Solitons and Relativistic Dynamics*

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1. Introduction

A soliton is a solution of a øeld equation whose energy travels as a localized packet and which preserves its form under perturbations. In this respect solitons have a particle-like behavior.

In this paper we study a Lorentz invariant equation in three space dimensions which has soliton like solutions. These solitons have some interesting properties since they behave as relativistic bodies, namely:

- they experience a relativisic contraction in the direction of the motion;

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- the rest mass is a scalar and not a tensor;
- the celebrated Einstein equation

$$E = mc^2 \tag{1.1}$$

holds;

- the mass increases with the velocity by the factor γ.

Moreover a topological invariant, consisting of a k-ple of integer numbers, is associated to these solitons. It allows to deøne the electric charge.

In the last two sections the interaction between the soliton and the electromagnetic øeld is analysed.

In this analysis we meet with many mathematical questions: some of them are solved rigorously, others require further investigations and in this paper we give just some heuristic arguments.

2. The equation

In our model a øeld is a function

$$\psi : \mathbf{R}^{3+1} \to \mathbf{R}^n - \Sigma, \qquad \psi = (\psi^1, ..., \psi^n).$$

where $\mathbf{R}^{3+1}$ is the space-time (the space and time coordinates will be respectively denoted by x and t) and the target space $\mathbf{R}^n - \Sigma$ is the internal parameters space; the set Σ is supposed to be closed and it will be called the singular set; moreover we suppose that $0 \notin \Sigma$. Since we require the Lorentz invariance, we shall consider Lagrangian densities of the form

$$\mathcal{L}(\psi, \sigma) = -\frac{1}{2}\alpha(\sigma) - V(\psi), \tag{2.1}$$

where

$$\sigma = c^2 \left|\nabla\psi\right|^2 - \left|\psi_t\right|^2$$

$\alpha : \mathbf{R} \to \mathbf{R}$ and the potential function V is deøned in $\mathbf{R}^n - \Sigma$. c is the velocity of the light and $\nabla\psi$, ψ_t denote respectively the Jacobian with respect to x and the derivative with respect to t.

The action functional related to (2.1) is

$$\mathcal{S}(\psi) = \int_{\mathbf{R}^{3+1}} \mathcal{L}(\psi, \sigma)\, dxdt = \int_{\mathbf{R}^{3+1}} \left[-\frac{1}{2}\alpha(\sigma) - V(\psi)\right] dxdt.$$

So the Euler-Lagrange equations are

$$\frac{\partial}{\partial t}\left[\alpha'(\sigma)\, \frac{\partial\psi}{\partial t}\right] - c^2\nabla \cdot \left[\alpha'(\sigma)\, \nabla\psi\right] + \frac{\partial V}{\partial \xi}(\psi) = 0\ , \tag{2.2}$$

where

$$\frac{\partial V}{\partial \xi} = \left(\frac{\partial V}{\partial \xi_1}, \ldots, \frac{\partial V}{\partial \xi_n}, \right).$$

and $\nabla \cdot \left[\alpha'(\sigma) \nabla \psi \right]$ denotes the vector whose $j-$th component is given by $div \left[\alpha'(\sigma) \nabla \psi^j \right]$.

When $\alpha(\sigma) = \sigma$ the equations (2.2) reduce to the semilinear wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - c^2 \Delta \psi + \frac{\partial V}{\partial \xi}(\psi) = 0, \tag{2.3}$$

The static solutions of the (2.2) solve the equation

$$-c^2 \nabla \cdot \left[\alpha'(\sigma) \nabla \psi \right] + \frac{\partial V}{\partial \xi}(\psi) = 0, \tag{2.4}$$

Let $u = u(x_1, x_2, x_3)$ be a solution of the (2.4), and consider a vector $\mathbf{v}$ with $|\mathbf{v}| < c$. For simplicity we take $\mathbf{v} = (v, 0, 0)$. Then it is easy to verify that

$$\psi_{\mathbf{v}}(x_1, x_2, x_3, t) = u \left(\frac{x_1 - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, x_2, x_3 \right). \tag{2.5}$$

is a solution of the equations (2.2).

Notice that the function u experiences a contraction of a factor

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

in the direction of the motion: and this is a consequence of the fact that (2.2) is Lorentz invariant.

Observe that the (2.4) are the Euler-Lagrange equation relative to the energy for functions independent of t. In fact we have (cf. e.g. [13] pg. 184 or [15])

$$T_{00} = \sum_{j=1}^{n} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \Psi^j}{\partial t} \right)} \frac{\partial \psi^j}{\partial t} - \mathcal{L}$$

$$= -\frac{1}{2} \sum_{j=1}^{n} \alpha'(\sigma) \frac{\partial \sigma}{\partial \left(\frac{\partial \psi^j}{\partial t} \right)} \frac{\partial \psi^j}{\partial t} + \frac{1}{2} \alpha(\sigma) + V(\psi)$$

$$= \alpha'(\sigma) \sum_{j=1}^{n} \left(\frac{\partial \psi^j}{\partial t} \right)^2 + \frac{1}{2} \alpha(\sigma) + V(\psi)$$

$$= \alpha'(\sigma) \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{1}{2} \alpha(\sigma) + V(\psi)$$

and so

$$E(\psi) = \int T_{00}\, dx = \int \left[\alpha'(\sigma) \left|\frac{\partial \psi}{\partial t}\right|^2 + \frac{1}{2}\alpha(\sigma) + V(\psi)\right] dx \qquad (2.6)$$

Thus, if ψ is independent of t, we have

$$E(\psi) = \int \left[\frac{1}{2}\alpha\left(c^2 |\nabla \psi|^2\right) + V(\psi)\right] dx. \qquad (2.7)$$

Moreover, if we make the following assumptions on α and V:

$$\alpha'(0) = 1$$

$$V''(0)[\xi, \xi] = \omega_0^2 |\xi|^2$$

the linearized equation at $\psi = 0$ of the (2.2) reduces to the Klein-Gordon equation:

$$\psi_{tt} - c^2 \Delta \psi + \omega_0^2 \psi = 0. \qquad (2.8)$$

3. The topological invariant

In this section we deøne a homotopic invariant. This invariant consists of an k-ple of integer numbers. First of all this invariant is useful in the proof of the existence of solitons and to classify them. Moreover, and this seems quite interesting, it provides a discrete invariant of the motion just as the quantum numbers related to the superselection rules (e.g. charge, leptonic number, barionic number, etc.).

We consider an homeomorphism

$$\pi : S^3 \to \mathbf{R}^3 \cup \{\infty\}$$

such that $\pi(N) = \infty$, where S^3 is the 3-dimensional sphere and N is its north pole. Now, set

$$\mathcal{C}_0(\mathbf{R}^3, \mathbf{R}^n - \Sigma) = \left\{ u \in \mathcal{C}(\mathbf{R}^3, \mathbf{R}^n - \Sigma) \mid \lim_{|x| \to \infty} u(x) = 0 \right\}.$$

For every function $u \in \mathcal{C}_0(\mathbf{R}^3, \mathbf{R}^n - \Sigma)$, we can deøne the continuous map

$$u \circ \pi : S^3 \to \mathbf{R}^n - \Sigma.$$

We denote with $u^{\#}$ the homotopy class of $u \circ \pi$, that is

$$u^{\#} = [u \circ \pi] \in \pi_3(\mathbf{R}^n - \Sigma).$$

Where $\pi_3(\mathbf{R}^n - \Sigma)$ denote the 3-homotopy group of $\mathbf{R}^n - \Sigma$.

Thus, to every øeld $u \in \mathcal{C}_0(\mathbf{R}^3, \mathbf{R}^n - \Sigma)$ is associated the homotopy class $u^{\#} \in \pi_3(\mathbf{R}^n - \Sigma)$.

Clearly this invariant is stable under uniform convergence.

In many cases (e.g. when the group is ønitely generated and torsion free)

$$\pi_3(\mathbf{R}^n - \Sigma) \cong \mathbf{Z}^k$$

with $k \in \mathbf{N}$. For example this situation occurs when

$$\mathbf{R}^n - \Sigma = \mathbf{R}^4 - \{\xi_1, \ldots, \xi_k\}.$$

So, if we øx a set of generators $\{e_1, ..., e_k\}$, every conøguration $u \in \mathcal{C}_0(\mathbf{R}^3, \mathbf{R}^n - \Sigma)$ is characterized by a k-tuple of integer numbers. If $u^{\#} = q_1 e_1 + ... + q_k e_k$ $(q_1, ..., q_k \in \mathbf{Z})$, then we set

$$q(u) = (q_1, ..., q_k).$$

If the øeld u has the energy concentrated in diœerent regions of the space $C_1, ..., C_\ell$, then it is possible to associate a k-ple of numbers $q(u, C_j) = (q_1, ..., q_k)$ to any of these components C_j in such a way that

$$q(u) = \sum_{j=1}^{\ell} q(u, C_j).$$

In order make these statements rigorous, it is necessary deøne a local topological invariant and more work is necessary.

Deønition 3.1. For every $u \in \mathcal{C}_0(\mathbf{R}^3, \mathbf{R}^n - \Sigma)$ we call δ-support of u the compact set deøned as follows

$$K_\delta(u) = \{x : |u(x)| \geq \delta\}. \tag{3.1}$$

where

$$\delta < \delta_0$$

and

$$\delta_0 = dist(0,\ \Sigma). \tag{3.2}$$

Now we decompose $K_\delta(u)$ into its connected components (which might be inønite but this fact does not create trouble):

$$K_\delta(u) = \bigcup_{j=1}^{\ell} C_j \tag{3.3}$$

and we want to to deøne

$$q(u, C_j) \in \pi_3(\mathbf{R}^n - \Sigma)$$

with suitable properties (ønite additivity and so on).

Since $\partial C_j \subset \partial K_\delta(u)$ and $\|u(x)\| = \delta$ for $x \in \partial K_\delta(u)$, by the continuity of u there exists an open neighborhood N of ∂C_j such that $u(N) \subset \mathbf{R}^n - \Sigma$. Then, by the Uryson lemma, there exists a continuous function

$$c_j : \mathbf{R}^3 \to [0,1]$$

such that

$$c_j(x) = \begin{cases} 1 & \text{on } C_j, \\ 0 & \text{on } \mathbf{R}^3 - (C_j \cup N). \end{cases}$$

By this construction, we have that the function

$$\tilde{\varphi}(x) := c_j(x) u(x)$$

takes its values in $\mathbf{R}^n - \Sigma$, i.e. extending $\tilde{\varphi}$ at the point at inønity, $\tilde{\varphi} \circ \pi \in \mathcal{C}\left(S^3, \mathbf{R}^n - \Sigma\right)$. Thus it makes sense to give the following deønition:

Deønition 3.2. We set

$$q(u, C_j) = \tilde{\varphi}^{\#}.$$

This deønition is well posed, i.e. it does not depend on $\tilde{\varphi}$ (see [3]).

The local invariant satisfy the ønite additivity property. Let C_1, C_2 be connected components of $K_\delta(u)$ with $C_1 \cap C_1 = \emptyset$, then we have

$$q(u, C_1 \cup C_2) = q(u, C_1) + q(u, C_2)$$

(the sum is meant, of course, in the group $\pi_3(\mathbf{R}^n - \Sigma)$).

4. Non existence results

The simplest equation of type (2.2) is obtained by choosing

$$\alpha(\sigma) = \sigma \tag{4.1}$$

Then the equation (2.4) for the static solutions and its energy functional (2.7) become respectively

$$-c^2 \Delta \psi + \frac{\partial V}{\partial \xi}(\psi) = 0$$

$$E(u) = \int_{\mathbf{R}^3} \frac{c^2}{2} |\nabla u|^2 + V(u)\, dx. \tag{4.2}$$

If $V \geq 0$, it is not diŒcult to show, by a rescaling argument, that any local minimizer u of (4.2) is necessarily trivial, i.e. it takes a constant value which is a minimum point of V (Derrick's Theorem, see [10]).

We shall deduce this non existence result by a more general lemma which will be used in the following to prove some dynamical properties of the solitons:

Lemma 4.1. Let u be a critical point of the energy (4.2) (and hence a weak solution of the equation (2.4)); then

$$\frac{1}{2}\int \alpha(|\nabla u|^2)\,dx + \int V(u)\,dx = \int \alpha'(|\nabla u|^2)\left|\frac{\partial u}{\partial x_i}\right|^2 dx,\ i=1,2,3 \tag{4.3}$$

and, consequently

$$\frac{1}{2}\int \alpha(|\nabla u|^2)\,dx + \int V(u)\,dx = \frac{1}{3}\int \alpha'(|\nabla u|^2)\,|\nabla u|^2\,dx. \tag{4.4}$$

whenever the integrals converge (here, to symplify the notation, we have assumed $c=1$).

Proof. Let

$$u_\lambda(x) = u(\lambda x_1, x_2, x_3);$$

then, setting $y=(\lambda x_1, x_2, x_3)$

$$E(u_\lambda) = \frac{1}{2}\int \alpha(|\nabla u_\lambda|^2)\,dx + \int V(u_\lambda)\,dx$$

$$= \frac{\lambda^{-1}}{2}\int \alpha\left(\lambda^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right) dy + \lambda^{-1}\int V(u)\,dy$$

Since u is a critical point of E,

$$\frac{d}{d\lambda}E(u_\lambda)\Big|_{\lambda=1} = 0$$

and since

$$\frac{d}{d\lambda}E(u_\lambda) = -\frac{\lambda^{-2}}{2}\int \alpha\left(\lambda^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right) dy$$

$$+\int \alpha'\left(\lambda^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right)\left(\frac{\partial u}{\partial x_1}\right)^2 dy$$

$$-\lambda^{-2}\int V(u)\,dy$$

we get

$$-\frac{1}{2}\int \alpha(|\nabla u|^2)\,dy + \int \alpha'(|\nabla u|^2)\left(\frac{\partial u}{\partial x_1}\right)^2 dy - \int V(u)\,dy = 0$$

which gives the 4.3. The 4.4 is obtained adding the 4.3 with $i=1,2,3$.

■

By the above lemma we get the Derrick's theorem.

Theorem 4.2. Let $V \geq 0$ and let u be a solution of the equation

$$-\Delta u + \frac{\partial V}{\partial \xi}(u) = 0, \tag{4.5}$$

with ønite energy. Then u is a constant c such that $V(c) = 0$.

Proof. If $\alpha(\sigma) = \sigma$ then the 4.4 gives

$$\frac{1}{2}\int |\nabla u|^2\, dx + \int V(u)\, dx = \frac{1}{3}\int |\nabla u|^2\, dx,$$

and so

$$\frac{1}{6}\int |\nabla u|^2\, dx + \int V(u)\, dx = 0.$$

■

From the above theorem we have:

Corollary 4.3. For any $\mathcal{C}^1$ potential V, the energy (4.2) does not have a smooth nontrivial local minimizer.

Proof. A smooth nontrivial local minimizer satisøes the equation (4.5) and hence, if $V \geq 0$, u is trivial by Th. 4.2. If $V(\xi) < 0$ for some $\xi \in \mathbf{R}^n$, then, it is easy to realise that

$$\inf E(\psi) = -\infty.$$

■

So, by the corollary 4.3, people have been forced to consider non positive potentials and to look for saddle points, instead of minima. However for these static solutions we have lack of stability . As an example we recall that, if we take scalar øelds u with potentials

$$V(\xi) = \frac{1}{2}\xi^2 - \frac{1}{4}\xi^4,\ \xi \in \mathbf{R},$$

critical points of the energy functional

$$E(u) = \int_{\mathbf{R}^3} \left[\frac{c^2}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4\right] dx.$$

have been found in [9] and [18] and for more general potentials in [8], [21]; but in [1] and [7] it has been proved that these static solutions are not stable (see also [22] and its references)

The diŒculties related to Derrick's Theorem can be avoided in other ways: if solutions with inønite energy (i.e. functions in $W^1_{loc}(\mathbf{R}^n)$) are allowed, then the equation (2.3) may have solutions. Each of these solutions $\psi(t,x)$ can be written as follows

$$\psi(t,x) = u_0(x) + w(t,x);$$

here $u_0(x) \in W^1_{loc}(\mathbf{R}^n)$ is a solution of the (4.5) which represents the vacuum and $w(t,x)$ is the physically meaningful part of the øeld. The energy density (whose integral diverges) is replaced by the normalized energy density which is deøned as follows:

$$\mathcal{E}(x) = \left[\frac{1}{2}\psi_t^2 + \frac{c^2}{2}\left|\nabla\psi\right|^2 + V\left(\psi\right)\right] - \left[\frac{1}{2}u_{0t}^2 + \frac{c^2}{2}\left|\nabla u_0\right|^2 + V\left(u_0\right)\right]$$

Thus the problem consists in ønding solutions such that

$$\int \mathcal{E}(x)\, dx < +\infty .$$

Most people in theoretical physics follow this approach (we refer to [23] for more details).

5. Existence results

To get the existence of solutions of the equation (2.4) which minimize the energy functional (2.7), we follow a diœerent approach and we look for functions α (see (2.1)) diœerent from the (4.1). Namely we consider α nonlinear

$$\alpha(\sigma) = \sigma + a_2\sigma^2 + a_3\sigma^3 + ...$$

and look for the simplest choice of α. A priori the simplest choice is

$$\alpha(\sigma) = \sigma + a_2\sigma^2,$$

however this is a bad choice for the evolution equation (2.2), in fact $\alpha'(\sigma)$ might be negative for some values of σ and this is a jdisasterj for (2.2). Then it is natural to consider the next simplest choice, namely

$$\alpha(\sigma) = \sigma + \frac{\varepsilon}{3}\sigma^3$$

being $\varepsilon > 0$. Of course other choices are possible if we do not assume that the lagrangian splits as in (2.1) as it is in the Skyrme model ([20]). Then (2.2) can be written

$$\frac{\partial}{\partial t}\left[\left(1+\varepsilon\sigma^2\right)\psi_t\right] - c^2\nabla\cdot\left[\left(1+\varepsilon\sigma^2\right)\nabla\psi\right] + \frac{\partial V}{\partial\xi}(\psi) = 0,$$

or

$$\Box\psi + \varepsilon\Box_6\psi + \frac{\partial V}{\partial\xi}(\psi) = 0 \qquad (5.1)$$

where

$$\Box_6\psi = \frac{\partial}{\partial t}\left[\left(c^2\left|\nabla\psi\right|^2 - \left(\psi_t\right)^2\right)^2\psi_t\right] - c^2\nabla\cdot\left[\left(c^2\left|\nabla\psi\right|^2 - \left(\psi_t\right)^2\right)^2\nabla\psi\right].$$

So the static solutions u solve the system of equations

$$-c^2\Delta u - \varepsilon c^6 \Delta_6 u + \frac{\partial V}{\partial \xi}(u) = 0, \tag{5.2}$$

where

$$\Delta_6 u = \nabla \cdot \left(|\nabla u|^4 \nabla u \right).$$

Then the energy functional becomes

$$E(u) = \int_{\mathbf{R}^3} \left(\frac{c^2}{2}|\nabla u|^2 + \varepsilon \frac{c^6}{6}|\nabla u|^6 + V(u) \right) dx \tag{5.3}$$

We make the following assumptions on V:

- $V \in C^2(\mathbf{R}^n - \Sigma, \mathbf{R})$;
- $V(\xi) \geqslant 0$ for every $\xi \in \mathbf{R}^n - \Sigma$ and $V(\xi) = 0 \Rightarrow \xi = 0$;
- there exist $k, r > 0$ such that,

$$\operatorname{dist}(\xi, \Sigma) < r \Rightarrow V(\xi) > \frac{k}{\operatorname{dist}(\xi, \Sigma)^6} \tag{5.4}$$

- $V''(0)\,[\xi, \xi] = \omega_0^2 |\xi|^2,\ \omega_0 > 0.$

A simple function V which satisøees the above assumptions is the following:

$$V(\xi) = \frac{1}{2} \cdot \frac{|\xi|^2}{\prod_j |\xi - \xi_j|^6};$$

in this case we have $\Sigma = \{\xi_1, ..., \xi_k\}$, $\xi_j \neq 0$ and

$$\omega_0 = \frac{1}{\prod_j |\xi_j|^3}.$$

Observe that $\alpha'(0) = 1$, then linearizing (2.2) at 0, we get the Klein-Gordon equation

$$\Box \psi + \omega_0 \psi = 0.$$

Of course we can consider our evolution equation 5.1 as a dynamical system. The conøguration space for this system is given by

$$\Lambda = \left\{ u \in W^{1,6}(\mathbf{R}^3, \mathbf{R}^n) \cap W^{1,2}(\mathbf{R}^3, \mathbf{R}^n) \mid u(\mathbf{R}^3) \subset \mathbf{R}^n - \Sigma \right\}$$

equipped with the norm

$$\|u\|_\Lambda = \|u\|_{W^{1,6}} + \|u\|_{W^{1,2}}.$$

where $W^{1,6}(\mathbf{R}^3,\mathbf{R}^n)$ and $W^{1,2}(\mathbf{R}^3,\mathbf{R}^n)$ are Sobolev spaces. Observe that the presence of the potential V, which is singular on Σ, implies that the functions, on which the energy is finite, take values on $\mathbf{R}^n-\Sigma$ (see [2]).

By the Sobolev embedding theorem, the functions in Λ are continuous and it is possible to prove that they decay to 0 at infinity (see e.g. [2] Prop. 2.1). So the topological invariant introduced in section 3 is well defined for the functions in Λ. For this reason, the algebraic structure of $\pi_3(\mathbf{R}^n-\Sigma)$ is reflected into a decomposition of the configuration space:

$$\Lambda=\bigcup_{q\in\pi_3(\mathbf{R}^n-\Sigma)}\Lambda_q \tag{5.5}$$

into connected components $\Lambda_q=\left\{u\in\Lambda : u^{\#}=q\right\}$ each of them will be called sector.

We point out that other models (see [20], [11], [12]) do not contain the singular potential and the topological classification of the fields follows from the fact that they take values in suitable compact manifolds.

It can be shown that the minimizers of E (see (5.3)) on Λ_q satisfy suitable stability properties (see [2]). Then we give the following definition of soliton:

Definition 5.1. A soliton is a function u which minimizes the energy E in a sector Λ_q; the set $K_\delta(u)$ (with $\delta\ll\delta_0=dist(0,\Sigma)$ which is assumed fixed once for ever) will be called δ-support of the soliton and its diameter will be called diameter of the soliton.

The definition of diameter of a soliton might appear unsatisfactory since it depends on the choice of δ; on the other hand, since u decays exponentially, the energy outside $K_\delta(u)$ (with δ small) is irrelevant for many problems and an appropriate choice of δ seems possible.

The proof of the existence of solitons is not achieved in a direct fashion since the problem is translation invariant. In fact, due to this invariance, the sectors Λ_q are not weakly closed and therefore we cannot straightforward minimize the energy on each Λ_q.

To overcome the difficulties arising from this lack of compactness we have used an argument in the spirit of the concentration-compactness principle for unbounded domains (see [16], [6]), and the following theorem can be proved (see [2] and [3]):

Theorem 5.2. Let V satisfy the assumptions (5.4) and set

$$\Xi=\left\{q\in\pi_3(\mathbf{R}^n-\Sigma)\mid E \text{ has a minimum in } \Lambda_q\right\}$$

Then the subgroup of $\pi_3(\mathbf{R}^n-\Sigma)$ generated by Ξ coincides with $\pi_3(\mathbf{R}^n-\Sigma)$ itself.

Since every local minimizer of E gives rise to a weak solution of the (5.2), then, if $\pi_3(\mathbf{R}^n - \Sigma)$ is not trivial, there exists at least a non trivial solution of the (5.2).

In particular, if $\pi_3(\mathbf{R}^n - \Sigma) \simeq \mathbf{Z}^k$, Theorem 5.2 implies that there exist at least k homotopically distinct static solutions of 5.1.

Also notice that, if $u(x) \in \Lambda_q$ solves the (5.2), also $u(-x) \in \Lambda_{-q}$ solves that equation and so, when $\pi_3(\mathbf{R}^n - \Sigma) \simeq \mathbf{Z}^k$, there are at least $2k$ solitons.

6. Mass and energy

In order to show the relativistic behaviour of this kind of solitons, we need to deøne their intertial mass. The most natural thing to do is to deøne it as the ratio between the momentum and the velocity of the soliton.

However this deønition is possible only if the momentum and the velocity are parallel; otherwise the mass is a tensor. So the ørst thing to do is to compute the momentum and then the mass tensor of our solitons.

Let $\psi(x,t) = \psi(x_1, x_2, x_3, t)$ be a solution of the 2.2 (where for simplicity we set $c = 1$). It is well known (see e.g. [13]) that the momentum of ψ is given by

$$p_k = \int T_{0,k}\, dx, \quad k = 1, 2, 3 \tag{6.1}$$

where $T_{i,k}$ is the energy-momentum tensor deøned by

$$T_{ki} = \sum_{j=1}^{n} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi_j}{\partial x_k}\right)} \frac{\partial \psi_j}{\partial x_i} - \delta_{ik}\mathcal{L}; \quad with\ x_0 := t \tag{6.2}$$

In our case, for $k \neq 0$, we have

$$T_{k,0} = -\alpha'(\sigma)\, \psi_{x_k} \cdot \psi_t \tag{6.3}$$

so we can easily compute the momentum of the soliton 2.5:

Lemma 6.1. Let $\psi_{\mathbf{v}}$ be the solution of 2.2 (with $c = 1$) given by the 2.5, then the momentum is given by

$$p_1(\psi_{\mathbf{v}}) = v\gamma \int \alpha'(|\nabla u|^2) \left(\frac{\partial u}{\partial x_1}\right)^2 dx$$

$$p_2(\psi_{\mathbf{v}}) = v \int \alpha'\left(|\nabla u|^2\right) \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} dx$$

$$p_3(\psi_{\mathbf{v}}) = v \int \alpha'\left(|\nabla u|^2\right) \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3} dx$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}}.$$

Proof. By the 6.3 we have

$$p_1(\psi_{\mathbf{v}}) = -\int \alpha'\left(|\nabla\psi_{\mathbf{v}}|^2 - \left(\frac{\partial\psi_{\mathbf{v}}}{\partial t}\right)^2\right)\frac{\partial\psi_{\mathbf{v}}}{\partial x_1}\frac{\partial\psi_{\mathbf{v}}}{\partial t}\,dx;$$

exploiting the Lorentz invariance of $|\nabla\psi_{\mathbf{v}}|^2 - \left(\frac{\partial\psi_{\mathbf{v}}}{\partial t}\right)^2$, we get

$$\left[|\nabla_x\psi_{\mathbf{v}}|^2 - \left(\frac{\partial\psi_{\mathbf{v}}}{\partial t}\right)^2\right](x_1,x_2,x_3,t) = |\nabla_\xi u|^2(\xi_1,\xi_2,\xi_3)$$

where

$$\begin{cases} \xi_1 = \gamma(x_1 - vt) \\ \xi_2 = x_2 \\ \xi_3 = x_3 \end{cases}$$

Thus

$$p_1(\psi_{\mathbf{v}}) = -\int \alpha'\left(|\nabla_\xi u|^2\right)\frac{\partial\psi_{\mathbf{v}}}{\partial x_1}\frac{\partial\psi_{\mathbf{v}}}{\partial t}\,dx_1\,dx_2\,dx_3$$

$$= -\int \alpha'\left(|\nabla_\xi u|^2\right)\frac{\partial u}{\partial\xi_1}\frac{\partial\xi_1}{\partial x_1}\frac{\partial u}{\partial\xi_1}\frac{\partial\xi_1}{\partial t}\,\frac{1}{\gamma}\,d\xi_1\,d\xi_2\,d\xi_3$$

$$= \gamma v\int \alpha'\left(|\nabla_\xi u|^2\right)\left(\frac{\partial u}{\partial\xi_1}\right)^2 d\xi_1\,d\xi_2\,d\xi_3.$$

Now let us compute p_2 :

$$p_2(\psi_{\mathbf{v}}) = -\int \alpha'\left(|\nabla\psi_{\mathbf{v}}|^2 - \left(\frac{\partial\psi_{\mathbf{v}}}{\partial t}\right)^2\right)\frac{\partial\psi_{\mathbf{v}}}{\partial x_2}\frac{\partial\psi_{\mathbf{v}}}{\partial t}\,dx$$

$$= -\int \alpha'\left(|\nabla_\xi u|^2\right)\frac{\partial\psi_{\mathbf{v}}}{\partial x_2}\frac{\partial\psi_{\mathbf{v}}}{\partial t}\,dx_1\,dx_2\,dx_3$$

$$= \int \alpha'\left(|\nabla_\xi u|^2\right)\frac{\partial u}{\partial\xi_2}\frac{\partial u}{\partial\xi_1}\gamma v\,\frac{1}{\gamma}\,d\xi_1\,d\xi_2\,d\xi_3$$

$$= v\int \alpha'\left(|\nabla_\xi u|^2\right)\frac{\partial u}{\partial\xi_2}\frac{\partial u}{\partial\xi_1}d\xi_1\,d\xi_2\,d\xi_3.$$

■

The mass matrix $\{m_{ij}\}$ of $\psi_{\mathbf{v}}$ is deøned by the following relation:

$$p_i(\psi_{\mathbf{v}}) = \sum_j m_{ij}(\psi_{\mathbf{v}})v_j,$$

then, by the previous lemma we get:

$$m_{11}(\psi_{\mathbf{v}}) = \gamma\int \alpha'(|\nabla u|^2)\left(\frac{\partial u}{\partial x_1}\right)^2 dx$$

$$m_{12}(\psi_{\mathbf{v}}) = \int \alpha'\left(|\nabla u|^2\right) \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} dx$$

$$m_{13}(\psi_{\mathbf{v}}) = \int \alpha'\left(|\nabla u|^2\right) \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3} dx.$$

and by simmetry arguments we get:

$$m_{ii}(\psi_{\mathbf{v}}) = \gamma \int \alpha'(|\nabla u|^2) \left(\frac{\partial u}{\partial x_i}\right)^2 dx \tag{6.4}$$

$$m_{ij}(\psi_{\mathbf{v}}) = \int \alpha'\left(|\nabla u|^2\right) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx, \quad for\ i \neq j \tag{6.5}$$

So the rest mass matrix is deøned as follows:

$$m_{ij}(u) = \int \alpha'\left(|\nabla u|^2\right) \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx,$$

Now let us compute the energy of a soliton. By the (6.2) we have

$$E(\psi_{\mathbf{v}}) = \int T_{0,0}\, dx \tag{6.6}$$

Since the direct computation of $T_{0,0}$ is quite involved, we start to compute the energy of a static solution; in this case (see (2.7)) we have that

$$E(u) = \int \left[\frac{1}{2}\alpha(|\nabla u|^2) + V(u)\right] dx \tag{6.7}$$

Theorem 6.2. If u is a solution of 2.4 with $c = 1$; then

$$m_{ii}(\psi_{\mathbf{v}}) = \gamma E(u)$$

$$m_{ij}(\psi_{\mathbf{v}}) = 0\ if\ i \neq j$$

Proof. Since the matrix $m_{ij}(\psi_{\mathbf{v}})$ is symmetric, we can put ourselves in a reference frame

$$x' = Tx$$

such that m'_{ij} is diagonal. In this reference frame (we shall omit the '), by the (6.4)

$$m_{ij}(\psi_{\mathbf{v}}) = \gamma\delta_{ij} \int \alpha'(|\nabla u|^2) \left(\frac{\partial u}{\partial x_i}\right)^2 dx$$

where δ_{ij} is the Kroneker symbol. Then, by the lemma 4.1 and the (6.7)

$$m_{ij}(\psi_{\mathbf{v}}) = \gamma E(u)\delta_{ij}$$

Thus $m_{ij}(\psi_{\mathbf{v}})$ is the identity marix times $\gamma E(u)$. This implies that $m_{ij}(\psi_{\mathbf{v}}) = \gamma E(u)\delta_{ij}$ in any reference frame.

■

By the above theorem we get the following two unexpected facts:

(i) since the mass matrix is a scalar times the identity, the mass is in fact a scalar;

(ii) the mass is equal to the rest energy times γ.

In particular, by the lemma 4.1, the mass gets the following form:

$$m(\psi_{\mathbf{v}}) = \frac{\gamma}{3} \int \alpha'(|\nabla u|^2)\, |\nabla u|^2 \; dx \tag{6.8}$$

and we have the equations

$$\mathbf{p}(\psi_{\mathbf{v}}) = m(\psi_{\mathbf{v}})\mathbf{v} \tag{6.9}$$

and

$$m(\psi_{\mathbf{v}}) = \gamma E(u) \tag{6.10}$$

Notice that

$$(\mathbf{p}(\psi_{\mathbf{v}}), E(\psi_{\mathbf{v}})) = \left(\int T_{0,1}\, dx,\ \int T_{0,2}\, dx,\ \int T_{0,3}\, dx,\ \int T_{0,0}\, dx \right)$$

is a 4-vector since the equation 2.2 is Lorentz invariant; so, its Lorentzian norm

$$|\mathbf{p}(\psi_{\mathbf{v}})|^2 - E(\psi_{\mathbf{v}})^2 \tag{6.11}$$

is independent of $\mathbf{v}$. Using this fact we get:

Theorem 6.3. If u is a solution of (2.4) (with $c = 1$) and $\psi_{\mathbf{v}}$ is the function (2.5), then

$$E(\psi_{\mathbf{v}}) = \gamma E(u) \tag{6.12}$$

Proof. Since the (6.11) is independent of $\mathbf{v}$, $|\mathbf{p}(\psi_{\mathbf{v}})|^2 - E(\psi_{\mathbf{v}})^2 = -E(u)^2$; then by (6.9) and Th. 6.2

$$E(\psi_{\mathbf{v}})^2 = E(u)^2 + |\mathbf{p}(\psi_{\mathbf{v}})|^2 = E(u)^2 + m(u)^2\gamma^2 v^2$$

$$= E(u)^2 + E(u)^2\gamma^2 v^2 = \left(1 + \frac{v^2}{1 - v^2}\right) E(u)^2 = \gamma^2 E(u)^2.$$

■

Comparing the (6.12) with the (6.10) we get the equality between mass and energy:

$$E(\psi_{\mathbf{v}}) = m(\psi_{\mathbf{v}}) \tag{6.13}$$

If $c \neq 1$, the above formula gives the celebrated Einstein equation

$$E(\psi_{\mathbf{v}}) = m(\psi_{\mathbf{v}})c^2$$

7. The dynamics of free solitons

Given $R \in SO(\mathbf{R}^3),\ \mathbf{v} \in T\mathbf{R}^3, \xi \in \mathbf{R}^3$, we set

$$g_{R,\mathbf{v},\xi}(t)\, x = g(t)\, x = RL_{\mathbf{v}}(x - \mathbf{v}t - \xi)$$

where $L_{\mathbf{v}}$ is the relativistic contraction in the direction of $\mathbf{v}$ of the factor $\gamma = \left(\sqrt{1-|\mathbf{v}|^2}\right)^{-1}$. Also we denote by $\mathcal{G}$ the set of such $g(t)$'s.

Now let $\mathcal{S}$ be the set of those solitons u, i.e. solutions u of the eq. (2.4), which have the baricenter $\beta(u)$ at the origin, where $\beta(u)$ is deøned by

$$\beta(u) = \frac{\int \alpha'(|\nabla u|^2)\, |\nabla u|^2\, x\, dx}{\int \alpha'(|\nabla u|^2)\, |\nabla u|^2\, dx}$$

We suppose that $\mathcal{S}$ contains one soliton for each sector Λ_q.

If $u \in \mathcal{S}$ and $g \in \mathcal{G}$, the function

$$\psi\,(t,x) = u(g(t)\, x)$$

is a solution of the equation (2.2); if we jaddj ℓ solitons, we obtain a function

$$\psi\,(t,x) = \sum_{k=1}^{\ell} u_k(g_k(t)\, x),\ u_k \in \mathcal{S} \text{ and } g_k \in \mathcal{G} \tag{7.1}$$

which we will call multisoliton (time-dependent) function. We will say that the solitons which jbuildj the $\psi\,(t,x)$ are located in $\xi_1(t), ..., \xi_\ell(t) \in \mathbf{R}^n$ if these points are the baricenters of the $u_k \circ g_k$'s.

$\psi\,(t,x)$ is not a solution of the (2.2) since this equation is nonlinear. However, since the u_k's have exponential decay, $\psi\,(t,x)$ represents a good approximation of the solution at least for those values of t such that the solitons are far from each other. For example this happens when $t \longrightarrow \pm\infty$ if

$$g(t)\, x = R_k L_{\mathbf{v}_k}\left[x - \mathbf{v}_k t - \xi_k\right]$$

and

$$\mathbf{v}_k \neq \mathbf{v}_h,\ k \neq h$$

If the solitons get close and even more if the solitons collide, the $\psi\,(t,x)$ is no longer a good approximation of a solution and an additional term must be added. To do this we ørst deøne

$$\Gamma = \left\{ h \in W^{1,6}(\mathbf{R}^n) \cap W^{1,2}(\mathbf{R}^n) \mid h(x) = \sum_{k=1}^{\ell} u_k(g_k\, x),\ u_k \in \mathcal{S} \text{ and } g_k \in \mathcal{G}_0 \right\}$$

where

$$\mathcal{G}_0 = \{g(0) \mid g(t) \in \mathcal{G}\}\,.$$

A generic element of Γ is called a multisoliton (time-independent) function.

For every soliton

$$u_k(g_k\, x) = u_k\left(R_k L_{\mathbf{v}_k}\left[x - \xi_k\right]\right)$$

its position ξ_k and its velocity $\mathbf{v}_k$ are well deøned (actually the velocity is deøned up to a sign).

Also, for $q \in \pi_3(\mathbf{R}^n - \Sigma)$, we set

$$\Gamma_q = \left\{ h \in \Gamma \mid h = \sum_{k=1}^{\ell} u_k \circ g_k \; and \; \sum_{k=1}^{\ell} q\left(u_k\right) = q \right\}$$

Now, given any function $u \in \Lambda_q$, we want to decompose it a joptimalj way as follows:

$$u = h + w \quad with \;\; h = \sum_{k=1}^{\ell} u_k \circ g_k \in \Gamma_q. \tag{7.2}$$

The following theorem holds:

Theorem 7.1. For any øxed $u_0 \in \Lambda_q$, there is $\bar{h} \in \Gamma_q$ which minimizes the functional

$$F(h) = m(u_0 - h);\; h \in \Gamma_q$$

where

$$m(u) = \frac{1}{3}\int \alpha'(|\nabla u|^2)\,|\nabla u|^2 \; dx$$

Thus, by the above theorem, for a generic $u \in \Lambda$ the decomposition (7.2) is uniquely determined; thus it makes sense to say that the conøguration u contains the ℓ solitons u_k.

Using this decomposition, any øeld $\psi\left(t, x\right)$ can be written as follows:

$$\psi\left(t, x\right) = \sum_{k=1}^{\ell(t)} u_k\left(R_k(t) L_{\mathbf{v}_k(t)}\left[x - \xi_k(t)\right]\right) + w(t, x) \tag{7.3}$$

Clearly this decomposition, which by theorem 7.1 can be always produced, is meaningful when $w(t, x)$ is suŒciently small and the distance of the baricenters of the solitons is bigger than their size in such a way that their supports do not intersect.

Since the (7.1) is an approximated solution of the (2.2), we expect to ønd solutions which have the form (7.3) with $w(t, x)$ small,

$$\frac{d}{dt} R_k(t) = 0$$

and

$$\mathbf{v}_k(t) \cong \frac{d}{dt}\xi_k(t)$$

When the solitons jcollidej, the functions $\xi_k(t)$ might loose the continuity and the number $\ell(t)$ of solitons contained in the øeld ψ might change. However, if $\psi(t,x)$ depends continuously on t, the function

$$t \mapsto q(\psi(t,\cdot))$$

is constant, i.e. $\psi(t,\cdot)$ does not change sector in Λ.

Thus the k-ple of integer numbers $q(\psi(t,\cdot))$ can be considered an integral of the motion. Then the numbers $q(\psi(t,\cdot))$ behave as the quantum numbers related to the superselection rules in the elementary particle theory; for this similarity we will call them with this name.

By the previous discussions, if $\psi(t,x)$ is a suŒciently regular solution of the (2.2) the following quantities are invariant of the motion:

- the global energy
- the global momentum
- the global angular momentum deøned by

$$\int M_{0kl}\,dx, \quad (k<l) \tag{7.4}$$

where M_{ikl} is the angular momentum tensor (cf. e. g. [13] pg. 185)

- the global quantum numbers.

If w in the (7.3) is small, as far as the solitons are far from each other, they move in a straight line since the øeld ψ decays exponentially out of their supports (i.e. they have a short range interaction with each other). If two or more solitons jcollidej and generate other solitons, by virtue of the conservation laws and the equality between mass and energy, the usual laws of the Relativistic Mechanics (cf. [15] pg.47-59) are satisøed, moreover the quantum numbers are preserved.

Thus this kind of solitons behave as relativistic particles as long as the jshort rangej interactions are involved. In the next sections, we will show that these solitons follow a relativistic dynamic even when they are embedded in long range øeld like the electromagnetic øeld.

8. The electric charge

In order to show that our solitons behave as relativistic particles under long range interactions, we let them to interact with the electromagnetic øeld. To carry out this program, we need ørst to deøne the electric charge. We recall that, when $\pi_3(\mathbf{R}^n-\Sigma)\simeq\mathbf{Z}^k$, to any conøguration $u\in\Lambda$ is associated a $k-$ple of quantum numbers

$$q(u)=(q_1(u),....,q_k(u))=q_1(u)e_1+...+q_k(u)e_k$$

where $e_1, ..., e_k$ is a set of generators (see section. 3). The ørst quantum number $q_1(u)$ will be called topological charge. Now we want to deøne the electric charge $Q(u)$ of a conøguration $u \in \Lambda$ in such a way that $Q(u) = Q_0 q_1(u)$ where Q_0 is a øxed constant.

e_1 is a nontrivial homotopy class in $\pi_3(\mathbf{R}^n - \Sigma)$; we assume that to this class corresponds a nontrivial cohomology class $[\eta]$ in the De Rham cohomology group $H^3_{DeRham}(\mathbf{R}^n - \Sigma)$; this means that there is a 3-diœerential form η (closed but non exact) deøned in $\mathbf{R}^n - \Sigma$.

In order to øx the ideas and to make this point understandable to those who are not too familiar with topology, we consider the simplest case and assume that,

$$\mathbf{R}^n - \Sigma = \mathbf{R}^4 - \{\bar{\xi}_1, ..., \bar{\xi}_k\}.$$

Also, in order to simplify the notation, we set

$$\zeta = \bar{\xi}_1.$$

In this case, η takes the following aspect

$$\eta = \sum_{1 \le i < j < k \le 4} \eta_{ijk}(\xi) d\xi^i \wedge d\xi^j \wedge d\xi^k$$

where

$$\eta_{ijk}(\xi) = \frac{(-1)^{l+1} \xi^l}{|\xi - \zeta|^4}$$

being l the unique index in $\{1, 2, 3, 4\}$ diœerent from i, j, k. If we consider any map $u \in \Lambda$, the diœerential form

$$u^*(\eta) = \sum_{1 \le i < j < k \le 4} \eta_{ijk}(u(x)) \det \frac{\partial(u^i, u^j, u^k)}{\partial(x_1, x_2, x_3)} dx^1 \wedge dx^2 \wedge dx^3,$$

is called pull-back of η by u and it can be shown that (see e.g. [17])

$$q_1(u) = \frac{1}{|S^3|} \int_{\mathbf{R}^3} u^*(\eta) = \frac{1}{|S^3|} \int_{u(\mathbf{R}^3)} \eta.$$

where $|S^3|$ denotes the measure of S^3. Thus we deøne the electric charge $Q(u)$ as

$$Q(u) = \frac{Q_0}{|S^3|} \int_{\mathbf{R}^3} u^*(\eta); \tag{8.1}$$

More in general we can deøne the electric charge contained in a region $\Omega \subset \mathbf{R}^3$ as follows:

$$Q(u, \Omega) = \frac{Q_0}{|S^3|} \int_{\Omega^+} u^*(\eta)$$

where Ω^+ is Ω equipped with the standard orientation of $\mathbf{R}^3$.

9. Solitons and electromagnetic øeld

In this section we derive the equations describing a soliton interacting with the elettromagnetic øeld.

We consider the electromagnetic 4-vector potential

$$(\mathbf{A}, \phi) = (A_1, A_2, A_3, \phi)$$

and the asociated 1-form in the space-time

$$\omega = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 - \phi dt.$$

The Lagrangian density of the system ¡soliton-electromagnetic øeld¡ is

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$

where

- $\mathcal{L}_1$ is the Lagrangian density of the free øeld ψ:

$$\mathcal{L}_1 = \mathcal{L}_1(\psi, \sigma) = -\frac{1}{2}\alpha(\sigma) - V(\psi)$$

with

$$\alpha(\sigma) = \sigma + \frac{\varepsilon}{3}\sigma^3$$

- $\mathcal{L}_2$ is the Lagrangian density of the electromagnetic øeld (E, H), namely

$$\mathcal{L}_2 = \frac{1}{8\pi}(|E|^2 - |H|^2)$$

where

$$E = A_t + \nabla\phi \tag{9.1}$$

$$H = \nabla \times A \tag{9.2}$$

- $\mathcal{L}_3$ is the Lagrangian density describing the interaction between ψ and the electromagnetic øeld, and it is deøned by

$$\mathcal{L}_3\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dt = \frac{Q_0}{|S^3|}\psi^*(\eta) \wedge \omega$$

where $\psi^*(\eta)$ is the pull back of η by the function $\psi(.,t)$, namely

$$\psi^*(\eta) = \sum_{\substack{1\le i<j<k\le 4 \\ 1\le l<m<n\le 4}} \eta_{ijk}(\psi(x,t)) \det \frac{\partial(\psi^i(x,t), \psi^j(x,t), \psi^k(x,t))}{\partial(x_l, x_m, x_n)} dx^l \wedge dx^m \wedge dx^n$$

Now we want to write the interaction $\mathcal{L}_3$ in a more physically meaningful way, namely

$$\begin{aligned} \mathcal{L}_3 &= (\mathbf{J}(\psi, \nabla\psi, \psi_t) \mid \mathbf{A}) - \rho(\psi, \nabla\psi)\phi \\ &= J_1 A_1 + J_2 A_2 + J_3 A_3 - \phi\rho \end{aligned} \tag{9.3}$$

where ρ will be interpreted as charge density and $\mathbf{J}$ as current density.

To do this we put

$$\frac{Q_0}{|S^3|}\psi^*(\eta) = J_1 dx^2 \wedge dx^3 \wedge dt - J_2 dx^1 \wedge dx^3 \wedge dt + J_3 dx^1 \wedge dx^2 \wedge dt - \rho dx^1 \wedge dx^2 \wedge dx^3$$

and so we get

$$J_i(\psi, \nabla\psi, \psi_t) = (-1)^i \frac{Q_0}{|S^3|} \sum_{\substack{1 \le a<b<c \le 4 \\ 1 \le l<m<n \le 4}} \eta_{abc}(\psi(x,t)) \det \frac{\partial(\psi^a, \psi^b, \psi^c)}{\partial(x_l, x_m, x_n)} \tag{9.4}$$

(being i diœerent from l, m, n) and

$$\rho(\psi, \nabla\psi) = \frac{Q_0}{|S^3|} \sum_{1 \le i<j<k \le 4} \eta_{ijk}(\psi(x,t)) \det \frac{\partial(\psi^i, \psi^j, \psi^k)}{\partial(x_1, x_2, x_3)}$$

Notice that, since the form $\psi^*(\eta)$ is closed, $d\psi^*(\eta) = 0$; this gives the continuity equation for $\mathbf{J}$ and ρ:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{9.5}$$

Moreover, by the (9.4), it is easy to realise that

$$\frac{\partial \psi}{\partial t} = 0 \Rightarrow \mathbf{J} = 0 \tag{9.6}$$

namely the current density vanishes whenever the charge carried by the soliton does not move.

This justiøes the interpretation of $\mathbf{J}$ as the current relative to ρ, which is indeed a density charge. In fact, by the (8.1)

$$\int \rho\, dx = \frac{Q_0}{|S^3|} \int \psi^*(\eta) = Q(\psi(.,t)) \tag{9.7}$$

is the electric charge. Observe that, if ψ depends continuously on t, the electric charge (like the other quantum numbers) is constant in t, since it is a topological invariant.

Notice that the action describing the interaction between ψ and the electromagnetic øeld

$$S_3(\psi, \mathbf{A}, \phi) = \int\int \mathcal{L}_3 dx dt$$

is invariant under the gauge transformation

$$\mathbf{A}'= \mathbf{A}+\nabla h, \quad \phi' = \phi - \frac{\partial h}{\partial t}$$

where $h = h(t, x)$ is an arbitrary smooth real function.

In fact, by using (9.3) and (9.5), we have

$$S_3(\psi, \mathbf{A}',\phi') = \int\int \left[(\mathbf{J} \mid \mathbf{A}') - \rho\phi'\right] dxdt$$

$$= \int\int \left[(\mathbf{J} \mid \mathbf{A} + \nabla h) - \rho(\phi - \frac{\partial h}{\partial t})\right] dxdt$$

$$= S_3(\psi, \mathbf{A},\phi) + \int\int \left[(\mathbf{J} \mid \nabla h) + \rho\frac{\partial h}{\partial t}\right] dxdt$$

$$= S_3(\psi, \mathbf{A},\phi) + \int\int (\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t})hdxdt = S_3(\psi, \mathbf{A},\phi)$$

Then we conclude that the total action of the system (soliton-electromagnetic field) is

$$\mathcal{S} = \mathcal{S}(\psi, A, \phi) = S_1(\psi) + S_2(\mathbf{A},\phi) + S_3(\psi, \mathbf{A},\phi) \tag{9.8}$$

$\mathcal{S} = \mathcal{S}(\psi, A, \phi) = S_1(\psi) + S_2(\mathbf{A},\phi) + S_3(\psi, \mathbf{A},\phi)$

where

$$S_i = \int\int \mathcal{L}_i dxdt, \ i = 1, 2, 3$$

The Euler-Lagrange equations

$$dS = 0 \tag{9.9}$$

give rise to a system whose unknown are ψ, A, ϕ.

Taking the variation of $\mathcal{S}(\psi, A, \phi)$ with respect to A and carring out the usual computations, we get

$$d\mathcal{S}[\delta A] = 0$$

if and only if

$$\nabla \times (\nabla \times A) = 4\pi J(\psi, \nabla\psi, \psi_t) + \frac{\partial}{\partial t}(A_t + \nabla\phi) \tag{9.10}$$

Taking the variation of $\mathcal{S}(\psi, A, \phi)$ with respect to ϕ, we get

$$d\mathcal{S}[\delta\phi] = 0$$

if and only if

$$\nabla \cdot (A_t + \nabla\phi) = 4\pi\, \rho(\psi, \nabla\psi) \tag{9.11}$$

Observe that, using (9.1) and (9.2), (9.10) and (9.11) give respectively

$$\nabla \times H = 4\pi J(\psi, \nabla\psi, \psi_t) + \frac{\partial E}{\partial t} \tag{9.12}$$

$$\nabla \cdot E = 4\pi\, \rho(\psi, \nabla\psi) \tag{9.13}$$

which complete the Maxwell equations (9.1) and (9.2).

Finally, if we take the variation with respect to ψ, and set

$$d\mathcal{S}[\delta\psi] = 0$$

we get

$$\frac{\partial}{\partial t}\left[\alpha'(\sigma)\ \frac{\partial\psi}{\partial t}\right] - c^2\nabla\cdot\left[\alpha'(\sigma)\,\nabla\psi\right] + \frac{\partial V}{\partial\xi}(\psi) = F \tag{9.14}$$

where F depends on A, ϕ, ψ and their derivatives. Notice that when A, ϕ are trivial F is zero and in this case (9.14) coincides with the equation of the free soliton (see (2.2)).

The existence of static solutions ψ, A, ϕ with $Q(\psi) \neq 0$ (i.e. with nontrivial charge) of the Euler-lagrange equation (9.9) has been proved in ([5]).

10. The dynamics of solitons in the electromagnetic øeld

Finally we can examine the behavior of solitons in an electromagnetic øeld. We suppose that they are suŒciently far from each other in such a way that, in the absence of electromagnetic øeld, the mutisoliton function (7.1) describes accurately the motion of the solitons. Now let us add a electromagnetic øeld whose energy is small compared to the mass of the solitons; under this assumption it is reasonable to assume that the shape of the solitons is not deformed in a sensitive way. Also we assume that they do not rotate, namely that

$$\frac{dR_k(t)}{dt} = 0$$

In this case the evolution of the øeld ψ is described by the following function:

$$\psi(t,x) = \sum_{k=1}^{\ell} u_k\left(R_k L_{\mathbf{v}_k(t)}\left[x - z_k(t)\right]\right) \tag{10.1}$$

with

$$\frac{dz_k(t)}{dt} = \mathbf{v}_k(t);$$

we recall that $z_k(t)$ is the baricenter of the soliton u_k.

Under these assumptions, we want to derive their dynamics, namely the equations which describe the evolution of the $z_k(t)$'s.

To carry out this program, we compute the action of the conøguration (10.1). We recall that the action is given by the (9.8). Let us begin to compute $\int\int \mathcal{L}_1 dxdt$; we need the following lemma.

Lemma 10.1. If

$$\psi(t,x) = u\left(RL_{\dot{z}(t)}\left[x - z(t)\right]\right)$$

then

$$\int\int \left[-\frac{1}{2}\alpha\left(|\nabla\psi|^2 - \psi_t^2\right) - V(\psi)\right] dxdt = -m(u)\int \sqrt{1-\dot{z}(t)^2}\, dt$$

Proof. First by the lorentz invariance we have that

$$|\nabla\psi(t,\xi)|^2 - \psi_t^2(t,\xi) = |\nabla u(\xi)|^2$$

where

$$\xi = RL_{\dot{z}(t)}\left[x - z(t)\right];$$

then, recalling the (6.7)

$$\begin{aligned}
&\int\int \left[-\frac{1}{2}\alpha\left(|\nabla\psi|^2 - \psi_t^2\right) - V(\psi)\right] dxdt \\
&= \int\int \left[-\frac{1}{2}\alpha\left(|\nabla u(\xi)|^2\right) - V(u(\xi))\right]\frac{1}{\gamma} d\xi dt \\
&= -\int \left[\frac{1}{2}\alpha\left(|\nabla u(\xi)|^2\right) + V(u(\xi))\right] d\xi \cdot \int \frac{1}{\gamma} dt \\
&= -E(u)\int \frac{1}{\gamma} dt
\end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1-\dot{z}(t)^2}},$$

The conclusion follows by Einstein formula (6.13).

■

Consider now the multisoliton øeld

$$\psi(t,x) = \sum_{k=1}^{\ell} u_k\left(R_k L_{\dot{z}_k}\left[x - z_k\right]\right) \tag{10.2}$$

The action of this øeld is given by

$$\int\int \mathcal{L}_1 dx\, dt = \int\int \left[-\frac{1}{2}\alpha\left(|\nabla\psi|^2 - \left|\frac{\partial\psi}{\partial t}\right|^2\right) - V(\psi)\right] dx\, dt$$

$$= \int\int \left[-\frac{1}{2}\alpha\left(\left|\nabla\left[\sum_{k=1}^{\ell} u_k\right]\right|^2 - \left|\sum_{k=1}^{\ell}\frac{\partial u_k}{\partial t}\right|^2\right) - V\left(\sum_{k=1}^{\ell} u_k\right)\right] dx\, dt.$$

If we assume that the solitons u_k are far from each other, since they decay to zero exponentially at inøinity, we can write

$$\int\int \mathcal{L}_1 dx\, dt \cong \sum_{k=1}^{\ell} \int\int \left[-\frac{1}{2}\alpha \left(|\nabla u_k|^2 - \left| \frac{\partial u_k}{\partial t} \right|^2 \right) - V(u_k) \right] dx\, dt;$$

ønally we can use the above lemma and we get

$$\int\int \mathcal{L}_1 dx\, dt = -\sum_{k=1}^{\ell} m(u_k) \int \sqrt{1-\dot{z}_k^2}\, dt = -\sum_{k=1}^{\ell} m_k \int \sqrt{1-\dot{z}_k^2}\, dt$$

where we have used the shortened notation $m(u_k) = m_k$.

Now let us compute $\int\int \mathcal{L}_3 dx dt$; here we shall carry out this computation under the assumption that the potentials $\mathbf{A}$ and ϕ are almost constant in balls $B_r(z(t))$ containing $K(\psi(t,\cdot))$. Under this assumption it is possible to see that,

$$\int\int \left[\rho(\psi, \nabla\psi)\phi - (\mathbf{J}(\psi, \nabla\psi, \psi_t) \mid \mathbf{A}) \right] dx dt \cong Q(u) \int \left[\phi - (\mathbf{A} \mid \dot{z}(t)) \right] dt; \quad (10.3)$$

In fact, let u be a solution of the (2.4). Since it decays exponentially with all its derivatives, outside of $B_r(z(t))$, we have:

$$\rho(\psi, \nabla\psi)\phi - (\mathbf{J}(\psi, \nabla\psi, \psi_t) \mid \mathbf{A}) \cong 0;$$

then

$$\int\int \left[\rho\phi - (\mathbf{J} \mid \mathbf{A}) \right] dx dt \cong \int \left[\int_{B_r(z(t))} \rho\phi\, dx - \int_{B_r(z(t))} (\mathbf{J} \mid \mathbf{A})\, dx \right] dt$$

Since A and ϕ are jalmostj constant in $B_r(z(t))$,

$$\int\int \left[\rho\phi - (\mathbf{J} \mid \mathbf{A}) \right] dx dt$$

$$\cong \int \left[\phi(z(t)) \int_{B_r(z(t))} \rho\, dx - \left(\mathbf{A}(z(t)) \mid \int_{B_r(z(t))} \mathbf{J}\, dx \right) \right] dt \quad (10.4)$$

Now we set

$$\tilde{\rho}(x) = \rho \circ u \circ R \circ L_{\mathbf{v}}$$

so that

$$\rho(\psi, \nabla\psi) = \tilde{\rho}(x - z(t)).$$

Then, by the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\tilde{\rho}(x - z(t)) = (\nabla\tilde{\rho} \mid \mathbf{v}) = \nabla \cdot (\tilde{\rho}\mathbf{v})$$

which gives

$$\mathbf{J} = \tilde{\rho}\mathbf{v} + \nabla \times \mathbf{k} = \rho\mathbf{v} + \nabla \times \mathbf{k}$$

where $\mathbf{k}$ is an arbirary vector øeld. Since in the frame where $\mathbf{v} = 0$, by the (9.6) we have $\mathbf{J} = 0$, it follows that $\nabla \times \mathbf{k} = 0$ and so

$$\mathbf{J} = \rho\mathbf{v}$$

Then

$$\int\int [\rho\phi - (\mathbf{J} \mid \mathbf{A})]\, dx dt \cong \int \left[\phi(z(t)) \int_{B_r(z(t))} \rho\, dx - \left(\mathbf{A}(z(t)) \mid \int_{B_r(z(t))} \rho\mathbf{v}\ dx \right) \right] dt$$

$$= \left(\int_{B_r(z(t))} \rho\, dx \right) \int [\phi(z(t)) - (\mathbf{A}(z(t)) \mid \mathbf{v})]\, dt$$

$$= Q(u) \int [\phi - (\mathbf{A} \mid \dot{z}(t))]\, dt.$$

Now let us compute $\int\int \mathcal{L}_3 dx dt$ for a multisoliton øeld. Let $\psi(t, x)$ be the multisoliton øeld (10.2), then, arguing as before, and using the (10.3), we can write

$$\int\int \mathcal{L}_3 dx\, dt = \int\int [\rho(\psi, \nabla\psi)\phi - (\mathbf{J}(\psi, \nabla\psi, \psi_t) \mid \mathbf{A})]\ dx\, dt$$

$$\cong \sum_{k=1}^{\ell} Q_k \int [\phi(z_k(t)) - (\mathbf{A}(z_k(t)) \mid \dot{z}_k(t))]\, dt$$

where we have used the shortened notation $Q(u_k) = Q_k$.

So we conclude that the total action can be approximated as follows:

$$\mathcal{S}(\psi, A, \phi) \cong \mathcal{S}(z_1, ..., z_\ell, A, \phi)$$

where

$$\mathcal{S}(z_1, ..., z_\ell, A, \phi) = \sum_{k=1}^{\ell} \int \left[-m_k \sqrt{1 - \dot{z}_k^2} + Q_k \left(\phi(z_k) - (\mathbf{A}(z_k) \mid \dot{z}_k) \right) \right] dt$$

$$+ \frac{1}{8\pi} \int\int (|E|^2 - |H|^2)\, dx\, dt.$$

This is the usual action of charged particles in an electromagnetic øeld (cf. e.g. [15]). Thus, as far as the approximations we have used are accurate, our solitons exibit the same dynamic of charged particles. Notice that our approximations are accurate if the δ-support $K(\psi(t, \cdot))$ of the solitons is so small that on it $\mathbf{A}$ and ϕ can be considered constant.

11. Conclusions

The øeld equation related to the Lagrangian (2.1) is probably the simplest Lorentz invariant equation, in three space dimensions, which has static solitons. Nevertheless, its soliton solutions have some interesting properties since they behave just as relativistic particles.

So some facts of Relativity theory (the space- contraction and time- dilation, the increasing of mass with the velocity, the equivalence between mass and energy) can be deduced, without extra assumptions, from a „mechanical" model described by a single equation. This equation might be interpreted as the equation of an „elastic medium" in a Newtonian space-time. Thus this model shows that, from a purely formal point of view, the main features of the special relativity can be deduced from a partial diœerential equation in a Newtonian space-time.

References

[1] D. Anderson, G. Derrick, Stability of time dependent particle like solutions in nonlinear øeld theories, J. Math Phys. 11 (1970), 1336-1346.

[2] V. Benci, D. Fortunato, L. Pisani, Soliton like solutions of a Lorentz invariant equation in dimension 3, preprint.

[3] V. Benci, D. Fortunato, L. Pisani, Remarks on topological solitons, Topological Methods in nonlinear Analysis, 7 (1996), 349-367.

[4] V. Benci, D. Fortunato, Solitons and particles, Proc. „Int. Conf. on Nonlinear Diœ. Equat. and Appl." Tata Inst. Fund.. Res., Bangalore. To appear.

[5] V. Benci, D. Fortunato, A. Masiello, L. Pisani, Solitons and electromagnetic øeld, preprint.

[6] Benci, G. Cerami, Positive Solutions of some Nonlinear Elliptic Problems in exterior Domains, Arc. Rational Mech. Anal., 99 (1987), 283-300

[7] H. Berestycki, T. Cazenave, Instabilit del etats stationnaires dans des equations de Schrdinger et de Klein-Gordon non linairs, C.R.A.S. Paris, srie I 293 (1981), 488-492.

[8] H. Berestycki, P.L. Lions, Existence d'ondes solitaires dans des problems non linairs du type Klein-Gordon, C.R.A.S. Paris, srie A 287 (1978), 503-506.

[9] M.S. Berger, On the Existence and Structure of Stationary States for a Nonlinear Klein-Gordon Equation, Jour. Funct. Analysis, 9 (1972), 249-261.

[10] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, London, New York, 1982.

[11] M.J. Esteban, A Direct Variational Approach to Skyrme's Model for Meson Fields, Comm. Math. Phys. 105 (1986), 571-591.

[12] M.J. Esteban, P.L. Lions, Skyrmions and Symmetry, Asymptotic Anal. 1 (1988), 187-192.

[13] I.M. Gelfand, S.V. Fomin, Calculus of Variations, Prentice-Hall, Englewood Cliœs, N.J., 1963

[14] S. Kichenassamy, Non linear wave equations, Marcel Dekker Inc., New York, Basel, Hong Kong (1996).

[15] L. Landau, E. Lifchitz, Thorie du Champ, ditions MIR, Moscou 1970

[16] P.L. Lions, The Concentration-Compactness Principle in the Calculus of Variations. The Limit case I - II, Revista Mat. Ibero-Americana 1.1 (1985) 145-200, 1.2 (1985), 45-121.

[17] B. Doubrovine, S. Novikov, A. Fomenko, Geometrie et Topologie des Varits, 2 partie, Editions Mir, Moscou (1985).

[18] S.I. Pohozaev, Eigenfunctions of the Equation $\Delta u + \lambda f(u) = 0$, Sov. Math. Dolk. 5 (1965), 1408-1411.

[19] R. Rajaraman, Solitons and instantons, North Holland, Amsterdam, Oxford, New York, Tokio, 1988.

[20] T.H.R. Skyrme, A non-linear øeld theory, Proc. Roy. Soc. A 260 (1961), 127-138.

[21] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977), 149-162.

[22] W.A. Strauss, Stable and unstable states of nonlinear wave equations, Contemporary Math. 17 (1983), 429-441.

[23] Strocchi S., Symmetry breking in classical dynamical systems, Preprint.

[24] G.B. Witham, Linear and nonlinear waves, John Wiley and Sons, New York, 1974