Solitons and the pilot wave theory

Vieri Benci
Dipartimento di Matematica Applicata "U. Dini"
and
Centro Interdisciplinare per lo Studio dei Sistemi Complessi
Università di Pisa, via Bonanno 25/b
56126 PISA (Italy)
e-mail: benci@dma.unipi.it

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Abstract

This work is part of a program which has the aim to investigate which phenomena can be explained by nonlinear effects in Classical Mechanics and which ones require the new axioms of Quantum Mechanics. In this paper, we construct a nonlinear field equation which admits soliton solutions. These solitons exibit a dynamics which is similar to that of quantum particles.

Contents

1 Introduction 1
1.1 The basic idea 1
1.2 The double solution theory 2
2 The model 4
2.1 The Lagrangian 4
2.2 The field w 5
2.3 Assumptions on α and V 6
3 Existence and properties of soliton solutions 8
3.1 Stationary solutions 8
3.2 Topological structure of the problem 8
3.3 The existence theorem 9
3.4 The De Broglie singular solutions 11
4 Structure of soliton solutions 12
4.1 The halo of a soliton 12
4.2 The core of a soliton 13
4.3 The support of a soliton 14
4.4 Size of the soliton 14
5 Dynamics of solitons 15

5.1 The equations of motion 15
5.2 The guidance formula 16
5.3 The Shrödinger equation 18
5.4 Bohmian mechanics 20
6 Conclusions 21

1. Introduction

1.1. The basic idea

In this paper we present a classic model which exhibit some of the features of Quantum Mechanics. By "classic model", we mean a field described by a partial differential equation of the second order, defined in the classical space-time (i.e. $\mathbf{R}^4$) and invariant for the Poincaré group. We assume that this equation is nonlinear; this nonlinearity is responsable of the existence of soliton-like solutions.

Roughly speaking a soliton is a solution of a field equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect solitons have a particle-like behavior.

A solution of our equation can be written as follows

$$u(x,t) = \psi(x,t) + \delta_\varepsilon(x - \mathbf{q}(t)) \tag{1.1}$$

where $\psi(x,t)$ can be considered as a wave and $\delta_\varepsilon(x - \mathbf{q}(t))$is our soliton: a bump of energy concentrated around the point $\mathbf{q}(t)$ in a ball of radius of the order ε. Considering this decomposition the solutions to our equation can be thought as a combination of a wave and a particle.

ε occurs in the equation as a parameter and it will be interpreted as the radius of the soliton; an other parameter occurring in our equation is c, the speed of light. The main purpose of this paper is to show that for ε sufficiently small and c sufficiently large, $\psi(x,t)$ satisfies the Schrödinger equation and $\mathbf{q}(t)$ satisfies the following equation:

$$\dot{\mathbf{q}}(t) = \frac{\hbar}{m} \operatorname{Im} \frac{\nabla\psi}{\psi}; \tag{1.2}$$

this is the fundamental equation of the Bohmian mechanics [7]; it is well known that a particle whose motion is governed by the (1.2) behaves as a quantum particle at least in many situations; according to Bell, "[The Bohmian version of Quantum Mechanics] is experimentally equivalent to the usual version insofar the latter is unambiguous" [1].

The obvious conclusion is the following one: there exist classic dynamical systems, described by nonlinear partial differential equations, whose behavior is equal (or at least very similar) to that of quantum systems.

In our model, it is possible to include as many particles as we want; however their interaction is essentially local. In other words, phenomena of EPR type cannot be reproduced by this model. And the reason why, is very clear. This

model is Lorentz invariant and hence it is local. If we like to describe the world by mean of differential equations in $\mathbf{R}^4$ in such a way that we can reproduce phenomena of EPR type, as far as I can see, we have to renounce to the Lorentz invariance and hence to the Special Relativity. Most likely we can reproduce the quantum nonlocal phenomena for n particles generalizing our equation in $\mathbf{R}^{4n}$ or in $\mathbf{R}^{3n+1}$, but the investigation of this possibility goes out of the aim of the present paper.

1.2. The double solution theory

In some sense this paper is a continuation of the work of De Broglie [8] about the double solution theory. We will spend a few word about this theory. Take for example the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - c^2\Delta\psi + \omega_0^2\psi = 0 \tag{1.3}$$

and consider, not only its regular solutions, but also the singular ones.

The plane waves are a class of regular solutions:

$$\psi(x,t) = be^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \text{ with } \omega^2 = \omega_0^2 + c^2\mathbf{k}^2, \ \ b \in \mathbf{C}; \tag{1.4}$$

also, there is a class of singular solutions related to the plane waves:

$$u(x,t) = \frac{b}{\sqrt{\left(\frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + x_2^2 + x_3^3}} e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} \text{ with } \mathbf{k} = \frac{\omega}{c^2}(v,0,0) \tag{1.5}$$

The ψ-wave (1.4) and the u-wave (1.5) are related to each other since they have the same phase: $\omega t - \mathbf{k}\cdot\mathbf{x}$.

By the great discovery of De Broglie, every particle of mass m and momentum $\mathbf{p}$ is associated to a wave of frequency $\omega = \frac{mc^2}{\hbar}$ and wave number $\mathbf{k} = \frac{\mathbf{p}}{\hbar}$; thus the velocity of the particle

$$\mathbf{v} = \frac{\mathbf{p}}{m} = c^2\frac{\mathbf{k}}{\omega} \tag{1.6}$$

depends *only* on the phase of the wave.

In the wave (1.5) the position of the singularity

$$\mathbf{q}(t) = (vt,\ 0,\ 0)$$

moves is a straight line with velocity given by the (1.6) and in general it moves according to the "guidance equation" which, in the nonrelativistic limit, reduces to the (1.2).

In the De Broglie theory the singularity is interpreted as the particle, and the u-wave determines the trajectory of the particle; since only the phase of the

wave is relevant for the motion of the particle, we can think that the particle is "guided" by the ψ-wave (the pilot wave) but this wave has no physical meaning: it is only a mathematical device to make computations easier.

One of the main objections to this theory is based on the fact that equations as the Klein-Gordon equation (1.3) and the Schrödinger equation are dispersive and a wave packet will survive only for a short time. If (1.5) is smoothed in order to make the energy not to diverge, it becomes a wave paket and hence it disperses. In any case it is not clear why (1.5) should be stable. However this objection disappears if we consider nonlinear equations and we replace the singularity with a soliton. De Broglie was very aware of this fact:

> Des considérasions sur lesquelles je rievendrai me conduisent aujord'hui à penser que le corpuscule doit être assimilé non pas à un véritable singularité punctuelle de u, mais à un très petite région singulière de l'espace où u prenderait une très grande valeur et obéirait à une équation non linéaire dont l'équation linéaire de la Mécanique ondulatoire ne serait qu'une forme approximative valable en dehor de la region singulière. L'idée que l'équation de propagation de u, contrairement à l'équation classique du Ψ, est en principe non linéaire m'apparaît même maintenant comme tout à fait essentielle. ([8], Ch. IX,1, page 95.).

But 40 years ago the methods of Nonlinear Analysis were not sufficiently developed to construct a mathematical model and to *prove* many of the intuitions of De Broglie.

I do not think that now it is too late.

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2. The model

We consider a nonlinear perturbation of the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - c^2\Delta\psi + \omega_0^2\psi = F_\psi \tag{2.1}$$

F_ψ is a nonlinear perturbation which depends on ψ and its first and second derivatives and which vanishes up to the second order terms. Thus, if ψ is a solution of (1.3) and if $\|\psi\| \ll 1$, then it is a good approximation of a solution of (2.1). However, where ψ is large, the non-linear term F_ψ becomes relevant and it might cause the formation of a stable structure which can be considered as a particle (namely a *soliton*).

The aim of this section is to determine the nonlinear term F_ψ; we require F_ψ to be Lorentz invariant and to be such that the equation (2.1) admits soliton solutions. Beside that we try to make F_ψ as simple as possible.

2.1. The Lagrangian

The Lagrangian density for the equation (1.3) is given by

$$\mathcal{L}(\nabla\psi, \psi_t, \psi) = -\frac{1}{2}\left(|\nabla\psi|^2 - \frac{1}{c^2}|\psi_t|^2 + \frac{\omega_0^2}{2c^2}|\psi|^2\right). \tag{2.2}$$

If we set

$$\psi(x,t) = u(x,t)e^{iS(x,t)/\hbar},\ u \in \mathbf{R},\ S \in S^1 \tag{2.3}$$

this Lagrangian takes the following form,

$$\mathcal{L}(\sigma_u, \sigma_S, u) = -\frac{1}{2}\left[\sigma_u + \frac{1}{\hbar^2}\sigma_S\, u^2\right] - \frac{\omega_0^2}{2c^2}u^2 \tag{2.4}$$

where

$$\sigma_u = |\nabla u|^2 - \frac{1}{c^2}\left|\frac{\partial u}{\partial t}\right|^2;\ \sigma_S = |\nabla S|^2 - \frac{1}{c^2}\left|\frac{\partial S}{\partial t}\right|^2. \tag{2.5}$$

A generalization of the Lagrangian (2.4) which gives nonlinear equations is given by

$$\mathcal{L}(\sigma_u, \sigma_S, u) = -\frac{1}{2}\left[\alpha(\sigma_u) + \frac{1}{\hbar^2}\sigma_S\, u^2\right] - \frac{\omega_0^2}{c^2}V(u); \tag{2.6}$$

here α and the potential V are smooth functions such that

$$\alpha(0) = 0,\ \alpha'(0) = 1;\ V(0) = 0,\ V'(0) = 0,\ V''(0) = 1$$

The action functional related to (2.6) is

$$\mathcal{J}(U) = \int_{\mathbf{R}^{3+1}} \mathcal{L}(\sigma_u, \sigma_S, u)\, dx\, dt$$

$$= -\int_{\mathbf{R}^{3+1}} \left[\frac{1}{2}\left(\alpha(\sigma_u) + \frac{1}{\hbar^2}\sigma_S\, u^2\right) + \frac{\omega_0^2}{c^2}V(u)\right] dx\, dt. \tag{2.7}$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}\left[\alpha'(\sigma_u)\,\frac{\partial u}{\partial t}\right] - c^2\nabla\cdot\left[\alpha'(\sigma_u)\,\nabla u\right] + \frac{c^2}{\hbar^2}\sigma_S\, u + \omega_0^2 V'(u) = 0,$$

$$\frac{\partial}{\partial t}\left[u^2\frac{\partial S}{\partial t}\right] - c^2\nabla\cdot\left[u^2\nabla S\right] = 0.$$

2.2. The field w

We want to impose conditions in order to have topological solitons namely solitons whose stability is guaranteed by topological constraints. For this reason, we introduce a new field $w : \mathbf{R}^3 \to \mathbf{R}^3$ coupled with the field $\psi = u(x,t)e^{iS(x,t)/\hbar}$ by the potential V and we set

$$U(x,t) = \begin{bmatrix} u(x,t) \\ S(x,t) \\ w(x,t) \end{bmatrix}.$$

Thus $U(x,t)$ is a functions which takes its values in a manifold $\mathcal{M} \subset \mathbf{R} \times S^1 \times \mathbf{R}^3$.

In order to develop our theory, we need the choose $\mathcal{M}$ such that the third homotopy group $\pi_3(\mathcal{M})$ is nontrivial.

One of the simplest possible choices for $\mathcal{M}$ is

$$\mathcal{M} = \left(\mathbf{R} \times S^1 \times \mathbf{R}^3\right) - \{p_0\}\,; \quad S^1 = \mathbf{R}/\left(2\pi\mathbf{Z}\right)$$

where

$$p_0 = (1,0,0) \in \mathbf{R} \times S^1 \times \mathbf{R}^3.$$

So we get the following Lagrangian:

$$\mathcal{L}(\sigma_u, \sigma_S, \sigma_w, u, w) = -\frac{1}{2}\left[\alpha\left(\sigma_u\right) + \alpha\left(\sigma_w\right) + \frac{1}{\hbar^2}\sigma_S\, u^2\right] - \frac{\omega_0^2}{c^2}V(u,w) \tag{2.8}$$

and the following evolution equations:

$$\frac{\partial}{\partial t}\left[\alpha'\left(\sigma_u\right)\frac{\partial u}{\partial t}\right] - c^2\nabla\cdot\left[\alpha'\left(\sigma_u\right)\nabla u\right] + \frac{c^2}{\hbar^2}\sigma_S\, u + \omega_0^2\frac{\partial V}{\partial u}(u,w) = 0\ , \tag{2.9}$$

$$\frac{\partial}{\partial t}\left[u^2\frac{\partial S}{\partial t}\right] - c^2\nabla\cdot\left[u^2\nabla S\right] = 0, \tag{2.10}$$

$$\frac{\partial}{\partial t}\left[\alpha'\left(\sigma_w\right)\frac{\partial w}{\partial t}\right] - c^2\nabla\cdot\left[\alpha'\left(\sigma_w\right)\nabla w\right] + \omega_0^2\frac{\partial V}{\partial w}(u,w) = 0. \tag{2.11}$$

2.3. Assumptions on α and V

Next, we are going to make an explicit choice of α and V.

First let us consider α. The simplest choice would be to take

$$\alpha\left(\tau\right) = \tau. \tag{2.12}$$

With this choice, the function F_ψ in the (2.1) will depend only on ψ but not on its derivatives. Unfortunately, with this choice, there are not soliton-like solutions (see Derrik's theorem in [10] or [3]).

To get the existence of soliton solutions of the equation (2.9) we need to consider functions α different from the (2.12). We write α as a power series

$$\alpha(\tau) = \tau + a_2\tau^2 + a_3\tau^3 + ...$$

and look for the simplest choice of the coefficients. A priori the simplest choice would be

$$\alpha(\tau) = \tau + a_2\tau^2;$$

however this is a bad choice for the equations (2.9) and (2.11) since $\alpha'(\tau)$ becomes negative when $\tau < \frac{2}{a_2}$. Then the next simple choice is

$$\alpha(\tau) = \tau + \frac{\varepsilon^4}{3}\tau^3 \tag{2.13}$$

being $\varepsilon > 0$ a parameter which has the dimension of space. As it has been proved in [4], this choice allows to prove the existence of solitons and hence from now on we will assume that α is given by the (2.13).

We assume that the potential V has the following form:

$$V(u,w) = \frac{1}{2}\left(u^2 + |w|^2\right) + V_1(u,w). \tag{2.14}$$

We make the following assumptions on V_1:

- (i) $V_1 \in C^2(\mathcal{M}, \mathbf{R})$;
- (ii) $V_1(0,0) = 0$; $V_1'(0,0) = 0$; $V_1''(0,0) = 0$.
- (iii) $V_1(u,w) \geq 0$;
- (iv) there exist $k, r > 0$ such that,

$$(u-1)^2 + |w|^2 < r \Rightarrow V_1(U) > \frac{k}{\left[(u-1)^2 + |w|^2\right]^3} \tag{2.15}$$

A function V_1 which satisfies the above assumption is the following:

$$V_1(u,w) = \left[\frac{u^2 + |w|^2}{(u-1)^2 + |w|^2}\right]^3$$

With the above choice of α and V, the equations (2.9), (2.10) and (2.11) take the following form

$$\frac{\partial}{\partial t}\left[\left(1 + \varepsilon^4\sigma_u^2\right)\frac{\partial u}{\partial t}\right] - c^2\nabla\cdot\left[\left(1 + \varepsilon^4\sigma_u^2\right)\nabla u\right] + \frac{c^2}{\hbar^2}\sigma_S u + \omega_0^2\left(u + \frac{\partial V_1}{\partial u}\right) = 0, \tag{2.16}$$

$$\frac{\partial}{\partial t}\left[u^2\frac{\partial S}{\partial t}\right] - c^2\nabla\cdot\left[u^2\nabla S\right] = 0\ . \tag{2.17}$$

$$\frac{\partial}{\partial t}\left[\left(1+\varepsilon^4\sigma_w^2\right)\frac{\partial w}{\partial t}\right] - c^2\nabla\cdot\left[\left(1+\varepsilon^4\sigma_w^2\right)\nabla w\right] + \omega_0^2\left(w+\frac{\partial V_1}{\partial w}\right) = 0, \tag{2.18}$$

The equation (2.16) can be rewritten as follows:

$$\Box u + \frac{1}{\hbar^2}\sigma_S u + \frac{\omega_0^2}{c^2}u = -\varepsilon^4\Box_6 u - \frac{\omega_0^2}{c^2}\frac{\partial V_1}{\partial u} \tag{2.19}$$

where

$$\Box_6 u = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\sigma_w^2\frac{\partial w}{\partial t}\right] - \nabla\cdot\left[\sigma_w^2\nabla w\right]$$

Using (2.3), the equations (2.19) and (2.17), take the form

$$\Box\psi + \frac{\omega_0^2}{c^2}\psi = -\varepsilon^4\Box_6 u - \frac{\omega_0^2}{c^2}\frac{\partial V_1}{\partial u} \tag{2.20}$$

Thus, the right hand side of (2.1) is given by

$$F_\psi = -\varepsilon^4 c^2\Box_6 u - \omega_0^2\frac{\partial V_1}{\partial u}$$

3. Existence and properties of soliton solutions

3.1. Stationary solutions

A stationary solution of equations (2.9), (2.10) and (2.11)is a solution of the form

$$U_0(x,t) = \begin{bmatrix} u(x) \\ \hbar\omega t \\ w(x) \end{bmatrix} \tag{3.1}$$

where $u(x)\in\mathbf{R},\ w(x)\in\mathbf{R}^3$.

Making this ansatz, the equation (2.10) is satisfied and the equations (2.9) and (2.11) become:

$$\nabla\cdot\left[\alpha'\left(|\nabla u|^2\right)\nabla u\right] + \frac{\omega^2}{c^2}u - \frac{\omega_0^2}{c^2}\frac{\partial V}{\partial u}(u,w) = 0\ , \tag{3.2}$$

$$\nabla\cdot\left[\alpha'\left(|\nabla w|^2\right)\nabla w\right] - \frac{\omega_0^2}{c^2}\frac{\partial V}{\partial w}(u,w) = 0\ , \tag{3.3}$$

These equations are the Euler-Lagrange equations relative to the functional

$$J_\omega(u,w) = \int\left[\frac{1}{2}\alpha\left(|\nabla u|^2\right) + \frac{1}{2}\alpha\left(|\nabla w|^2\right) + \frac{\omega_0^2}{c^2}V(u,w) - \frac{\omega^2}{2c^2}u^2\right]dx$$

$$= \int \left[\frac{1}{2}\alpha \left(|\nabla u|^2 \right) + \frac{1}{2}\alpha \left(|\nabla w|^2 \right) + \frac{\omega_0^2 - \omega^2}{2c^2} u^2 + \frac{\omega_0^2}{2c^2} |w|^2 + \frac{\omega_0^2}{c^2} V_1(u,w) \right] dx$$

We will refer to this functional as the *reduced action* functional.

Every critical point of the reduced action functional $J_\omega(u,w)$ is a solution of the equations (3.2) and (3.3) and hence it gives a periodic solution of (2.9), (2.10) and (2.11) having the form (3.1).

3.2. Topological structure of the problem

Now, we analyze the topological structure of our problem. The couple of functions (u,w) takes values in the manifold

$$\mathcal{M}_4 \stackrel{def.}{=} \mathbf{R} \times \mathbf{R}^3 - \{(1,0)\}$$

We consider continuous functions $\varphi : \mathbf{R}^4 \to \mathcal{M}_4$ with the asymptotic condition

$$\lim_{|x| \to \infty} \varphi(x) = 0. \tag{3.4}$$

If we identify $\mathbf{R}^3 \cup \{\infty\}$ with S^3 with a distinguished point, each of these functions identifies a homotopy class $[\varphi] \in \pi_3(\mathcal{M}_4)$. Since $\mathcal{M}_4$ is homotopically equivalent to S^3, we have that

$$\pi_3(\mathcal{M}_4) \simeq \pi_3(S^3) \simeq \mathbf{Z}.$$

Thus, it is possible to associate an integer number $[\varphi] \in \mathbf{Z} \simeq \pi_3(\mathcal{M}_4)$ to every continuous function φ satisfying the condition (3.4); this number labels the homotopy class of φ.

Now we consider functions

$$\varphi(x) = (u(x), w(x)) \in \mathbf{R}^4, \ x \in \mathbf{R}^3$$

which belong to the Sobolev space $W^{1,p}(\mathbf{R}^3, \mathbf{R}^4)$; these spaces are Banach spaces equipped with the norm

$$\|\varphi\|_{W^{1,p}} = \left[\int \left(|\nabla \varphi|^p + |\varphi|^p \right) dx \right]^{\frac{1}{p}}.$$

We recall that if $p > 3$, the functions in $W^{1,p}(\mathbf{R}^3, \mathbf{R}^4)$ are continuous; moreover, if $p \geq 6$, they satisfy the asymptotic condition (3.4).

Now let us define the following function space

$$\Lambda = \left\{ \varphi \in W^{1,6}\left(\mathbf{R}^3, \mathbf{R}^4\right) \cap W^{1,2}\left(\mathbf{R}^3, \mathbf{R}^4\right) \mid \varphi(\mathbf{R}^3) \subset \mathcal{M}_4 \right\}$$

By the above remarks, for every function $\varphi \in \Lambda$ the integer number $[\varphi]$ is well defined; thus Λ can be decomposed into connected components as follows:

$$\Lambda = \bigcup_{q \in \mathbf{Z}} \Lambda_q \tag{3.5}$$

where

$$\Lambda_q = \{\varphi \in \Lambda \mid [\varphi] = q\}\,.$$

Each connected component Λ_q is called *sector.*

3.3. The existence theorem

Using the methods developed in [4], it is possible to obtain the following theorem:

Theorem 3.1. *Let us fix $\omega \in [0, \omega_0]$. Then, if ε is sufficiently small, for some $q \neq 0$*

$$\min_{\varphi \in \Lambda_q} J_\omega(\varphi)$$

is achieved and the minimizer (u, w) is a solution of class $\mathcal{C}^2$ of the equations (3.2) and (3.3).

Proof. The proof of this theorem can be found in [4].

■

The function (3.1) allows to define a travelling soliton $U_{\mathbf{v}}$ just making a Lorentz transformation.

Let us compute the explicit form of $U_{\mathbf{v}}$. Taking for simplicity $\mathbf{v} = (v, 0, 0)$ the Lorentz transformations are:

$$x_1' = \frac{x_2 - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x_2' = x_2$$

$$x_3' = x_2$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

By the Lorentz invariance of the equation (2.9), the function

$$U_v(x, t) := U_0(x', t')$$

is a solution. Explicitly, $U_v(x, t)$ takes the following form

$$U_v(x,t) = \begin{bmatrix} u_v(x,t) \\ S_v(x,t) \\ w_v(x,t) \end{bmatrix} = \begin{bmatrix} u\left(\frac{x_1 - vt}{\sqrt{1 - v^2 c^{-2}}}, x_2, x_3\right) \\ \hbar\left(\omega_v t - \mathbf{k}_v \cdot \mathbf{x}\right) \\ w\left(\frac{x_1 - vt}{\sqrt{1 - v^2 c^{-2}}}, x_2, x_3\right) \end{bmatrix} \tag{3.6}$$

with

$$\omega_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\omega \tag{3.7}$$

and

$$\mathbf{k}_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\omega}{c^2}\mathbf{v}. \tag{3.8}$$

The "barycenter" of this soliton is located at the point

$$\mathbf{q}(t) = \mathbf{v}t$$

and so this solution represents a soliton travelling with a constant speed $\mathbf{v}$.

Notice that, for any constant $A \in \mathbf{C}$, the function

$$Ae^{i(\omega_v t - \mathbf{k}_v \cdot \mathbf{x})} \tag{3.9}$$

is a plane wave which solves the Klein-Gordon equation; the phase velocity of this wave is given by

$$v_{ph} = \frac{\omega_v}{|\mathbf{k}_v|},$$

but its energy travels with the velocity

$$\mathbf{v} = c^2\frac{\mathbf{k}_v}{\omega_v}; \tag{3.10}$$

so the soliton travels with the same speed at which travels the energy of a "associated" plane wave. Notice that $\mathbf{v}$ is also the group velocity of a wave packet of the (1.3); however a wave packet does not behave like a particle, since after some time it disperses its energy. But it is very interesting to note that the laws of the geometric optic and of the linear waves are followed not only by the wave packets, but also *by the stable bumps created by the nonlinearity.*

The stationary solution (3.1) has the frequency ω, but our problem presents a second frequency ω_0 which is the frequency of small stationary waves. We suspect that, in the stable solutions, the nonlinear effects of the equation (2.9) make these two frequencies to coincide:

$$\omega = \omega_0. \tag{3.11}$$

In any case the stationary solutions of the (2.9) with frequency ω_0 appear somewhat simpler and in the following we will be mainly concerned with these solutions.

3.4. The De Broglie singular solutions

For $x \to \infty$, we have that $u_v(x,t) \to 0$; then, for x sufficiently large, the function

$$\psi_v(x,t) = u_v(x,t)\, e^{iS_v(x,t)/\hbar}$$

satisfies the Klein-Gordon equation up to higher order terms. In particular the corresponding $u(x)$ must (approximately) satisfy the equation (3.2) linearized around 0 :

$$-\Delta u + \frac{\omega_0^2 - \omega^2}{c^2}u \ = 0.$$

Assuming (3.11), we get

$$-\Delta u \ = 0,$$

If we take a constant solution of the above equation, namely $u(x) = A \in \mathbf{C}$, the corresponding travelling solution (obtained by mean of (3.6)) is given by (3.9). But this solution has infinite energy outside of any ball of arbitrarily large radius and hence is not related to the solution obtained via theorem 3.1. We must choose a function which decays at infinity. Then $u(x)$ must have the following form

$$u(x) \cong \frac{b}{|x|} \tag{3.12}$$

where b is an arbitrary constant. Then for x large we have

$$\psi_0(x,t) \cong b\frac{e^{i\omega_0 t}}{|x|}; \tag{3.13}$$

and

$$\psi_v(x,t) \cong \frac{b}{\sqrt{\left(\frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + x_2^2 + x_3^3}} e^{i(\omega_v t - \mathbf{k}_v \cdot \mathbf{x})}. \tag{3.14}$$

Notice that (3.14) is nothing else but the singular solution of De Broglie (1.5) (see [8], page. 97). The singularity of (3.14) is located at the point $\mathbf{q}(t) = \mathbf{v}t$ and it travels with speed $\mathbf{v}$.

Thus, our theory has replaced the singularity of (1.5) with a soliton.

Remark 1. *(3.14) holds if we assume (3.11); if $\omega < \omega_0$ the $u(x)$ has an asymptotic behavior given by a De Broglie singular solution with exponential decay:*

$$u(x) \cong \frac{be^{-\frac{|x|}{c}\sqrt{\omega_0^2 - \omega^2}}}{|x|}; \tag{3.15}$$

Thus, the corresponding travelling solution takes the following form:

$$\psi_v(x,t) \cong \frac{b_0}{a} e^{-\frac{a}{c}\sqrt{\omega_0^2-\omega^2} + i(\omega_v t - \mathbf{k}_v \cdot \mathbf{x})}; \tag{3.16}$$

with

$$a = \sqrt{\left(\frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + x_2^2 + x_3^3}$$

Some comments about these solution can be found in ([8], page 98).

4. Structure of soliton solutions

4.1. The halo of a soliton

We define the *halo* of the soliton as the "très petite région singulière de l'espace où u prenderait une très grande valeur" ([8]). More precisely, we define the halo as the region where u is so big that the effects of the nonlinear terms are relevant. Clearly, the relevance of the nonlinear effects depends on the desired degree of accuracy and this fact makes this region fuzzy unless, somewhat arbitrarily, we choose an accuracy parameter ℓ.

These considerations lead to the following definition:

Definition 4.1. *For $U \in \Lambda$, we call halo of a function $U \in \Lambda$ the compact set*

$$H(U(\cdot,t)) = \left\{ x \in \mathbf{R}^3 \mid |U(x,t)| \geq \ell \right\}$$

where $\ell \ll 1$ is an accuracy parameter fixed once for ever.

In $\mathbf{R}^3 - H(U)$

$$V(u,w) \simeq \frac{\omega_0^2}{2c^2} \left(|u|^2 + |w|^2 \right). \tag{4.1}$$

Thus, unless ∇U is very large, in the set

$$\left\{ (x,t) \mid x \in \mathbf{R}^3 - H(U(\cdot,t)) \right\}$$

the Klein-Gordon equation is a good approximation of the nonlinear equation (2.9).

If we consider stationary solutions, the halo of U_0 does not depend on time since

$$|U_0(x,t)| = \sqrt{|u(x)|^2 + |w(x)|^2}.$$

In the set $\mathbf{R}^3 - H(U_0)$, the function $u_0(x)$ can be approximated by the (3.12). In this region the function $w_0(x)$ satisfy the equation

$$-\Delta w + \frac{\omega_0^2}{c^2} w = 0,$$

so that w has exponential decay:

$$w(x) = b_1 e^{-\frac{\omega_0}{c}|x|}$$

Thus, if the (3.11) is satisfied, out of the halo of the soliton, $w_0 \ll u_0$ and it can be considered essentially null. In general, unless we choose very special initial conditions, w is a field whose effects are relevant only at a very short distance from the barycenter, namely "inside" the soliton; thus the dynamic of the soliton, considered as a particle, can be influenced only by the field u.

4.2. The core of a soliton

Now let us analyze more closely the structure of the soliton inside the halo $H(U_0)$.

First we want to define the *core* namely a region where the energy reaches it maximum. For this purpose we consider the following submanifold of $\mathcal{M}_4$

$$\mathcal{B}_0 = \left\{(u,w) \in \mathbf{R} \times \mathbf{R}^3 \mid \ u > 1;\ w = 0\right\}. \tag{4.2}$$

This is a region where the potential V takes high values. The topological constraints imposed by the fact that $U(x) \in \Lambda_q$, $U(x) = (u(x), w(x))$, $q \neq 0$, imply that

$$U(\mathbf{R}^3) \cap \mathcal{B}_0 \neq 0.$$

Thus the set

$$\Gamma(U) = U^{-1}(\mathcal{B}_0)$$

is not empty. If $U(\mathbf{R}^3)$ and $\mathcal{B}_0$ intersect transversally, this set is a 0-dimesional manifold, and most likely a single point. By construction, this is the point around which the energy tends to concentrate.

The set $\Gamma(U)$ will be called *core* of the soliton.

4.3. The support of a soliton

The high energy region which surrounds $\Gamma(U)$ will be called the support of U and it is defined as follows:

$$K(U) = U^{-1}(\mathcal{B})$$

where

$$\mathcal{B} = \left\{(u,w) \in \mathbf{R} \times \mathbf{R}^3 \mid u^2 + \ |w|^2 \geq 1\right\}. \tag{4.3}$$

Notice that the halo of U can be defined as follows

$$H(U) = U^{-1}(\mathcal{B}_\ell)$$

where

$$\mathcal{B}_\ell = \left\{(u,w) \in \mathbf{R} \times \mathbf{R}^3 \mid \ u^2 + \ |w|^2 \geq \ell\right\}. \tag{4.4}$$

Since

$$\mathcal{B}_0 \subset \mathcal{B} \subset \mathcal{B}_\ell$$

we have that

$$\Gamma(U) \subset K(U) \subset H(U).$$

The importance of the support of a soliton relies in the fact that it carries the *topological charge* namely a topological invariant which characterizes the soliton itself. Since this fact will not be used in the following, we will not explain it here and we refer to [4] and [5]. Also we refer to [6] and [3] where the authors study a model in which the topological charge is identified with the electric charge.

4.4. Size of the soliton

Now let us make some considerations on the size of $H(U)$.

In order to analize this issue, it is useful to choose ε and ω_0 related by the following equation:

$$\omega_0 = \frac{c}{\varepsilon}, \tag{4.5}$$

Here ε is the parameter defined in the (2.13).

By (4.5), the equations (2.16), (2.17) and (2.18) can be written as follows

$$\Box u + \varepsilon^4 \Box_6 u + \frac{c^2}{\hbar^2}\sigma_S u + \frac{1}{\varepsilon^2}\frac{\partial V}{\partial u} = 0 \tag{4.6}$$

$$\frac{\partial}{\partial t}\left[u^2 \frac{\partial S}{\partial t}\right] - c^2 \nabla \cdot \left[u^2 \nabla S\right] = 0 \tag{4.7}$$

$$\Box w + \varepsilon^4 \Box_6 w + \frac{1}{\varepsilon^2}\frac{\partial V}{\partial w} = 0 \tag{4.8}$$

were

$$\Box_6 f = \frac{1}{c^2}\frac{\partial}{\partial t}\left[\sigma_f^4 \frac{\partial f}{\partial t}\right] - \nabla \cdot \left[\sigma_f^4 \nabla f\right].$$

Let

$$\Phi(x,t) = \begin{bmatrix} u(x)e^{i\omega t} \\ \hbar\omega t \\ w(x) \end{bmatrix}$$

be a stationary solution of the equations (4.6), (4.7), (4.8) with $\varepsilon = 1$; then, it is easy to check that

$$U(x,t) = \Phi\left(\frac{x}{\varepsilon}\right) e^{i\frac{\omega}{\varepsilon}t} \tag{4.9}$$

is a solution of (4.6) (4.7), (4.8). Thus ε, which has the dimension of a length, is proportional to the diameter of $K(U)$. Letting $\varepsilon \to 0$, the support $K(U)$ becomes smaller while the frequencies ω becomes larger.

If we use the (4.5), the Planck law and the Einstein equation, we get

$$\varepsilon = \frac{c}{\omega_0} = \frac{c\hbar}{E} = \frac{c\hbar}{mc^2} = \frac{\hbar}{mc};$$

for example, if we take m equal to the mass of the proton, we get

$$\varepsilon = \frac{6.6 \cdot 10^{-27}/2\pi}{(1.7 \cdot 10^{-24}) \cdot (3 \cdot 10^{10})} \cong 2 \cdot 10^{-14} cm.$$

which is a value of the order of nuclear distances.

5. Dynamics of solitons

5.1. The equations of motion

By the remarks at the end of the section 4.1 the equation (2.11) does not influence the motion of the solitons (unless two solitons "collide" with each other); thus we will not consider it.

If $u \neq 0$, the equation (2.9) can be written as follows:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 = |\nabla S|^2 + \frac{\hbar^2}{u}\left(\frac{\omega_0^2}{c^2}\frac{\partial V}{\partial u} + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\alpha'(\sigma_u)\frac{\partial u}{\partial t}\right] - \nabla\cdot\left[\alpha'(\sigma_u)\nabla u\right]\right)$$

and using (2.14), we get

$$\left(\frac{\partial S}{\partial t}\right)^2 = c^2|\nabla S|^2 + \hbar^2\omega_0^2\left(1 + \frac{\partial V_1}{\partial(u^2)}\right) + 2\hbar\omega_0 Q \tag{5.1}$$

where we have set

$$Q = \frac{\hbar c^2}{2\omega_0 u}\left[\frac{1}{c^2}\frac{\partial}{\partial t}\left(\alpha'(\sigma_u)\frac{\partial u}{\partial t}\right) - \nabla\cdot\left(\alpha'(\sigma_u)\nabla u\right)\right] \tag{5.2}$$

This equation is nothing else but the relativistic Hamilton-Jacobi equation relative to the Hamiltonian function

$$H(\mathbf{p}) = \sqrt{\hbar^2\omega_0^2 + c^2\mathbf{p}^2 + 2\hbar\omega_0 Q + \hbar^2\omega_0^2\frac{\partial V_1}{\partial(u^2)}}.$$

If, according to the equations of Plank and Einstein, we set

$$m_0 = \frac{\hbar\omega_0}{c^2}$$

we obtain

$$H(\mathbf{p}) = m_0c^2\sqrt{1 + \frac{\mathbf{p}^2}{m_0^2c^2} + \frac{2}{m_0c^2}Q + \frac{\partial V_1}{\partial(u^2)}}$$

$$\cong m_0c^2 + \frac{\mathbf{p}^2}{2m_0} + Q + \frac{1}{2}\frac{\partial V_1}{\partial(u^2)}$$

where

$$Q = \frac{\hbar^2}{2m_0 u}\left[\frac{1}{c^2}\frac{\partial}{\partial t}\left(\alpha'(\sigma_u)\frac{\partial u}{\partial t}\right) - \nabla\cdot\left(\alpha'(\sigma_u)\nabla u\right)\right]$$

The equation (2.10) is the continuity equation for the "fluid" $\rho = u^2\frac{\partial S}{\partial t}$ which moves accoding to the velocity field

$$\mathbf{v}(x,t) = -c^2\frac{\nabla S(x,t)}{\frac{\partial S}{\partial t}(x,t)}; \tag{5.3}$$

in particular, the quantity

$$L = \int \frac{\partial S}{\partial t} u^2 \, dx \tag{5.4}$$

is an integral of the motion.

For the solutions of type (3.6), (3.14) holds out of the halo and we have $S(x,t) = \hbar\,(\omega t - \mathbf{k}\cdot\mathbf{x})$; hence we get

$$\mathbf{v}(x,t) \equiv c^2 \frac{\mathbf{k}}{\omega} \tag{5.5}$$

and this is consistent with (3.10) and the remarks which we made soon after.

5.2. The guidance formula

Now consider a soliton located in the point $\mathbf{q}(t) \in \mathbf{R}^3$ and which moves under the influence of a perturbation of the field. Then its trajectory will not be given by

$$\mathbf{q}(t) = \mathbf{x}_0 + \mathbf{v}t = \mathbf{x}_0 + c^2 \frac{\mathbf{k}}{\omega} t$$

but it will be more complicated. First we define more precisely the position $\mathbf{q}(t)$ of the "barycenter" of the soliton as follows:

$$\mathbf{q}(t) = \frac{\int \mathbf{x} \frac{\partial S}{\partial t} u^2 \, dx}{\int \frac{\partial S}{\partial t} u^2 \, dx} = \frac{1}{L} \int \mathbf{x} \frac{\partial S}{\partial t} u^2 \, dx \tag{5.6}$$

where L is defined by (5.4). Clearly $\mathbf{q}(t)$ is well defined for any solution of our equation; in particular, if we have a soliton-like solution, $\mathbf{q}(t)$ can be interpreted as the position of the soliton. Hence, the dynamic of the soliton considered as a particle is described by the function $t \mapsto \mathbf{q}(t)$.

Lemma 5.1. *The following identity holds*

$$L\dot{\mathbf{q}}(t) = -c^2 \int \nabla S \; u^2 \, dx \tag{5.7}$$

Proof. We have

$$L\dot{\mathbf{q}}(t) = \frac{d}{dt} \int \mathbf{x} \frac{\partial S}{\partial t} u^2 \, dx = \int \mathbf{x} \frac{\partial}{\partial t} \left(\frac{\partial S}{\partial t} u^2 \right) \, dx$$

(using (2.10))

$$= c^2 \int \mathbf{x} \nabla \cdot \left(\nabla S \, u^2 \right) \, dx = -c^2 \int \nabla S \, u^2 \; dx.$$

■

Now, assume to have a soliton-solution, namely a situation such that

$$\int_{\mathbf{R}^3} u^2(x,t) \, dx \cong \int_{H(u)} u^2(x,t) \, dx. \tag{5.8}$$

If we assume that the wavelenght is very large with respect to the dimension of the halo $H(u)$, we get the De Broglie guidance formula: in fact, under this assumption, by (5.7), (5.8) and the definition of L, we get

$$L\dot{\mathbf{q}}(t) = -c^2 \int \nabla S(x,t)\, u^2(x,t)\, dx \cong -c^2 \int_{H(u)} \nabla S(x,t) u^2(x,t)\, dx$$

$$\cong -c^2 \nabla S(\mathbf{q}(t)) \int_{H(u)} u^2(x,t)\, dx \cong -c^2 \nabla S(\mathbf{q}(t)) \int u^2(x,t)\, dx$$

$$L = \int \frac{\partial S}{\partial t}(x,t)\, u^2(x,t)\, dx \cong \int_{H(u)} \frac{\partial S}{\partial t}(x,t)\, u^2(x,t)\, dx$$

$$\cong \frac{\partial S}{\partial t}(\mathbf{q}(t)) \int_{H(u)} u^2(x,t)\, dx \cong \frac{\partial S}{\partial t}(\mathbf{q}(t)) \int u^2(x,t)\, dx$$

and hence

$$\dot{\mathbf{q}}(t) = -c^2 \frac{\int \nabla S\, u^2\, dx}{\int \frac{\partial S}{\partial t} u^2\, dx} \cong -c^2 \frac{\nabla S\,(\mathbf{q}(t))}{\frac{\partial S}{\partial t}\,(\mathbf{q}(t))}. \tag{5.9}$$

Notice that (5.9) is consistent with (5.3) but they have a quite different interpretation. The equation (5.3) is the definition of the quantity $\mathbf{v}(x,t)$ which can be interpreted as a velocity field; the equation (5.9) defines an ordinary differential equation for the motion of the soliton.(5.3) is always true, (5.9) is an approximation which is accurate when the dimension of the soliton is small compared to the wavelenghts, namely, with respect to ∇S. By the remark at the end of section 4.4 the size of the soliton is of the order of $10^{-14} cm.$; hence, the (5.9) is accurate for waves with wavelenght $\lambda \gg 10^{-14} cm.$ associated with paricles with momentum

$$|\mathbf{p}| = \frac{h}{\lambda} \ll 10^{-14} g \cdot cm \cdot sec^{-1}.$$

We refer to [8] page 104 for further comments on this point.

The evolution of S is determined by the Hamilton-Jacobi equation (5.1). Since we have assumed that the space variation of S is small if compared with the dimensions of the halo, the evolution of S is governed by the equation (5.1) out of the halo. In this region we have $\alpha'(\sigma_{S,R}) \cong 1$ and $V_1 \cong 0$; thus (5.1) takes the form

$$\left(\frac{\partial S}{\partial t}\right)^2 = c^2 |\nabla S|^2 + \hbar^2 \omega_0^2 + 2\hbar\omega_0 Q \tag{5.10}$$

and the quantum potential Q defined by (5.2) becomes

$$Q = \frac{\hbar^2}{2m_0} \frac{\Box u}{u} \tag{5.11}$$

which is the form considered by De Broglie.

As long as the (5.9) ia accurate, i.e., as long as the halo is small with respect to the space variation of S, our soliton moves as a relativistic particle submitted to the force given by the potential Q.

5.3. The Shrödinger equation

Now let us consider the nonrelativistic limit. In this case it is interesting to consider a potential which depends also on the space variable x; thus we replace ω_0 in (2.14) with

$$\Omega_0(x) = \omega_0 + \frac{W(x)}{\hbar} = \frac{m_0c^2 + W(x)}{\hbar};$$

the constant $\hbar$ makes W to have the dimension of an energy.

Also we may assume that our equation is defined in some open set $D \subseteq \mathbf{R}^3$; in this case, boundary Dirichlet conditions are assumed, namely

$$u(t,x) = 0 \quad \text{for every } x \in \partial D \text{ and for every } t \in \mathbf{R}.$$

Now, suppose that the action $S(x,t)$ is a small perturbation of the stationary action $\hbar\omega_0 t$. If we set

$$S(x,t) = \hbar\omega_0 t - s(x,t) = m_0c^2 t - s(x,t) \tag{5.12}$$

we get

$$\sigma_S = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla S)^2 = m_0^2c^2 - 2m_0\frac{\partial s}{\partial t} + \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla s)^2$$

$$= m_0^2c^2 - 2m_0\frac{\partial s}{\partial t} + \sigma_s$$

With this substitution, we get

$$\frac{c^2}{\hbar^2}\sigma_S\, u + \omega_0^2\frac{\partial V}{\partial u}$$

$$= \frac{c^2}{\hbar^2}\, u\left(m_0^2c^2 - 2m_0\frac{\partial s}{\partial t} + \sigma_s\right) + \left(\frac{m_0c^2 + W(x)}{\hbar}\right)^2\left(u + \frac{\partial V_1}{\partial u}\right)$$

$$= \frac{c^2}{\hbar^2}\left(2m\frac{\partial s}{\partial t}s + \sigma_s\right)u + \frac{2m_0c^2W(x) + W(x)^2}{\hbar^2}\left(u + \frac{\partial V_1}{\partial u}\right);$$

the equations (2.9) and (2.10) become

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left[\alpha'(\sigma_u)\,\frac{\partial u}{\partial t}\right] - \nabla\cdot\left[\alpha'(\sigma_u)\,\nabla u\right] + \frac{1}{\hbar^2}\left(2m_0\frac{\partial s}{\partial t} + \sigma_s\right)u$$

$$+ \left(\frac{2m_0W}{\hbar^2} + \frac{W^2}{c^2\hbar^2}\right)\left(u + \frac{\partial V_1}{\partial u}\right) = 0,$$

$$\frac{\partial}{\partial t}\left[u^2\left(1 - \frac{1}{m_0c^2}\frac{\partial s}{\partial t}\right)\right] + \nabla\cdot\left[u^2\frac{\nabla s}{m_0}\right] = 0,$$

Now, if we let $c \to \infty$, these equations give:

$$-\nabla \cdot \left[\alpha'\left(|\nabla u|^2\right)\nabla u\right] + \frac{1}{\hbar^2}\left(2m_0\frac{\partial s}{\partial t} + |\nabla s|^2\right)u + \frac{2m_0 W}{\hbar^2}\left(u + \frac{\partial V_1}{\partial u}\right) = 0,$$

$$\frac{\partial}{\partial t}\left(u^2\right) + \nabla \cdot \left(u^2\frac{\nabla s}{m_0}\right) = 0. \tag{5.13}$$

If $u \neq 0$, the first equation becomes the nonrelativistic Hamilton-Jacobi equation:

$$\frac{\partial s}{\partial t} + \frac{1}{2m_0}|\nabla s|^2 + W(x)\left(u + \frac{\partial V_1}{\partial u}\right) + Q = 0 \tag{5.14}$$

where the quantum potential is given by

$$Q = -\frac{\hbar^2}{2m_0}\frac{\nabla \cdot \left[\alpha'\left(|\nabla u|^2\right)\nabla u\right]}{u}$$

The Lagrangian density for the equations (5.13) and (5.14) is given by

$$\mathcal{L} = \left(\frac{\partial s}{\partial t} + \frac{1}{2m_0}|\nabla s|^2 + W(x)\right)u^2 + \frac{\hbar^2}{2m_0}\alpha\left(|\nabla u|^2\right) + W(x)V_1(u)$$

Out of the halo, the equation (5.14) take thes form

$$\frac{\partial s}{\partial t} + \frac{1}{2m_0}|\nabla s|^2 + W(x)u - \frac{\hbar^2}{2m_0}\frac{\Delta u}{u} = 0 \tag{5.15}$$

If we set

$$\psi(x,t) = u(x,t)e^{is(x,t)/\hbar} \tag{5.16}$$

the equations (5.13) and (5.15) are equivalent to the Scrödinger equation:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + W\psi \tag{5.17}$$

Now, remember that this equation holds in $D - H_\varepsilon(u)$ where $H_\varepsilon(u)$ is the halo of the soliton; we have written $H_\varepsilon(u)$ to enphasize that it depends on ε; in fact, by the discussion of section 4.4, it shrinks to a point as $\varepsilon \to 0$.

If we denote be ψ_ε the solution of (5.17) with the bondary conditions

$$\begin{cases} \psi_\varepsilon = 0 & \text{on } \partial D \\ \psi_\varepsilon = g & \text{on } H_\varepsilon(u) \end{cases}$$

where g is any function such that $|g| = \ell$, it is a well known fact that as $\varepsilon \to 0$ the solution ψ_ε converges to a function ψ_0 which satisfies (5.17) in all D with the Dirichlet boundary conditions. In other words, the limit of the ψ_ε forgets of the presence of the soliton. And this is true since in three space dimension a pointwise singularity is removable.

Thus if ε is very small, the function u satisfy the Schrödinger equation in $D - H_\varepsilon(u)$ up to an high degree of accuracy and it is very close to ψ_0. Hence, if we set

$$\delta_\varepsilon(x - \mathbf{q}(t)) = u(x,t) - \psi_0(x,t),$$

the function $\delta_\varepsilon(x - \mathbf{q}(t))$ is very close to 0 out of the halo: we have obtained the equation (1.1).

5.4. Bohmian mechanics

By (5.9) and (5.12), the barycenter of the soliton, $\mathbf{q}(t)$, satisfies the following equation:

$$\dot{\mathbf{q}}(t) = -c^2 \frac{\nabla \left[mc^2 t - s(\mathbf{q}(t))\right]}{mc^2 - \frac{\partial s}{\partial t}}$$

and taking the limit for $c \to \infty$, we obtain

$$\dot{\mathbf{q}}(t) = \frac{\nabla s(\mathbf{q}(t))}{m} \tag{5.18}$$

which, by (5.16), is equivalent to (1.2).

The Schrödinger equation (5.17) and the guidance formula in the form (5.18) are the equations of the Bohmian dynamics. Notice that when ε gets smaller, the De Broglie guidance formula (5.9) becomes more accurate and hence also the equation (5.18) becomes more accurate.

By the discussion at the end of the previous subsection, we know that, for ε very small, the function ψ_ε does not see the presence of the soliton; this fact is one of the basic (and controversial point) of the Bohmian mechanics: the evolution of ψ is independent of the presence of particles. In our approach, the evolution of ψ feels the presence of particles, since the (5.17) is only approximately true; however, for ε very small, the influence of the particles on ψ is negligible. On the other hand, the soliton, which is not a real particle, but a bump of the field, is piloted by the wave via the formula (5.18).

It is well known that the Bohmian mechanics is experimentally equivalent to the conventional quantum mechanics [7]. Thus, as far as one particle is involved, our model presents all the feature of quantum mechanics. In particular it presents the intersection phenomena in the double slit experiment, the energetic level of the hydrogen atom, the Heisenberg uncertainly principle etc. We refer to Bohm and Hiley [7] for a description and the interpretation of these phenomena in the light of the equation (5.18).

However, in our model, the (5.17) and the (5.9) are only approximations; if we think of the actual solution u as composed by a wave ψ and a particle $\mathbf{q}(t)$ (cf. (1.1)), there is an exchange of energy and momentum between ψ and $\mathbf{q}(t)$ even if the time-average of this exchange is essentially null. Moreover, an empty wave ψ carries some energy which might eventually have some effect on the matter. This considerations might provide a new explanation for "strange"

quantum phenomena as the Elitzur-Vaidman bomb-testing method [11] where an "empty" wave is used to perform the test.

6. Conclusions

The first conclusion which comes in mind is that De Broglie was right in the sense that his theory could not be falsified at least if we consider the quantum phenomena which were known at that time: it is possible to explain the double-slit experiment, the stability of atoms, and some of the quantization phenomena by mean of a wave associated with a particle which is nothing else but a bump due to nonlinear effects. It is true that in the 20's (and even later in the 50's when he came back to this problem), the nonlinear analysis was not sufficiently developed and he was not able to produce formalized models and to prove his statements; nevertheless his idea was simple and clear and now it can be modelled in a rigorous way. On the other hand, its direct antagonist, the *complementary principle* was unclear then and it is unclear now and probably it will never be formalized (cf. the remarks of Bell [1] page 190). From this point of view, our paper enforces the idea that the success of the Copenhagen interpretation was due to a philosophical choice rather than to physical evidence.

However, the main point of this paper is to show that it is possible to investigate why, when and to which extent the *Axioms* of Quantum Physics are necessary to describe nature.

Quantum Mechanics, in its standard interpretation, has never been fully accepted by the community of physicists and many different interpretations appeared and will appear. We do not want to give an other interpretation of QM. Our aim is not so ambitious. We just want to understand which phenomena can be explained by nonlinear effects in Classical Mechanics and which ones require new axioms. And in order to justify this project, it is appropriate to recall the words of Bell ([1] page 166):

> If you make axioms, rather than definitions and theorems [...] then you commit redundancy and risk inconsistency.

Many phenomena, which in the past appeared unclear and even paradoxical, today can be explained by nonlinear models. I think that making a rigorous mathematical analysis of the nonlinear field models, we can understand quite a lot of things and in particular which are the experimental facts which *really* force us to abandon the classical mechanics.

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