Solitons on Manifolds

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Abstract

We prove existence and multiplicity result for solutions of a class of Lorentz invariant equations on a compact Riemannian manifold $\mathcal{M}$. Such solutions, that arise as extremals for a suitable positive functional on the space of smooth functions from $\mathbb{R}^3$ to $\mathcal{M}$, are classiøed by the elements of the third homotopy group of $\mathcal{M}$. A physical interpretation of this invariant is given in terms of the quantic numbers of a traveling particle.

1 Preliminaries and Statement of the Results

The aim of this paper is to study the existence and the multiplicity of solutions of a Lorentz invariant equation on a compact Riemannian manifold. Such solutions have the energy traveling as localized packets, preserving their form under perturbation, and so they describe a particle-like behavior.

This kind of solutions occurs naturally in several fields of mathematical physics, like classical and quantum field theory, non linear optics, fluid mechanics and plasma physics. Details may be found in the references [12, 15, 23, 26].

The solitonic behavior may be found when there exist infinitely many first integral of the motion, like for instance in the KdV equation, or when there exist topological constraint in the state space. In this paper we will deal exclusively with the second case, by considering topological solitons in a curved manifold of dimension greater than one.

A classical 1-dimensional model for a soliton solution is given by the so-called sine-Gordon equation, given by:

$$u_{tt} - c^2\, u_{xx} + \sin u = 0, \tag{1}$$

$$\lim_{x \to -\infty} u(x,t) = 0, \tag{2}$$

$$\lim_{x \to +\infty} u(x,t) = 2\,k\,\pi, \quad k \in Z, \tag{3}$$

where c is the velocity of light. The asymptotic conditions (1) and (2) allow to have solutions with finite energy $E(u)$ given by:

$$E(u) = \frac{1}{2} \int_{-\infty}^{+\infty} \left(u_t^2 + c^2\, u_x^2\right)\, \mathrm{d}x + \int_{-\infty}^{+\infty} (1 - \cos u)\, \mathrm{d}x.$$

The static solutions $u(x,t) = \phi(x)$ of the sine-Gordon equation above solve the elliptic equation:

$$-c^2\, \ddot{\phi} + \sin \phi = 0, \tag{4}$$

$$\lim_{x \to -\infty} \phi(x) = 0, \tag{5}$$

$$\lim_{x \to +\infty} \phi(x) = 2\,k\,\pi, \quad k \in Z. \tag{6}$$

Any solution of the static problem defines a solution of the sine-Gordon

equation traveling at the constant velocity v, by setting:

$$u(x,t) = \phi\left(\frac{x - v\,t}{\sqrt{1-\frac{v^2}{c^2}}}\right).$$

The solutions to the equation (4) have a variational characterization, namely they are critical point for the energy functional:

$$E(\phi) = \int_{-\infty}^{+\infty}\left(\frac{1}{2}\,c^2\dot{\phi}^2 + 1 - \cos\phi\right)\,\mathrm{d}x,$$

deøned on a suitable function space that will be denoted by Λ

The asymptotic conditions (6) split the function space Λ into inønitely many connected components, denoted by Λ_k, where:

$$\Lambda_k = \left\{\phi \in \Lambda \;\middle|\; \lim_{x\to+\infty}\phi(x) = 2\,k\,\pi\right\}, \quad k \in Z.$$

One can prove that E possess local minima on the connected components Λ_1 and Λ_1 and these solutions have a solitonic behavior.

The situation is drastically diœerent for dimensions of the base space higher than 1, where the existence of non trivial minimal point for the energy functional E fails. For instance, let's consider the 3-dimensional analogue of (4), given by the scalar equation:

$$u_{tt}(x,t) - c^2\Delta u(x,t) + \nabla u(x,t) = 0,$$

where Δ and ∇ denote respectively the Laplacian and the gradient operator with respect to the space variable $x \in I\!R^3$.

The corresponding energy functional for the static solutions has the form:

$$E(\phi) = \int_{I\!R^3}\left(\frac{c^2}{2}\,|\nabla\phi| + V(\phi)\right)\,\mathrm{d}x. \tag{7}$$

If V is bounded from below, Derrick's Theorem (see [12]) says that every minimal point for (7) is necessarily trivial, i.e. constant.

On the other hand, if one considers more general potentials V which are unbounded from below (see for instance [8, 25]), then the typical critical points for E are saddle points, and this kind of extremizers fails to be stable

by small perturbations, hence they do not represent an observable physical situation (see [2, 7]).

In order to overcome the lack of stability for the critical points, in [6] the authors consider a correction of the functional (7) given by the positive functional:

$$E(\phi) = \int_{\mathbb{R}^3} \left[\frac{c^2}{2} |\nabla \phi|^2 + \varepsilon \frac{c^6}{6} |\nabla \phi|^6 + V(\phi) \right] \mathrm{d}x. \tag{8}$$

Under some regularity and growth conditions on the function V, the authors prove the existence of sufficiently many critical points for (8) in a suitable space Λ of functions $\phi : \mathbb{R}^3 \longmapsto \Omega$, where Ω is an open, connected subset of $\mathbb{R}^n$.

The purpose of this paper is to extend the results of Benci, Fortunato and Pisani to the case where one replaces the open set Ω with any compact Riemannian manifold $\mathcal{M}$. We will consider manifolds $\mathcal{M}$ with a distinguished base point p, that plays the role of the 0 in the open set Ω. In our approach to the problem, the disadvantage of losing the linear structure on the target space is compensated by the fact that, using the non-flat metric and the topological structure of $\mathcal{M}$, we are able to prove the same multiplicity results of [6] without using any potential function V. We will consider a functional E of the form:

$$E(\phi) = \int_{\mathbb{R}^3} \beta(\|\mathrm{d}\phi\|^2)\, \mathrm{d}s, \tag{9}$$

where $\beta(\lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3$ is a polynomial of the third degree ($a_3 > 0$), $\phi : \mathbb{R}^3 \longmapsto \mathcal{M}$ is an absolutely continuous function vanishing at infinity varying in a suitable space, denoted by $H_{2,6}(\mathbb{R}^3, \mathcal{M})$, where E is naturally defined. The space $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ will be defined formally in Section 3, where we will discuss the entire variational structure of our problem. For simplicity, in our treatment we will consider the polynomial $\beta(\lambda) = \lambda + \lambda^3$.

For $k \in \mathbb{N}$, we denote by S^k the unit sphere in $\mathbb{R}^{k+1}$, with ω distinguished point in S^k, and by $\pi_k(\mathcal{M})$ the k-th homotopy group of the manifold $\mathcal{M}$, which is the group of homotopy classes of all maps in the set:

$$C_{\omega,p}(S^k, \mathcal{M}) = \left\{ \phi : S^k \longmapsto \mathcal{M} \text{ continuous} : \phi(\omega) = p \right\}.$$

The space $H_{2,6}(\mathbb{R}^3, \mathcal{M}) \subset C_0(\mathbb{R}^3, \mathcal{M})$ has the same homotopy type of $C_0(\mathbb{R}^3, \mathcal{M}) \simeq C_{\omega,p}(S^k, \mathcal{M})$ (see Proposition A.5). Hence, every $\phi \in$

$H_{2,6}(\mathbb{R}^3,\mathcal{M})$ identiøes an element $\phi^{\#}$ of $\pi_3(\mathcal{M})$; for $\alpha\in\pi_3(\mathcal{M})$, we denote by Λ_α the set of all maps $\phi\in H_{2,6}(\mathbb{R}^3,\mathcal{M})$ that belong to the homotopy class α.

Let $A(\mathcal{M})$ be the subset of $\pi_3(\mathcal{M})$ consisting of all elements α such that the energy functional E attains its minimum on Λ_α.

The main result of the paper is the following:

Theorem 1.1. The set $A(\mathcal{M})$ is a generator set for the group $\pi_3(\mathcal{M})$.

2 Topological Invariants.

The Quantic Numbers

In this section we deøne a local topological invariant for the functions in $H_{2,6}(\mathbb{R}^3,\mathcal{M})$, that comes from the restriction onto suitable closed subsets of $\mathbb{R}^3$. The model for this invariant is the topological degree of maps.

We will not use any embedding of $\mathcal{M}$ into $\mathbb{R}^N$; $\mathcal{M}$ will be thought as a metric space, with metric $\mathrm{dist}(\cdot,\cdot)$.

We øx a normal, convex subset V of $p\in\mathcal{M}$; in particular, V is contractible in $\mathcal{M}$. Let $\delta_0>0$ be such that the closure $\overline{B(p,\delta_0)}$ of the δ_0-ball around p satisøes:

$$\overline{B(p,\delta_0)}\subset V,$$

and let $\rho>0$ be deøned by:

$$\rho=\mathrm{dist}(\partial V,\overline{B(p,\delta_0)})>0.$$

Finally, we denote by U the open ball $B(p,\rho+\frac{\delta_0}{2})$; since $U\subset V$, then for every point $q\in U$ there exists a unique minimal geodesic between p and q, and it is entirely contained in U.

For each $q\in U$, we denote by $\gamma_q:[0,1]\longmapsto U$ the unique minimal geodesic (parameterized on the interval $[0,1]$ by a multiple of the arc-length) such that $\gamma_q(0)=p$ and $\gamma_q(1)=q$. Using the exponential map, it is easy to see that the map $q\longmapsto\gamma_q\in C([0,1],U)$ is continuous in the compact˜open topology, i.e., if $q_n\longmapsto q$, then γ_{q_n} tends to γ_q uniformly.

Deønition 2.1. For $\phi\in H_{2,6}(\mathbb{R}^3,\mathcal{M})$, we deøne the support of ϕ, denoted by $K(\phi)$ the closed set:

$$K(\phi)=\overline{\{x\in\mathbb{R}^3:\phi(x)\notin V\}}. \tag{10}$$

Since V is contractible, and every map with values in a contractible space is homotopically trivial, if $\phi^{\#} \neq 0$ then $K(\phi) \neq \emptyset$.

We denote by $\simeq$ the homotopy equivalence of functions.

The homotopical invariant $\phi^{\#}$ is localized in $K(\phi)$ in the following sense:

Lemma 2.2. Let $\phi_1, \phi_2 \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$. If $K(\phi_1) = K(\phi_2)$ and $\phi_1\big|_{K(\phi_1)} \equiv \phi_2\big|_{K(\phi_2)}$, then $\phi_1 \simeq \phi_2$.

Proof. Define A to be the open set $\phi_1^{-1}(V) \subset \mathbb{R}^3$; by definition $A \cup K(\phi_1)^\circ = \mathbb{R}^3$, where $^\circ$ denotes the interior.

The restriction $\phi_1\big|_A$ and $\phi_2\big|_A$ are homotopic, because they are both homotopic to the constant map $\phi \equiv p$.

Clearly, $\phi_1\big|_{K(\phi_1)} \simeq \phi_2\big|_{K(\phi_2)}$, and $\phi_1 \simeq \phi_2$ on $\mathbb{R}^3$. □

For all $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$ (or more in general for all $\phi \in C_0(\mathbb{R}^3, \mathcal{M})$), we denote by $\mathcal{K}(\phi)$ a collection of closed subsets of $\mathbb{R}^3$, given by:

$$\mathcal{K}(\phi) = \left\{ K \subset \mathbb{R}^3 \text{ closed} \ : \phi(\partial K) \subset \overline{V} \right\}.$$

Since $\lim_{|x| \to +\infty} \phi(x) = p \in V$, then every ball in $\mathbb{R}^3$ centered in 0 and with radius sufficiently large belongs to $\mathcal{K}(\phi)$. Moreover, every connected component of the support $K(\phi)$ belongs to $\mathcal{K}(\phi)$.

Let $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$ be fixed and $K \in \mathcal{K}(\phi)$. By continuity, there exists an open neighborhood N of ∂K such that $\phi(N) \subset U$. Define $\tilde{K}$ to be the open set:

$$\tilde{N} = K \bigcup N.$$

By the Uryson Lemma, there exists a continuous function $c : \mathbb{R}^3 \longmapsto [0,1]$ such that:

$$c(x) = \begin{cases} 1, & \text{if } x \in K; \\ 0, & \text{if } x \notin \tilde{N}. \end{cases}$$

Now, we define a function $\tilde{\phi} : \mathbb{R}^3 \longmapsto \mathcal{M}$ by:

$$\tilde{\phi}(x) = \begin{cases} \gamma_{\phi(x)}(c(x)), & \text{if } \phi(x) \in \tilde{N}; \\ p & \text{if } \phi(x) \notin \tilde{N}. \end{cases}$$

The continuity of $\tilde{\phi}$ is easily established, considering that $q \longmapsto \gamma_q$ is continuous in the compact-open topology, that c is continuous and that $\gamma_{\phi(x)}(0) = p$ for all x. Moreover, $\tilde{\phi} \equiv \phi$ in K, because $c \equiv 1$ in K and $\gamma_{\phi(x)}(1) = \phi(x)$ for all $x \in U$.

Observe that $\tilde{\phi} \in C_0(I\!R^3, \mathcal{M})$, hence its homotopy class defines an element $\tilde{\phi}^{\#}$ of $\pi_3(\mathcal{M})$.

Definition 2.3. For all $\phi \in H_{2,6}(I\!R^3, \mathcal{M})$ and every $K \in \mathcal{K}(\phi)$, we set:

$$(\phi\big|_K)^{\#} = \tilde{\phi}^{\#}$$

The main properties of the localized invariant $(\phi\big|_K)^{\#}$ are described in the following:

Proposition 2.4. (1) $(\phi\big|_K)^{\#})$ is well defined, i.e., it does not depend on the choice of the open neighborhood N of ∂K and the continuous function c;

(2) for all closed set $K \subset I\!R^3$ such that $K(\phi) \subset K$, it is $K \in \mathcal{K}(\phi)$ and $(\phi\big|_K)^{\#} = \phi^{\#}$. In particular, $(\phi\big|_{K(\phi)})^{\#} = \phi^{\#}$;

(3) if $K_1, K_2 \in \mathcal{K}(\phi)$ and $K_1^{\circ} \cap K_2^{\circ} = \emptyset$, then

$$(\phi\big|_{K_1 \cup K_2})^{\#} = (\phi\big|_{K_1})^{\#} + (\phi\big|_{K_2})^{\#}.$$

Proof. To prove part (1), let $\tilde{\phi}_i$, $i = 1, 2$, be two different extensions of ϕ outside K corresponding to two different choices of pairs (N_1, c_1) and (N_2, c_2) as in the definition of the local invariant $(\phi\big|_K)^{\#}$.

We show that $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are homotopic, hence they define the same element in $\pi_3(\mathcal{M})$.

Observe indeed that $\tilde{\phi}_1 = \tilde{\phi}_2 = \phi$ on K, and $\tilde{\phi}_1 = \tilde{\phi}_2 = p$ outside U. For $a, b \in U$, let $\gamma_{a,b} : [0,1] \longmapsto U$ denote the unique minimal geodesic from a to b, parameterized by a multiple of arc length.

Again, using the exponential map it is easy to see that $(a, b) \longmapsto \gamma_{a,b} \in C([0,1], U)$ is continuous in the compact open topology.

We define a homotopy $H : [0,1] \times \mathcal{M}$ by:

$$H(\lambda, x) = \gamma_{\tilde{\phi}_1(x), \tilde{\phi}_2(x)}(\lambda) \quad \forall\, x \in U,\ \lambda \in [0,1],$$

and

$$H(\lambda, x) = \tilde{\phi}_1(x) = \tilde{\phi}_2(x), \quad \text{if } x \notin U.$$

The map H is clearly continuous. Moreover, it is:

$$H(0, x) = \tilde{\phi}_1(x), \quad H(1, x) = \tilde{\phi}_2(x), \quad \forall\, x \in U,$$

hence H is a homotopy between $\tilde{\phi}_1$ and $\tilde{\phi}_2$.

Part (2) follows from the fact that, if $K(\phi) \subset K$, then, by definition of $\tilde{\phi}$, we have that $K(\tilde{\phi}) = K(\phi)$ and $\tilde{\phi} \equiv \phi$ on $K(\phi)$.

Finally, for part (3), observe that if $K_1^\circ \cap K_2^\circ = \emptyset$, then we can find the corresponding open neighborhoods N_1 and N_2 disjoint, in such a way that also $\tilde{N}_1 = K_1 \cup N_1$ and $\tilde{N}_2 = K_2 \cup N_2$ are disjoint open sets. The function $\tilde{\phi}$ corresponding to the union $K_1 \cup K_2$ is then given by $\tilde{\phi}\big|_{\tilde{N}_i} = \tilde{\phi}_i$, $i = 1, 2$, and $\tilde{\phi}(x) \equiv p$ if $x \notin \tilde{N}_1 \cup \tilde{N}_2$. Since p is the base point for the maps representatives of elements in $\pi_3(\mathcal{M})$, we have $\tilde{\phi}^\# = \tilde{\phi}_1^\# + \tilde{\phi}_2^\#$ in $\pi_3(\mathcal{M})$. □

3 The Variational Framework

Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_x)$ be an m-dimensional ($m \geq 1$) complete Riemannian manifold of class C^4, and let p denote a distinguished point in $\mathcal{M}$. Using Nash's embedding Theorem (see [20]), we can assume that $\mathcal{M}$ is isometrically embedded in some Euclidean space, $\iota : \mathcal{M} \longmapsto \mathbb{R}^N$, and we will also assume that $\iota(p) = 0$.

Even though some of the formulas written in this Section depend on the particular choice of the embedding ι, due to the compactness of $\mathcal{M}$ one can check (see Appendix A) that passing to a different embedding does not change the differentiable structure of our framework. Moreover, the regularity results proven Theorem 3.2 ensure that the extremals of our variational problem satisfy an intrinsic differential equation, defined only in terms of the Riemannian structure of $\mathcal{M}$ (see (18)).

For $x \in \mathbb{R}^N$ and $\rho \in \mathbb{R}^+$, $B(x, \rho)$ will denote the open ball centered in x and of radius ρ in $\mathbb{R}^N$.

If $\phi : \mathbb{R}^{n_1} \longmapsto \mathbb{R}^{n_2}$ is an absolutely continuous map, we denote by $\mathrm{d}\phi(x) : \mathbb{R}^{n_1} \longmapsto \mathbb{R}^{n_2}$, the linear map given by the derivative of ϕ at the point x. We

denote by $\|\mathrm{d}\phi(x)\|$ the Hilbert-Schmidt norm of $\mathrm{d}\phi(x)$ as a linear operator from $\mathbb{R}^{n_1}$ to $\mathbb{R}^{n_2}$ with respect to the Euclidean norms of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$:

$$\|\mathrm{d}\phi(x)\|^2 = \sum_{i=1}^{n_1} \|\mathrm{d}\phi(x)\, e_i\|^2,$$

where $\{e_i\}_i$ is any orthonormal basis of $\mathbb{R}^{n_1}$. In particular, for $e_i = \frac{\partial}{\partial x_i}$, we have:

$$\|\mathrm{d}\phi\|^2 = \sum_{i=1}^{n_1} \|\frac{\partial \phi}{\partial x_i}\|^2.$$

We consider the Banach space H given by the completion of the vector space $C_c^\infty(\mathbb{R}^3, \mathbb{R}^N)$ of all smooth maps maps from $\mathbb{R}^3$ to $\mathbb{R}^N$ with compact support, with respect to the norm :

$$\|\phi\|_{2,6} = \|\mathrm{d}\phi\|_{L^2} + \|\mathrm{d}\phi\|_{L^6}, \tag{11}$$

where

$$\|\mathrm{d}\phi\|_{L^p} = \left(\int_{\mathbb{R}^3} \|\mathrm{d}\phi(x)\|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

By deøinition of $\|\cdot\|_{2,6}$, the derivative $\mathrm{d}\phi$ belongs to the intersection

$$L^2(\mathbb{R}^3, \mathbb{R}^{3N}) \cap L^6(\mathbb{R}^3, \mathbb{R}^{3N}),$$

where we identify $\mathbb{R}^{3N}$ with the set of all linear maps from $\mathbb{R}^3$ to $\mathbb{R}^N$. By the inequality of Sobolev, Gagliardo and Nirenberg (see [10]), there exists a positive constant C_1 such that, for $\phi \in H$ it is:

$$\|\phi\|_{L^6} \leq C_1 \cdot \|\mathrm{d}\phi\|_{L^2}.$$

This implies that $H \subset W^{1,6}(\mathbb{R}^3, \mathbb{R}^N)$ with continuous immersion. Then, by Morrey's inequality (see [10]), it is also:

$$H \subset L^\infty(\mathbb{R}^3, \mathbb{R}^N) \tag{12}$$

with continuous injection; moreover, there exists a positive constant C_2 such that, for every $\phi \in H$ it is:

$$|\phi(x) - \phi(y)| \leq C_2 \cdot \|\mathrm{d}\phi\|_{L^6} \cdot |x - y|^{\frac{1}{2}}. \tag{13}$$

If we denote by $C^{0,\alpha}(\mathbb{R}^3, \mathbb{R}^N)$ the space of all Hölder continuous maps with Hölder constant α, then by (13) we also have a continuous injection:

$$H \subset C^{0,\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^N). \tag{14}$$

Observe also that, by (12), for all $\phi \in H$ it is:

$$\lim_{|x|\to\infty} \phi(x) = 0,$$

because every $\phi \in H$ is uniform limit of smooth functions with compact support. We denote by $C_0(\mathbb{R}^3, \mathbb{R}^N)$ the set of continuous functions from $\mathbb{R}^3$ to $\mathbb{R}^N$ vanishing at inøinity; we also have:

$$H \subset C_0(\mathbb{R}^3, \mathbb{R}^N), \tag{15}$$

so that $\phi(\mathbb{R}^3) \cup \{0\}$ is a compact subset of $\mathbb{R}^N$ for all ϕ in H.

Remark 3.1. Observe that the combination of the Sobolev–Gagliardo–Nirenberg inequality with Morrey's theorem is a very special coincidence due to the dimension 3 of the base space. This fact is crucial for the proof of our results without the need of a potential in our variational problem. The three–dimensional case, which is the one with more physical interest, appears to be mathematically the most natural environment for the study of solitonic waves. Diœerent results were found also in the one dimensional case, where the problem of the existence of homoclinic orbits is studied by global variational methods (see [17] and the references therein).

The inøinite dimensional space where our problem is studied is given by:

$$H_{2,6}(\mathbb{R}^3, \mathcal{M}) = \left\{ u \in H \;\middle|\; u(\mathbb{R}^3) \subset \mathcal{M} \right\}.$$

As shown in the Appendix A, the space $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ has the structure of a Banach manifold. For $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$, the tangent space $T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M})$ can be identiøed with the Banach space:

$$T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M}) = \left\{ \zeta \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^N) \;\middle|\; \zeta(x) \in T_{\phi(x)}\mathcal{M} \;\forall\, x \right\}.$$

We consider the following functional on $H_{2,6}(\mathbb{R}^3, \mathcal{M})$:

$$E(\phi) = \int_{\mathbb{R}^3} \left(\|\mathrm{d}\phi(x)\|^2 + \|\mathrm{d}\phi(x)\|^6 \right) \,\mathrm{d}x. \tag{16}$$

If Ω is an open subset of $\mathbb{R}^3$, we define on $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ a localized functional E_Ω by restricting the integral in (16) to Ω:

$$E_\Omega(\phi) = \int_\Omega \left(\|\mathrm{d}\phi(x)\|^2 + \|\mathrm{d}\phi(x)\|^6 \right) \, \mathrm{d}x.$$

We also define the pseudo-distance d_Ω in $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ by:

$$\mathrm{d}_\Omega(\phi_1, \phi_2) = \left(\int_\Omega \|\mathrm{d}\phi_1(x) - \mathrm{d}\phi_2(x)\|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_\Omega \|\mathrm{d}\phi_1 - \mathrm{d}\phi_2\|^6 \, \mathrm{d}x \right)^{\frac{1}{6}}.$$

It is proven in Appendix A that different embeddings of $\mathcal{M}$ into $\mathbb{R}^N$ give rise to equivalent metrics d_Ω.

Observe that, if $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$, for almost every $x \in \mathbb{R}^3$ the differential $\mathrm{d}\phi(x)$ belongs to space $\mathcal{L}(\mathbb{R}^3, T_{\phi(x)}\mathcal{M}) \simeq (\mathbb{R}^3)^* \otimes T_{\phi(x)}\mathcal{M} \simeq \mathbb{R}^3 \otimes T_{\phi(x)}\mathcal{M}$ of all linear maps from $\mathbb{R}^3$ to $T_{\phi(x)}\mathcal{M}$. Here the $*$ denotes the dual space. This space is endowed with a natural Hilbert structure induced by the Euclidean structure on $\mathbb{R}^3$ and the Riemannian structure on $T_{\phi(x)}\mathcal{M}$; the inner product in $(\mathbb{R}^3)^* \otimes T_{\phi(x)}\mathcal{M}$ will be denoted by $\langle \cdot, \cdot \rangle_0$. The norm relative to this product is the Hilbert–Schmidt norm of a linear operator from $\mathbb{R}^3$ to $T_{\phi(x)}\mathcal{M}$.

Using standard computations (see Appendix A) in Calculus of Variations on Riemannian manifolds, one proves that E is a functional of class C^2. Moreover, for $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$ and for $\zeta \in T_\phi(\mathbb{R}^3, \mathcal{M})$, the Gateaux derivative $E'(\phi)[\zeta]$ is given by:

$$E'(\phi)[\zeta] = \int_{\mathbb{R}^3} \left(2 + 6\,\|\mathrm{d}\phi(x)\|^2\right) \cdot \left\langle \nabla^\phi \zeta(x), \mathrm{d}\phi(x) \right\rangle_0 \mathrm{d}x. \tag{17}$$

In (17), $\nabla^\phi \zeta$ denotes the covariant derivative of the vector field ζ along the function ϕ induced by the Levi–Civita connection of $\mathcal{M}$. For the reader's convenience, a fairly detailed discussion about the Banach manifold structure, the differentiability of the functional E, together with some basic notions from covariant calculus along functions, is postponed to the Appendix A.

Using local coordinates on $\mathcal{M}$, if $\Gamma^k_{ij}(p)$ denote the Christoffel symbols for the metric of $\mathcal{M}$, the Euler–Lagrange equations of the functional E for the function $\phi = (\phi^1, \ldots, \phi^m)$ can be written as:

$$\begin{aligned} &\Delta\phi^k(x) \;+\; 3\,\mathrm{div}\left(\|\mathrm{d}\phi(x)\|^4 \cdot \nabla\phi^k(x)\right) \\ &\qquad + \sum_{a,b=1}^{m} \sum_{i,j=1}^{3} \Gamma^k_{ab}(\phi(x)) \, \frac{\partial \phi^a}{\partial x_i}(x) \, \frac{\partial \phi^b}{\partial x_j}(x) = 0, \quad k = 1, 2, \ldots, m \end{aligned} \tag{18}$$

where Δ is the Laplacian operator in $\mathbb{R}^3$ and $\nabla\phi^k$ denotes the gradient of ϕ^k (see Proposition A.7).

An integration by parts in (17) shows that if ϕ is a strong solution (in the C^2-sense) of (18), then ϕ is a critical point of E in $H_{2,6}(\mathbb{R}^3, \mathcal{M})$, i.e.

$$E'(\phi)[\zeta] = 0, \quad \forall\, \zeta \in T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M}). \tag{19}$$

The converse is given by the following regularity result, whose proof is basically contained in [1] and [18]:

Theorem 3.2. Assume that ϕ satisøees (19). Then ϕ is a map of class C^2, and therefore it satisøees (18).

Proof. Computing the Hessian of E in local coordinates and using well known embedding theorems shows that every critical point ϕ of E minimizes locally E. This means that every point $x_0 \in \mathbb{R}^3$ has an open neighborhood Ω such that $E_\Omega(\phi) \leq E_\Omega(\phi')$ for all $\phi' \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$.

By (13), ϕ is Hlder continuous and we can apply Theorem 4.3 of [18] in local coordinates to get that $\mathrm{d}\phi$ is also Hlder continuous. Finally, the methods used in [1] allow to prove that $\mathrm{d}\phi$ is of class C^1. □

Remark 3.3. Some estimates on the Hessian of the functional E at $\phi \equiv 0$ show that, if $\rho \in \mathbb{R}^+$ is chosen suŒ ciently small, then the only critical point of E with image in $B(0, \rho_0)$ is the null map. Clearly, this remains true for the localized functional $E_{B(0,\rho_0)}$. This observation is essential for the study of the behavior of the Palais˜Smale sequences which is done in Section 4.

4 Palais˜Smale Sequences for the functional E

In this section we will study the Palais˜Smale sequences for the functional E deøned in (16). We recall that, if $c \in \mathbb{R}$, a sequence $\{\phi_n\}_n \subset H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is said to be a Palais˜Smale sequence at the level c for the functional E if:

$$\lim_{n\to\infty} E(\phi_n) = c, \quad \text{and} \quad \lim_{n\to\infty} ||E'(\phi_n)|| = 0,$$

where $||E'(\phi_n)||$ is the norm in the dual Banach space of $T_{\phi_n} H_{2,6}(\mathbb{R}^3, \mathcal{M})$.

Before stating the main result of the section, the following observation is needed:

Remark 4.1. As an immediate consequence of the continuous embedding of $H_{2,6}(\mathbb{R}^3,\mathcal{M})$ is L^∞, we have the existence of two positive constants δ_0 and Δ_* such that, for all $\phi\in\Lambda$, if for some $x\in\mathbb{R}^3$ it happens that $\phi(x)\notin B(0,\delta_0)$, then $E(\phi)\geq\Delta_*$. It follows that:

$$\inf_{\Lambda^*} E\geq\Delta_*>0. \tag{20}$$

The main result of the section is contained in the following Proposition, and it is about the behavior of the Palais-Smale sequences of the functional E.

Proposition 4.2. Let $b>\Delta_*$ and $\{\phi_n\}_n\subset H_{2,6}(\mathbb{R}^3,\mathcal{M})$ be a Palais-Smale sequence at the level b for E.

Then, there exists $\rho\in\mathbb{N}$, $\overline{\phi}_1,\dots,\overline{\phi}_\rho\in\Lambda_*$ critical points of E, $x_n^1,\dots,x_n^l$ in $\mathbb{R}^3$ and $R_n^1,\dots,R_n^l>0$ such that, up to passing to subsequences, the following facts hold:

$$\lim_{n\to\infty} R_n^i=+\infty,\quad\forall\, i=1,\dots,l; \tag{21}$$

$$\lim_{n\to\infty} \mathrm{d}_{B(x_n^i,R_n^i)}(\phi_n,\overline{\phi}(\cdot+x_n^i))=0,\quad\forall\, i=1,\dots,l; \tag{22}$$

$$\lim_{n\to\infty} |x_n^i-x_n^j|=0,\quad \text{if } i\neq j; \tag{23}$$

$$\lim_{n\to\infty} E_{B(x_n^i,R_n^i)}(\phi_i)=E(\overline{\phi}_i),\quad\forall\, i=1,\dots,l; \tag{24}$$

$$\sum_{i=1}^{l} E(\overline{\phi}_i)=b; \tag{25}$$

$$\lim_{n\to\infty} \mathrm{d}_{\mathbb{R}^3\setminus\left(\bigcup_{i=1}^l B(x_n^i,R_n^i)\right)}(\phi_n,0)=0. \tag{26}$$

Remark 4.3. From (26) of Proposition 4.2 it follows that, if ϕ_n is a Palais-Smale sequence for E, then:

$$\lim_{n\to\infty}\sup\left\{\mathrm{d}(\phi_n(x),0):x\in\mathbb{R}^3\setminus\bigcup_{i=1}^{l}B(x_n^i,R_n^i)\right\}=0.$$

Then, it is not too difficult to see that, for n sufficiently large, $\phi_n^\#$ is generated by $\overline{\phi}_i^\#$, $i=1,2,\dots,l$. (see [5])

To carry out the proof of Proposition 4.2, the following preliminaries Lemmas are needed.

Lemma 4.4. Let u_n be a sequence in H. Let R_n be a sequence in $\mathbb{R}^+ \cup \{+\infty\}$ such that $\lim_{n\to\infty} R_n = R_0 \in \mathbb{R}^+$. Assume that:

$$\lim_{n\to\infty} E'_{B(0,R_n)}(u_n) = 0, \tag{27}$$

$$E_{B(0,R_n)}(u_n) \leq c \quad \text{eventually}, \tag{28}$$

where c is a constant independent of n.

Then there exists $u \in H(\overline{B(0,R_0)})$ such that:

$$\lim_{n\to\infty} \mathrm{d}_{B(0,R)}(u_n, u) = 0, \quad \forall\, R \in]0, R_0[; \tag{29}$$

$$E'_{B(0,R_0)}(u)[V] = 0, \quad \forall\, V \in T_u H \text{ with } \mathrm{supp}(V) \subset B(0,R_0), \tag{30}$$

$$E_{B(0,R_0)}(u) \leq \liminf_{n\to\infty} E_{B(0,R_n)}(u_n). \tag{31}$$

Lemma 4.5. Let u_n a sequence in H and R_n a sequence in $\mathbb{R}^+ \cup \{+\infty\}$ be as in Lemma 3.2. Assume that (27) and (28) hold, and that the following two conditions hold:

$$\sup_{x\in B(0,R_n)} \mathrm{d}(u_n(x), 0) \leq \rho_0, \quad \forall\, n \in \mathbb{N}, \tag{32}$$

$$\lim_{n\to\infty} \left(\sup\left\{\mathrm{d}(u_n(x), 0) : x \in \partial B(0,R_n)\right\}\right) = 0. \tag{33}$$

Then $\lim_{n\to\infty} \mathrm{d}(u_n, 0) = 0$.

Proof of Lemma 4.4. By (28), u_n can be extended to a bounded sequence $\tilde{u}_n$ in $H(\overline{B(0,R_0)})$. Then, there exists $u \in H(\overline{B(0,R_0)})$ such that u_n converges weakly to u.

The extension $\tilde{u}_n$ can be done in such a way that

$$\lim_{n\to\infty} E_{B(0,R_0)\setminus B(0,R_n)}(\tilde{u}_n) = 0,$$

from which (31) follows immediately. Since E is a functional of class C^1, (30) follows from (27) and (29). Thus, we are left with the proof of (27).

Fix $R \in]0, R_0[$. By (27) we have:

$$|E'(u_n)[\xi]| = \left| \int_{B(0,R)} (1 + 3\,\|\mathrm{d}u_n\|^2 \cdot \left\langle \mathrm{d}u_n, \nabla^{u_n}\xi \right\rangle \,\mathrm{d}x \right| \leq \varepsilon_n \cdot \|\nabla^{u_n}\xi\|, \quad (34)$$

for all $\xi \in H_{2,6}(I\!R^3, I\!R^N)$ with $\xi\big|_{\partial B(0,R)} = 0$.

For all $y \in \mathcal{M}$, let $P(y)$ denote the orthogonal projection from $I\!R^N$ to the tangent space $T_y\mathcal{M}$.

Now, let u be as above and deøne

$$\rho_n = R - \sqrt{\|u_n - u\|_{L^\infty}},$$

$$c_n(x) = \theta_n(|x|) \cdot (u_n(x) - u(x)),$$

where

$$\theta_n(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq \rho_n; \\ \frac{s-\rho_n}{R-\rho_n} & \text{if } \rho_n < s \leq R; \\ 1 & \text{if } s > R. \end{cases}.$$

Since u_n tends to u uniformly, then c_n tends to 0 uniformly and so does $\mathrm{d}c_n$. Finally, we consider:

$$\xi_n = P(u_n)(u_n - u - c_n).$$

By the above construction, we have that $\xi_n \equiv 0$ on $\partial B(0,R)$, and $\|\xi_n\|$ is bounded independently of n. Then, by (34), we get:

$$\lim_{n\to\infty} \int_{B(0,R)} (1 + 3\,\|\mathrm{d}u_n\|^2) \cdot \left\langle \mathrm{d}u_n, \mathrm{d}P(u_n)[\mathrm{d}u_n]u_n - u - c_n \right\rangle \mathrm{d}x + \quad (35)$$
$$\int_{B(0,R)} (1 + 3\,\|\mathrm{d}u_n\|^2) \left\langle \mathrm{d}u_n, P(u_n)(\mathrm{d}u_n - \mathrm{d}u - \mathrm{d}c_n \right\rangle \mathrm{d}x = 0,$$

because u_n is uniformly bounded. Since $u_n - u - c_n$ tends to 0 uniformly, from (35) we deduce that:

$$\lim_{n\to\infty} \int_{B(0,R)} (1 + 3\,\|\mathrm{d}u_n\|^2) \cdot \left\langle \mathrm{d}u_n, P(u_n)(\mathrm{d}u_n - \mathrm{d}u - \mathrm{d}c_n) \right\rangle \mathrm{d}x = 0.$$

Now,

$$\big\langle \mathrm{d}u_n, P(u_n)(\mathrm{d}u_n - \mathrm{d}u - \mathrm{d}c_n)\big\rangle = \big\langle \mathrm{d}u_n, \mathrm{d}u_n - \mathrm{d}u - \mathrm{d}c_n\big\rangle,$$

and $\mathrm{d}c_n$ tends to 0 uniformly. Then,

$$\lim_{n\to\infty} \int_{B(0,R)} (1 + 3\,\|\mathrm{d}u_n\|^2)\big\langle \mathrm{d}u_n, \mathrm{d}u_n - \mathrm{d}u\big\rangle = 0. \tag{36}$$

Moreover, since u_n tends to u weakly, we have:

$$\lim_{n\to\infty} \int_{B(0,R)} (1 + 3\,\|\mathrm{d}u\|^2)\big\langle \mathrm{d}u, \mathrm{d}u_n - \mathrm{d}u\big\rangle = 0. \tag{37}$$

Now, there exists $\nu = \nu(k)$[1] such that, for every pair v_1, v_2 it is:

$$\big\langle (1 + 3\,\|\mathrm{d}v_2\|^2)\, v_2 - (1 + 3\,\|\mathrm{d}v_1\|^2)\, v_1, v_2 - v_1\big\rangle \geq \nu(k)(1 + 3\,\|\mathrm{d}v_1 - \mathrm{d}v_2\|^2).$$

Hence, combining (36) and (37) gives (29). □

Proof of Lemma 4.5. It will be sufficient to prove that:

$$\limsup_{n\to\infty} \big\{\mathrm{d}(u_n(x), 0) : x \in B(0, R_n) = 0.\big\} \tag{38}$$

Indeed, if (38) holds, by the same arguments used in the proof of Lemma 4.4, it is possible to conclude that

$$\lim_{n\to\infty} \int_{B(0,R_n)} \|\mathrm{d}u_n\|^6 \,\mathrm{d}x = 0.$$

Up to passing to a subsequence, we can assume that $\lim_{n\to\infty} R_n = R_0 > 0$. By Lemma 4.4 there exists $u \in H(\overline{B(0,R_0)})$ such that (29) is satisfied.

Now, if $R_0 < +\infty$, by (33) we have:

$$\lim_{|x|\to R_0} u(x) = 0, \tag{39}$$

while if $R_0 = +\infty$ (39) is a consequence of (28). Moreover, by Lemma 4.4, u is a critical point of $E_{B(0,R_0)}$, and by Remark 3.3 we get a contradiction.[2] □

[1] chi k?

[2] che cos' che d una contradizione?

Proof of Proposition 4.2. If u is a strong solution of (18) on Ω and its behavior on the straight lines .....[3], it is not diŒcult to see that, if u vanishes on a ball, then it vanishes on the whole set Ω. Then, the uniform Hlder continuity of the curves in H, together with Lemmas 3.2 and 3.3, allows to repeat the argument used in the proof of Proposition 3.2 of [17] (concerning the one dimensional case). □

5 Proof of the Results

A The Manifold Structure of $\boldsymbol{H_{2,6}(\mathbb{R}^3, \mathcal{M})}$. Smoothness of the Functional $\boldsymbol{E}$. The Euler˜Lagrange Equations.

The purpose of this Appendix is to give a formal presentation of regularity property of our main function space and the energy functional E introduced in Section 1. These results and the techniques used to prove them are of technical nature, and they may be skipped without jeopardizing the general understanding of the main issue discussed in the paper. Nevertheless, the problem of determining a Hilbert manifold structure for a family of maps deøned on a non-compact base space is by no means secondary. Even though some authors may øndit a bit awkward, it must be said that the regularity problem for maps on a non-compact base in its full generality has not been satisfactorily investigated in the classical literature (see [13, 22]).

We will assume throughout this section that $\mathcal{M}$ is a compact, m-dimensional Riemannian manifold, with $m \geq 4$, and $p \in \mathcal{M}$ is a distinguished point. Whenever $\mathcal{M}$ is embedded (isometrically) into some Euclidean space $\mathbb{R}^N$ we will implicitly assume that $0 = p \in \mathcal{M}$. Also, all the maps from $\mathcal{M}$ to itself will be assumed to øx p.

We øx some notation ørst. If V_1 and V_2 are (real) vector spaces, we denote by $\mathcal{L}(V_1, V_2)$ the real vector space of all linear maps from V_1 to V_2. By V_1^* we mean the dual space of V_1; then $\mathcal{L}(V_1, V_2)$ is canonically isomorphic to the tensor product $V_1^* \otimes V_2$. If V_1 and V_2 are endowed respectively with the (positive deønite) inner product $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, then $V_1^* \otimes V_2$ inherits inner product structure $\langle \cdot, \cdot \rangle_{HS}$, which, in terms of linear operators T_1 and T_2 from

[3] cosa ci va qui?

V_1 to V_2 is written as:

$$\langle T_1, T_2 \rangle_{HS} = \sum_{i=1}^{\dim(V_1)} \langle T_1 e_i, T_2 e_i \rangle_2 = \sum_{i=1}^{\dim(V_1)} \langle T_2^* T_1 e_i, e_i \rangle_1 = \mathrm{tr}(T_2^* T_1),$$

where $\{e_i\}_i$ is any orthonormal basis of V_1, T_2^* is the adjoint map and tr is the usual trace. The product $\langle \cdot, \cdot \rangle_{HS}$ and the associated norm are called the Hilbert–Schmidt product and norm of linear operators between inner product spaces.

If V_1 is an inner product space, V_2 is vector space and $B : V_1 \otimes V_1 \longmapsto V_2$ is a bilinear map, one defines the trace $\mathrm{tr}(B)$ as the vector in V_2:

$$\mathrm{tr}(B) = \sum_{i=1}^{\dim(V_1)} B(e_i, e_i), \tag{40}$$

where $\{e_i\}_i$ is any orthonormal basis of V_1. As for the usual trace, it is easily seen that the right hand side of (40) does not depend on the choice of the orthonormal basis of V_1.

We recall that the topology of $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ was defined in Section 3 using an isometric embedding of $\mathcal{M}$ into some Euclidean space $\mathbb{R}^N$. The first step in the proof of the manifold structure for $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is to show that the topological structure does not depend on the particular choice of an embedding.

Lemma A.1. The topological structure of $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ does not depend on the isometric embedding of $\mathcal{M}$ into a Euclidean space.

Proof. Given an (isometric) embedding i of $\mathcal{M}$ into a Euclidean space $\mathbb{R}^N$, we define a distance in the space $C_c^\infty(\mathbb{R}^3, \mathcal{M})$ of all smooth function with compact support from $\mathbb{R}^3$ to $i(\mathcal{M})$ by:

$$\mathrm{dist}(u, v) = \left(\int_{\mathbb{R}^3} \|\mathrm{d}u(x) - \mathrm{d}v(x)\|^2 \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^3} \|\mathrm{d}u(x) - \mathrm{d}v(x)\|^6 \right)^{\frac{1}{6}}, \tag{41}$$

where $\|\cdot\|$ denotes the Hilbert–Schmidt norm of a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^N$. Observe that the metric (41) does indeed depend on the choice of the embedding; however, we will show that two different embeddings give rise to equivalent metrics. Before doing that, we show that the topological structure

of $H_{2,6}(I\!R^3,\mathcal{M})$ as a subset of $H_{2,6}(I\!R^3,I\!R^N)$ is completely determined by the distance (41).

Indeed, we claim that the closure $C_{2,6}(I\!R^3,\mathcal{M})$ of $C_c^\infty(I\!R^3,\mathcal{M})$ with respect to (41) gives $H_{2,6}(I\!R^3,\mathcal{M})$. To see this, ørst observe that clearly $C_{2,6}(I\!R^3,\mathcal{M})\subseteq H_{2,6}(I\!R^3,\mathcal{M})$.

To prove the opposite inclusion, observe that if $u_n : I\!R^3 \longmapsto I\!R^N$ is a sequence of smooth functions with compact support that converges in the norm (11) to a function $u\in H_{2,6}(I\!R^3,\mathcal{M})$, then, by the Morrey's inequality, in particular u_n converges uniformly to u. Hence, by the compactness of $\mathcal{M}$, for n large enough the image $u_n(I\!R^3)$ is contained in an open tubular neighborhood U of $\mathcal{M}$ in $I\!R^N$, which may assumed to be contractible onto $\mathcal{M}$.

If $r : U \longmapsto \mathcal{M}$ is any smooth retraction, then the sequence $v_n = r\circ u_n$ is in $C_c^\infty(I\!R^3,\mathcal{M})$ and converges to u in the distance (41). This says that $C_{2,6}(I\!R^3,\mathcal{M})\supseteq H_{2,6}(I\!R^3,\mathcal{M})$.

To conclude the proof, it suŒces to show that if $i_i : \mathcal{M}\longmapsto I\!R^{N_i}$, $i=1,2$ are two diœerent isometric embeddings, then the corresponding distances d_i given by (41) are equivalent.

To this aim, observe ørst that we can assume $N_1=N_2=N$. Moreover, if N is taken suŒciently large, by the stable equivalence (see [3]) we can assume that the normal bundles of $i_1(\mathcal{M})$ and $i_2(\mathcal{M})$ in $I\!R^N$ are isomorphic. Then, since $\mathcal{M}$ is compact, the map $i_1\circ i_2^{-1}$ can be extended to a diœeomorphism I from a tubular neighborhood U_2 of $i_2(M)$ to a tubular neighborhood U_1 of $i_1(\mathcal{M})$. To see this, one simply extends $i_1\circ i_2^{-1}$ isometrically in the directions which are normal to $\mathcal{M}$.

Since I is an isometry on $\mathcal{M}$ (so that $\|\mathrm{d}I\|\equiv 1$ on $\mathcal{M}$), then we can assume that $\|\mathrm{d}I\|\in[\frac12,\frac32]$ in U_2. This implies immediately that if $u_i : I\!R^3\longmapsto i_2(\mathcal{M})$, $i=1,2$, are smooth functions, setting $\tilde u_i = i_1\circ i_2^{-1}\circ u_i : I\!R^3\longmapsto i_1(\mathcal{M})$, one has:

$$\frac12\,\mathrm{d}_2(u_1,u_2)\le \mathrm{d}_1(\tilde u_1,\tilde u_2)\le \frac32\,\mathrm{d}_2(u_1,u_2).$$

Thus, d_1 and d_2 are equivalent and we are done. □

By similar arguments, i.e. considering a diœeomorphism of open sets containing the image of two diœerent isometric embeddings of $\mathcal{M}$ into $I\!R^N$, we will prove later that also the diœerential structure of $\mathcal{M}$ is independent of the embedding (see Proposition A.3).

In order to prove the differential structure of our function space and the regularity of our functional, we give here a very brief account of the theory of covariant differentiation along maps. A detailed exposition of the theory may be found in classical textbooks of differential geometry, like for instance [11].

We recall that E is a vector bundle over the manifold X, a connection ∇ on E is an $\mathbb{R}$-linear map $\nabla : \Gamma(E) \longmapsto \Gamma(TX^* \otimes E)$ satisfying the product rule:

$$\nabla(f \cdot W) = \mathrm{d}f \otimes W + f \cdot \nabla W, \tag{42}$$

where f is any smooth function on X, $W \in \Gamma(E)$, $\Gamma(\cdot)$ denotes the space of (smooth) sections of E, and TX^* is the cotangent bundle of X. Here, $TX^* \otimes E$ denotes the vector bundle over X with fiber $(TX^* \otimes E)_x = T_xX^* \otimes E_x \simeq \mathcal{L}(T_xX, E_x)$.

Roughly speaking, a connection ∇ over a vector bundle E allows to compute the directional derivative $\nabla_v W$, where v is a tangent vector to X and W is a section of E, i.e. a vector field on X with values in E. The condition (42) is then translated into the familiar product rule for derivatives:

$$\nabla_v(f \cdot W) = \mathrm{d}f[v] \cdot W + f\, \nabla_v W.$$

If M is a Riemannian manifold and TM is the tangent bundle, the Levi-Civita connection of $\mathcal{M}$ is the unique symmetric connection on TM which is compatible with the metric of $\mathcal{M}$ (see [24]).

If $\phi : \mathbb{R}^3 \longmapsto \mathcal{M}$ is a smooth map, we denote by $\phi^*(T\mathcal{M})$ the vector bundle on $\mathbb{R}^3$ given by:

$$\phi^*(T\mathcal{M}) = \Big\{(x, v) \in \mathbb{R}^3 \times T\mathcal{M} : v \in T_{\phi(x)}\mathcal{M}\Big\}, \tag{43}$$

with projection $\pi(x, v) = x$. A section W of the bundle $\phi^*(T\mathcal{M})$ is a vector field along the map ϕ, i.e. a smooth map $W : \mathbb{R}^3 \longmapsto T\mathcal{M}$ such that $W(x) \in T_{\phi(x)}\mathcal{M}$ for all $x \in \mathbb{R}^3$. These vector fields are also called infinitesimal variations of the map ϕ: every such field W defines a variation ϕ_t of ϕ, with $t \in]-\delta, \delta[$, by $\phi_t(x) = \exp(t \cdot W(x))$, where $\exp$ is the exponential map of $\mathcal{M}$.

Observe that if W is a vector field on $\mathcal{M}$, i.e. a smooth section of the bundle $T\mathcal{M}$, then the composition $W \circ \phi$ defines a smooth section of the

bundle $\phi^*(T\mathcal{M})$; however, if ϕ is not an immersion, not all the sections of $\phi^*(T\mathcal{M})$ are of this form.

It can be proven by elementary methods that, if ∇ is the Levi-Civita connection of the Riemannian manifold $\mathcal{M}$, then there exists a unique connection ∇^ϕ in the vector bundle $\phi^*(T\mathcal{M})$ such that:

$$\nabla^\phi_V(W\circ\phi) = \nabla_{\mathrm{d}\phi[V]}W, \tag{44}$$

for all vector field W on $\mathcal{M}$ and all vector fields V in $\mathbb{R}^3$. Namely, the two conditions (42) and (44) imply that locally ∇^ϕ has the form:

$$\nabla^\phi_v W = \sum_{i=1}^{\dim(\mathcal{M})} \left(\mathrm{d}f_i[v]\cdot W_i\circ\phi + f_i\cdot\nabla_{\mathrm{d}\phi[v]}W_i\right) \tag{45}$$

where $v\in T\mathcal{M}$, $W=\sum f_i\cdot W_i$, $\{W_i\}_i$ is a (local) basis of smooth sections of $T\mathcal{M}$, and f_i are smooth functions. Conversely, it is easily checked that (45) defines a connection ∇^ϕ with the required properties.

If $\phi:\mathbb{R}^3\longmapsto\mathcal{M}$ is a smooth map, then the differential $\mathrm{d}\phi$ maps vector fields in $\mathbb{R}^3$ into vector fields along ϕ, hence it is a section of the bundle $(\mathbb{R}^3)^*\otimes\phi^*(T\mathcal{M})$. The connection ∇^ϕ on $\phi^*(T\mathcal{M})$ induces a connection $\overline{\nabla}$ on $(\mathbb{R}^3)^*\otimes\phi^*(T\mathcal{M})$, given by:

$$(\overline{\nabla}_V\omega)[\zeta] = \nabla^\phi_V(\omega[\zeta]) - \omega(\nabla^{\mathbb{R}^3}_V\zeta), \tag{46}$$

where V is a vector field in $\mathbb{R}^3$, $\zeta:\mathbb{R}^3\longmapsto T\mathcal{M}$ is a vector field along ϕ, $\omega(x):\mathbb{R}^3\longmapsto T_{\phi(x)}\mathcal{M}$ is a linear map for all $x\in\mathbb{R}^3$ and $\nabla^{\mathbb{R}^3}$ is the Levi-Civita connection of the Euclidean space $\mathbb{R}^3$. For $x\in\mathbb{R}^3$ and ζ a section of $\phi^*(T\mathcal{M})$, when we omit the lower index $\nabla^\phi\zeta(x)$ we mean the map in $\mathcal{L}(\mathbb{R}^3,T_{\phi(x)}\mathcal{M})$ given by $v\longmapsto(\nabla^\phi_v\zeta)(x)$. The symbol $\langle\cdot,\cdot\rangle_0$ will denote the Hilbert-Schmidt inner product in $\mathcal{L}(\mathbb{R}^3,T_{\phi(x)}\mathcal{M})$.

The covariant derivative $\overline{\nabla}\,\mathrm{d}\phi$ is called the second fundamental form of ϕ. We will think of $\overline{\nabla}\,\mathrm{d}\phi$ as a bilinear map:

$$\overline{\nabla}\,\mathrm{d}\phi\,[V_1,V_2],$$

that produces a vector field along ϕ when V_1 and V_2 are vector fields in $\mathbb{R}^3$. Observe that, by (46), $\overline{\nabla}\,\mathrm{d}\phi$ is $C^\infty(\mathbb{R})$-linear in both variables.

We define a vector field $\tau(\phi)$ along ϕ by taking the trace of $\overline{\nabla}\,\mathrm{d}\phi$:

$$\tau(\phi) = \mathrm{tr}(\overline{\nabla}\,\mathrm{d}\phi) = \sum_{i=1}^{3} \overline{\nabla}\,\mathrm{d}\phi\,[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}] = \sum_{i=1}^{3} \nabla^{\phi}_{\partial_i}\,\mathrm{d}\phi\,[\frac{\partial}{\partial x_i}] = \sum_{i=1}^{3} \nabla^{\phi}_{\partial_i}\,\frac{\partial \phi}{\partial x_i}. \tag{47}$$

The vector field $\tau(\phi)$ is called the tension of ϕ. If we choose local charts $(y_1, y_2, \ldots, y_m)$ on $\mathcal{M}$ around the point $\phi(x)$, then, denoting by ϕ^k and $\tau^k(\phi)$ respectively the k-th components of ϕ and $\tau(\phi)$, it is easily computed:

$$\tau^k(\phi)(x) = \Delta\phi^k(x) + \sum_{a,b=1}^{m} \sum_{i,j=1}^{3} \Gamma^k_{ab}(\phi(x))\,\frac{\partial \phi^a}{\partial x_i}(x)\,\frac{\partial \phi^b}{\partial x_j}(x), \tag{48}$$

where Δ is the Laplace operator in $\mathbb{R}^3$ and Γ^k_{ab} are the Christoffel symbols of the Riemannian metric of $\mathcal{M}$.

If ϕ is an absolutely continuous map, one extends the concepts of the bundle $\phi^*(T\mathcal{M})$ and the connection ∇^{ϕ} in a canonical way. Similarly, one defines the second fundamental form $\overline{\nabla}\mathrm{d}\phi$ and the tension field $\tau(\phi)$ for C^1-functions having absolutely continuous derivative.

For $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$, we denote by $V(\phi)$ the set of all absolutely continuous vector fields along ϕ, i.e. the set of all absolutely continuous sections of the vector bundle $\phi^*(\mathcal{M})$.

Before proving the main result of the Section, we need the following simple criterion for the differentiability of maps between Banach spaces, which is basically contained in [22], Theorem 11.3.

If E and F are Banach spaces, we denote by $\mathcal{L}(E, F)$ the set of all continuous linear maps from E to F; if Λ is a family of linear maps from F to some other vector space, we say that Λ separates the points of F if for all $x, y \in F$, with $x \neq y$, there exists $\lambda \in \Lambda$ such that $\lambda(x) \neq \lambda(y)$. Equivalently, Λ separates the points of F if for all $x \neq 0$ in F there exists $\lambda \in \Lambda$ such that $\lambda(x) \neq 0$.

Lemma A.2. Let E and F be real Banach spaces, $f : E \longmapsto F$ a map, and $g : F \longmapsto \mathcal{L}(E, F)$ a continuous map. Suppose that there exists a family Λ of continuous linear maps from F to $\mathbb{R}^s$ that separates the points of F and such that, for all $x, v \in E$ and for all $\lambda \in \Lambda$, it is:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda(f(x + tv)) = \lambda(g(x)v).$$

Then, f is a map of class C^1 and $\mathrm{d}f = g$.

Proof. Let $x \in E$ be øxed. For all $h \in E$, we set $r(h) = f(x+h) - f(x) - g(x)h$; since $g(x)$ is linear and continuous, to prove that f is diœerentiable at x and $\mathrm{d}f(x) = g(x)$ it suŒces to show that $\lim_{h\to 0} r(h) \cdot \|h\|^{-1} = 0$.

To this aim, consider the curve $\gamma(t) = g(x+th)h$ in F. Since γ is continuous, then there exists the F-valued integral:

$$I = \int_0^1 \gamma(t)\,\mathrm{d}t \in F.$$

We claim that $I = f(x+h) - f(x)$. Indeed, if it were $I \neq f(x+h) - f(x)$, by the separability condition on Λ there would exist $\lambda \in \Lambda$ such that

$$\lambda(I) \neq \lambda(f(x+h)) - \lambda(f(x)). \tag{49}$$

By the continuity of λ, it is:

$$\lambda(I) = \lambda\left(\int_0^1 \gamma(t)\,\mathrm{d}t\right) = \int_0^1 \lambda(\gamma(t))\,\mathrm{d}t = \int_0^1 \lambda(g(x+th)h)\,\mathrm{d}t. \tag{50}$$

On the other hand, our diœerentiability condition on the function $q(t) = \lambda(f(x+th))$ implies that $q'(t) = \lambda(g(x+th)h)$, hence, from (50) we get:

$$\begin{aligned} \lambda(f(x+h)) - \lambda(f(x)) &= q(1) - q(0) = \int_0^1 q'(t)\,\mathrm{d}t = \\ &= \int_0^1 \lambda(g(x+th)h)\,\mathrm{d}t = \lambda(I). \end{aligned} \tag{51}$$

The identity (51) contradicts (49), hence $r(h) = \int_0^1 (g(x+th)h - g(x)h)\ \mathrm{d}t$. Since the evaluation at h is a continuous function in $\mathcal{L}(E,F)$, then we have:

$$r(h) = \left(\int_0^1 (g(x+th) - g(x))\ \mathrm{d}t\right) h,$$

from which we get:

$$\|r(h)\| \leq \|h\| \int_0^1 \|g(x+th) - g(x)\|\,\mathrm{d}t.$$

By the continuity of g, the latter integral tends to 0 as $h \to 0$, so that $\|r(h)\| \cdot \|h\|^{-1}$ goes to 0 as $h \to 0$. This implies that $\mathrm{d}f(x) = g(x)$. Since g is continuous, then f is C^1 and we are done. □

Observe that in the proof of Lemma A.2 we have never used the fact that the functions in Λ have image in $\mathbb{R}^s$, that could be substituted in the statement of the Lemma by any other Banach space G. As a matter of fact, it is also unnecessary to fix the image space of the functions in Λ, and it would be enough to have a different space G_λ for each λ in Λ.

We are now ready to describe the differentiable structure of $H_{2,6}(\mathbb{R}^3, \mathcal{M})$.

Proposition A.3. Let $\mathcal{M}$ be an m-dimensional compact Riemannian manifold of class C^n, with $n \geq 3$. The set $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ has the structure of a complete Banach manifold of class C^{n-2} modeled on the Banach space $H_{2,6}(\mathbb{R}^3, \mathbb{R}^m)$.

For $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$, the tangent space $T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is identified with the space:

$$T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M}) = \tag{52}$$
$$= \left\{ \zeta \in V(\phi) \;\middle|\; \int_{\mathbb{R}^3} \left(\left\langle \nabla^\phi \zeta, \nabla^\phi \zeta \right\rangle_0 + \left\langle \nabla^\phi \zeta, \nabla^\phi \zeta \right\rangle_0^3 \right) \mathrm{d}x < +\infty \right\}.$$

If $\mathcal{M}$ is isometrically embedded in the Euclidean space $\mathbb{R}^N$, then the tangent space at ϕ can be written as:

$$T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M}) = \left\{ \zeta \in T_\phi H_{2,6}(\mathbb{R}^3, \mathbb{R}^N) \;\middle|\; \zeta(x) \in T_{\phi(x)}\mathcal{M} \;\forall\, x \right\}. \tag{53}$$

Proof. Using the Nash embedding theorem, we assume temporarily that $\mathcal{M}$ is isometrically embedded in $\mathbb{R}^N$ as a submanifold of class C^n, and we assume that $0 \in \mathcal{M}$. Let $k = N - m$ be the codimension of $\mathcal{M}$ in $\mathbb{R}^N$. We consider the Banach space $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$.

By the compactness of $\mathcal{M}$, there exists an open tubular neighborhood U of $\mathcal{M}$ in $\mathbb{R}^N$ and a map of class C^n $r : U \longmapsto \mathcal{M}$ which is a retraction, i.e. $r(x) = x$ for all $x \in \mathcal{M}$. The map r is an orthogonal projection onto $\mathcal{M}$.

For $A \subset \mathbb{R}^N$ we denote by $\overline{A}$ the closure of A in $\mathbb{R}^N$. The set:

$$H_{2,6}(\mathbb{R}^3, \mathbb{R}^N; U) = \left\{ \phi \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^N) : \overline{\phi(\mathbb{R}^3)} \subset U \right\},$$

is an open subset of $\mathbb{R}^3$. Namely, since $\overline{\phi(\mathbb{R}^3)}$ is a compact subset of $\mathbb{R}^N$ for all $\phi \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$, then $\inf_{x \in \mathbb{R}^3} \mathrm{dist}(\phi(x), \partial U) > 0$. So, the same is true for all $\tilde{\phi}$ in a small neighborhood of ϕ in $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$, because of the continuous injection of $H_{2,6}$ in L^∞ (see (12)). It follows that $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N; U)$ is an open submanifold of $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$.

We need the following:

Lemma A.4. Let U be an open subset of $\mathbb{R}^N$ that contains 0. Let $\Theta : U \longrightarrow \mathbb{R}^k$ be a map of class C^n ($n \geq 3$) such that $\Theta(0) = 0$ and such that $\|\mathrm{d}^k\Theta\|$ admits a continuous extension to the closure $\overline{U}$ for all $k = 0, 1, \ldots, n$. Then, the map $T : H_{2,6}(\mathbb{R}^3, \mathbb{R}^N; U) \longmapsto H_{2,6}(\mathbb{R}^3, \mathbb{R}^k)$ defined by:

$$T(\phi) = \Theta \circ \phi$$

is of class C^{n-2}, and

$$\mathrm{d}T(\phi)[\zeta] = \mathrm{d}\Theta(\phi) \circ \zeta, \tag{54}$$

for all $\phi \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^N; U)$ and all $\zeta \in T_\phi H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$.

Proof of Lemma A.4. First of all we observe that, by the regularity of Θ and the fact that $\Theta(0) = 0$ and $\|\mathrm{d}\Theta\|$ is bounded on U, the map T is well defined, i.e. $T(\phi) \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^k)$ for all $\phi \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^N; U)$.

By our hypothesis, we can extend Θ to a map of class C^n defined on $\mathbb{R}^N$ and such that $\|\mathrm{d}^k\Theta\|$ is bounded on $\mathbb{R}^N$ for all $k = 0, 1, \ldots, n$. In this way, we can extend the map T to the entire space $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$; we will prove the Lemma for this extension, and the conclusion will follow easily by restricting T on the open subset $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N; U)$.

To prove the continuity of T, we consider the Banach space

$$B = H_{2,6}(\mathbb{R}^3, \mathbb{R}^k) \oplus \mathcal{C},$$

where $\mathcal{C}$ is the set of constant functions from $\mathbb{R}^3$ to $\mathbb{R}^k$, and the norm of B is given by:

$$\|(\phi, y)\|_B = \|\phi\|_{2,6} + \|y\|,$$

where $y \in \mathbb{R}^k$ is the value of the constant function.

We define a map $\tilde{T} : C^\infty_c(\mathbb{R}^3, \mathbb{R}^N) \longmapsto B$ by:

$$\tilde{T}(\phi) = (\Theta \circ \phi - \Theta(0), \Theta(0)). \tag{55}$$

We temporarily ignore the fact that $\Theta(0) = 0$ and we prove the C^1-regularity of an extension of the map $\tilde{T}$ without using this fact. Observe indeed that (55) defines $\tilde{T}$ independently of the value $\Theta(0)$. This technicality is needed to be able to use the induction on the derivatives of Θ, that in general do not satisfy the equality $\mathrm{d}^k\Theta(0) = 0$ for $k \geq 1$.

Since $C_c^\infty(\mathbb{R}^3,\mathbb{R}^N)$ is dense in $H_{2,6}(\mathbb{R}^3,\mathbb{R}^N)$, to prove that $\tilde{T}$ admits a continuous extension to $H_{2,6}(\mathbb{R}^3,\mathbb{R}^N)$ we show that $\tilde{T}$ is locally uniformly continuous (observe that $\tilde{T}$ is not linear).

To this aim, let $\phi_1,\phi_2 \in C_c^\infty(\mathbb{R}^3,\mathbb{R}^N)$ be øxed, and let K denote a compact convex subset of $\mathbb{R}^N$ containing the images of ϕ_1 and ϕ_2. Using the triangular inequality and the Mean Value Theorem, one sees easily that, for all $x\in\mathbb{R}^3$, the following inequality holds:

$$\begin{aligned}\|\mathrm{d}T(\phi_1(x))\,\mathrm{d}\phi_1(x)-\mathrm{d}T(\phi_2(x))\,\mathrm{d}\phi_2(x)\| \le \\ \le\ c_1\cdot\|\mathrm{d}\phi_1(x)-\mathrm{d}\phi_2(x)\|+c_2\cdot\|\mathrm{d}\phi_2(x)\|\cdot\|\phi_1(x)-\phi_2(x)\|,\end{aligned} \tag{56}$$

where $c_1=\max_K\|\mathrm{d}T\|$ and $c_2=\max_S\|\mathrm{d}^2T\|$, and S is the segment between $\phi_1(x)$ and $\phi_2(x)$. From (56) it follows immediately that $\tilde{T}$ is locally uniformly continuous, and so it admits a continuous extension to $H_{2,6}(\mathbb{R}^3,\mathbb{R}^N)$.

Finally, we deøne $\overline{T}:B\longmapsto B$ by:

$$\overline{T}(\phi,y)=(\Theta\circ\phi-\Theta(0),\Theta(0)+\Theta(y)).$$

As $\tilde{T}$ is continuous, then also $\overline{T}$ is continuous, and this was established using the C^2-regularity of Θ and the boundedness of the ørst two derivatives of Θ, without using the assumption that $\Theta(0)=0$.

To prove that $\overline{T}$ is of class C^1, we use the criterion of Lemma A.2, by exhibiting explicitly the diœerential of $\overline{T}$. Namely, for each $(\phi,y)\in B$, we consider the map $g(\phi,y):B\longmapsto B$ given by:

$$g(\phi,y)(\zeta,w)=(\mathrm{d}\Theta(\phi(\cdot))\circ\zeta,\mathrm{d}\Theta(y)[w]). \tag{57}$$

Since $\mathrm{d}\Theta$ is a map of class C^{n-1} and its ørst two derivatives are bounded in $\mathbb{R}^3$, then, by the previous argument, the map $(\phi,y)\longmapsto g(\phi,y)$ is continuous from B to $\mathcal{L}(B,B)$.

Let $\Lambda=\{\lambda_x : x\in\mathbb{R}^3\}$ be the family of linear maps $\lambda_x : B\longmapsto\mathbb{R}^4$ deøned by:

$$\lambda_x(\phi,y)=(\phi(x),y).$$

Clearly, each λ_x is linear and continuous in B. Infact, the evaluation at the point x is continuous in $H_{2,6}(\mathbb{R}^3,\mathbb{R}^N)$, because it is continuous in the C^0-topology. Moreover, Λ separates the points of B, because the evaluations at the points of $\mathbb{R}^3$ clearly separates the points of $H_{2,6}(\mathbb{R}^3,\mathbb{R}^N)$.

For every $x \in I\!R^3$, we have:

$$\lambda_x(g(\phi,y)(\zeta,w)) = (\mathrm{d}\Theta(\phi(x))[\zeta(x)], \mathrm{d}\Theta(y)[w]), \tag{58}$$

for all (ϕ,y) and (ζ,w) in B.

On the other hand, using the deønition of $\overline{T}$ and the diœerentiability of Θ, we have:

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda_x\Big((\phi,y)+t(\zeta,w)\Big) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda_x\left((\phi+t\zeta, y+tw)\right) = \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \lambda_x\big(\Theta\circ(\phi+t\zeta), \Theta(0)+\Theta(y+tw)\big) = \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\Theta(\phi(x)+t\zeta(x)) - \Theta(0), \Theta(0)+\Theta(y+tw)\big) = \\
&= \big(\mathrm{d}\Theta(\phi(x))[\zeta(x)], \mathrm{d}\Theta(y)[w]\big).
\end{aligned} \tag{59}$$

From (58) e (59), using Lemma A.2, we have that $\overline{T}$ is a C^1-map, whose diœerential is given by $\mathrm{d}\overline{T}(\phi,y) = g(\phi,y)$ of (57).

By induction, we have that $\overline{T}$ is of class C^{n-2}, and so is the map T of the hypothesis, which is obtained by composing $\overline{T}$ and the projection onto the ørst coordinate from B to $H_{2,6}(I\!R^3, I\!R^N)$.

Observe that the diœerential of T is obtained by taking the ørst coordinate of the diœerential of $\overline{T}$, which gives (54) and concludes the proof. □

Going back to the proof of Proposition A.3, we assume for simplicity that the normal bundle of $\mathcal{M}$ in $I\!R^3$ is trivial; the general case will follow from this considering that the normal bundle of $\mathcal{M}$ can be written as the clutching of a ønite number of trivial bundles over open submanifolds of $\mathcal{M}$. The set U can be considered an open subset of the normal bundle of $\mathcal{M}$, with $r : U \longmapsto \mathcal{M}$ the projection. We choose a trivialization of U given by a map of class C^n $\psi = (\psi^1, \psi^2) : U \longmapsto \mathcal{M} \times I\!R^k$, satisfying

$$\psi(x+\lambda v) = (x, \lambda\psi^2(v)),$$

for all $x \in \mathcal{M}$, v orthogonal to $T_x\mathcal{M}$ and λ real number suŒciently small. By construction, $\psi^1 = r$. Up to shrinking the open set U, we can assume that the map $\psi^2 : U \longmapsto I\!R^k$ satisøes the continuity hypotheses of Lemma A.4, because ψ^2 is of class C^n in U, $0 \in \mathcal{M}$ and $\psi(0) = (0,0)$. It follows that the map $\mathcal{F} : H_{2,6}(I\!R^3, I\!R^N; U) \longmapsto H_{2,6}(I\!R^3, I\!R^k)$ deøned by:

$$\mathcal{F}(\phi) = \psi^2 \circ \phi,$$

is of class C^{n-2}, and

$$\mathrm{d}\mathcal{F}(\phi)[\zeta](x) = \mathrm{d}\psi^2(\phi(x))[\zeta(x)], \tag{60}$$

for all $\phi \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$, all $\zeta \in T_\phi H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$ and all $x \in \mathbb{R}^3$.

Observe that, by construction, $H_{2,6}(\mathbb{R}^3, \mathcal{M}) = \mathcal{F}^{-1}(0)$.

By the Implicit Function Theorem, in order to prove that $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is a C^{n-2}-submanifold of $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$ it suffices to show that $\mathrm{d}\mathcal{F}$ is surjective on the points of $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ and that $\mathrm{Ker}\big(\mathrm{d}\mathcal{F}(\phi)\big)$ has a closed complement in $T_\phi H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$.

To settle the first question, we simply observe that, by construction, the restriction of $\mathrm{d}\phi^2(z)$ to the orthogonal space $T_z\mathcal{M}^\perp$ gives an isomorphism with image $\mathbb{R}^k$, for all $z \in \mathcal{M}$.

Then, if $w \in H_{2,6}(\mathbb{R}^3, \mathbb{R}^k) \simeq T_{\mathcal{F}(\phi)} H_{2,6}(\mathbb{R}^3, \mathbb{R}^k)$, setting:

$$\zeta(x) = \big(\mathrm{d}\psi^2(\phi(x))\big)^{-1}\,[w(x)],$$

we have that $\zeta \in T_\phi H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$ and, from (60), $\mathrm{d}\mathcal{F}(\phi)[\zeta] = w$.

As to the splitting condition on the kernel of $\mathrm{d}\mathcal{F}$, observe that this space consists of vector fields that are tangent to $\mathcal{M}$, and it is complemented by the vector fields that are normal to $\mathcal{M}$. This proves that $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is a C^{n-2}-submanifold of $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$.

At each point $\phi \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$, by the Implicit Function Theorem the tangent space $T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is given by the kernel of the differential $\mathrm{d}\mathcal{F}(\phi)$, as we have observed:

$$\begin{aligned} T_\phi H_{2,6}(\mathbb{R}^3, \mathcal{M}) &= \mathrm{Ker}(\mathrm{d}\mathcal{F}(\phi)) = \\ &= \Big\{\zeta \in T_\phi H_{2,6}(\mathbb{R}^3, \mathbb{R}^N) : \zeta(x) \in T_{\phi(x)}\mathcal{M},\ \forall\, x \in \mathbb{R}^3\Big\}, \end{aligned}$$

obtaining (53). Observe that, since $\dim(T_{\phi(x)}\mathcal{M}) = \dim(\mathcal{M}) = m$, then the Banach space model for $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is $H_{2,6}(\mathbb{R}^3, \mathbb{R}^m)$. Observe also that, arguing as in the proof of Lemma A.1 and using the fact that, by Lemma A.4, the composition on the left with a diffeomorphism is a C^1-map, we have that the differentiable structure of $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ does not depend on the isometric embedding of $\mathcal{M}$ in $\mathbb{R}^N$. Then, (52) is an intrinsic rewriting of (53).

Finally, the completeness of $H_{2,6}(\mathbb{R}^3, \mathcal{M})$ follows easily from the completeness of $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$ and the fact that $\mathcal{M}$ is closed. Namely, every Cauchy sequence $u_n \in H_{2,6}(\mathbb{R}^3, \mathcal{M})$ is convergent to u in $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$. Since the convergence is also uniform and $\mathcal{M}$ is closed, then $u(\mathbb{R}^3) \subset \mathcal{M}$ and u_n converges to u in $H_{2,6}(\mathbb{R}^3, \mathcal{M})$. □

The continuous embedding of $H_{2,6}(\mathbb{R}^3,\mathcal{M})$ into $C_0(\mathbb{R}^3,\mathcal{M})$ implies that the two spaces have the same homotopy type:

Proposition A.5. The inclusion map $J : H_{2,6}(\mathbb{R}^3,\mathcal{M}) \longmapsto C_0(\mathbb{R}^3,\mathcal{M})$ is a homotopy equivalence.

Proof. The proof goes along the same lines of the proof of Theorem 13.14 of [22]. Namely, using the fact that and so $\mathbb{R}^3 \times \mathcal{M}$ is closed in $\mathbb{R}^3 \times \mathbb{R}^N$, one reduces the proof to the case where $\mathcal{M}$ is isometric to a Euclidean space. Observe that the compactness of $\mathcal{M}$ plays no role in this proof. Then, the map J is a continuous map between Banach spaces with dense image, because $J(H_{2,6}(\mathbb{R}^3,\mathcal{M}))$ contains the set of compactly supported smooth functions. Hence, J is a homotopy equivalence (see Lemma 13.12 of [22]). □

We recall the deøinition of the functional $E : H_{2,6}(\mathbb{R}^3,\mathcal{M}) \longmapsto \mathbb{R}^+$ given in Section 1:

$$E(\phi) = \int_{\mathbb{R}^3} \beta\left(\left\langle \mathrm{d}\phi(x), \mathrm{d}\phi(x)\right\rangle_0\right)\, \mathrm{d}x, \tag{61}$$

where $\beta(\lambda) = \lambda + \lambda^3$.

Proposition A.6. The functional $E : H_{2,6}(\mathbb{R}^3,\mathcal{M}) \longmapsto \mathbb{R}^3$ is of class C^{n-2}. For $\phi \in H_{2,6}(\mathbb{R}^3,\mathcal{M})$ and $\zeta \in T_\phi H_{2,6}(\mathbb{R}^3,\mathcal{M})$, the Gateaux derivative of E at the point ϕ and in the direction ζ is given by:

$$E'(\phi)[\zeta] = 2\int_{\mathbb{R}^3} \left(1 + 3\left\langle \mathrm{d}\phi(x), \mathrm{d}\phi(x)\right\rangle_0^2\right) \cdot \left\langle \nabla^\phi \zeta(x), \mathrm{d}\phi(x)\right\rangle_0 \,\mathrm{d}x. \tag{62}$$

Proof. We identify the space $\mathbb{R}^{3N}$ with the space $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^N)$ endowed with the Hilbert˜Schmidt norm of linear operators. Let's consider the Banach space $L_{2,6}(\mathbb{R}^3,\mathbb{R}^{3N})$ given by the completion of $C_c^\infty(\mathbb{R}^3,\mathbb{R}^N)$ with respect to the norm:

$$||u||_{L_{2,6}} = ||u||_{L^2} + ||u||_{L^6}.$$

The map $\mathrm{d} : \phi \longmapsto \mathrm{d}\phi$ is clearly a smooth map from $H_{2,6}(\mathbb{R}^3,\mathbb{R}^N)$ to $L_{2,6}(\mathbb{R}^3,\mathbb{R}^{3N})$, because it is linear and continuous.

For $p > 1$, the map $E_p : L^p(\mathbb{R}^3,\mathbb{R}^{3N}) \longmapsto \mathbb{R}^+$ given by $E_p(u) = ||u||_{L^p}^p$ is smooth, hence $E_{2,6} = E_2 + E_6$ is smooth on $L^2 \cap L^6$. Since $L_{2,6}(\mathbb{R}^3,\mathbb{R}^{3N})$ is a linear subspace of $L^2(\mathbb{R}^3,\mathbb{R}^{3N}) \cap L^6(\mathbb{R}^3,\mathbb{R}^{3N})$ (not closed), and the

topology of $L_{2,6}$ is generated by the topologies induced by L^2 and L^6, then the restriction of $E_{2,6}$ to $L_{2,6}(\mathbb{R}^3, \mathbb{R}^{3N})$ is also smooth.

So, the composite map $E_{2,6} \circ \mathrm{d} = E$ is smooth on the Banach space $H_{2,6}(\mathbb{R}^3, \mathbb{R}^N)$, and, by Proposition A.3, E is a functional of class C^{n-2} on the submanifold $H_{2,6}(\mathbb{R}^3, \mathcal{M})$.

Since (62) deøne a formula which is $H_{2,6}$-continuous in ϕ and in ζ, in order to prove that (62) holds it suŒces to consider the case of $\phi \in C_0^\infty(\mathbb{R}^3, \mathcal{M})$ and ζ a smooth section of $\phi^*(T\mathcal{M})$ with compact support.

To this aim, we consider the functional $\tilde{E}$ given by:

$$\tilde{E}(\phi) = \int_{\mathbb{R}^3} \left\langle \mathrm{d}\phi(x), \mathrm{d}\phi(x) \right\rangle_0 \mathrm{d}x.$$

We have just proven the regularity of $\tilde{E}$; if $\phi_t(x) = \exp(t \cdot \zeta(x))$ is the variation associated to the vector øeld ζ, then we have:

$$\tilde{E}'(\phi)[\zeta] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{E}(\phi_t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{\mathbb{R}^3} \left\langle \mathrm{d}\phi_t(x), \mathrm{d}\phi_t(x) \right\rangle_0 \mathrm{d}x..$$

Since ϕ and ζ have compact support, we have:

$$\begin{aligned} \tilde{E}'(\phi)[\zeta] &= \int_{\mathbb{R}^3} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left\langle \mathrm{d}\phi_t, \mathrm{d}\phi_t \right\rangle_0 \mathrm{d}x = \\ &= 2 \int_{\mathbb{R}^3} \left\langle \nabla^\phi \frac{\mathrm{d}}{\mathrm{d}t}\phi_t, \mathrm{d}\phi_t \right\rangle_0 \Big|_{t=0} \mathrm{d}x = 2 \int_{\mathbb{R}^3} \left\langle \nabla^\phi \zeta(x), \mathrm{d}\phi(x) \right\rangle_0 \mathrm{d}x. \end{aligned} \tag{63}$$

Finally, (62) follows immediately from (61) and (63). □

Proposition A.7. The critical points of E satisfy the Euler˜Lagrange equation of the functional E, given by:

$$\tau(\phi) + 3 \operatorname{div} \left(\left\langle \mathrm{d}\phi, \mathrm{d}\phi \right\rangle_0^2 \cdot \mathrm{d}\phi \right) = 0, \tag{64}$$

where $\tau(\phi)$ is the tension of the map ϕ (see (47) and (48)) and $\left\langle \mathrm{d}\phi, \mathrm{d}\phi \right\rangle_0$ is the Hilbert˜Schmidt inner product given by:

$$\left\langle \mathrm{d}\phi, \mathrm{d}\phi \right\rangle_0 = \sum_{i=1}^{3} \left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle.$$

Using local coordinates on the manifold $\mathcal{M}$, the diœerential equation (64) is written as:

$$\Delta\phi^k(x) \;+\; \sum_{a,b=1}^{m}\sum_{i,j=1}^{3}\Gamma^k_{ab}(\phi(x))\,\frac{\partial\phi^a}{\partial x_i}(x)\,\frac{\partial\phi^b}{\partial x_j}+ \\ +3\,\mathrm{div}\,\left(\|\mathrm{d}\phi(x)\|^4\cdot\nabla\phi^k(x)\right)=0,\quad k=1,\dots,m, \tag{65}$$

where Δ and ∇ are respectively the Laplacian and the gradient operators in $I\!R^3$ and Γ^k_{ab} are the Christoœel symbols of the metric of $\mathcal{M}$.

Proof. The regularity for the critical points of E is proven in Theorem 3.2, hence they are solutions of the Euler˜Lagrange equations of E in a strong sense.

To obtain an explicit form of the Euler˜Lagrange equations for the functional E, we need an integration by parts in (62). By deønition of $\langle\cdot,\cdot\rangle_0$, we have:

$$\begin{aligned}\int_{I\!R^3}\left\langle\nabla^\phi\zeta(x),\mathrm{d}\phi(x)\right\rangle_0\,\mathrm{d}x &= \sum_{i=1}^{3}\int_{I\!R^3}\left\langle\nabla^\phi_{\partial_i}\zeta(x),\mathrm{d}\phi(x)\frac{\partial}{\partial x_i}\right\rangle\,\mathrm{d}x= \\ &= \sum_{i=1}^{3}\int_{I\!R^3}\left\langle\nabla^\phi_{\partial_i}\zeta(x),\frac{\partial\phi}{\partial x_i}(x)\right\rangle\,\mathrm{d}x.\end{aligned} \tag{66}$$

Now, since ζ has compact support, for every $i=1,2,3$ it is:

$$0=\int_{I\!R^3}\frac{\partial}{\partial x_i}\Big\langle\zeta,\frac{\partial\phi}{\partial x_i}\Big\rangle\,\mathrm{d}x=\int_{I\!R^3}\left(\Big\langle\nabla^\phi_{\partial_i}\zeta,\frac{\partial\phi}{\partial x_i}\Big\rangle+\Big\langle\zeta,\nabla^\phi_{\partial_i}\frac{\partial\phi}{\partial x_i}\Big\rangle\right)\,\mathrm{d}x.$$

Recalling (47), we therefore get:

$$\int_{I\!R^3}\left\langle\nabla^\phi\zeta(x),\mathrm{d}\phi(x)\right\rangle_0\,\mathrm{d}x=-\int_{I\!R^3}\left\langle\zeta(x),\tau(\phi)(x)\right\rangle\,\mathrm{d}x. \tag{67}$$

By a similar argument, we get:

$$\begin{aligned}\int_{I\!R^3}\left\langle\mathrm{d}\phi,\mathrm{d}\phi\right\rangle_0^2\left\langle\nabla^\phi\zeta,\mathrm{d}\phi\right\rangle_0\,\mathrm{d}x &= -\sum_{i=1}^{3}\int_{I\!R^3}\left\langle\zeta,\nabla^\phi_{\partial_i}\left(\left\langle\mathrm{d}\phi,\mathrm{d}\phi\right\rangle_0^2\cdot\frac{\partial\phi}{\partial x_i}\right)\right\rangle\,\mathrm{d}x= \\ &= -\int_{I\!R^3}\left\langle\mathrm{d}\phi,\mathrm{d}\phi\right\rangle_0^2\cdot\left\langle\zeta,\tau(\phi)\right\rangle\,\mathrm{d}x+ \\ &\quad -\sum_{i=1}^{3}\int_{I\!R^3}\left\langle\zeta,\frac{\partial}{\partial x_i}\left(\left\langle\mathrm{d}\phi,\mathrm{d}\phi\right\rangle_0^2\right)\cdot\frac{\partial\phi}{\partial x_i}\right\rangle\,\mathrm{d}x.\end{aligned} \tag{68}$$

From (62), (67) and (68), passing to a strong equality we obtain the Euler–Lagrange equations of E:

$$\left(1+3\left\langle \mathrm{d}\phi,\mathrm{d}\phi\right\rangle_0^2\right)\tau(\phi)+3\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\left\langle \mathrm{d}\phi,\mathrm{d}\phi\right\rangle_0^2\right)\cdot\frac{\partial\phi}{\partial x_i}=0. \tag{69}$$

The equations (64) and (65) follow immediately from (69) and (48)). □

References

[1] E. Acerbi, N. Fusco, Regularity for Minimizers of Non-quadratic Functionals: The Case $1<p<2$., J. Math. An. Appl. 140, No. 1 (1989), 115–135.

[2] D. Anderson, G. Derrick, Stability of Time Dependent Particle-like Solutions in Nonlinear Field Theories, J. Math. Phys. 11 (1970), 1336–1346.

[3] M. Atiyah, K-Theory, W. A. Benjamin, New York, 1967.

[4] V. Benci, G. Cerami, Positive Solutions of Some Nonlinear Elliptic Problems in Exterior Domains, Arc. Rational Mech. Anal. 99 (1987), 283–300.

[5] V. Benci, D. Fortunato, L. Pisani, Soliton-like Solutions of a Lorentz Invariant Equation in Dimension 3, Rapporti Dip. Matematica Univ. Bari 35/96.

[6] V. Benci, D. Fortunato, L. Pisani, Remarks on Topological Solitons, preprint 1996.

[7] H. Berestici, T. Cazenave, Instabilit del Etats Stationnaire dans des Equations de Schrdinger et de Klein–Gordon non Linairs, CRAS Paris, srie I, 293 (1981), 488–492.

[8] H. Berestici, P. L. Lions, Existence d'Ondes Solitaires dans des Problems non Linairs du Type Klein–Gordon, CRAS Paris, srie A, 287 (1978), 503–506.

[9] M. S. Berger, On the Existence and the Structure of Stationary States for a Nonlinear Klein–Gordon Equation, J. Funct. An. 9 (1972), 246–261.

[10] H. Brezis, Analyse Fonctionelle, Thorie et Applications, Masson, Paris-New York, 1983.

[11] M. do Carmo, Geometria Riemanniana, 2nd edition, IMPA, Rio de Janeiro, 1988.

[12] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, H. C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, London-New York, 1982.

[13] J. Eells, A Setting for Global Analysis, Bull. A.M.S. 72, vol. 5 (1966), 751˜807.

[14] J. Eells, A. Ratto, Harmonic Maps and Minimal Immersions with Symmetries, Princeton University Press, Princeton, 1993.

[15] S. Kichenassamy, Nonlinear Wave Equations, Marcel Dekker Inc., New York-Basel-Hong Kong, 1996.

[16] N. Fusco, J. Hutchinson, Partial Regularity for Minimisers of Certain Functionals Having Nonquadratic Growth, Ann. Mat. Pura Appl. (IV), Vol. 155 (1989), 1˜24.

[17] F. Giannoni, P. H. Rabinowitz, On the Multiplicity of Homoclinic Orbits on Riemannian Manifolds for a Class of Second Order Hamiltonian Systems, NoDEA 1 (1994), 1˜46.

[18] M. Giaquinta, G. Modica, Remarks on the Regularity of the Minimizers of Certain Degenerate Functionals, Manuscripta Math. 57 (1986), 55˜99.

[19] R. Hardt, F. H. Lin, Mappings Minimizing the L^p˜Norm of the Gradient, Comm. Pure Appl. Math. 40 (1987), 555˜588.

[20] J. Nash, The Embedding Problem for Riemannian Manifolds, Ann. Math. 63, 1956, 20˜63.

[21] R. Palais, Morse Theory on Hilbert Manifolds, Topology 2 (1963), 299˜340.

[22] R. Palais, Foundations of Global Nonlinear Analysis, W. A. Benjamin, 1968.

[23] R. Rajaraman, Solitons and Instantons, North Holland, Amsterdam ˜ Oxford˜New York˜Tokio, 1988.

[24] M. Spivak, A Comprehensive Introduction to Diœerential Geometry, vol. 2 (second edition), Publish or Perish, Inc., Wilmington, Delaware.

[25] W. A. Strauss, Existence of Solitary Waves in Higher Dimension, Comm. Math. Phys. 55 (1977), 149˜162.

[26] G. B. Witham, Linear and Nonlinear Waves, J. Wiley and Sons, New York, 1974.