2. Statement of the Main Result

In this paper we consider the case $n = 3$. We propose a model equation like 2.2 whose static solutions can be topologically classified, as well as the static solutions of the sin-Gordon equation. Then we shall prove the existence of such a solution $\overline{\varphi}$, obtained as minimum of the energy functional.

We have seen that α linear implies that the 2.2 does not have static soliton; thus we are forced to take α nonlinear:

$$\alpha(\rho) = \rho + a_2\rho^2 + a_3\rho^3 + ..;$$

now we look for the simplest choice of α. *A priori* the simplest choice is

$$\alpha(\rho) = \rho + a_2\rho^2$$

however this is a bad choice for the evolution equation 2.2, in fact $\alpha'(\rho)$ might be negative for some values of ρ and this is a "disaster" for the equation 2.2. Then it is natural to consider the next simple choice namely

$$\alpha(\rho) = \rho + a_3\rho^3;$$

this choice of α will be considered in this paper. However, from the proof of our theorem, it is not difficult to realize that more general choices are possible. We have decided to restrict ourselves to the simplest case since our aim is just to show that static solitons in dimension 3 do exist.

Other choices are possible if we do not assume that the Lagrangian splits as in 2.1 as it is in the Skyrme model [17].

So we consider maps

$$u : \mathbb{R}^{3+1} \to \mathbb{R}^4$$

(i.e. $n = 3$, $k = 4$) and we assume

$$\alpha(\rho) = \rho + \frac{\varepsilon}{3}\rho^3,$$

being $\varepsilon > 0$. Then 2.2 can be written

$$\frac{\partial}{\partial t}\left[\left(1+\varepsilon\rho^2\right)u_t\right] - c^2\nabla\left[\left(1+\varepsilon\rho^2\right)\nabla u\right] + V'(u) = 0, \tag{2.8}$$

or

$$\Box u + \varepsilon\Box_6 u + V'(u) = 0 \tag{2.9}$$

where

$$\Box_6 u = \frac{\partial}{\partial t}\left[\left(c^2\left|\nabla u\right|^2 - \left|u_t\right|^2\right)^2 u_t\right] - c^2\nabla\left[\left(c^2\left|\nabla u\right|^2 - \left|u_t\right|^2\right)^2 \nabla u\right].$$

So the static solutions φ solve the equation

$$-c^2\Delta\varphi - \varepsilon c^6\nabla\left(\left|\nabla\varphi\right|^4\nabla\varphi\right) + V'(\varphi) = 0,$$

which can be written

$$-c^2\Delta\varphi - \varepsilon c^6\Delta_6\varphi + V'(\varphi) = 0. \tag{2.10}$$

Then the energy functional becomes

$$E(\varphi) = \int_{\mathbb{R}^3}\left(\frac{c^2}{2}|\nabla\varphi|^2 + \varepsilon\frac{c^6}{6}|\nabla\varphi|^6 + V(\varphi)\right)\,dx \tag{2.11}$$

As the function V, we assume that it is defined on

$$\Omega = \mathbb{R}^4 \setminus \{\bar{\xi}\}$$

and that it is positive and diverges for $\xi \to \bar{\xi}$. More precisely we assume

- $V \in C^2(\Omega, \mathbb{R})$;
- $V(x) \geqslant V(0) = 0$ for every $x \in \Omega$;
- there exist $c, r > 0$ such that,

$$\text{if } |\xi| < r, \text{then } V(\bar{\xi}+\xi) \geqslant c|\xi|^{-6} \tag{2.12}$$

- for sake of simplicity $|\bar{\xi}| = 1$.

We observe that, for $j = 1, .., 4$,

$$\alpha'(0) = 1$$

and, since 0 is a minimum for V, we can choose a base in $\mathbb{R}^4$ which diagonalizes $V''(0)$ so that

$$V''(0) = \begin{pmatrix} m_1^2 & & 0 \\ & \ddots & \\ 0 & & m_4^2 \end{pmatrix}$$

Then, linearizing 2.9 at 0, we get a system of Klein-Gordon equations

$$\begin{aligned} \Box u^j + m_j^2 u^j &= 0, \\ (1 &\leqslant j \leqslant 4). \end{aligned}$$

The presence of Δ_6 in 2.10 implies that the functions on which E is finite are continuous and decay to 0 at infinity; the presence of the singualr term $V'(u)$ implies that such maps take value in Ω. So the non trivial topological properties of Ω (namely $\pi_3(\Omega) = \mathbb{Z}$) permit, as in the sin-Gordon equation, to give a topological classification of static configurations. This classification is carried out by means of a topological invariant, the charge (see Subsection 3.2), which depends only on the region where the function is concentrated, namely the support (see Definition 3.2).

We point out that in other models (see [17], [8], [9]), the topological classification follows from the fact that the field U takes value in suitable manifolds.

Now we state our existence result, whose precise statement will be given in Section 4, Theorem 4.2.

Theorem 2.2. *There exists $\overline{\varphi}$ weak solution of 2.10 (i.e. a static solution of 2.9), which is obtained as minimum of the energy functional in the class of maps which are not homotopic to the null map, that is with charge different from 0.*

Remark 1. *The functional E exibits an invariance for the simmetry group*

$$G = \mathbb{O}(3) \times \mathbb{R}^3$$

of rotations and traslations; indeed, for every function φ and $g \in G$, if we set

$$\varphi_g(x) = \varphi(gx),$$

we have immediately

$$E(\varphi_g) = E(\varphi).$$

Then our theorem gives the existence of an orbit of minimum solutions. This orbit consists of two connected components, which are identified respectively by $\overline{\varphi}$ *and*

$$\overline{\varphi}_{op}(x) = \overline{\varphi}(-x).$$

Since $n = 3$ *is odd,* $\overline{\varphi}$ *and* $\overline{\varphi}_{op}$ *have opposite topological charge.*

3. Functional Setting

3.1. The space H and the open set Λ

In order to choose a suitable Banach space where we are going to minimize the functional 2.11, first we consider the parts of E which do not contain V. We call internal energy of $\varphi: \mathbb{R}^3 \to \mathbb{R}^4$, the functional

$$E_i(\varphi) = \frac{c^2}{2} \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx + \varepsilon \frac{c^6}{6} \int_{\mathbb{R}^3} |\nabla \varphi|^6 \, dx. \tag{3.1}$$

Let H denote the closure of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^4)$ with respect to the norm

$$\|\varphi\| = \|\nabla \varphi\|_{L^2} + \|\nabla \varphi\|_{L^6}\,; \tag{3.2}$$

where we mean

$$\begin{aligned} \|\nabla \varphi\|_{L^2} &= \left(\sum_{j=1}^{4} \left\| \nabla \varphi^j \right\|_{L^2}^2 \right)^{1/2}; \\ \|\nabla \varphi\|_{L^6} &= \left(\sum_{j=1}^{4} \left\| \nabla \varphi^j \right\|_{L^6}^6 \right)^{1/6}. \end{aligned}$$

Now let us recall some properties of the space H.

Proposition 3.1. *There exists a continuous imbedding of* H *in* $W^{1,6}(\mathbb{R}^3, \mathbb{R}^4)$.

The proof easily follows from the Sobolev inequality,

$$\exists C > 0 \quad \text{s.t.} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^4) \qquad \|\varphi\|_{L^6} \leqslant C \, \|\nabla \varphi\|_{L^2}\,. \tag{3.3}$$

By Proposition 3.1 and well known Sobolev embeddings, we get easily some useful properties of the Banach space H.

1. There exist two costants $C_0, C_1 > 0$ such that, for every $\varphi \in H$

$$\|\varphi\|_\infty \leqslant C_0 \|\varphi\|, \tag{3.4}$$

and

$$|\varphi(x) - \varphi(y)| \leqslant C_1 |x-y|^{1/2} \|\nabla\varphi\|_{L^6}. \tag{3.5}$$

2. For every $\varphi \in H$,

$$\lim_{|x|\to\infty} \varphi(x) = 0. \tag{3.6}$$

3. If $\{\varphi_n\} \subset H$ converges weakly in H to φ, then it converges uniformly on every compact set contained in $\mathbb{R}^3$.

Since the functions in H are continuous, we can consider the set

$$\Lambda = \left\{\varphi \in H \mid \forall x \in \mathbb{R}^3 \quad \varphi(x) \neq \overline{\xi}\right\},$$

which is open in H; in fact, if $\varphi \in \Lambda$, by 3.6, we have

$$0 < d = \inf_{x\in\mathbb{R}^3} |\varphi(x) - \bar{\xi}|;$$

then, by using 3.4, we deduce that there exists a small neighborhood of φ in H contained in Λ. The boundary of Λ is given by

$$\partial\Lambda = \left\{\varphi \in H \mid \exists \bar{x} \in \mathbb{R}^3 \text{ s.t. } \varphi(\bar{x}) = \bar{\xi}\right\}.$$

We can show that Λ has a rich topological structure, more precisely it consists of infinitely many connected components. These components are identified by the topological charge we are going to introduce.

3.2. Topological Charge and Connected Components of Λ

In the open set $\Omega = \mathbb{R}^4 \setminus \{\bar{\xi}\}$ we consider the 3-sphere centered at $\bar{\xi}$

$$\Sigma = \left\{\xi \in \mathbb{R}^4 \mid |\xi - \overline{\xi}| = 1\right\}.$$

On Σ we consider the north and the south pole, denoted by ξ_N and ξ_S, with respect to the axis joining the origin with $\bar{\xi}$. Since $|\overline{\xi}| = 1$, we have

$$\xi_N = 2\bar{\xi}, \qquad \xi_S = 0.$$

We consider also the projection $P : \Omega \to \Sigma$

$$\forall \xi \in \Omega \qquad P(\xi) = \bar{\xi} + \frac{\xi - \bar{\xi}}{|\xi - \bar{\xi}|}$$

Remark 2. *For every* $\xi \in \Omega$

$$P(\xi) = 2\bar{\xi} \Longleftrightarrow \xi = (1 + |\xi - \bar{\xi}|)\,\bar{\xi},$$

therefore

$$P(\xi) = 2\bar{\xi} \Rightarrow |\xi| > 1.$$

Using 3.6 and the previous remark we can give the following definition.

Definition 3.2. *For* $\varphi \in \Lambda$, *we call support of* φ *the compact set*

$$K(\varphi) = \overline{\{x \in \mathbb{R}^3 \mid 1 < |\varphi(x)|\}},$$

(the value 1 depends on the norm of the singularity $\bar{\xi}$*). Then we define (topological) charge of* φ *the Brower degree of* $P \circ \varphi$ *in the support of* φ *with respect to the north pole of* Σ, *namely*

$$car(\varphi) = \deg(P \circ \varphi, \mathrm{int}(K(\varphi)), 2\bar{\xi}).$$

Clearly the support of φ is the minimal intrinsic subset where the degree of $P \circ \varphi$ with respect to $2\bar{\xi}$ stabilizes. More precisely the following proposition holds and it will be useful in order to study the properties of the charge.

Proposition 3.3. *For every* $\varphi \in \Lambda$ *and for every* $R > 0$ *such that* $K(\varphi) \subset B_R(0)$,

$$car(\varphi) = \deg(P \circ \varphi, B_R(0), 2\bar{\xi}). \tag{3.7}$$

Proof. Using the excision property of the degree, we have

$$\deg(P \circ \varphi, B_R(0), 2\bar{\xi}) = car(\varphi) + \deg(P \circ \varphi, B_R(0) \setminus \overline{\mathrm{int}(K(\varphi))}, 2\bar{\xi}). \tag{3.8}$$

where the second term on the right hand side is 3.8 is 0. Indeed, by definition of $K(\varphi)$, for every $x \in \mathbb{R}^3 \setminus K(\varphi)$,

$$|\varphi(x)| \leqslant 1$$

and, by Remark 2,

$$P(\varphi(x)) \neq 2\bar{\xi}.$$

Then 3.7 easily follows. ■

On the other hand we notice that, if $K(\varphi)$ consists of m connected components, $K_1, ..., K_m$, we can define also

$$car(\varphi, K_j) = \deg(P \circ \varphi, K_j, 2\bar{\xi}),$$

then, by the additivity of the degree,

$$car(\varphi) = \sum_{j=1}^{m} car(\varphi, K_j).$$

In order to use some properties of the topological degree, we need to show the following result.

Proposition 3.4. *If a sequence $\{\varphi_n\} \subset \Lambda$ converges to $\varphi \in \Lambda$ uniformly on $A \subset \mathbb{R}^3$, then also $P \circ \varphi_n$ converges to $P \circ \varphi$ uniformly on A.*

Proof. A simple calculation shows that, for every $\xi, \xi_1 \in \Omega$ and $\delta > 0$,

$$\begin{aligned} |\xi_1 - \xi| &\leqslant \delta < \left|\xi - \bar{\xi}\right| \Rightarrow \\ |P(\xi_1) - P(\xi)| &\leqslant \sqrt{2}\sqrt{1 - \sqrt{1 - \frac{\delta^2}{\left|\xi - \bar{\xi}\right|^2}}}. \end{aligned} \tag{3.9}$$

Let $\varphi \in \Lambda$ and consider $\delta > 0$ such that

$$0 < \delta < d = \inf_{x \in \mathbb{R}^3} \left|\varphi(x) - \bar{\xi}\right|.$$

Let $\{\varphi_n\}$ be a sequence in Λ converging to φ, uniformly on A. For n sufficiently large, we have

$$\forall x \in A \qquad |\varphi_n(x) - \varphi(x)| \leqslant \delta \leqslant \left|\varphi(x) - \bar{\xi}\right|.$$

Then, by 3.9, we get, for such n

$$\begin{aligned} \sup_{x \in A} |P(\varphi_n(x)) - P(\varphi(x))| &\leqslant \sqrt{2}\sqrt{1 - \sqrt{1 - \frac{\delta^2}{\left|\varphi(x) - \bar{\xi}\right|^2}}} \\ &\leqslant \sqrt{2}\sqrt{1 - \sqrt{1 - \frac{\delta^2}{d^2}}}. \end{aligned}$$

Then, for every $\eta > 0$, we can choose δ sufficiently small so that

$$\sup_{x \in A} |P(\varphi_n(x)) - P(\varphi(x))| < \eta,$$

for n large enough. ■

Proposition 3.4 permits to prove the continuity of the charge with respect to the uniform convergence. For every $A \subset \mathbb{R}^3$ we set

$$\mathsf{C}A = \mathbb{R}^3 \setminus A.$$

Theorem 3.5. *For every $\varphi \in \Lambda$ there exists $r = r(\varphi) > 0$ such that, for every $\psi \in \Lambda$*

$$\|\psi - \varphi\|_\infty \leqslant r \Rightarrow car(\psi) = car(\varphi).$$

Proof. Let $\{\varphi_n\} \subset \Lambda$ be uniformly convergent to $\varphi \in \Lambda$; we shall show that, for n sufficiently large, $car(\varphi_n) = car(\varphi)$.

By 3.6, there exists $R > 0$ such that

$$\forall x \in \mathsf{C}B_R(0) \qquad |\varphi(x)| \leqslant 1/4;$$

then

$$car(\varphi) = \deg(P \circ \varphi, B_R(0), 2\bar{\xi}). \tag{3.10}$$

Since φ_n uniformly converges to φ, for n sufficiently large,

$$\forall x \in \mathsf{C}B_R(0) \qquad |\varphi_n(x)| \leqslant 1/2,$$

which implies

$$car(\varphi_n) = \deg(P \circ \varphi_n, B_R(0), 2\bar{\xi}). \tag{3.11}$$

Now, using the previous Proposition and the continuity of topological degree with respect to the uniform convergence, we get, for n large,

$$\deg(P \circ \varphi_n, B_R(0), 2\bar{\xi}) = \deg(P \circ \varphi, B_R(0), 2\bar{\xi}).$$

So the conclusion follows by 3.10 and 3.11. ■

Now, for every $q \in \mathbb{Z}$, we set

$$\mathcal{A}_q = \{\varphi \in \Lambda \mid car(\varphi) = q\}.$$

By Theorem 3.5, it follows that $\mathcal{A}_q$ is open in H; since we have also

- $\Lambda = \bigcup_{q \in \mathbb{Z}} \mathcal{A}_q$;
- $\mathcal{A}_p \cap \mathcal{A}_q = \emptyset \quad \text{if} \quad p \neq q$;

we conclude that every $\mathcal{A}_q$ is a connected component of Λ.

We observe that for every $q \in \mathbb{Z}$, the component $\mathcal{A}_q$ is isomorphic to the component $\mathcal{A}_{-q}$. Indeed, if we set

$$\varphi_{op}(x) = \varphi(-x), \tag{3.12}$$

it is easy to see that

$$\varphi \in \mathcal{A}_q \Rightarrow \varphi_{op} \in \mathcal{A}_{-q}. \tag{3.13}$$

We point out that 3.13 is due the fact that the space dimension is odd.

Now we can define the set on which we are going to show the existence of a nontrivial minimum of the energy functional; we set

$$\Lambda^* = \bigcup_{q \neq 0} \mathcal{A}_q$$

Finally we observe that, from Remark 2, it follows that

$$\forall \varphi \in \Lambda^* : \qquad \|\varphi\|_\infty > 1. \tag{3.14}$$

which implies that $K(\varphi)$ is not empty.

3.3. Properties of the Energy Functional

Clearly the energy functional E defined in 2.11 in coercive in the H norm, namely

$$\lim_{\|\varphi\| \to \infty} E(\varphi) = +\infty.$$

Moreover, as immediate consequence of the continuous embedding of H in L^∞ (see 3.4), we have the following property.

Proposition 3.6. *There exists $\Delta^* > 0$ such that, for every $\varphi \in \Lambda$,*

$$\|\varphi\|_\infty \geqslant 1 \Rightarrow E(\varphi) \geqslant \Delta^*. \tag{3.15}$$

Now we are going to study the properties of the second piece of the energy functional

$$\int_{\mathbb{R}^3} V(\varphi)\, dx.$$

Since $\mathbb{R}^3$ is unbounded, it is obvious that this integral can diverge, even if we choose $\varphi \in \Lambda$.

Remark 3. *It is not difficult to see that E takes real values and it is continuous on*

$$M = \Lambda \cap L^2(\mathbb{R}^3, \mathbb{R}^4).$$

Nevertheless we cannot choose $H \cap L^2$ as "basic" function space since E is not coercive, in general, with respect to the norm $\|\cdot\| + \|\cdot\|_{L^2}$. Such coercivity property holds if 0 is the unique zero of the potential V and it is non degenerate.

First we are going to study the behaviour of E when φ approachs the boundary of Λ, in the spirit of a well known result of Gordon (see [11]), concerning strongly attractive potentials.

Lemma 3.7. *Let $\{\varphi_n\} \subset \Lambda$ be bounded in the H norm and weakly converging to $\varphi \in \partial\Lambda$, then*

$$\int_{\mathbb{R}^3} V(\varphi_n)\, dx \to +\infty.$$

Proof. Since $\varphi \in \partial\Lambda$, there exists $\bar{x} \in \mathbb{R}^3$ such that $\varphi(\bar{x}) = \bar{\xi}$; since V is nonnegative, it is sufficient to show that there exists a small ball centered at $\bar{x}$ such that

$$\int_{B_\rho(\bar{x})} V(\varphi_n)\, dx \to +\infty. \tag{3.16}$$

By the uniform convergence on compact sets we have

$$\lim_{n\to\infty} \varphi_n(\bar{x}) = \bar{\xi}. \tag{3.17}$$

Now we show that there exists $\rho > 0$ such that, for every $x \in B_\rho(\bar{x})$ and for n sufficiently large,

$$|\varphi_n(x) - \bar{\xi}| < r, \tag{3.18}$$

where r has been introduced in 2.12.

Since $\{\varphi_n\}$ is bounded in H, then, in particular $\{\nabla\varphi_n\}$ is bounded in L^6. Using 3.5 and the boundedness of $\{\nabla\varphi_n\}$ in L^6, we have

$$|\varphi_n(x) - \varphi_n(\bar{x})| \leqslant \text{const } |x - \bar{x}|^{1/2}, \tag{3.19}$$

for every $x \in \mathbb{R}^3$. Then 3.18 easily follows from 3.17 and 3.19. We have also

$$|\varphi_n(x) - \bar{\xi}| \leqslant \text{const } |x - \bar{x}|^{1/2} + o(1). \tag{3.20}$$

Now, using 3.18 and 2.12, for every $x \in B_\rho(\bar{x})$, we have

$$V(\varphi_n(x)) \geqslant \frac{c}{|\varphi_n(x) - \bar{\xi}|^6};$$

then, using 3.20, we obtain

$$V(\varphi_n(x)) \geqslant \frac{c}{\text{const } |x - \bar{x}|^3 + o(1)}.$$

Integrating on $B_\rho(\bar{x})$, we get 3.16. ■

From this lemma we immediately deduce that the sublevels of E are complete and that the following proposition holds.

Proposition 3.8. *Let $\{\varphi_n\} \subset \Lambda$ be weakly converging to φ and such that $E(\varphi_n)$ is bounded, then $\varphi \in \Lambda$.*

From Proposition 3.8 we get also the following result.

Proposition 3.9. *For every $a > 0$, there exists $d > 0$ such that, for every $\varphi \in \Lambda$*

$$E(\varphi) \leqslant a \Rightarrow \min_{x \in \mathbb{R}^3} |\varphi(x) - \bar{\xi}| \geqslant d.$$

Proof. Arguing by contradiction, assume that there exist $a > 0$ and a sequence $\{\varphi_n\} \subset \Lambda$ such that

$$\begin{aligned} E(\varphi_n) &\leqslant a \\ \min_{x \in \mathbb{R}^3} |\varphi_n(x) - \bar{\xi}| &< 1/n. \end{aligned} \tag{3.21}$$

For every $n \in \mathbb{N}$, by 3.6, there exists $x_n \in \mathbb{R}^3$ such that

$$|\varphi_n(x_n) - \bar{\xi}| = \min_{x \in \mathbb{R}^3} |\varphi_n(x) - \bar{\xi}|$$

Then we can consider the sequence

$$\psi_n = \varphi_n(\cdot + x_n).$$

Since

$$E(\psi_n) = E(\varphi_n) \leqslant a, \tag{3.22}$$

we have that $\{\psi_n\}$ is bounded in H, then, up to subsequence, it weakly converges to ψ.

Now, from the definition of ψ_n and 3.21, we obtain

$$\psi(0) = \lim_{n \to \infty} \psi_n(0) = \bar{\xi};$$

therefore $\psi \in \partial\Lambda$. Taking into account 3.22 we have got a contradiction with Proposition 3.8. ■

Now we prove the weakly lower semicontinuity of the energy functional E.

Proposition 3.10. *For every $\varphi \in \Lambda$ and for every sequence $\{\varphi_n\} \subset \Lambda$, if $\{\varphi_n\}$ weakly converges to φ, then*

$$\liminf_{n \to \infty} E(\varphi_n) \geqslant E(\varphi).$$

Proof. The result is obvious when

$$\liminf_{n \to \infty} E(\varphi_n) = +\infty,$$

then we have to study the case

$$\liminf_{n\to\infty} E(\varphi_n) < +\infty.$$

Clearly

$$\liminf_{n\to\infty} E_i(\varphi_n) \geqslant E_i(\varphi). \tag{3.23}$$

where E_i is defined in 3.1. Now we have to study

$$\int_{\mathbb{R}^3} V(\varphi_n)\ dx.$$

Since $\{\varphi_n\}$ converges to φ uniformly on every compact set, we fix a sphere $B_R(0)$ and we have

$$\lim_{n\to\infty} \int_{B_R(0)} V(\varphi_n)\ dx = \int_{B_R(0)} V(\varphi)\ dx.$$

On the other hand, since V is nonnegative, we have

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} V(\varphi_n)\ dx \geqslant \liminf_{n\to\infty} \int_{B_R(0)} V(\varphi_n)\ dx = \int_{B_R(0)} V(\varphi)\ dx,$$

and taking the limit for $R \to \infty$, we obtain

$$\liminf_{n\to\infty} \int_{\mathbb{R}^3} V(\varphi_n)\ dx \geqslant \int_{\mathbb{R}^3} V(\varphi)\ dx. \tag{3.24}$$

By 3.23 and 3.24 the Proposition is completely proved. ■

Theorem 3.11. *The minimum points $\varphi \in \Lambda$ for the functional E are weak solutions of the system 2.10.*

Proof. Let $\varphi \in \Lambda$ be a minimum point of E and $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{R})$. Let $\mathbf{e}_j$ denote the j-th vector of the canonical base in $\mathbb{R}^4$.

If $s > 0$ is sufficiently small, then $\varphi + s\psi\mathbf{e}_j \in \Lambda$ and $E(\varphi + s\psi\mathbf{e}_j) < +\infty$. Since φ is a minimum point of E, differentiating $E(\varphi + s\psi\mathbf{e}_j)$ with respect to s, we have

$$\begin{aligned} 0 &= \frac{d}{ds} E(\varphi + s\psi\mathbf{e}_j)\Big|_{s=0} = \\ &= \int_{\mathbb{R}^3} c^2 \nabla\varphi^j \cdot \nabla\psi + \varepsilon c^6 \left|\nabla\varphi\right|^4 \nabla\varphi^j \cdot \nabla\psi + \frac{\partial V}{\partial \xi_j}(\varphi)\psi\ dx. \end{aligned}$$

■

4. Existence Theorems

4.1. The splitting lemma

The proof of our main result is based on the following proposition, in the spirit of Concentration-Compactness principle for unbounded domains (see [2], [13])

Lemma 4.1 (Splitting lemma). *Let* $\{\varphi_n\} \subset \Lambda^*$ *such that*

$$E(\varphi_n) \leqslant a. \tag{4.1}$$

There exists $l \in \mathbb{N}$,

$$1 \leqslant l \leqslant a/\Delta^*, \tag{4.2}$$

(Δ^* *has been introduced in Proposition 3.6) and there exist* $\overline{\varphi}_1, ..., \overline{\varphi}_l \in \Lambda$, $\{x_n^1\}, ..., \{x_n^l\} \subset \mathbb{R}^3$, $R_1, ..., R_l > 0$ *such that, up to subsequence,*

$$\varphi_n(\cdot + x_n^i) \rightharpoonup \overline{\varphi}_i; \tag{4.3}$$

$$\|\overline{\varphi}_i\|_\infty \geqslant 1; \tag{4.4}$$

$$|x_n^i - x_n^j| \to \infty \qquad \textit{for} \qquad i \neq j; \tag{4.5}$$

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n\to\infty} E(\varphi_n); \tag{4.6}$$

$$\forall x \in \mathsf{C}\left(\bigcup_{i=1}^{l} B_{R_i}(x_n^i)\right) : \quad |\varphi_n(x)| \leqslant 1. \tag{4.7}$$

Then we have also

$$car(\varphi_n) = \sum_{i=1}^{l} car(\overline{\varphi}_i); \tag{4.8}$$

$$\limsup_{n\to\infty} \left\| \varphi_n - \sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \right\|_\infty \leqslant 1. \tag{4.9}$$

Remark 4. *We notice that, from 4.4, it follows*

$$E(\overline{\varphi}_i) \geqslant \Delta^*. \tag{4.10}$$

Proof. The proof is divided in two parts. In the first part, with an iterative procedure, we prove the existence of $l \in \mathbb{N}$, $\overline{\varphi}_1, ..., \overline{\varphi}_l \in \Lambda, \{ x_n^1\}, ..., \{x_n^l\} \subset \mathbb{R}^3, R_1, ..., R_l > 0$ such that 4.2-4.7 are satisfied; in the second part from these properties we shall easily deduce 4.8 and 4.9. For the sake of simplicity, whenever it is necessary, we shall tacitly consider a subsequence of $\{\varphi_n\}$.

First of all we arbitrarily choose $\gamma \in]0, 1[$.

Let $x_n^1 \in \mathbb{R}^3$ be a maximum point for $|\varphi_n|$; by 3.14 we have $\left|\varphi_n\left(x_n^1\right)\right| > 1$. We set

$$\varphi_n^1 = \varphi_n(\cdot + x_n^1)$$

and we obtain

$$\left\|\varphi_n^1\right\|_\infty = \left|\varphi_n^1(0)\right| > 1. \tag{4.11}$$

Since $E(\varphi_n^1) = E(\varphi_n)$ and the functional E is coercive, then the sequence $\{\varphi_n^1\}$ is bounded in H and we have

$$\varphi_n^1 \rightharpoonup \overline{\varphi}_1 \in H, \tag{4.12}$$

From 4.11 it follows $\|\overline{\varphi}_1\|_\infty \geqslant 1$.

Since $\{\varphi_n^1\} \subset \Lambda$ and $E(\varphi_n^1)$ is bounded, by 4.12 and Proposition 3.8, we get $\overline{\varphi}_1 \in \Lambda$.

Since E is weakly lower semicontinuous, we have

$$E(\overline{\varphi}_1) \leqslant \liminf_{n\to\infty} E(\varphi_n^1) = \liminf_{n\to\infty} E(\varphi_n) \tag{4.13}$$

Now, using 3.6, we consider $R_1 > 0$ such that

$$\forall x \in \mathsf{C}B_{R_1}(0) \qquad |\overline{\varphi}_1(x)| \leqslant \gamma; \tag{4.14}$$

for simplicity we set

$$B_n^1 = B_{R_1}(x_n^1).$$

Now we distinguish two cases: either

A1) for n sufficiently large

$$\forall x \in \mathsf{C}B_n^1 : \quad |\varphi_n(x)| \leqslant 1;$$

or

B1) eventually passing to a subsequence,

$$\exists x \in \mathsf{C}B_n^1 \quad \text{s.t.} \quad |\varphi_n(x)| > 1.$$

In the case **A**1) the first part of Proposition is proved with $l = 1$; let us consider the case **B**1).

Let x_n^2 be a maximum point for $|\varphi_n|$ in $\mathbb{R}^3 \setminus B_n^1$; we have that $\left|\varphi_n\left(x_n^2\right)\right| > 1$. We set

$$\varphi_n^2 = \varphi_n(\cdot + x_n^2)$$

and we obtain

$$\left\|\varphi_n^2\right\|_\infty = \left|\varphi_n^2(0)\right| > 1.$$

As for $\{\varphi_n^1\}$, we have that

$$\varphi_n^2 \rightharpoonup \overline{\varphi}_2 \in \Lambda, \tag{4.15}$$

with

$$\|\overline{\varphi}_2\|_\infty \geqslant 1. \tag{4.16}$$

Now we have to show that

$$\left|x_n^2 - x_n^1\right| \to \infty. \tag{4.17}$$

We set

$$y_n = x_n^2 - x_n^1$$

and, arguing by conctradiction, we assume that the sequence $\{y_n\}$ is bounded in $\mathbb{R}^3$; then, up to subsequence, we have that

$$y_n \to \tilde{y}.$$

Since $|y_n| = \left|x_n^2 - x_n^1\right| \geqslant R_1$, we have $|\tilde{y}| \geqslant R_1$; then, using 4.14,

$$|\overline{\varphi}_1(\tilde{y})| \leqslant \gamma < 1. \tag{4.18}$$

On the other hand we have

$$1 \leqslant \left|\varphi_n(x_n^2)\right| = \left|\varphi_n(y_n + x_n^1)\right| = \left|\varphi_n^1(y_n)\right|,$$

then, by 4.18,

$$\begin{aligned} 0 \;<\; & 1 - |\overline{\varphi}_1(\tilde{y})| \leqslant \left|\varphi_n^1(y_n)\right| - |\overline{\varphi}_1(\tilde{y})| \leqslant \left|\varphi_n^1(y_n) - \overline{\varphi}_1(\tilde{y})\right| \leqslant \\ \leqslant\; & \left|\varphi_n^1(y_n) - \overline{\varphi}_1(y_n)\right| + |\overline{\varphi}_1(y_n) - \overline{\varphi}_1(\tilde{y})| \leqslant \\ \leqslant\; & \left(\sup_{|y-\tilde{y}|\leqslant 1} \left|\varphi_n^1(y) - \overline{\varphi}_1(y)\right|\right) + |\overline{\varphi}_1(y_n) - \overline{\varphi}_1(\tilde{y})|. \end{aligned}$$

Taking the limit for $n \to \infty$ we get a contradiction.

Now we show that

$$E(\overline{\varphi}_1) + E(\overline{\varphi}_2) \leqslant E(\varphi_n). \tag{4.19}$$

Hereafter, for sake of simplicity, we set, for every $\varphi \in \Lambda$ and $A \subset \mathbb{R}^3$

$$E_{|A}(\varphi) = \int_A \left(\frac{c^2}{2} |\nabla \varphi|^2 + \varepsilon \frac{c^6}{6} |\nabla \varphi|^6 + V(\varphi) \right) \, dx.$$

For a fixed $\eta > 0$, there exists $\rho > 0$ such that

$$E_{|\complement B_\rho(0)}(\overline{\varphi}_1) < \eta/2 \qquad \text{and} \qquad E_{|\complement B_\rho(0)}(\overline{\varphi}_2) < \eta/2.$$

From 4.17 it follows that the spheres $B_\rho(x_n^1)$ and $B_\rho(x_n^2)$ are disjoint for n sufficiently large, then we get:

$$\begin{aligned}
\liminf_{n\to\infty} E(\varphi_n) &\geqslant \liminf_{n\to\infty} \left(E_{|B_\rho(x_n^1)}(\varphi_n) + E_{|B_\rho(x_n^2)}(\varphi_n) \right) \geqslant \\
&\geqslant \liminf_{n\to\infty} E_{|B_\rho(x_n^1)}(\varphi_n) + \liminf_{n\to\infty} E_{|B_\rho(x_n^2)}(\varphi_n) = \\
&= \liminf_{n\to\infty} E_{|B_\rho(0)}(\varphi_n^1) + \liminf_{n\to\infty} E_{|B_\rho(0)}(\varphi_n^2) \geqslant \\
&\geqslant E_{|B_\rho(0)}(\overline{\varphi}_1) + E_{|B_\rho(0)}(\overline{\varphi}_2) > \\
&> E(\overline{\varphi}_1) + E(\overline{\varphi}_2) - \eta.
\end{aligned}$$

From the arbitrariness of η, we get 4.19.

Finally, as well as for $\overline{\varphi}_1$, from 3.6 we get $R_2 > 0$, such that

$$\forall x \in \complement B_{R_2}(0) \qquad |\overline{\varphi}_2(x)| \leqslant \gamma$$

and we set

$$B_n^2 = B_{R_2}(x_n^2)$$

Also in this second step we have an alternative: either

A2) for n sufficiently large,

$$\forall x \in \complement(B_n^1 \cup B_n^2) : \quad |\varphi_n(x)| \leqslant 1;$$

or

B2) up to a subsequence,

$$\exists x \in \complement(B_n^1 \cup B_n^2) \quad \text{s.t.} \quad |\varphi_n(x)| > 1.$$

If case **A**2) holds true, the first part of Proposition is proved with $l = 2$; in the case **B**2) we consider a maximum point of $|\varphi_n|$ in $\complement(B_n^1 \cup B_n^2)$ and we repeat the same argument used in the case **B**1)**.**

This alternative process terminates in a finite number of steps. Indeed, using 4.10, 4.6 and 4.1, we get 4.2; we notice that this estimate is independent on the sequence $\{\varphi_n\}$.

Now we prove 4.8. We consider n sufficiently large so that 4.7 holds and

$$B_n^i \cap B_n^j = \emptyset \qquad \text{for} \qquad i \neq j. \tag{4.20}$$

Then we have, by the additivity property of the topological degree,

$$\begin{aligned} car(\varphi_n) &= \deg\left(P \circ \varphi_n, \bigcup_{i=1}^{l} B_n^i, 2\bar{\xi}\right) = \sum_{i=1}^{l} \deg\left(P \circ \varphi_n, B_n^i, 2\bar{\xi}\right) = \\ &= \sum_{i=1}^{l} \deg\left(P \circ \varphi_n^i, B_{R_i}(0), 2\bar{\xi}\right). \end{aligned} \tag{4.21}$$

On the other hand, for every $i \in \{1, ..., l\}$, since $\{\varphi_n^i\}$ converges uniformly to $\overline{\varphi}_i$ on $B_{R_i}(0)$ and

$$\forall x \in \mathsf{C}B_{R_i}(0) \qquad |\overline{\varphi}_i(x)| \leqslant \gamma < 1,$$

we obtain, for n large enough,

$$\deg\left(P \circ \varphi_n^i, B_{R_i}(0), 2\bar{\xi}\right) = \deg\left(P \circ \overline{\varphi}_i, B_{R_i}(0), 2\bar{\xi}\right) = car(\overline{\varphi}_i).$$

Then, substituting in 4.21, we obtain 4.8.

Finally, in order to prove 4.9, we assume that, for every $i \in \{1, ..., l\}$,

$$\forall x \in B_n^i : \qquad \left|\varphi_n(x) - \overline{\varphi}_i(x - x_n^i)\right| < \gamma. \tag{4.22}$$

We shall prove that, for n large enough,

$$\forall x \in \mathbb{R}^3 : \qquad \left|\varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i)\right| < 1 + l\gamma. \tag{4.23}$$

Indeed, if $x \in \bigcup_{i=1}^{l} B_n^i$, then, by 4.20, there exists a unique index $j \in \{1, ..., l\}$ such that $x \in B_n^j$, then

$$\begin{aligned} \left|\varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i)\right| &\leqslant \left|\varphi_n(x) - \overline{\varphi}_j(x - x_n^j)\right| + \sum_{i \neq j} \left|\overline{\varphi}_i(x - x_n^i)\right| \leqslant \\ &< \gamma + (l-1)\gamma = l\gamma < 1 + l\gamma. \end{aligned} \tag{4.24}$$

On the other hand, if $x \notin \bigcup_{i=1}^{l} B_n^i$, then, by 4.7,

$$\begin{aligned}\left|\varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i)\right| &\leqslant |\varphi_n(x)| + \sum_{i=1}^{l} \left|\overline{\varphi}_i(x - x_n^i)\right| \leqslant \\ &\leqslant 1 + l\gamma.\end{aligned}$$

Now fix $\eta > 1$; choosing γ sufficiently small we have

$$1 + l\gamma < \eta \tag{4.25}$$

(taking in account 4.2, this kind of choice can be made a priori in the proof). Substituting 4.25 in 4.23, we get

$$\forall x \in \mathbb{R}^3 : \qquad \left|\varphi_n(x) - \sum_{i=1}^{l} \overline{\varphi}_i(x - x_n^i)\right| < \eta,$$

and, by the arbitrariness of $\eta > 1$, we obtain 4.9. ■

Remark 5. *Consider the function $\sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i)$ which has been introduced in 4.9; using 3.6 and 4.5, it is not difficult to show that, for n large enough,*

$$\sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i) \in \Lambda,$$

$$car\left(\sum_{i=1}^{l} \overline{\varphi}_i(\cdot - x_n^i)\right) = \sum_{i=1}^{l} car(\overline{\varphi}_i).$$

4.2. Existence of minima in the connected components of Λ

Finally we can give the proof of Theorem 2.2. We set

$$E^* = \inf E(\Lambda^*).$$

By 3.14 it follows

$$\Delta^* \leqslant E^*.$$

Theorem 4.2. *There exists $\overline{\varphi} \in \Lambda^*$ such that $E(\overline{\varphi}) = E^*$.*

Proof. We consider a minimizing sequence $\{\varphi_n\} \subset \Lambda^*$. It has obviously bounded energy; then we can apply Proposition 4.1. There exist $l \in \mathbb{N}$ and $\overline{\varphi}_1, ..., \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n\to\infty} E(\varphi_n) = E^*; \tag{4.26}$$

$$car(\varphi_n) = \sum_{i=1}^{l} car(\overline{\varphi}_i). \tag{4.27}$$

Since $car(\varphi_n) \neq 0$, from 4.27 we deduce that there exists $i \in \{1, ..., l\}$, for sake of simplicity $i = 1$, such that $car(\overline{\varphi}_1) \neq 0$.

Then, by 4.26, we obtain

$$E^* \geqslant \sum_{i=1}^{l} E(\overline{\varphi}_i) \geqslant E(\overline{\varphi}_1) \geqslant E^*;$$

so we get $E(\overline{\varphi}_1) = E^*$. ■

Remark 6. *We recall that, for every $\varphi \in \Lambda$,*

$$car(\varphi_{op}) = -car(\varphi),$$
$$E(\varphi_{op}) = E(\varphi),$$

where $\varphi_{op}(x) = \varphi(-x)$. Then Theorem 4.2 can be restated in two different ways:

- *there exists $\overline{\varphi} \in \Lambda_+^*$, such that*

  $$E(\overline{\varphi}) = \inf E(\Lambda_+^*),$$

  where

  $$\Lambda_+^* = \bigcup_{q>0} \mathcal{A}_q;$$

- *there exists at least two solutions of 2.10 having opposite topological charge (more precisely, taking in account Remark 1, in Subsection 2.2, we have the existence of two connected manifolds of solutions).*

For every $q \in \mathbb{N}$, we set

$$E_q = \inf E(\mathcal{A}_q) = \inf E(\mathcal{A}_{-q})$$

Now we want to study when the minima E_q are attained. It is obvious that the functional E takes its absolute minimum 0 in the class $\mathcal{A}_0$. On the other hand, by Theorem 4.2, there exists at least $\bar{q} = car(\overline{\varphi}) \neq 0$ such that $E_{\bar{q}}$ is attained.

We can show the following properties.

Proposition 4.3. *For every $q \neq 0$, a sufficient condition to guarantee that E_q is attained is that there exists a minimizing sequence in $\mathcal{A}_q$ which satisfies the properties of Proposition 4.1 with $l = 1$. So, if E_q is not attained, then every minimizing sequence in $\mathcal{A}_q$ satisfies the properties of Proposition 4.1 with $l \geqslant 2$.*

Proof. Let $\{\varphi_n\}$ be a minimizing sequence in $\mathcal{A}_q$ which satisfies the properties of Proposition 4.1 with $l = 1$. From 4.8, we have

$$car(\overline{\varphi}_1) = car(\varphi_n) = q,$$

which implies

$$E(\overline{\varphi}_1) \geqslant E_q.$$

On the other hand, by 4.6,

$$E_q = \liminf_{n\to\infty} E(\varphi_n) \geqslant E(\overline{\varphi}_1).$$

So we conclude $E(\overline{\varphi}_1) = E_q$. ■

Proposition 4.4. *For every $q \neq 0$, if $E_q < 2\,E^*$, then the value E_q is attained in $\mathcal{A}_q$.*

Proof. Let $\{\varphi_n\}$ be a minimizing sequence in $\mathcal{A}_q$. For n sufficiently large we have

$$E(\varphi_n) < 2E^*. \tag{4.28}$$

Using Proposition 4.1, there exist $l \in \mathbb{N}$ and $\overline{\varphi}_1, ..., \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n\to\infty} E(\varphi_n) = E_q; \tag{4.29}$$

$$q = car(\varphi_n) = \sum_{i=1}^{l} car(\overline{\varphi}_i). \tag{4.30}$$

By 4.30 there exists $i \in \{1, ..., l\}$ such that $car(\overline{\varphi}_i) \neq 0$; by 4.28 and 4.29 such index is unique; for sake of simplicity we mean $i = 1$. Then we have

$$car(\overline{\varphi}_1) = car(\varphi_n) = q$$

and, substituting in 4.29,

$$E_q \geqslant \sum_{i=1}^{l} E(\overline{\varphi}_i) \geqslant E_q + \sum_{i=2}^{l} E(\overline{\varphi}_i).$$

So we obtain

$$\sum_{i=2}^{l} E(\overline{\varphi}_i) = 0$$

and we have necessarily $l = 1$. Then we can apply Proposition 4.3. ■

Let

$$S = \{q \in \mathbb{N}^* \mid E_q \text{ is attained}\}$$

Theorem 4.2 states that S is not empty; the following Theorem provides a more precise information.

Theorem 4.5. *The ideal spanned by S coincides with $\mathbb{Z}$.*

Remark 7. *From Theorem 4.5 we deduce the following alternative. Either the set S contains 1, that is E attains its minimum in the classes $\mathcal{A}_{\pm 1}$; or the set S contains at least two elements, that is E has two (pairs of) minima, with different charge.*

Proof. Arguing by contradiction, assume that

$$A = \mathbb{Z} \setminus \text{span}\{S\} \neq \emptyset.$$

Then we set

$$\begin{aligned} \mathcal{A} &= \bigcup_{q \in A} \mathcal{A}_q \\ E_A &= \inf E(\mathcal{A}) \end{aligned}$$

Let $\{\varphi_n\} \subset \mathcal{A}$ be such that $E(\varphi_n) \to E_A$. Since $\mathcal{A} \subset \Lambda^*$, we can apply Proposition 4.1: there exist $l \in \mathbb{N}$, $\overline{\varphi}_1, ..., \overline{\varphi}_l \in \Lambda$ such that, up to a subsequence,

$$E(\overline{\varphi}_i) \geqslant \Delta^* > 0; \tag{4.31}$$

$$\sum_{i=1}^{l} E(\overline{\varphi}_i) \leqslant \liminf_{n \to \infty} E(\varphi_n) = E_A; \tag{4.32}$$

$$car(\varphi_n) = \sum_{i=1}^{l} car(\overline{\varphi}_i). \tag{4.33}$$

For simplicity we set

$$\sum_{i=1}^{l} car(\overline{\varphi}_i) = \bar{q}; \tag{4.34}$$

substituting in 4.33, we get

$$car(\varphi_n) = \bar{q}, \tag{4.35}$$

then, since $\{\varphi_n\} \subset \mathcal{A}$, it follows

$$\bar{q} \in A. \tag{4.36}$$

Then, since $\mathcal{A}_{\bar{q}} \subset \mathcal{A}$ and using 4.35, we get

$$E_A \leqslant E_{\bar{q}} \leqslant E(\varphi_n).$$

So we conclude that $\{\varphi_n\}$ is a minimizing sequence in $\mathcal{A}_{\bar{q}}$.

From 4.34 and 4.36, we deduce that there exists $i \in \{1, ..., l\}$, for simplicity we mean $i = 1$, such that $car(\overline{\varphi}_1) \in A$; indeed, arguing by contradiction, if $car(\overline{\varphi}_i) \notin A$ for all $i \in \{1, ..., l\}$, *i.e.* $car(\overline{\varphi}_i) \in \text{span}\{S\}$, then by 4.34, $\bar{q} \in \text{span}\{S\}$, which contradicts 4.36.

On the other hand $E_{\bar{q}}$ is not attained, since $\bar{q} \in A \subset \mathbb{Z}\backslash S$, then, by Proposition 4.3, we get $l \geqslant 2$. Therefore, using again 4.32 and 4.31, we get the contradiction

$$E_A \geqslant E(\overline{\varphi}_1) + \sum_{i=2}^{l} E(\overline{\varphi}_i) \geqslant E_A + (l-1)\Delta^* > E_A.$$

■

Remark 8. *We notice that our static problem 2.10 have some similarities with the problem of finding homoclinic solutions for dynamical systems with a singular potential; for this latter problem the results we know are confined to planar dynamical systems, where the solutions are classified by the winding number (see [15],[6] and therein references).*

5. Mass and energy

5.1. The mass matrix

In order to show the relativistic behaviour of solitons defined in Section 2, we need to define their inertial mass. The most natural thing to do is to define it as the ratio between the momentum and the velocity of the soliton.

However this definition is possible only if the momentum and the velocity are parallel; otherwise the mass is a tensor. So the first thing to do is to compute the momentum and then the mass tensor of our solitons.

Let $u(x,t) = u(x_1, x_2, x_3, t)$ be a solution of the 2.2 (where for simplicity we set $c = 1$). It is well known (see e.g. [10]) that the momentum of u is given by

$$p_k = \int T_{0,k}\, dx, \quad k = 1, 2, 3 \tag{5.1}$$

where $T_{i,k}$ is the energy-momentum tensor defined by

$$T_{ki} = \sum_{j=1}^{n} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial u_j}{\partial x_k}\right)} \frac{\partial u_j}{\partial x_i} - \delta_{ik}\mathcal{L}; \quad with\ x_0 := t \tag{5.2}$$

In our case, for $k \neq 0$, we have

$$T_{k,0} = -\alpha'(\rho)\, u_{x_k} \cdot u_t \tag{5.3}$$

so we can easily compute the momentum of the soliton 2.4.

Lemma 5.1. *Let $u_{\mathbf{v}}$ be the solution of 2.2 (with $c = 1$) given by the 2.4, then the momentum is given by*

$$\begin{aligned} p_1(u_{\mathbf{v}}) &= v\gamma \int \alpha'(|\nabla\varphi|^2) \left(\frac{\partial\varphi}{\partial x_1}\right)^2 dx \\ p_2(u_{\mathbf{v}}) &= v \int \alpha'\left(|\nabla\varphi|^2\right) \frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial x_2} dx \\ p_3(u_{\mathbf{v}}) &= v \int \alpha'\left(|\nabla\varphi|^2\right) \frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial x_3} dx \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

Proof. By the 5.3 we have

$$p_1(u_{\mathbf{v}}) = -\int \alpha'\left(|\nabla u_{\mathbf{v}}|^2 - \left(\frac{\partial u_{\mathbf{v}}}{\partial t}\right)^2\right) \frac{\partial u_{\mathbf{v}}}{\partial x_1}\frac{\partial u_{\mathbf{v}}}{\partial t}\, dx;$$

exploiting the Lorentz invariance of $|\nabla u_{\mathbf{v}}|^2 - \left(\frac{\partial u_{\mathbf{v}}}{\partial t}\right)^2$, we get

$$\left[|\nabla_x u_{\mathbf{v}}|^2 - \left(\frac{\partial u_{\mathbf{v}}}{\partial t}\right)^2\right](x_1, x_2, x_3) = |\nabla_\xi \varphi|^2 (\xi_1, \xi_2, \xi_3)$$

where

$$\begin{cases} \xi_1 = \gamma(x_1 - vt) \\ \xi_2 = x_2 \\ \xi_3 = x_3 \end{cases}$$

Thus

$$
\begin{aligned}
p_1(u_{\mathbf{v}}) &= -\int \alpha'\left(|\nabla_\xi \varphi|^2\right)\, \frac{\partial u_{\mathbf{v}}}{\partial x_1}\frac{\partial u_{\mathbf{v}}}{\partial t}\, dx_1\, dx_2\, dx_3 \\
&= -\int \alpha'\left(|\nabla_\xi \varphi|^2\right)\, \frac{\partial \varphi}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1}\frac{\partial \varphi}{\partial \xi_1}\frac{\partial \xi_1}{\partial t}\, \frac{1}{\gamma}\, d\xi_1\, d\xi_2\, d\xi_3 \\
&= \gamma v \int \alpha'\left(|\nabla_\xi \varphi|^2\right)\, \left(\frac{\partial \varphi}{\partial \xi_1}\right)^2 d\xi_1\, d\xi_2\, d\xi_3.
\end{aligned}
$$

Now let us compute p_2 :

$$
\begin{aligned}
p_2(u_{\mathbf{v}}) &= -\int \alpha'\left(|\nabla u_{\mathbf{v}}|^2 - \left(\frac{\partial u_{\mathbf{v}}}{\partial t}\right)^2\right)\frac{\partial u_{\mathbf{v}}}{\partial x_2}\frac{\partial u_{\mathbf{v}}}{\partial t}\, dx \\
&= -\int \alpha'\left(|\nabla_\xi \varphi|^2\right)\, \frac{\partial u_{\mathbf{v}}}{\partial x_2}\frac{\partial u_{\mathbf{v}}}{\partial t}\, dx_1\, dx_2\, dx_3 \\
&= \int \alpha'\left(|\nabla_\xi \varphi|^2\right)\, \frac{\partial \varphi}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_1}\gamma v\, \frac{1}{\gamma}\, d\xi_1\, d\xi_2\, d\xi_3 \\
&= v \int \alpha'\left(|\nabla_\xi \varphi|^2\right)\, \frac{\partial \varphi}{\partial \xi_2}\frac{\partial \varphi}{\partial \xi_1} d\xi_1\, d\xi_2\, d\xi_3.
\end{aligned}
$$

■

The mass matrix $\{m_{ij}\}$ of $u_{\mathbf{v}}$ is defined by the following relation:

$$
p_i(u_{\mathbf{v}}) = \sum_{j=1}^{4} m_{ij}(u_{\mathbf{v}}) v_j,
$$

then, by the previous lemma we get:

$$
\begin{aligned}
m_{11}(u_{\mathbf{v}}) &= \gamma \int \alpha'(|\nabla \varphi|^2)\left(\frac{\partial \varphi}{\partial x_1}\right)^2 dx \\
m_{12}(u_{\mathbf{v}}) &= \int \alpha'\left(|\nabla \varphi|^2\right)\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_2} dx \\
m_{13}(u_{\mathbf{v}}) &= \int \alpha'\left(|\nabla \varphi|^2\right)\frac{\partial \varphi}{\partial x_1}\frac{\partial \varphi}{\partial x_3} dx.
\end{aligned}
$$

and by simmetry arguments we get:

$$
m_{ii}(u_{\mathbf{v}}) = \gamma \int \alpha'(|\nabla \varphi|^2)\left(\frac{\partial \varphi}{\partial x_i}\right)^2 dx \tag{5.4}
$$

$$
m_{ij}(u_{\mathbf{v}}) = \int \alpha'\left(|\nabla \varphi|^2\right)\frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_j} dx, \quad for\ i \neq j \tag{5.5}
$$

So the rest mass matrix is defined as follows:

$$m_{ij}(u_{\mathbf{v}}) = \int \alpha'\left(|\nabla\varphi|^2\right)\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx.$$

5.2. Equivalence between mass and energy

Now let us compute the energy of a soliton. By the (5.2) we have

$$E(u_{\mathbf{v}}) = \int T_{0,0}\,dx \tag{5.6}$$

Since the direct computation of $T_{0,0}$ is quite involved, we start to compute the energy of a static solution; in this case it is easy to see that

$$E(\varphi) = \int T_{0,0}\,dx = \int \left[\frac{1}{2}\alpha(|\nabla\varphi|^2) + V(\varphi)\right]\,dx \tag{5.7}$$

At this point we need the following lemma

Lemma 5.2. *Let φ be a critical point of the energy (2.7) (and hence a weak solution of the equation (2.3)); then*

$$\frac{1}{2}\int \alpha(|\nabla\varphi|^2)\,dx + \int V(\varphi)\,dx = \int \alpha'(|\nabla\varphi|^2)\left(\frac{\partial\varphi}{\partial x_i}\right)^2 dx, i = 1,2,3 \tag{5.8}$$

and, consequently

$$\frac{1}{2}\int \alpha(|\nabla\varphi|^2)\,dx + \int V(\varphi)\,dx = \frac{1}{3}\int \alpha'(|\nabla\varphi|^2)\,|\nabla\varphi|^2\,dx. \tag{5.9}$$

whenever the integrals converge (as before, we have assumed $c = 1$).

Proof. Let

$$\varphi_\lambda(x) = \varphi(\lambda x_1, x_2, x_3);$$

then, setting $y = (\lambda x_1, x_2, x_3)$

$$\begin{aligned} E(\varphi_\lambda) &= \frac{1}{2}\int \alpha(|\nabla\varphi_\lambda|^2)\,dx + \int V(\varphi_\lambda)\,dx \\ &= \frac{\lambda^{-1}}{2}\int \alpha\left(\lambda^2\left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2 + \left(\frac{\partial\varphi}{\partial x_3}\right)^2\right)\,dy + \lambda^{-1}\int V(\varphi)\,dy \end{aligned}$$

Since φ is a critical point of E,

$$\frac{d}{d\lambda}E(\varphi_\lambda)\Big|_{\lambda=1} = 0$$

and since

$$\begin{aligned}\frac{d}{d\lambda}E(\varphi_\lambda) = \quad & -\frac{\lambda^{-2}}{2}\int \alpha\left(\lambda^2\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\left(\frac{\partial\varphi}{\partial x_2}\right)^2+\left(\frac{\partial\varphi}{\partial x_3}\right)^2\right)dy \\ & +\int \alpha'\left(\lambda^2\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\left(\frac{\partial\varphi}{\partial x_2}\right)^2+\left(\frac{\partial\varphi}{\partial x_3}\right)^2\right)\left(\frac{\partial\varphi}{\partial x_1}\right)^2 dy \\ & -\lambda^{-2}\int V(\varphi)\,dy\end{aligned}$$

we get

$$-\frac{1}{2}\int \alpha(|\nabla\varphi|^2)\,dy+\int \alpha'(|\nabla\varphi|^2)\left(\frac{\partial\varphi}{\partial x_1}\right)^2 dy-\int V(\varphi)\,dy=0$$

which gives the 5.8. The 5.9 is obtained adding the 5.8 with $i=1,2,3$.

■

By the above lemma we get the following theorem.

Theorem 5.3. *If φ is a solution of 2.3 (with $c=1$); then*

$$m_{ii}(u_{\mathbf{v}}) = \gamma E(\varphi)$$

$$m_{ij}(u_{\mathbf{v}}) = 0 \quad \textit{if} \quad i \neq j$$

Proof. Since the matrix $m_{ij}(u_{\mathbf{v}})$ is symmetric, we can put ourselves in a new reference frame

$$x' = Tx$$

such that m'_{ij} is diagonal. In this reference frame (where we omit the $'$), by the (5.4)

$$m_{ij}(u_{\mathbf{v}}) = \gamma\delta_{ij}\int \alpha'(|\nabla\varphi|^2)\left(\frac{\partial\varphi}{\partial x_i}\right)^2 dx$$

where δ_{ij} is the Kroneker symbol. Then, by the lemma 5.2 and the (5.7)

$$m_{ij}(u_{\mathbf{v}}) = \gamma E(\varphi)\delta_{ij}$$

Thus $m_{ij}(u_{\mathbf{v}})$ is the identity matrix times $\gamma E(\varphi)$. This implies that $m_{ij}(u_{\mathbf{v}}) = \gamma E(\varphi)\delta_{ij}$ in any reference frame.

■

By the above lemma we get the following facts:

i) since the mass matrix is a scalar times the identity, the mass is in fact a scalar;

ii) the mass is equal to the rest energy times γ.

In particular, by the lemma 5.2, the mass gets the following form:

$$m(u_{\mathbf{v}}) = \frac{\gamma}{3} \int \alpha'(|\nabla\varphi|^2)\, |\nabla\varphi|^2 \; dx \tag{5.10}$$

and we have the equations

$$\mathbf{p}(u_{\mathbf{v}}) = m(u_{\mathbf{v}})\mathbf{v} \tag{5.11}$$

and

$$m(u_{\mathbf{v}}) = \gamma E(\varphi) \tag{5.12}$$

Notice that

$$(\mathbf{p}(u_{\mathbf{v}}), E(u_{\mathbf{v}})) = \left(\int T_{0,1}\, dx,\ \int T_{0,2}\, dx,\ \int T_{0,3}\, dx,\ \int T_{0,0}\, dx \right)$$

is a 4-vector since the equation 2.2 is Lorentz invariant; so, its Lorentzian norm

$$|\mathbf{p}(u_{\mathbf{v}})|^2 - E(u_{\mathbf{v}})^2 \tag{5.13}$$

is independent of $\mathbf{v}$. Using this fact we get:

Theorem 5.4. *If φ is a solution of 2.3 (with $c = 1$) and $u_{\mathbf{v}}$ is the function 2.4, then*

$$E(u_{\mathbf{v}}) = \gamma E(\varphi) \tag{5.14}$$

Proof. Since the 5.13 is independent of $\mathbf{v}$, $|\mathbf{p}(u_{\mathbf{v}})|^2 - E(u_{\mathbf{v}})^2 = -E(\varphi)^2$; then by 5.11 and Th. 5.3

$$\begin{aligned} E(u_{\mathbf{v}})^2 &= E(\varphi)^2 + |\mathbf{p}(u_{\mathbf{v}})|^2 = E(\varphi)^2 + m(\varphi)^2\gamma^2\, |\mathbf{v}|^2 \\ &= E(\varphi)^2 + E(\varphi)^2\gamma^2\, |\mathbf{v}|^2 = \left(1 + \frac{|\mathbf{v}|^2}{1 - |\mathbf{v}|^2}\right) E(\varphi)^2 = \gamma^2 E(\varphi)^2. \end{aligned}$$

■

Comparing the 5.14 with the 5.12 we get the equality between mass and energy:

$$E(u_{\mathbf{v}}) = m(u_{\mathbf{v}}) \tag{5.15}$$

If $c \neq 1$, the above formula gives the celebrated Einstein equation

$$E(u_{\mathbf{v}}) = m(u_{\mathbf{v}})c^2$$

It is interesting to note that this formula is a consequence not only of the Lorentz invariance of the Langrangian $\mathcal{L}$ but also of the fact that $u_{\mathbf{v}}$ is a soliton (and hence the lemma 5.2 holds).

6. Conclusions

The 2.8 probably is the simplest Lorentz invariant equation which has static solitons. Nevertheless, these solitons have some interesting properties since they behave as relativistic bodies, namely:

- they experience a relativisic contraction in the direction of the motion;
- the rest mass is a scalar and not a tensor;
- the mass equals the energy;
- the mass increases with the velocity by the factor γ.

Our Lagrangian is Lorentz invariant, thus it is reasonable to expect at least some of these features. However, it is somewhat surprising that they can be deduced from a single equation without extra assumptions. Moreover, this equation might be interpreted as the equation of an "elastic medium" in a Newtonian space-time. Thus, this model shows, from a purely formal point of view, that the main features of the special relativity can be deduced from a partial differential equation in a Newtonian space-time.

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