CLASSIFICATION OF DISCRETE WEAK KAM SOLUTIONS ON LINEARLY REPETITIVE QUASI-PERIODIC SETS

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Abstract. A discrete weak KAM solution is a potential function that highlights the ground state configurations at zero temperature of an infinite chain of atoms interacting with a periodic or quasi-periodic substrate. It is well known that weak KAM solutions exist for periodic substrates as in the Frenkel-Kontorova model. Weak solutions may not exist in the almost periodic setting as in the theory of stationary ergodic Hamilton-Jacobi equations (where they are called correctors). For linearly repetitive quasi-periodic substrates, we show that equivariant interactions that fulfill a twist condition and a non-degenerate property always admit sublinear weak KAM solutions. We moreover classify all possible types of weak KAM solutions and calibrated configurations according to an intrinsic prefered order. The notion of prefered order is new even in the classical periodic case.

1. Introduction

We consider a generalized model of Frenkel-Kontorova type on the real line. The model describes the states at equilibrium of chains of atoms interacting with their nearest neighbors and with an underlying one-dimensional substrate. The interaction between two successive atoms at the positions (x_n, x_{n+1}) has the general form $E(x_n, x_{n+1})$ for some continuous function $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, called subsequently *interaction model*.

The seminal Frenkel-Kontorova model [19] was proposed to represent the dislocation in a crystal. This simple model has appeared to be universal to describe several physical problems [3, 5]. Its standard version may be given as

$$E(x, y) = \frac{1}{2\tau}(y - x)^2 - \lambda(y - x) + \tau K V(x), \tag{1.1}$$

where the constants $\lambda, K \in \mathbb{R}$, $\tau > 0$, are the parameters of the model. Here E has the physical dimension of an action assuming the mass is equal

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to 1. The constant τ plays the role of a discretized time, $\tau\lambda$ is a inter-distance between two successive atoms (a positive constant $\lambda > 0$ forces the chain $(x_n)_{n\in\mathbb{Z}}$ to be increasing), K is a dimensionless constant that measures the strength of the interaction between one atom of the chain and the substrate, and V is a periodic potential of unit size describing a periodic external environment. Our main motivation is to understand interaction models of Frenkel-Kontorova type without assuming V periodic.

A central part of the theory consists in studying the set of configurations at the ground state, called hereinafter *Mañé calibrated* configurations, that is the set of positions of the atoms in a chain $(x_n)_{n\in\mathbb{Z}}$ that minimize, in a sense to be defined, the total action

$$\arg\min_{(x_n)_{n\in\mathbb{Z}}} \sum_{n\in\mathbb{Z}} E(x_n, x_{n+1}).$$

A key tool for this study is the notion of weak KAM solution at an appropriate action level. Let us recall these concepts. The *ground action* or atomic mean action is a particular choice of action level defined as

$$\bar{E} := \lim_{n\to+\infty} \inf_{x_0,\dots,x_n\in\mathbb{R}} \frac{1}{n}\sum_{k=0}^{n-1} E(x_k, x_{k+1}). \tag{1.2}$$

It is finite by the choice of the interaction model we are going to make. A continuous function $u : \mathbb{R} \to \mathbb{R}$ is called *weak KAM solution* if it satisfies

$$\begin{cases} \forall\, x, y \in \mathbb{R}, \quad u(y) - u(x) \le E(x,y) - \bar{E}, \\ \forall\, y, \;\; \exists\, x \quad \text{s.t.} \quad u(y) - u(x) = E(x,y) - \bar{E}. \end{cases} \tag{1.3}$$

The relevance of these solutions has been highlighted by Fathi in [13, 14, 15] in the context of the Hamilton-Jacobi equations on compact manifolds, and by Contreras, Iturriaga, Paternain, Paternain [10], Contreras [9], Fathi, Maderna [17] for non compact manifolds. In [2], Arnaud and Zavidovique describe precisely the pseudograph set of $\frac{du}{dx}$ for exact symplectic twists maps of the annulus, where $E(x,y)$ is understood as a global generating function. These works are influenced by the viewpoint proposed by Mather [27, 28] for studying minimizing orbits of Lagrangian systems.

A discretization of weak KAM theory applied to optimal transportation also enables to relate deep results of existence of optimal transport maps [16, 4]. More recently, the discrete analogue of the Hamilton-Jacobi equations has been studied by many authors – see, for instance, [23, 21, 33]. In particular, it is known there exists a weak KAM solution u for periodic interaction models, that is, for models satisfying

$$\forall x, y \in \mathbb{R}, \quad E(x+1, y+1) = E(x,y),$$

as is the case of the standard Frenkel-Kontorova model (for details see [8, 21]). Moreover, such a solution u may be chosen periodic and hence bounded.

Regarding the atomic interpretation, an interest of weak KAM solution lies in the concept of *u-calibrated* configuration $(x_n)_{n\in\mathbb{Z}}$, that is:

$$\forall\, m,n \in \mathbb{Z},\ m<n, \qquad \sum_{k=m}^{n-1} \big(E(x_k,x_{k+1})-\bar{E}\big) = u(x_n)-u(x_m). \tag{1.4}$$

A weak KAM solution u plays the role of a fictitious potential needed to keep a finite block of atoms of a configuration in equilibrium by applying opposing forces at both ends, simulating the remaining action of the chain outside the block [8]. However the effect of u as a fictitious potential does not occur everywhere in space. A u-calibrated configuration enables to materialize these positions.

A weaker notion of calibration may be defined which we call Mañé calibration. More precisely, the configuration $(x_n)_n$ is said to be *Mañé calibrated* if it satisfies $\forall\, m,n \in \mathbb{Z}$, $n \geq 1$,

$$\sum_{k=m}^{m+n-1} \big(E(x_k,x_{k+1})-\bar{E}\big) = S(x_m,x_{m+n}), \tag{1.5}$$

where S denotes the *Mañé potential*,

$$S(x,y) = \inf_{n\geq 1} \inf_{x=x_0,\dots,x_n=y} \sum_{k=0}^{n-1} \big(E(x_k,x_{k+1})-\bar{E}\big).$$

In words, the Mañé potential between two sites measures the minimal reduced action necessary to go from one site to another. A Mañé calibrated configuration is an infinite chain such that each finite sub-chain realizes the smallest reduced action between its two endpoints.

We focus on the ground action $\bar{E}$, defined in (1.2). Other calibration levels may be chosen as in [33], where weak KAM solutions are constructed for some non-explicit constants. In order to stay in the class of weak KAM solutions that are sublinear at infinity, $\bar{E}$ is the only possible level (for details see appendix B). The interaction model E is supposed to satisfy usual properties: C^2, twist, and superlinear as described more precisely in hypotheses 2. Actually, our main results link sublinear growths of weak KAM solutions to a non-degeneracy condition: $\inf_{x\in\mathbb{R}} E(x,x) > \bar{E}$. The complementary case $\inf_{x\in\mathbb{R}} E(x,x) = \bar{E}$ is treated in section 6. Previous works assume the substrate to be periodic. The purpose of this article is to extend the theory of calibrated configurations and weak KAM solutions to quasi-periodic substrates. Here the interaction model E is supposed to be *equivariant with respect to a linearly repetitive quasi-periodic set* (see definitions 3 and 4), a property which naturally appears when considering the classical examples of quasicrystals. We classify thus all weak KAM solutions in three types depending of their growths at infinity: type I, the sublinear case, type II, the linear case, and type III, a mix of these two growths.

Theorem 1. *Any superlinear and weakly twist interaction E fulfilling $\inf_x E(x,x) > \bar{E}$ always admits a Lipschitz weak KAM solution.*

If E is in addition pattern equivariant with respect to a linearly repetitive quasi-periodic set, then there is a selected half-space (either the positive or the negative reals), depending only on E, such that any weak KAM solution u belongs to one of three non-empty disjoint classes, namely,

- *type I: u has a sublinear growth on the real line,*
- *type II: u has a linear growth on the real line,*
- *type III: u has a mixed linear/sublinear growth: u grows sublinearly on the selected half-space and grows linearly on the complementary half-space.*

Weak KAM solutions of type III are the only ones for which there is no bi-infinite calibrated configurations as in (1.4).

Besides, any two weak KAM solutions of the same type lie at uniform distance from each other.

Our framework includes significant and classical quasicrystals such as Fibonacci, substitutive ones or typical cut-and-project quasicrystals [1]. One-dimensional quasicrystals are essentially discrete sets for which any finite pattern repeats in space in a syndetic manner and with some pattern-dependent density. For models on quasicrystals, the atoms of the chain are supposed to interact with an underlying aperiodic substrate, a structure having no translational symmetry but exhibiting a long-range order [31]. This context is analogous but more rigid than the almost-periodic scenery (e.g. a Frenkel-Kontorova model with a sum of incommensurable periodic functions as potential V). A major difference between the two models is that, in our setting, the interaction reaches its minimum. This avoids pathological cases.

Actually this quasi-periodic context fall into the framework of a topological stationary setting (see [26, 20, 22]). Initial studies (see, for instance, [20, 32, 11]) showed that some classical properties of the periodic case still hold when models on quasicrystals are taken into account, in particular the links between the minimizing configurations and their rotation numbers. Notably, unbounded Mañé calibrated configurations do exist as shown in [22]. However all these previous works did not answer to the existence problem of a weak KAM solution.

Our proof techniques remain mainly in the territory of classical analysis. Partial results where obtained in [22] using ergodic tools and Mather measures. The present work is actually independent from [22]; it gives a simpler proof of the existence of calibrated configuration, and above all, it constructs and classifies all weak KAM solutions.

In the Hamilton-Jacobi framework, a way to obtain weak KAM solutions on a non-compact space pass through the use of Busemann functions associated with an appropriate potential as in [9]. However, this method adapted to the present context does not guarantee the existence of a discrete weak KAM solution, since the second condition in (1.3) is not *a priori* satisfied. We overcome this problem by studying a Lax-Oleinik operator on a suitable space of functions associated to a bi-infinite Mañé-calibrated configuration. The properties of such configurations leads to the classification. A precise description of these relations and the classification is given in theorem 5, providing qualitative criteria to distinguish to which case a weak KAM solution belongs.

In the next section, we detail our assumptions and more rigorously formulate our central results. Note that theorem 1 also provides a classification on the asymptotic behavior of weak KAM solutions in the periodic case, as summarized in theorem 6. At our knowledge, this is a new result in this classical setting.

2. Main results

We consider a general interaction model $E(x,y)$ (in which each variable describes a position on the real line).

Hypotheses 2. An interaction model is a C^0 function $E:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ that fulfills the following three conditions

(H1) $E(x,y)$ is *locally uniformly bounded* in the sense that

$$\inf_{x,y\in\mathbb{R}} E(x,y) > -\infty, \qquad \sup_{x\in\mathbb{R}} E(x,x) < +\infty,$$

(H2) $E(x,y)$ is *locally uniformly Lipschitz* for every $R>0$, there exists a constant $C^E_{\text{Lip}}(R)>0$ such that, for every $x,y,z\in\mathbb{R}$,

$$\text{if } |y-x|\le R,\ |z-x|\le R \text{ then } |E(x,z)-E(x,y)|\le C^E_{\text{Lip}}(R)|z-y|,$$
$$\text{if } |z-x|\le R,\ |z-y|\le R \text{ then } |E(y,z)-E(x,z)|\le C^E_{\text{Lip}}(R)|y-x|,$$

(H3) $E(x,y)$ is *uniformly superlinear*:

$$\lim_{R\to+\infty}\ \inf_{|y-x|\ge R} \frac{E(x,y)}{|y-x|} = +\infty.$$

An interaction model $E(x,y)$ is said to be *weakly twist* if it is a C^2 function such that

(H4) $\forall\, x\in\mathbb{R},\ \dfrac{\partial^2 E}{\partial x\partial y}(x,\cdot)<0$ a.e., and $\forall\, y\in\mathbb{R},\ \dfrac{\partial^2 E}{\partial x\partial y}(\cdot,y)<0$ a.e.

Note that the hypothesis (H4) is slightly more general than the usual twist condition stated for any x,y and not only for almost every x,y. This gives us the advantage to treat, for instance, a model of the form $E(x,y)=W(y-x)-\lambda(y-x)+V(x)$ with $W(s)=s^4/4$.

Here we are interested in models that take into account the interaction between the atoms of the chain and the underlying environment that will be modeled by a quasi-periodic substrate ω. In particular, we focus on *pattern equivariant* interactions, a notion that not only captures the dependence with respect to the quasi-periodic environment, but also introduces the concept of short-range interaction in this context.

By a one-dimensional quasi-periodic set we mean a discrete set $\omega \subset \mathbb{R}$ which has *finite local complexity* and is *repetitive*. To introduce these notions, we will need the one of *pattern*, namely, a set $\mathtt{P}$ of the form $\omega \cap I$ for some bounded open interval I. Two patterns $\mathtt{P}$ and $\hat{\mathtt{P}}$ are *equivalent* whenever one is the translated of the other one, that is: $\mathtt{P} + t = \hat{\mathtt{P}}$ for some $t \in \mathbb{R}$. A discrete set $\omega \subset \mathbb{R}$ is said to be *quasi-periodic* when the following properties are satisfied:

finite local complexity: the set ω possesses only finitely many equivalence classes of patterns of cardinality 2;

repetitivity: for any $R > 0$, there is a constant $M(R)$ such that for any open interval I of length at least $M(R)$, $\omega \cap I$ contains a representative from each class of patterns of diameter less than R.

In particular, we observe that the finite local complexity condition implies the quasi-periodic set is (uniformly) discrete. Moreover, ω, as a set, is unbounded from above and below by the repetitivity condition. By this condition, each type of pattern occurs infinitely many times along the real line with uniformly bounded gaps between the occurences. A very representative class of quasi-periodic sets is formed by quasicrystals. Such sets are quasi-periodic ones with an additional density property on the occurences of the patterns (see appendix A). The repetitivity condition can be interpreted as a weak homogeneity property in a topological sense. For a dynamical explanation, see [20]. Of course periodic lattices are quasi-periodic sets, but there also exist aperiodic examples (*i.e.* that are invariant under no translation). The simplest ones are constructed by iteration of a procedure so called substitution, or in a geometrical way by a cut and project scheme (see [1, 30]). The Fibonacci quasicrystal is a famous one that can be obtained by both methods. Note that in the substitutive case, the parameter $M(R)$ of the repetitivity condition can be taken with a growth at most linear in R. Such quasi-periodic sets are called *linearly repetitive*, in the following sense.

Definition 3. We say that a quasi-periodic set ω is *linearly repetitive* if the repetitivity parameter $M(R)$ has at most linear growth as a function of the upper bound R for pattern diameters.

Most of quasicrystals obtained by cut and project are linearly repetitive. In the geometrical, combinatorial and dynamical senses, they are the simplest examples of aperiodic quasicrystals. We refer to [1] for a survey of their properties.

A quasi-periodic set ω on the real line models the underlying substrate to be considered. We now describe the kind of interaction energy E between the chain and the substrate we are interested in. We say that an interaction potential V is pattern equivariant with respect to ω if two potentials are the same, $V(x) = V(y)$, at two distinct positions $x \neq y$

provided the relative structures $\omega - x$ and $\omega - y$ coincide locally. We generalize that idea to interaction energies in the following definition.

Definition 4. We say that an interaction $E(x, y)$ is *pattern equivariant* with respect to the quasi-periodic set ω if there exists $\varsigma_0 > 0$ such that for patterns $\mathtt{P}$ (of diameter greater than $2\varsigma_0$), whenever $\mathtt{P} + t$ is again a pattern of ω,

$$E(x, y) = E(x + t, y + t) \qquad \forall\, x, y \in \big[\min \mathtt{P} + \varsigma_0, \, \max \mathtt{P} - \varsigma_0\big].$$

We refer the reader to [20, 22] and to appendix A for examples of pattern equivariant interactions. We precise our core results (theorem 1) in the following statement. For a weakly twist interaction model that is pattern equivariant with respect to a linearly repetitive quasi-periodic set, we not only show the existence but we completely classify all possible types of weak KAM solutions.

Theorem 5. *Let E be an interaction fulfilling the assumptions (H1-4) of hypotheses 2. Suppose that $\inf_x E(x, x) > \bar{E}$. Then there exist positive constants K and $r < R$ such that the following holds.*

(i) There exist Mañé calibrated configurations. All Mañé calibrated configurations are strictly monotone and satisfy

$$\forall\, k \in \mathbb{Z}, \ r \le |x_{k+1} - x_k| \le R.$$

(ii) There exist weak KAM solutions. Every weak KAM solution u is Lipschitz with $\mathrm{Lip}(u) \le K$ and satisfies

$$\forall\, y \in \mathbb{R}, \quad \arg\min\{u(\cdot) + E(\cdot, y)\} \subset [y - R, y + R].$$

Assume moreover that E is pattern equivariant with respect to a linearly repetitive quasi-periodic set. Then there exists $\gamma > 0$ such that the following holds.

(iii) There exists a preferred ordering of $\mathbb{R}$ ($\epsilon = 1$ for the standard ordering, $\epsilon = -1$ for the reversed one) such that every weak KAM solution u belongs to one of the following three types:

- *type I: every u-calibrated configuration $(x_n)_{n\in\mathbb{Z}}$ is such that $(\epsilon x_n)_{n\in\mathbb{Z}}$ is increasing, and*

$$\lim_{x\to\pm\infty} \frac{u(x)}{x} = 0;$$

- *type II: every u-calibrated configuration $(x_n)_{n\in\mathbb{Z}}$ is such that $(\epsilon x_n)_{n\in\mathbb{Z}}$ is decreasing, and*

$$\limsup_{x\to+\infty} \frac{u(\epsilon x)}{|x|} \le -\gamma, \quad \liminf_{x\to-\infty} \frac{u(\epsilon x)}{|x|} \ge \gamma;$$

- *type III: there is no bi-infinite u-calibrated configuration and*

$$\limsup_{x\to+\infty} \frac{u(\epsilon x)}{|x|} \le -\gamma, \quad \lim_{x\to-\infty} \frac{u(\epsilon x)}{|x|} = 0.$$

(iv) *There exist weak KAM solutions u of the three types previously described. In type I and II, there exist u-calibrated configurations. In type III, there is no u-calibrated configurations.*

(v) *Any two weak KAM solutions u and v of the same type lie at uniform distance from each other:* $\sup_x |u(x) - v(x)| < +\infty$.

To illustrate theorem 5, the figures 1, 2, 3, present different possible asymptotic behaviors of weak KAM solutions.

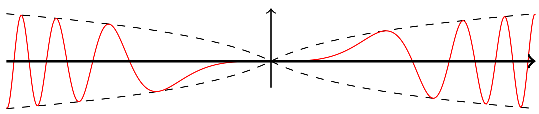

FIGURE 1. Type I: A sublinear growth at $\pm\infty$.

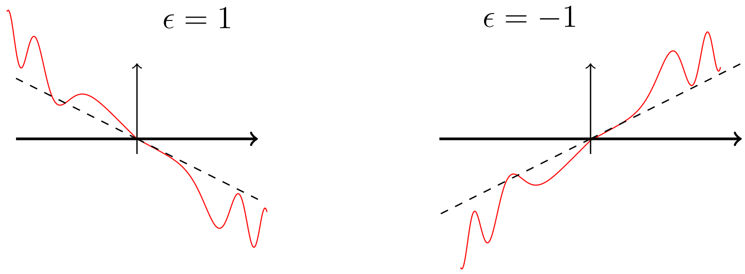

FIGURE 2. Type II: On the left hand side, a growth at $\pm\infty$ at least negative. On the right hand side, a growth at $\pm\infty$ at least positive.

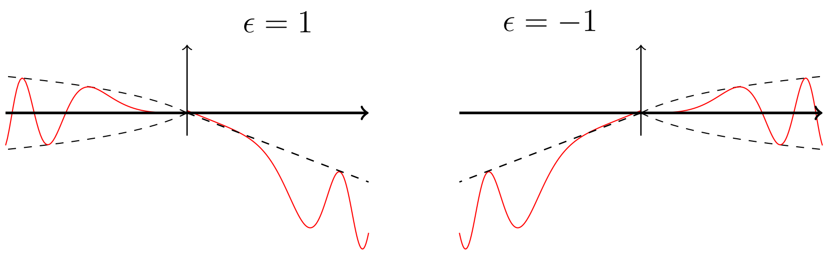

FIGURE 3. On the left hand side, a sublinear growth at $-\infty$ and a growth at $+\infty$ at least negative. On the right hand side, a sublinear growth at $+\infty$ and a growth at $-\infty$ at least positive.

At the opposite, the figures 4, 5, 6, present impossible asymptotic behaviors for weak KAM solutions.

We provide families of interactions fulfilling the hypotheses of this theorem in appendix A. Assumption $\inf_x E(x,x) > \bar{E}$ is a non-degeneracy hypothesis. This condition roughly indicates that a chain formed by atoms

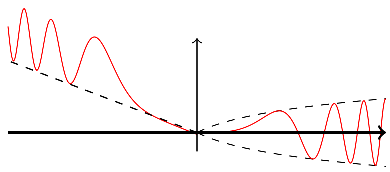

FIGURE 4. At least (negative) linear growth at $-\infty$ and sublinear growth a $+\infty$. The symmetric case with respect to the vertical axe is also impossible.

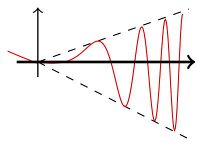

FIGURE 5. The limits inferior and superior of linear growth at $+\infty$ (or at $-\infty$) have different signs.

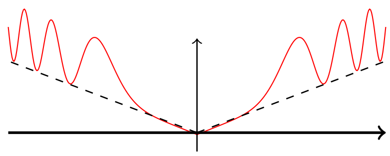

FIGURE 6. At least negative (resp. positive) linear growth at $-\infty$ (resp. $+\infty$). The symmetric case with respect to the horizontal axe is also impossible.

very close to each other cannot be strongly minimizing, so that Mañé calibrated configurations have an intrinsic interspacing.

The above theorem has implications for the periodic case. As already mentioned, the existence of periodic (and therefore bounded) weak KAM solutions is well established. To the best of our knowledge, however, weak KAM solutions with linear or mixed behavior in the periodic context are not reported in the literature.

Theorem 6. *For a weakly twist and periodic interaction model E such that $\inf_x E(x,x) > \bar{E}$, there are exactly three types of weak KAM solutions: those that are bounded, those that have linear growth along the real line, and those that are simultaneously bounded on a selected half-line and have linear growth along the complementary half-line.*

Moreover, two weak KAM solutions belonging to the same class remain at a uniform distance from each other along the real line.

Of course a weak KAM solution with linear growth provided by theorem 6 is not periodic. Actually, for periodic interaction, we do not known if any bounded weak KAM solution is also periodic.

In the degenerate case, where the infimum of self-interactions ($E(x,x)$, $x \in \mathbb{R}$) equals the ground action, behaviors of weak KAM solutions may be very different from those observed in theorem 5, even in the periodic case. As an illustration, we propose the following result.

Theorem 7. *There is a periodic interaction $E(x,y)$, with $\inf_x E(x,x) = \bar{E}$, for which $x \mapsto S(0,x)$ defines a weak KAM solution of linear growth but all Mañé calibrated configurations are (uniformly) bounded.*

As noted by one of the referees, more information about the preferred ordering can be proved for periodic interaction models.

Proposition 8. *Let E_0 be a periodic interaction model satisfying (H1) and (H3). With respect to the one-parameter family $E_\lambda(x,y) := E_0(x,y) - \lambda(y-x)$, the following items hold.*

(i) E_λ is non-degenerate for large $|\lambda|$.
(ii) For positive large λ, the preferred ordering for E_λ is increasing.
(iii) For negative large λ, the preferred ordering for E_λ is decreasing.

If in addition E_0 is supposed to fulfill (H1-4), denote

$$\Lambda^+ := \big\{\lambda \in \mathbb{R} : E_\lambda \textit{ is non-degenerate and has an increasing ordering}\big\},$$
$$\Lambda^- := \big\{\lambda \in \mathbb{R} : E_\lambda \textit{ is non-degenerate and has an decreasing ordering}\big\}.$$

(iv) Then there exist $\lambda_- \le \lambda_+$ such that

$$\Lambda_+ = (\lambda_+, +\infty), \qquad \Lambda_- = (-\infty, \lambda_-),$$

and for every $\lambda \in [\lambda_-, \lambda_+]$ the interaction model E_λ is degenerate.

A classical approach to get weak KAM solution is through the study of the action of the Lax-Oleinik operator on a suitable space of functions. Related to our context, an interesting space is the one formed by continuous functions $u\colon \mathbb{R} \to \mathbb{R}$ with *at most linear growth*, namely, fulfilling $\sup_x |u(x)|/(|x|+1) < +\infty$. For E an interaction satisfying assumptions (H1-3) of hypotheses 2, recall that the *backward Lax-Oleinik operator* is the non-linear operator acting on the space of continuous functions with at most linear growth as

$$\forall\, y \in \mathbb{R}, \quad T[u](y) := \inf_{x\in\mathbb{R}} \big\{u(x) + E(x,y)\big\}. \tag{2.1}$$

Then weak KAM solutions are functions that satisfy $T[u] = u + \bar{E}$.

Our purpose in the next sections is to detail the proofs of the above results. The rest of the paper is organized as follows. In section 3, from a Mañé calibrated configuration that traverses the entire real line, we define a family of localized Lax-Oleinik operators preserving a suitable sequence of functional spaces. We then show they admit additive eigenfunctions and moreover their accumulation points are indeed Lipschitz weak KAM solutions (theorem 12). In section 4, for weakly twist models E that fulfill $\inf_x E(x,x) > \bar{E}$, proposition 16 ensures that calibrated configurations are always monotone and have successive jumps bounded (in a uniform way) from above as well as from below. This provides the existence of a Mañé calibrated configuration as required to apply theorem 12, which gives theorem 19 that corresponds to the first statement of theorem 5. The linearly repetitive quasi-periodic case is studied in details in section 5. Thanks to the specific properties of repetitions of the pattern in this quasi-periodic setting, we are able to identify, as $|x-y| \to +\infty$, distinct behaviors of the Mañé potential according to whether $x<y$ or $x>y$ (proposition 26). Actually, the model introduces a preferred ordering on the real line ($\epsilon = 1$ for the standard ordering, $\epsilon=-1$ for the reversed one), and a dichotomy on the type of growth of the Mañé potential $S(x,y)$ as $|x-y| \to +\infty$: a sublinear growth of $S(\epsilon x, \epsilon y)$ when $x<y$ and a positive linear growth of $S(\epsilon x, \epsilon y)$ when $x>y$. Such a feature is the key piece that allows the classification of all weak KAM solutions. Hence, proposition 31, corollary 32 and proposition 33 are the results that complete the statement of theorem 5. The periodic example proving theorem 7 is studied in proposition 36, in section 6. Also in this section is the proof of proposition 8.

3. Mañé calibration and weak KAM solutions

We prove in this section the existence of weak KAM solutions (1.3) under the assumption there exists a Mañé calibrated configuration unbounded at $\pm\infty$ with uniformly bounded jumps. Note that we do not require this configuration to be monotone.

During this section, we suppose that E is an interaction model satisfying assumptions (H1-3) of hypotheses 2, no additional condition is required.

We recall first some of main definitions mentioned in the introduction.

Definition 9.

(i) We call *ground action* the quantity

$$\bar{E} := \lim_{n\to+\infty} \inf_{x_0,x_1,\ldots,x_n} \frac{1}{n}\sum_{k=0}^{n-1} E(x_k, x_{k+1})$$

(ii) We call *Mañé potential* the function defined on $\mathbb{R}\times\mathbb{R}$ as

$$S(x,y) := \inf_{n\geq 1} \inf_{x=x_0,x_1,\ldots,x_n=y} \sum_{k=0}^{n-1} \big(E(x_k,x_{k+1}) - \bar{E}\big).$$

(iii) A subconfiguration $(x_k)_{k=p}^{q}$, $p<q$, is said to be *Mañé calibrated* if

$$\forall\, p \leq m < n \leq q, \quad S(x_m,x_n) = \sum_{k=m}^{n-1} \big(E(x_k,x_{k+1}) - \bar{E}\big).$$

To simplify the notations we will use the convention

$$E(x_0,x_1,\ldots,x_n) := \sum_{k=0}^{n-1} E(x_k,x_{k+1}).$$

We observe the following simple properties.

Remark 10.

(i) $\bar{E} = \sup_{n\geq 1} \inf_{x_0,x_1,\ldots,x_n} \frac{1}{n} E(x_0,x_1,\ldots,x_n)$,
(ii) $\inf_{x,y\in\mathbb{R}} E(x,y) \leq \bar{E} \leq \inf_{x\in\mathbb{R}} E(x,x)$,
(iii) $\forall\, x_0 \in \mathbb{R},\ \forall\, n \geq 1, \quad \bar{E} \leq \inf_{x_1,\ldots,x_{n-1}} \frac{1}{n} E(x_0,x_1,\ldots,x_{n-1},x_0)$
(iv) $\inf_{x,y\in\mathbb{R}} S(x,y) \leq 0 \leq \inf_{x\in\mathbb{R}} S(x,x)$,
(v) $\forall\, x,y,z \in \mathbb{R},\ S(x,z) \leq S(x,y) + S(y,z)$,
(vi) $\forall\, x,y \in \mathbb{R},\ \bar{E} - E(y,x) \leq S(x,y) \leq E(x,y) - \bar{E}$.

Property (i) is a consequence of Fekete's lemma and the super-additivity of $[n \mapsto \inf_{x_0,\ldots,x_n} E(x_0,\ldots,x_n)]$. Property (ii) is obtained by bounding from above $\inf_{x_0,\ldots,x_n} E(x_0,\ldots,x_n)$ by computing the action on configurations of the form $(x,x,\ldots,x)$. Property (iii) follows from Fekete's lemma and the sub-additivity of $[n \mapsto \inf_{x_1,\ldots,x_{n-1}} E(x_0,x_1,\ldots,x_{n-1},x_0)]$. Property (iv) is a consequence of (iii) for the right hand side, and a consequence of the definition of $\bar{E}$ and the inequality $S(x,y) \leq E(x,y) - \bar{E}$ for the left hand side. Property (v) follows by concatenation of configurations. Finally, property (vi) follows from the inequality $S(x,y) \leq E(x,y) - \bar{E}$ obtained by taking a simple configuration (x,y), and then from (v) using the second inequality of (iv).

We show in the following lemma that any weak KAM solution is Lipschitz and that any backward minimizer in the definition of the Lax-Oleinik operator (2.1) has a uniform bounded jump.

Lemma 11. *There exist constants $K, R \geq 0$ (depending only on the interaction model E) such that for every weak KAM solution u*

(i) u is Lipschitz continuous and $\mathrm{Lip}(u) \leq K$,
(ii) $\forall\, y \in \mathbb{R},\ \arg\min\{u(\cdot) + E(\cdot,y)\} \subset [y-R, y+R]$.

Proof. Let u be a weak KAM solution, that is, $T[u] = u + \bar{E}$, where T is the Lax-Oleinik operator associated to E defined as (2.1).

Step 1. We show an *a priori* linear growth of u. Denote

$$\tilde{K} := \sup_{|y-x|\leq 1} E(x,y) - \inf_{x,y\in\mathbb{R}} E(x,y).$$

We claim that $|u(y) - u(x)| \leq \tilde{K}(|y-x|+1)$ for all $x,y \in \mathbb{R}$. Indeed, either one has $|y-x| \leq 1$, and then

$$u(y) - u(x) \leq E(x,y) - \bar{E} \leq \sup_{|y-x|\leq 1} E(x,y) - \inf_{x,y\in\mathbb{R}} E(x,y) \leq \tilde{K}, \tag{3.1}$$

which clearly implies $u(y) - u(x) \leq \tilde{K}(|y-x|+1)$. Or otherwise for some $n \geq 2$, $n-1 < |y-x| \leq n$. In this case, consider $x_k := x + \frac{k}{n}(y-x)$, $k = 0,\ldots,n$, a sequence of points spaced apart by at most 1. Then from (3.1) $u(x_k) - u(x_{k-1}) \leq \tilde{K}$, so that

$$u(y) - u(x) \leq n\tilde{K} \leq \tilde{K}(|y-x|+1).$$

Step 2. We show item ii. By the superlinearity, there exists $R \geq \tilde{K}$ such that $E(x,y) > \tilde{K}|x-y| + \bar{E} + \tilde{K}$ whenever $|x-y| > R$. Suppose

$x, y \in \mathbb{R}$ fulfill $u(y) - u(x) = E(x, y) - \bar{E}$. Assume by contradiction that $|y - x| > R$. From the first step, $|u(y) - u(x)| \leq \tilde{K}(|y - x| + 1)$. However, from the choice of R, we see that $E(x, y) - \bar{E} > \tilde{K}(|y - x| + 1)$. We thus obtain a contradiction and conclude that $|y - x| \leq R$.

Step 3. We show item i. Let $y, z \in \mathbb{R}$ and $x \in \arg\min\{u(\cdot) + E(\cdot, y)\}$. Hence,

$$u(y) = u(x) + E(x, y) - \bar{E} \qquad \text{and} \qquad u(z) \leq u(x) + E(x, z) - \bar{E}.$$

Either $|z - y| \geq 1$, and therefore

$$u(z) - u(y) \leq \tilde{K}(|z - y| + 1) \leq 2\tilde{K}|z - y|,$$

or $|z - y| < 1$ so that, as $|y - x| \leq R$ and $|z - x| \leq R + 1$, using the constant C^E_{Lip} in (H2),

$$u(z) - u(y) \leq E(x, z) - E(x, y) \leq C^E_{Lip}(R + 1)|z - y|.$$

We obtain that u is K-Lipschitz with $K := \max\{2\tilde{K}, C^E_{Lip}(R + 1)\}$. □

We highlight the key result of this section.

Theorem 12. *Assume there exists a configuration $(x_k)_{k\in\mathbb{Z}}$ fullfiling*

- *$(x_k)_{k\in\mathbb{Z}}$ is Mañé calibrated,*
- *it has bounded jumps, namely, $\sup_{k\in\mathbb{Z}} |x_{k+1} - x_k| < +\infty$,*
- *it is unbounded from above and below in the sense that either*

$$\limsup_{k\to+\infty} x_k = +\infty, \ \liminf_{k\to-\infty} x_k = -\infty,$$

or

$$\liminf_{k\to+\infty} x_k = -\infty, \ \limsup_{k\to-\infty} x_k = +\infty.$$

Then there exists a Lipschitz weak KAM solution v such that for $m < n$, the configuration $(x_k)_{k\in\mathbb{Z}}$ satisfies

$$v(x_n) - v(x_m) = S(x_m, x_n).$$

We first prove an *a priori* linear growth of the Mañé potential. Note that, thanks to hypotheses (H1) and (H2),

$$\forall R > 0, \qquad \sup_{|y-x|\leq R} E(x, y) < +\infty.$$

Lemma 13. *There exists a constant $C > 0$ such that*

$$\forall x, y \in \mathbb{R}, \quad |S(x, y)| \leq C(|y - x| + 1).$$

Proof. Define

$$C := \sup_{|y-x|\leq 1} |E(x, y) - \bar{E}|.$$

Choose $n \geq 1$ such that $n-1 \leq |y-x| < n$, and denote $t_k := x + \frac{k}{n}(y-x)$. Then

$$|S(x, y)| \leq \sum_{k=1}^{n} |S(t_{k-1}, t_k)| \leq nC \leq C(|y - x| + 1).$$ □

Proof of theorem 12. Suppose that

$$\limsup_{k\to+\infty} x_k = +\infty \quad \text{and} \quad \liminf_{k\to-\infty} x_k = -\infty.$$

The other case follows from this one by introducing $\hat{E}(x,y) := E(-x,-y)$ and noticing that $(\hat{x}_k)_k := (-x_k)_k$ is calibrated with respect to $\hat{S}(x,y) = S(-x,-y)$.

Step 1. The idea of the proof is to construct (in a uniformly Lipschitz way) approximated weak KAM solution on an exhausting sequence of compact intervals $B_N := [x_{i_N}, x_{j_N}]$, $i_N \to -\infty$, $j_N \to +\infty$. The difference between the standard Lax-Oleinik operator and the approximated one is that we impose on the latter a fixed boundary condition on the set of solutions outside B_N.

We define inductively two sequences of indices

$$\cdots \le i_2 \le i_1 < 0 < j_1 \le j_2 \le \cdots$$

such that for every $N \ge 1$,

$$x_{i_N - 1} < x_0 - N < x_0 + N < x_{j_N+1} \text{ and}$$
$$\forall\, i_N \le k \le j_N,\ x_k \in [x_0 - N, x_0 + N].$$

Let $B_N := [x_{i_N}, x_{j_N}]$ and T_N be the operator acting on $C^0(B_N)$ by

$$\forall\, y \in B_N, \quad T_N[u](y) = \min_{x\in\mathbb{R}} \big[\tilde{u}(x) + E(x,y) - \bar{E}\big],$$

where $\tilde{u}$ is the extension of u on $\mathbb{R}$ defined as

$$\forall\, x \notin B_N, \quad \tilde{u}(x) = S(x_{i_N-2}, x).$$

Note that T_N is well defined thanks to the superlinearity of the interaction and the sublinearity of S. We show there exists a constant $K > 0$ such that for every N, T_N preserves the following functional space

$$\begin{aligned}\mathcal{H}_N := \big\{u \in C^0(B_N) : \forall\, i_N \le k \le j_N,\ u(x_k) = S(x_{i_N-2}, x_k),\\ \forall\, x \in B_N,\ u(x) \ge S(x_{i_N-2}, x),\\ \forall\, x, y \in B_N \text{ with } |x-y| < 1,\ |u(y) - u(x)| \le K|y-x|\big\},\end{aligned}$$

Note that for $u \in \mathcal{H}_N$,

$$\forall\, x \in \mathbb{R},\ \tilde{u}(x) \ge S(x_{i_N-2}, x), \quad \forall\, k \in \mathbb{Z},\ \tilde{u}(x_k) = S(x_{i_N-2}, x_k).$$

To prove the invariance of $\mathcal{H}_N$ under T_N, observe for every $y \in B_N$ and $x \in \mathbb{R}$,

$$\tilde{u}(x) + E(x,y) - \bar{E} \ge S(x_{i_N-2}, x) + S(x,y) \ge S(x_{i_N-2}, y),$$

and for every $i_N \le k \le j_N$, thanks to the calibration of $(x_k)_{k\in\mathbb{Z}}$,

$$\tilde{u}(x_{k-1}) + E(x_{k-1}, x_k) - \bar{E} = S(x_{i_N-2}, x_{k-1}) + S(x_{k-1}, x_k) = S(x_{i_N-2}, x_k),$$

which implies

$$T_N[u](y) \geq S(x_{i_N-2}, y), \ \forall\, y \in B_N \quad \text{and}$$
$$T_N[u](x_k) = S(x_{i_N-2}, x_k), \ \forall\, i_N \leq k \leq j_N.$$

Let $y \in B_N$. We prove that the infimum in the definition of $T_N[u](y)$ is attained at some $x \in \mathbb{R}$ satisfying $|y-x| \leq R$ for some uniform constant $R>0$. Define $\rho := \sup_{k\in\mathbb{Z}} |x_{k+1}-x_k|$. On the one hand, if $x_k \in B_N$ is chosen such that $|y-x_k| \leq \rho$, then

$$\begin{aligned} T_N[u](y) &\leq \tilde{u}(x_k) + E(x_k, y) - \bar{E} = S(x_{i_N-2}, x_k) + E(x_k, y) - \bar{E} \\ &\leq S(x_{i_N-2}, y) + S(y, x_k) + E(x_k, y) - \bar{E} \\ &\leq S(x_{i_N-2}, y) + 2 \sup_{|x-x'|\leq\rho} \big|E(x,x') - \bar{E}\big|, \end{aligned}$$

On the over hand, by the superlinearity of the interaction, one can find $x \in \mathbb{R}$ such that

$$T_N[u](y) = \tilde{u}(x) + E(x,y) - \bar{E} \geq S(x_{i_N-2}, x) + E(x,y) - \bar{E}.$$

Combining both inequalities, one obtains

$$E(x,y) - \bar{E} \leq S(x,y) + D,$$

with $D := 2\sup_{|y-x|\leq\rho} |E(x,y)-\bar{E}|$. Using again the superlinearity of E and the constant C from lemma 13, one gets for some constant $B>0$,

$$(C+1)|y-x| - B \leq E(x,y) - \bar{E} \leq S(x,y) + D \leq C(|y-x|+1) + D,$$
$$|y-x| \leq R \quad \text{with} \quad R := B + C + D.$$

We prove that $T_N[u]$ is Lipschitz continuous. Consider $y_1, y_2 \in B_N$ with $|y_2 - y_1| \leq 1$. Then there exists $x \in \mathbb{R}$, $|y_1 - x| \leq R$, such that

$$T_N[u](y_1) = \tilde{u}(x) + E(x, y_1) \quad \text{and} \quad T_N[u](y_2) \leq \tilde{u}(x) + E(x, y_2).$$

Using the constant $C^E_{\text{Lip}}(R)$ as in (H2) and denoting $K := C^E_{\text{Lip}}(R+1)$, one obtains

$$|T_N[u](y_2) - T_N[u](y_1)| \leq K|y_2 - y_1|.$$

In conclusion, $\mathcal{H}_N$ is a compact convex subset of $C^0(B_N)$ for the uniform topology. The non linear operator $T_N : \mathcal{H}_N \to \mathcal{H}_N$ is 1-Lipschitz. By Schauder-Tychonoff theorem, T_N admits a fixed point u_N.

Step 2. Define

$$v_N(y) := u_N(y) - u_N(x_0), \quad \forall\, y \in B_N.$$

Then, for N sufficiently large, $x_{i_N} < x_0 - R < x_0 + R < x_{j_N}$,

- $v_N(x_0) = 0$,
- $\forall\, x, y \in B_N$ with $|x-y| < 1$, $|v_N(y) - v_N(x)| \leq K|y-x|$,
- $\forall\, x, y \in B_N$, $v_N(y) \leq v_N(x) + E(x,y) - \bar{E}$,
- $\forall\, y \in [x_{i_N} + R, x_{j_N} - R]$, $\exists\, x \in B_N$ such that $|y-x| \leq R$ and $v_N(y) = v_N(x) + E(x,y) - \bar{E}$,
- $\forall\, i_N \leq k < l \leq j_N$, $v_N(x_l) - v_N(x_k) = S(x_k, x_l)$.

By using a diagonal procedure of extraction, there exists a subsequence of $(v_N)_N$ that converges uniformly on any compact interval to a K-Lipschitz function $v : \mathbb{R} \to \mathbb{R}$ that is a weak KAM solution calibrating $(x_k)_{k\in\mathbb{Z}}$. □

4. Non-degenerate and weakly twist models

The main result of this section guarantees that, for weakly twist models (*i.e.*, interactions fulfilling all the assumptions (H1-4) of hypotheses 2) that satisfy the non-degenerate condition $\inf_x E(x,x) > \bar{E}$, there are always weak KAM solutions. In order to apply theorem 12, we prove in lemma 18 the existence of increasing as well as decreasing Mañé calibrated configurations with bounded jumps and unbounded from above and below. Actually we improve a result obtained in [22] for which the environment is supposed to be a quasi-crystal – in particular, it possesses a uniquely ergodic hull and the interaction is pattern equivariant. On the contrary lemma 18 does not require any particular assumption on the structure of an underlying substrate.

We first gather results that have been proved in [22].

Lemma 14. *Let E be a weakly twist interaction. Then*

(i) $\forall\, x < y,\ \ S(x,y) = \inf_{x=x_0<x_1<\cdots<x_n=y} \big\{E(x_0,\ldots,x_n) - n\bar{E}\big\}$,

(ii) $\forall\, x > y,\ \ S(x,y) = \inf_{x=x_0>x_1>\cdots>x_n=y} \big\{E(x_0,\ldots,x_n) - n\bar{E}\big\}$,

(iii) $\forall\, x \in \mathbb{R},\ \ S(x,x) = E(x,x) - \bar{E}$.

Moreover, if the interaction is pattern equivariant with respect to a quasi-periodic set, then the Mañé potential is also pattern equivariant.

Proof. See proposition 24 in [22]. □

We assume from now on that $\inf_{x\in\mathbb{R}} E(x,x) > \bar{E}$. We choose once for all $\eta_0 > 0$ such that

$$\forall\, x,y \in \mathbb{R},\ |y-x| < \eta_0 \ \Rightarrow\ E(x,y) - \bar{E} > \eta_0. \tag{4.1}$$

Lemma 15. *Assume that E is a weakly twist interaction model satisfying $\inf_x E(x,x) > \bar{E}$. Then, there exist constants $A_0, B_0 > 0$ such that for any $(x_0,\ldots,x_n) \in \mathbb{R}^{n+1}$, with $x_0 = x$ and $x_n = y$,*

$$n \le A_0|y-x| + B_0 \sum_{k=1}^{n} \big(E(x_{k-1},x_k) - \bar{E}\big), \tag{4.2}$$

where $A_0 = (\bar{E} - \inf_{x,y\in\mathbb{R}} E(x,y) + \eta_0)/\eta_0^2$ and $B_0 = 1/\eta_0$.

Proof. If $(x_0,\ldots,x_n)$ is not monotone, then by lemma 23 in [22] there exists a subset of distinct indices $\{i_0,\ldots,i_\sigma\}$ of $\{0,\ldots,n\}$, with $i_0 = 0$, $i_\sigma = n$, such that $(x_{i_0},\ldots,x_{i_\sigma})$ is strictly monotone, and

$$E(x_0,\ldots,x_n) - n\bar{E} > E(x_{i_0},\ldots,x_{i_\sigma}) - \sigma\bar{E} + \sum_{j\notin\{i_0,\ldots,i_\sigma\}} \big(E(x_j,x_j) - \bar{E}\big). \tag{4.3}$$

If $(x_0,\dots,x_n)$ is monotone, we choose $\sigma = n$ and $i_k = k$ for all k. In both cases, we obtain

$$E(x_0,\dots,x_n) - n\bar{E} \geq E(x_{i_0},\dots,x_{i_\sigma}) - \sigma\bar{E} + (n-\sigma)\eta_0.$$

We now consider the set of indices $I \subseteq \{0,\dots,\sigma-1\}$ such that $k \in I$ if and only if $|x_{i_k} - x_{i_{k+1}}| \geq \eta_0$. If $k \in I$, we use an *a priori* lower bound

$$E(x_{i_k}, x_{i_{k+1}}) - \bar{E} \geq E_{min} - \bar{E}$$

where $E_{min} = \inf_{x,y\in\mathbb{R}} E(x,y)$. If $k \notin I$, the definition of η_0 gives

$$E(x_{i_k}, x_{i_{k+1}}) - \bar{E} \geq \eta_0.$$

Hence, we have

$$E(x_{i_0},\dots,x_{i_\sigma}) - \sigma\bar{E} \geq |I|(E_{min} - \bar{E}) + (\sigma - |I|)\eta_0.$$

Combining the estimates above, we obtain

$$E(x_0,\dots,x_n) - n\bar{E} \geq |I|(E_{min} - \bar{E} - \eta_0) + n\eta_0.$$

By monotonicity of $(x_{i_0},\dots,x_{i_\sigma})$, clearly $|I| \leq |x_{i_\sigma} - x_{i_0}|/\eta_0 = |x_n - x_0|/\eta_0$, so that

$$n \leq \frac{\bar{E} - E_{min} + \eta_0}{\eta_0^2}|y-x| + \frac{1}{\eta_0}\sum_{k=1}^{n}\big(E(x_{k-1},x_k) - \bar{E}\big).$$

□

We show in the following lemma that the infimum in the definition in $S(x,y)$ is actually a minimum, and that the number of points realizing the minimum is bounded from above by a quantity proportional to $|y-x|$. Besides, we prove that the successive jumps of Mañé calibrated configurations are uniformly bounded from above and from below.

Proposition 16. *Suppose that E is a weakly twist interaction model fulfilling $\inf_x E(x,x) > \bar{E}$.*

(i) For every $x \neq y$, there are an integer $n \geq 1$ and a strictly monotone configuration $(x_0,\dots,x_n)$, with $x_0 = x$ and $x_n = y$, fulfilling

$$S(x,y) = \sum_{k=1}^{n} E(x_{k-1},x_k) - n\bar{E}.$$

(ii) There exist constants $A > 0$ and $B \geq 0$ such that, for every pair of points $x,y \in \mathbb{R}$, if $(x_0\dots,x_n) \in \mathbb{R}^{n+1}$ satisfies $x_0 = x$, $x_n = y$, and $S(x,y) = \sum_{k=1}^n E(x_{k-1},x_k) - n\bar{E}$, then

$$S(x_p,x_q) = \sum_{k=p+1}^{q} E(x_{k-1},x_k) - (p-q)\bar{E}, \quad \forall\, 0 \leq p < q \leq n,$$

$$(x_0,\dots,x_n) \text{ is monotone} \quad \text{and} \quad n \leq A|y-x| + B.$$

Besides, for $n \geq 2$, $(x_0,\dots,x_n)$ is strictly monotone.

(iii) The Mañé potential admits a negative sublinear lower bound in the following sense:

$$\forall\, \alpha > 0,\ \exists\, \beta \ge 0,\ \forall\, x, y \in \mathbb{R},\ S(x,y) \ge -\alpha|y-x| - \beta.$$

(iv) There exist constants $0 < r < R$ such that every Mañé calibrated subconfiguration $(x_p, \ldots, x_q)$ with $q - p \ge 2$ is strictly monotone and satisfies

$$\forall\, p \le k < q, \qquad r \le |x_{k+1} - x_k| \le R.$$

Proof. We assume in all items $x \le y$. The other case $x \ge y$ is similar. We start by proving item ii.

Item ii. Let $(x_0, \ldots, x_n)$ be a configuration satisfying

$$S(x,y) = \sum_{k=1}^{n} E(x_{k-1}, x_k) - n\bar{E}.$$

Obviously for $0 \le p < q \le n$,

$$S(x_0, x_q) + \sum_{k=p+1}^{q} E(x_{k-1}, x_k) - (p-q)\bar{E} + S(x_q, x_n) \le$$

$$\le \sum_{k=1}^{n} E(x_{k-1}, x_k) - n\bar{E} = S(x_0, x_n) \le S(x_0, x_q) + S(x_p, x_q) + S(x_q, x_n)$$

implies that $S(x_p, x_q) = \sum_{k=p+1}^{q} E(x_{k-1}, x_k) - (p-q)\bar{E}$. Inequality (4.3) shows that $(x_0, \ldots, x_n)$ must be monotone, since otherwise one could decrease strictly the Mañé potential

$$S(x,y) = E(x_0, \ldots, x_n) - n\bar{E} > E(x_{i_0}, \ldots, x_{i_\sigma}) - \sigma\bar{E} \ge S(x,y).$$

For $n \ge 2$, the configuration is actually strictly monotone, since otherwise there would exist $1 \le j \le n$ such that $x_{j-1} = x_j$ and we would obtain the same contradiction as above

$$\begin{aligned} S(x,y) &= \Big[\sum_{k \ne j} E(x_{k-1}, x_k) - (n-1)\bar{E}\Big] + \big[E(x_{j-1}, x_j) - \bar{E}\big] \\ &\ge S(x,y) + \inf_{x \in \mathbb{R}} E(x,x) - \bar{E}. \end{aligned}$$

The estimate (4.2) thus implies $n \le A_0|y-x| + B_0 S(x,y)$. We conclude using the a priori sublinearity estimate for the Mañé potential in lemma 13, so that $n \le A|y-x| + B$, with $A = A_0 + B_0 C$ and $B = B_0 C$.

Item i. Let us consider a sequence $S_\ell > S(x,y)$ converging to $S(x,y)$. Item i of lemma 14 shows there exists a strictly increasing configuration $(x_0^\ell, \ldots, x_{n_\ell}^\ell)$, with $x_0^\ell = x$ and $x_n^\ell = y$, such that

$$S_\ell > \sum_{k=1}^{n_\ell} \big(E(x_{k-1}, x_k) - n_\ell \bar{E}\big).$$

The estimate (4.2) implies $n_\ell \le A_0|y-x|+B_0 S_\ell$. As $S_\ell \to S(x,y)$, we may assume $n_\ell = n$ is constant. We then extract a subsequence of $(x_0^\ell, \dots, x_n^\ell)$ converging to some $(x_0, \dots, x_n)$ satisfying

$$S(x,y) \ge \sum_{k=1}^{n} \big(E(x_{k-1}, x_k) - n\bar{E}\big) \ge S(x,y).$$

The previous item shows that $(x_0, \dots, x_n)$ is strictly monotone.

Item iii. Let $\alpha > 0$ and $\alpha' = \alpha/A$, where A is the constant obtained in the first item. Thanks to item i of definition 9, there exists $\beta' \ge 0$ such that

$$\forall\, n \ge 1, \quad \forall\, (x_0, \dots, x_n), \quad E(x_0, \dots, x_n) - n\bar{E} \ge -\alpha' n - \beta'.$$

Items i and ii of the present proposition ensure that there is a particular configuration $(x_0, \dots, x_n)$ such that

$$\begin{aligned} S(x,y) &= E(x_0, \dots, x_n) - n\bar{E} \ge -\alpha' n - \beta' \\ &\ge -\alpha'(A|y-x| + B) - \beta' = -\alpha|y-x| - \beta, \end{aligned}$$

with $\beta := \alpha' B + \beta'$.

Item iv. Let $(x_p, \dots, x_q)$ be a Mañé calibrated subconfiguration. It is strictly monotone for $q - p \ge 2$ as a consequence of item ii. From lemma 13, we have

$$S(x_k, x_{k+1}) \le C(|x_{k+1} - x_k| + 1)$$

for some constant C. From the superlinearity of the interaction, there exists a constant $B > 0$ such that

$$(C+1)|x_{k+1} - x_k| - B \le E(x_k, x_{k+1}) - \bar{E} = S(x_k, x_{k+1}).$$

Therefore, $|x_{k+1} - x_k| \le B + C := R$. With respect to the lower bound, let first $\eta_0 > 0$ be defined as in (4.1). Note then that for $p \le k < q-1$,

$$\begin{aligned} S(x_k, x_{k+1}) &= S(x_k, x_{k+2}) - S(x_{k+1}, x_{k+2}) \\ &\le E(x_k, x_{k+2}) - E(x_{k+1}, x_{k+2}) \\ &\le C_{\mathrm{Lip}}^E(2R)\, |x_{k+1} - x_k|. \end{aligned} \tag{4.4}$$

We claim that $|x_{k+1} - x_k| > \eta_0 / \big(C_{\mathrm{Lip}}^E(2R) + 1\big) =: r$. Indeed, otherwise by the very definition of η_0 we would have $\eta_0 < E(x_k, x_{k+1}) - \bar{E} = S(x_k, x_{k+1})$, but (4.4) shows that $S(x_k, x_{k+1}) \le C_{\mathrm{Lip}}^E(2R)\, \eta_0 / \big(C_{\mathrm{Lip}}^E(2R) + 1\big) < \eta_0$, and we would reach a contradiction. The equality $S(x_{q-1}, x_q) = S(x_{q-2}, x_q) - S(x_{q-2}, x_{q-1})$ allows to discuss the case of the last index in a similar way. □

The regularity of the Mañé potential is an immediate consequence of the previous proposition.

Corollary 17. *For a weakly twist interaction E fulfilling $\inf_x E(x,x) > \bar{E}$, the Mañé potential is Lipschitz continuous.*

Proof. Let $I, J \subset \mathbb{R}$ be both open intervals of length 1. It is enough to argue that $S|_{I\times J}$ is Lipschitz. Items i and ii of proposition 16 guarantees there exists $N = N(I,J) > 0$ such that

$$S(a,b) = \min_{1\le n\le N} \min_{(a=x_0,\dots,x_n=b)} \big[E(x_0,\dots,x_n) - n\bar{E}\big],$$

for all $a \in I$ and $b \in J$. Therefore, given $x, \hat{x} \in I$ and $y, \hat{y} \in J$, items i and iv of proposition 16 provide the estimate

$$|S(x,y) - S(\hat{x},\hat{y})| \le \max\Big\{ \max_{\substack{|x-a|\le R \\ |y-b|\le R}} |E(x,a) - E(\hat{x},a) + E(b,y) - E(b,\hat{y})|\,, \max_{\substack{|\hat{x}-\hat{a}|\le R \\ |\hat{y}-\hat{b}|\le R}} \big|E(x,\hat{a}) - E(\hat{x},\hat{a}) + E(\hat{b},y) - E(\hat{b},\hat{y})\big| \Big\}.$$

Since E is locally uniformly Lipschitz, we conclude that

$$|S(x,y) - S(\hat{x},\hat{y})| \le C^E_{\text{Lip}}(R+1)\big(|x-\hat{x}| + |y-\hat{y}|\big).$$

□

The existence of Mañé calibrated configurations such as those required among the hypotheses of theorem 12 was actually proved in [22] by adopting a viewpoint focused on minimizing Mather measures, a similar strategy to the one inaugurated by Mather [27]. We actually do not want to discuss minimizing measures in the present paper and prove the existence of Mañé calibrated configurations in a more direct way.

Lemma 18. *Assume that E is a weakly twist interaction model such that $\inf_x E(x,x) > \bar{E}$. Then E admits increasing as well as decreasing Mañé calibrated configurations which have bounded jumps and are unbounded from above and below.*

Proof. We make use of $R, r > 0$, the constants that bound the successive jumps of calibrated configurations according to item iv of proposition 16. In particular, given $A \in \mathbb{R}$ with $|A| > R$, from proposition 16 we consider a subconfiguration $(x^A_p, \dots, x^A_q)$, with $p < 0 < q$, such that

$$\begin{aligned}
&|x^A_0| \le R, \\
&r \le |x^A_{k+1} - x^A_k| \le R \quad \forall\, k, \\
&S(-A, A) = E_{q-p}(x^A_p, \dots, x^A_q) - (q-p)\bar{E}.
\end{aligned}$$

Note that $(x^A_p, \dots, x^A_q)$ is increasing for $A > 0$ and decreasing for $A < 0$. By denoting $x^A_k := x^A_p = -A$ for all $k \le p$ and $x^A_k = x^A_q = A$ for all $k \ge q$, we have a configuration $(x^A_k)_{k\in\mathbb{Z}}$ that belongs to the compact set $\prod_{i\in\mathbb{Z}}\big[-(|i|+1)R, (|i|+1)R\big]$. Hence, as either $A \to +\infty$ or $A \to -\infty$, we are able to obtain an accumulation point $(y_k)_{k\in\mathbb{Z}}$. By taking into account a suitable subfamily, we may suppose that (y_k) is the limit of (x^A_k). Obviously (y_k) fulfills for all $k \in \mathbb{Z}$, $r \le |y_{k+1} - y_k| \le R$. It only remains to show that it is a calibrated configuration. However, thanks to the continuity of the Mañé potential, from item ii of proposition 16, it follows for all $i < j$,

$$\begin{aligned} S(y_i, y_j) &= \lim_A S(x_i^A, x_j^A) \\ &= \lim_A \left[E_{j-i}(x_i^A, \dots, x_j^A) - (j-i)\bar{E}\right] \\ &= E_{j-i}(y_i, \dots, y_j) - (j-i)\bar{E}. \end{aligned}$$

□

From the previous lemma, one has the following consequence of theorem 12.

Theorem 19. *Let $E(x, y)$ be a weakly twist interaction model satisfying $\inf_x E(x, x) > \bar{E}$. Then there exist a weak KAM solution u and a u-calibrated (and thus Mañé calibrated) configuration $(x_k)_{k\in\mathbb{Z}}$. There exist constants $K > 0$ and $0 < r < R$ such that every weak KAM solution u is Lipschitz with $\mathrm{Lip}(u) \leq K$, and every Mañé calibrated configuration $(x_k)_{k\in\mathbb{Z}}$ is strictly monotone and satisfies*

$$\forall\, k \in \mathbb{Z},\ r \leq |x_{k+1} - x_k| \leq R.$$

The results in the previous theorem and lemma 11 constitute the first two statements of theorem 5.

5. Linearly repetitive quasi-periodic sets

Our main goal is to show that, in the context of linearly repetitive quasi-periodic sets (see definition 20 below), all weak KAM solutions are of one three types, which may be described according to their kind of growth (linear *versus* sublinear), to the ordering of their calibration, or to the existence or not of calibrated configuration traversing the real line. Proposition 31 and corollary 32 gather the core of this classification. Along with proposition 33, they complete the statement of theorem 5. One of the essential ingredients to reach the classification is the notion of *fundamental configuration*, that is, a finite configuration that performs the minimum sum of a certain fixed number of interactions. In the linearly repetitive framework, with the hypothesis of non-degeneracy $\inf_{x\in\mathbb{R}} E(x, x) > \bar{E}$, all these fundamental configurations are shown to be ordered in the same way as long as a large enough number of interactions is considered. This define a preferred ordering. The linear repetitivity and the non-degeneracy hypotheses allow us to show that the Mañé potential has a sublinear growth according to this preferred ordering (see proposition 26). In fact, the understanding of the behavior of the Mañé potential against the ordering introduced by these sufficiently large fundamental configurations is the key element for the study of the possible types of weak KAM solutions.

We begin by reestablishing repetitivity, now in more quantitative terms.

Definition 20. A discrete set $\omega \subset \mathbb{R}$, is said *repetitive* if for every $R > 0$, there exists $M(R) > 0$ such that, for any open interval J of length at least

$M(R)$ and any pattern $\mathtt{P}$ of diameter at most R, there is $t \in \mathbb{R}$ for which $\mathtt{P}+t$ is a pattern of $\omega \cap J$. Besides, whenever there are positive constants A and B such that $M(R) \leq AR + B$ for all R, ω is said to be *linearly repetitive*.

First note that the repetitivity implies that the quasi-periodic set ω is *relatively dense*, *i.e.* there is no arbitrary large gap between consecutive elements. More quantitatively, there is a constant $R_\omega > 0$ such that

$$\omega \cap I \neq \emptyset \quad \text{for any interval } I \text{ of length greater than } R_\omega. \tag{5.1}$$

We assume from now on that the interaction is pattern equivariant with respect to a quasi-periodic set.

Definition 21. For a given interaction E, a *fundamental configuration of size* $n \geq 1$ is a finite sequence $(z_0, \ldots, z_n)$ such that

$$E(z_0, \ldots, z_n) = \min_{x_0, \ldots, x_n \in \mathbb{R}} E(x_0, \ldots, x_n).$$

We denote by $\Gamma_n(E) \subset \mathbb{R}^{n+1}$ the set of fundamental configurations of size n.

The above minimum exists because of the superlinearity of E. Moreover, by definition of $\bar{E}$, for any sequence of fundamental configurations $(z_0^n, \ldots, z_n^n)$

$$E(z_0^n, \ldots, z_n^n) \leq n\bar{E},$$

$$\lim_{n \to +\infty} \frac{1}{n} E(z_0^n, \ldots, z_n^n) = \sup_{n \to +\infty} \frac{1}{n} E(z_0^n, \ldots, z_n^n) = \bar{E}.$$

We first recall the notion of minimizing configurations which is weaker than Mañé calibration. In particular, there is no need to introduce the ground action $\bar{E}$.

Definition 22. Let E be an interaction model, $n \geq 1$, and $(x_0, \ldots, x_n)$ be a sequence of (possibly unordered) points of $\mathbb{R}$. The finite configuration $(x_0, \ldots, x_n)$ is said to be *minimizing* if, for every configuration $(y_0, \ldots, y_n)$ of the same cardinality and same extremities, $x_0 = y_0$ and $x_n = y_n$, one has

$$E(x_0, x_1, \ldots, x_n) \leq E(y_0, y_1, \ldots, y_n).$$

We show in the next lemma that minimizing configurations are strictly monotone provided that their endpoints are sufficiently far apart from each other. Recall that ς_0 is the constant that characterizes the pattern equivariance of $E(x, y)$ (see definition 4).

Lemma 23. *Let E be a weakly twist interaction model that is pattern equivariant with respect to a quasi-periodic set ω. Then there exists a constant $L > 0$ such that every minimizing configuration $(x_0, \ldots, x_n)$ satisfying $|x_n - x_0| \geq L$ is strictly monotone.*

Proof. The proof is by contradiction. To simplify, assume $x_0 < x_n$. In the case the configuration is not monotone, by lemma 23 in [22] there are indices $i_0 = 0 < i_1 < \cdots < i_\sigma = n$ such that $(x_{i_0}, x_{i_1}, \ldots, x_{i_\sigma})$ is strictly monotone and

$$\sum_{k=0}^{n-1} E(x_k, x_{k+1}) \geq \sum_{k=0}^{\sigma-1} E(x_{i_k}, x_{i_{k+1}}) + \sum_{j \notin \{i_0,\ldots,i_\sigma\}} E(x_j, x_j). \tag{5.2}$$

In the case the configuration is monotone but not strictly monotone, we obtain the existence of σ and $(i_0, \ldots, i_\sigma)$ as above but with an equality in (5.2) instead of an inequality. We now use the pattern equivariance of E to transport the points x_j, $j \notin \{i_0, \ldots, i_\sigma\}$, to new locations $\tilde{x}_j$ inside the interval (x_0, x_n). Let $\mathcal{P}$ be the set of all patterns of the form $\mathsf{P}_x := (x - R_\omega - \varsigma_0, x + R_\omega + \varsigma_0) \cap \omega$ where x is any point. By repetitivity, there exists $L > 0$ such that any interval of length L contains a translate of any pattern in $\mathcal{P}$. Then, for $x_n - x_0 \geq L$, there are $\tilde{x}_j$, $j \notin \{i_0, \ldots, i_\sigma\}$, such that

$$\mathsf{P}_{\tilde{x}_j} \subset (x_0, x_n) \qquad \text{and} \qquad \mathsf{P}_{\tilde{x}_j} - \tilde{x}_j = \mathsf{P}_{x_j} - x_j.$$

By pattern equivariance $E(x_j, x_j) = E(\tilde{x}_j, \tilde{x}_j)$.

Whenever $\tilde{x}_j \in (x_{i_s}, x_{i_{s+1}})$ for some j and s, by Aubry crossing Lemma

$$E(x_{i_s}, x_{i_{s+1}}) + E(\tilde{x}_j, \tilde{x}_j) > E(x_{i_s}, \tilde{x}_j) + E(\tilde{x}_j, x_{i_{s+1}}).$$

We may re-index the new set $\{x_{i_0}, \ldots, x_{i_\sigma}\} \cup \{\tilde{x}_j\}$ as $\{\tilde{x}_{i_0}, \ldots, \tilde{x}_{i_{\sigma+1}}\}$ and again apply again Aubry crossing Lemma to other points $\tilde{x}_k$ distinct from $\{\tilde{x}_{i_0}, \ldots, \tilde{x}_{i_{\sigma+1}}\}$. We finally obtain a new monotone sequence $(\tilde{x}_0, \ldots, \tilde{x}_n)$, with $\tilde{x}_0 = x_0$ and $\tilde{x}_n = x_n$, satisfying

$$\sum_{k=0}^{n-1} E(x_k, x_{k+1}) > \sum_{k=0}^{n-1} E(\tilde{x}_k, \tilde{x}_{k+1}).$$

The strict inequality shows that $(x_0, \ldots, x_n)$ is not minimizing. We have obtained a contradiction.

We are led to consider the situation in which all the new points $\tilde{x}_j$ belong to $\{x_{i_1}, \ldots, x_{i_{\sigma-1}}\}$. By re-indexing, one obtains a monotone but not strictly monotone configuration $x_0 = \tilde{x}_0 < \tilde{x}_1 \leq \ldots \leq \tilde{x}_{n-1} < \tilde{x}_n = x_n$ fulfilling

$$\sum_{k=0}^{n-1} E(x_k, x_{k+1}) \geq \sum_{k=0}^{n-1} E(\tilde{x}_k, \tilde{x}_{k+1}).$$

Proposition 25 of [22] implies that $(\tilde{x}_0, \ldots, \tilde{x}_n)$ is not minimizing, and therefore $(x_0, \ldots x_n)$ is not minimizing. We have reached again a contradiction. ☐

Note that $(z_0, \ldots, z_n) \in \Gamma_n(E)$ implies $E(z_0, \ldots, z_n) = \min_x T^n[0](x)$, where T stands for the Lax-Oleinik operator introduced in (2.1). In the next lemma, we guarantee that any configuration $(y_{-n}, \ldots, y_0)$, with endpoints sufficiently apart from each other, such that $E(y_{-n},\dots,y_0) = T^n[0](y_0)$ has (uniformly) bounded jumps.

Lemma 24. *Let E be a weakly twist interaction model that is pattern equivariant with respect to a quasi-periodic set ω. Then there exist constants $L > R > 0$ such that, for every configuration $(y_{-n},\dots,y_0)$ satisfying $|y_{-n} - y_0| \geq L$ and $E(y_{-n},\dots,y_0) = T^n[0](y_0)$,*

$$|y_{-k+1} - y_{-k}| \leq R, \quad \forall\, 1 \leq k \leq n. \tag{5.3}$$

Proof. Although we deal with a more general context, the proof is very similar to the one of proposition 39 in [22].

Part 1. We prove first an intermediate result: there exists a constant $R' > 0$ such that

$$|y_{-n+1} - y_{-n}| \leq R', \qquad |y_{-n+2} - y_{-n+1}| \leq R'.$$

For the first estimate, denoting $E^{sup} := \sup_{x\in\mathbb{R}} E(x,x)$, since $T^n[0](y_0) \leq E(y_{-n+1}, y_{-n+1}, y_{-n+2},\dots,y_0)$, one has

$$E(y_{-n},y_{-n+1}) - E^{sup} \leq E(y_{-n+1},y_{-n+1}) - E^{sup} \leq 0.$$

With respect to the second estimate, introducing $E^{inf} := \inf_{x,y\in\mathbb{R}} E(x,y)$, note that $T^n[0](y_0) \leq E(y_{-n+2}, y_{-n+2}, y_{-n+2}, y_{-n+3},\dots,y_0)$ obviously implies $E(y_{-n},y_{-n+1},y_{-n+2}) \leq E(y_{-n-2},y_{-n+2},y_{-n+2})$, so that

$$E(y_{-n+1},y_{-n+2}) - E^{sup} \leq E^{sup} - E^{inf}.$$

Superlinearity ensures there is $R' > E^{sup} - E^{inf}$ such that

$$|x-y| > R' \;\Rightarrow\; E(x,y) - E^{sup} > |x-y|.$$

Therefore, we necessarily have $|y_{-n+1} - y_{-n}| \leq R'$ and $|y_{-n+2} - y_{-n+1}| \leq R'$.

Part 2. Lemma 23 shows that $(y_{-n},\dots,y_0)$ is strictly monotone. To fix ideas, suppose that $(y_{-n},\dots,y_0)$ is increasing. Let I denote the interval $(y_{-n} - R_\omega - \varsigma_0, y_{-n+2} + R_\omega + \varsigma_0)$ and

$$s := 2R' + 2R_\omega + 2\varsigma_0 \geq |y_{-n+2} - y_{-n}| + 2R_\omega + 2\varsigma_0.$$

By repetitivity, any interval of length at least $M(s)$ contains a translate $I + t$, $\omega \cap (I+t) = (\omega \cap I) + t$, and by pattern equivariance,

$$\forall\, x,y \in [y_{-n}, y_{-n+2}], \qquad E(x+t,y+t) = E(x,y).$$

Define $R := M(s)$. We claim that $|y_{-k+1} - y_{-k}| \leq R$ for every $1 \leq k \leq n-3$. The proof is by contradiction. Indeed, if this is not the case, based on the foregoing there exists $t \geq 0$ such that

$$\begin{cases} [y_{-n} - R_\omega - \varsigma_0, y_{-n+2} + R_\omega + \varsigma_0] + t \subseteq (y_{-k}, y_{-k+1}), \\ \forall\, x,y \in [y_{-n}, y_{-n+2}], \quad E(x+t,y+t) = E(x,y). \end{cases}$$

Aubry crossing lemma (lemma 22 of [22]) shows that

$$\begin{aligned}E(y_{-k}, y_{-k+1}) + E(y_{-n}, y_{-n+1}, y_{-n+2}) &= \\ &= E(y_{-k}, y_{-k+1}) + E(y_{-n} + t, y_{-n+1} + t, y_{-n+2} + t) \\ &> E(y_{-k}, y_{-n+1} + t, y_{-k+1}) + E(y_{-n} + t, y_{-n+2} + t) \\ &= E(y_{-k}, y_{-n+1} + t, y_{-k+1}) + E(y_{-n}, y_{-n+2}).\end{aligned}$$

Shifting y_{-n+1} to the position $y_{-n+1} + t$, one obtains

$$\begin{aligned}T^n[0](y_0) &= E(y_{-n}, y_{-n+1}, y_{-n+2}, \dots, y_{-k}, y_{-k+1}, \dots, y_0) \\ &> E(y_{-n}, y_{-n+2}, \dots, y_{-k}, y_{-n+1} + t, y_{-k+1}, \dots, y_0).\end{aligned}$$

We have obtained a configuration of $n+1$ points ending at y_0 that decreases strictly $T^n[0](y_0)$. That contradicts the optimality of $(y_{-n}, \dots, y_0)$. □

We gather in the following lemma several conclusions that are proved in the lemmas 41 and 42 in [22]. We actually simplify the proof and we only use the results of that work exclusively related to the twist condition to obtain the lemma below in a more general framework.

Lemma 25. *Let E be a weakly twist interaction that is pattern equivariant with respect to a quasi-periodic set ω. Assume $\inf_x E(x,x) > \bar{E}$. Then there exist constants $\phi > 0$, $R > 0$ and an integer $N > 0$ such that, for every $n \geq 1$, for every $(z_0, \dots, z_n) \in \Gamma_n(E)$,*

(i) $|z_n - z_0| \geq n\phi$,
(ii) $\forall\, 0 \leq i < n,\ |z_{i+1} - z_i| \leq R$,
(iii) if $n \geq N$, then $(z_0, \dots, z_n)$ is strictly monotone.

Proof.
Item i. Inequality (4.2) shows that

$$n \leq A_0|z_n - z_0| + B_0\big[E(z_0, \dots, z_n) - n\bar{E}\big].$$

As $(z_0, \dots, z_n)$ is a fundamental configuration, one has $E(z_0, \dots, z_n) \leq n\bar{E}$, so that $|z_n - z_0| \geq n\phi$, with $\phi = 1/A_0$.

Item ii. For $L > 0$ as in lemma 24, denote $N := \lceil \frac{L}{\phi} \rceil$. If $n \geq N$, then $|z_n - z_0| \geq L$ and item ii is a consequence of lemma 24. If $n \leq N$, let $E^{sup} := \sup_x E(x,x)$ and $E^{inf} := \inf_{x,y} E(x,y)$. The superlinearity provides the existence of $R > 0$ such that $|x - y| > R$ implies

$$E(x,y) > N(E^{sup} - E^{inf}) + E^{sup}.$$

By contradiction, assume $|z_{-k+1} - z_{-k}| > R$ for some $1 \leq k \leq n$. In particular, $E(z_{-k}, z_{-k+1}) - E^{sup} > N(E^{sup} - E^{inf})$. Then, using the *a priori* bound $E(z_{-\ell}, z_{-\ell+1}) \geq E^{inf}$ for $\ell \neq k$, as well as $\bar{E} \leq E^{sup}$, we obtain the contradiction

$$0 \geq E(z_0, \dots, z_n) - n\bar{E} > (N - n + 1)(E^{sup} - E^{inf}) \geq 0.$$

Item iii. If $n \geq N$, then $|z_n - z_0| \geq L$ and $(z_0, \ldots, z_n)$ is strictly monotone thanks to lemma 24. □

In the next result, we highlight fundamental properties of the growth of the Mañé potential when the quasi-periodic set is linearly repetitive (recall definition 20). Notice that item iii introduces a dichotomy on the order of fundamental configurations of large size.

Proposition 26. *Let $E(x, y)$ be a weakly twist interaction that is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω. Assume that $\inf_x E(x, x) > \bar{E}$. Suppose there exists a sequence $\{K(n)\}_n$ of positive integers diverging to infinite for which, associated with each n, there is an increasing fundamental configuration $\underline{z}^n = (z_0^n, \ldots, z_{K(n)}^n)$. Then*

(i) the Mañé potential has sublinear growth for positively ordered variables

$$\forall\, \alpha > 0,\ \exists\, \beta \geq 0,\ \forall\, x \leq y \in \mathbb{R},\ |S(x, y)| \leq \alpha|y - x| + \beta;$$

(ii) the Mañé potential grows linearly for negatively ordered variables

$$\exists\, \gamma > 0,\ \delta \geq 0,\ \forall\, x \geq y,\ S(x, y) \geq \gamma|y - x| - \delta.$$

(iii) for m large enough, any fundamental configuration of size m is increasing.

Similarly if there exists a sequence of decreasing fundamental configurations whose sizes tend to infinity, then the Mañé potential has sublinear growth for negatively ordered variables

$$\forall\, \alpha > 0,\ \exists\, \beta \geq 0,\ \forall\, x \geq y \in \mathbb{R},\ |S(x, y)| \leq \alpha|y - x| + \beta,$$

the Mañé potential grows linearly for positively ordered variables

$$\exists\, \gamma > 0,\ \delta \geq 0,\ \forall\, x \leq y,\ S(x, y) \geq \gamma|y - x| - \delta,$$

and all sufficiently long fundamental configuration is decreasing.

Proof. We prove the case where fundamental configurations are increasing. The other case will be deduced from the symmetric interaction $\hat{E}(x, y) = E(-x, -y)$.

Item i. Proposition 16 shows that S admits negative sublinear lower bounds. It is enough to show that S also admits positive sublinear upper bounds.

Let $C > 0$ be the constant in lemma 13 that gives *a priori* growth of S, that is,

$$\forall\, x, y \in \mathbb{R},\ |S(x, y)| \leq C(|y - x| + 1).$$

The repetitivity assumption says there exist constants $A, B > 0$ such that

$$\forall\, R > 0,\ M(R) \leq AR + B,$$

where M is the repetitivity function introduced in definition 20. As we will avoid working with overlaps, there is no loss of generality in assuming $A > 1$. Denote

$$\alpha_* := \inf\big\{\alpha > 0 : \exists\, \beta > 0,\ \forall\, x \le y,\ S(x,y) \le \alpha|y-x| + \beta\big\}.$$

We want to show that $\alpha_* = 0$. By contradiction, assume $\alpha_* > 0$. Let $\alpha \in (\alpha_*, 2\alpha_*)$. We will reach an absurd by considering a large index n (to be completely defined later) and a corresponding fundamental configuration $(z_0^n, \dots, z_{K(n)}^n)$. Initially, applying lemma 25, we require that n be large enough so that $(z_0^n, \dots, z_{K(n)}^n)$ is strictly increasing and, for some $\phi > 0$ (see Lemma 25 for the definition of ϕ),

$$z_{K(n)}^n - z_0^n \ge \phi K(n) > 2\varsigma_0.$$

Denote

$$I_0 := (z_0^n, z_{K(n)}^n) \quad \text{and} \quad \mathtt{P} := (z_0^n - R_\omega, z_{K(n)}^n + R_\omega) \cap \omega.$$

Note that the pattern $\mathtt{P}$ has diameter $\ell = z_{K(n)}^n - z_0^n + 2R_\omega$ greater than $2\varsigma_0$. By repetitivity, we may find a sequence $(t_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ such that, for all k, $\mathtt{P} + t_k$ is a pattern of ω and

$$0 < \min(\mathtt{P} + t_{k+1}) - \max(\mathtt{P} + t_k) \le M(\ell) - \ell.$$

Define $(a_k, b_k) = I_k := I_0 + t_k$ and let $I_k' = (b_k, a_{k+1})$ be the interval in between I_k and I_{k+1}. Then

$$\begin{aligned} |I_k'| &\le M(\ell) - \ell + 2(\varsigma_0 + R_\omega) \\ &\le (A-1)\ell + B + 2(\varsigma_0 + R_\omega) = A'|I_k| + B', \end{aligned}$$

where $A' = A - 1$ and $B' = 2A(\varsigma_0 + R_\omega) + B$. Note thus that, for $p < q$,

$$|a_q - a_p| = \sum_{k=p}^{q-1} \big(|I_k| + |I_k'|\big) \le A \sum_{k=p}^{q-1} |I_k| + B'(q-p).$$

By pattern equivariance,

$$\begin{aligned} S(a_k, b_k) &\le E(z_0^n + t_k, \dots, z_{K(n)}^n + t_k) - K(n)\bar{E} \\ &= E(z_0^n, \dots, z_{K(n)}^n) - K(n)\bar{E} \le 0. \end{aligned}$$

By sub-additivity of S,

$$S(a_p, a_q) \le \sum_{k=p}^{q-1} [S(a_k, b_k) + S(b_k, a_{k+1})] \le \sum_{k=p}^{q-1} S(b_k, a_{k+1}).$$

By the choice of α, there is $\beta > 0$ such that

$$S(b_k, a_{k+1}) \le \alpha|I_k'| + \beta.$$

These estimates provide

$$\begin{aligned} S(a_p, a_q) + \frac{\alpha}{A}|a_q - a_p| &\le \sum_{k=p}^{q-1} S(b_k, a_{k+1}) + \alpha \sum_{k=p}^{q-1} |I_k| + \frac{\alpha B'}{A}(q-p) \\ &\le \alpha|a_q - a_p| + \Big(\frac{\alpha B'}{A} + \beta\Big)(q-p). \end{aligned}$$

Since $|a_q - a_p| \ge \sum_{k=p}^{q-1} |I_k| = (q-p)|I_0|$, we obtain

$$S(a_p, a_q) \le \Big(\frac{\alpha A'}{A'+1} + \Big(\frac{\alpha B'}{A'+1} + \beta\Big)\frac{1}{|I_0|}\Big)|a_q - a_p|.$$

The distance between a_k and a_{k+1} is at most

$$|I_k| + |I'_k| \le (A'+1)|I_0| + B' := H_0.$$

If $y - x > H_0$ are any given points, we choose $p < q$ such that $x \in \overline{I_p} \cup I'_p$ and $y \in \overline{I_{q-1}} \cup I'_{q-1}$. Hence, by the sub-additivity and the *a priori* growth of the Mañé potential,

$$\begin{aligned} S(x,y) &\le S(x, a_p) + S(a_p, a_q) + S(a_q, y) \\ &\le \Big(\frac{\alpha A'}{A'+1} + \Big(\frac{\alpha B'}{A'+1} + \beta\Big)\frac{1}{|I_0|}\Big)|a_q - a_p| + 2C(H_0 + 1) \\ &\le \alpha'|a_q - a_p| + \beta', \end{aligned}$$

with

$$\alpha' = \frac{\alpha A'}{A'+1} + \Big(\frac{2\alpha_* B'}{A'+1} + \beta\Big)\frac{1}{|I_0|} \quad \text{and} \quad \beta' = 2C(H_0+1).$$

Suppose first $y - x > nH_0$. Then, using $0 \le x - a_p \le H_0$ and $0 \le a_q - y \le H_0$, we see that $|a_q - a_p| \le \big(1 + \frac{2}{n}\big)|y - x|$ and therefore

$$S(x,y) \le \alpha'\Big(1 + \frac{2}{n}\Big)|y - x| + \beta'.$$

If however $y - x \le nH_0$, then

$$S(x,y) \le C(nH_0 + 1) \le C(nH_0 + 1) + \beta' =: \beta''.$$

We focus on a strictly bigger constant $\frac{A'}{A'+1} < \frac{2A'}{2A'+1} < 1$ to choose α sufficiently close to α_* and then n large enough so that

$$\alpha'' := \Big(\frac{A'}{A'+1}\alpha + \Big(\frac{2\alpha_* B'}{A'+1} + \beta\Big)\frac{1}{\phi K(n)}\Big)\Big(1 + \frac{2}{n}\Big) < \frac{2A'}{2A'+1}\alpha_*.$$

We have obtained two constants $0 < \alpha'' < \alpha_*$ and $\beta'' > 0$ such that

$$\forall\, x \le y,\ S(x,y) \le \alpha''|y - x| + \beta''.$$

The existence of α'' contradicts the definition of α_*. We have thus proved that $\alpha_* = 0$.

Item ii. Let $x > y$. Item i of proposition 16 shows that there exist $n \ge 1$ and a strictly decreasing sequence $(y_0, \ldots, y_n)$, with $y_0 = x$ and $y_n = y$, such that $S(x,y) = E(y_0, \ldots, y_n)$. Lemma 18 shows that there exists a strictly increasing Mañé calibrated configuration $(x_k)_{k\in\mathbb{Z}}$ with bounded jumps that is unbounded from above and bellow. Let $x_i \le y$ be the largest point of this configuration less than or equal to y. Let $x_j \le x$ be defined similarly. Then $i \le j$. If $i = j$, item iii of lemma 14 provides $S(x_i, x_j) = E(x_i, x_j) - \bar{E}$. Otherwise, by calibration $S(x_i, x_j) = E(x_i, \dots, x_j) - (j-i)\bar{E}$. Consider now the configuration $(y_0, \dots, y_n, x_i, \dots, x_j, y_0)$. Then lemma 23 in [22] guarantees that

$$S(x,y) + \big(E(y,x_i) - \bar{E}\big) + S(x_i,x_j) + \big(E(x_j,x) - \bar{E}\big) \ge \\ \ge (n+j-i+2)\big(\inf_{x\in\mathbb{R}} E(x,x) - \bar{E}\big).$$

Since S has sublinear growth for positively ordered variables thanks to the previous item, for $\alpha > 0$ (to be chosen later), there exists $\beta \ge 0$ such that

$$S(x_i, x_j) \le \alpha |x_j - x_i| + \beta.$$

As the jumps are bounded, $|x_j - x_i| \le R(j-i)$, we thus have

$$S(x,y) + \alpha|x_j - x_i| + \beta \ge 2\gamma |x_j - x_i| - \delta',$$

with $2\gamma := \inf_{x\in\mathbb{R}}(E(x,x) - \bar{E})/R$ and $\delta' := 2\sup_{|y-x|\le R}\big(E(x,y) - \bar{E}\big)$. We conclude by choosing $\alpha = \gamma$ and $\delta = \delta' + \beta + \gamma R$, so that

$$S(x,y) \ge \gamma|y-x| - \delta.$$

Item iii. Let $m \ge N$ (where N is given in lemma 25) and $(z_0, \dots, z_m)$ be a decreasing fundamental configuration. Let $R > 0$ be the constant given in lemma 25 and

$$\alpha := \frac{1}{2R}\big(\inf_{x\in\mathbb{R}} E(x,x) - \bar{E}\big).$$

Then on the one hand, thanks to item i, there exists $\beta \ge 0$ such that

$$S(z_m, z_0) \le \alpha|z_0 - z_m| + \beta.$$

On the other hand, thanks to item i of proposition 16, one can find an increasing configuration $(x_0, \dots, x_n)$, with $x_0 = z_m$ and $x_n = z_0$, such that

$$S(z_m, z_0) = E(x_0, \dots, x_n) - n\bar{E}.$$

Using item 1 of lemma 23 in [22], we obtain

$$\begin{aligned} S(z_m, z_0) &\ge S(z_m, z_0) + E(z_0, \dots, z_m) - m\bar{E} \\ &= E(x_0, \dots, x_n, z_1, \dots, z_m) - (m+n)\bar{E} \\ &\ge (m+n)\big(\inf_{x\in\mathbb{R}} E(x,x) - \bar{E}\big) \ge m\big(\inf_{x\in\mathbb{R}} E(x,x) - \bar{E}\big). \end{aligned}$$

As item ii of lemma 25 implies $|z_0 - z_m| \le mR$, the choice of α is contradicted by the inequality

$$\alpha m R + \beta \ge m\big(\inf_{x\in\mathbb{R}} E(x,x) - \bar{E}\big)$$

for m large enough. □

Note that the proof makes extensive use of the fact that the growth rate at infinity of the repetititvity function is linear whereas that of the Mañé potential S is at most linear. We leave open the cases of other growth orders for the repetitivity function, which would, *a priori*, need stronger conditions on the Mañé potential to get a similar result. Recall nevertheless the linear growth rate of the repetitivity function is the smallest one among aperiodic sets [1].

The previous result makes it clear that, in the linearly repetitive context, the ordering of arbitrarily long fundamental configurations plays a key role, thus introducing a preferential ordering to the model.

Definition 27. Suppose that $E(x, y)$ is a weakly twist interaction which fulfills $\inf_x E(x, x) > \bar{E}$, and is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω.

- (i) *The preferred ordering* is the ordering given by sufficiently long fundamental configurations.
- (ii) A monotone configuration is said to be *compatible with the preferred ordering* or for short *compatible* if the configuration is ordered as any fundamental configurations of size sufficiently large. Otherwise, the configuration is said to be *anti-compatible with the preferred ordering* or for short *anti-compatible*.

We now classify the set of weak KAM solutions u. There are three approaches: a classification using the type of growth (sublinear *versus* linear), a classification using the ordering of u-calibrated subconfigurations, and a classification using bi-infinite u-calibrated configurations. We recall that any calibrated configuration for a weak KAM solution is also Mañé calibrated and therefore strictly monotone with a minimal spacing as stated by proposition 16.

Lemma 28. *Let $E(x, y)$ be a weakly twist interaction that is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω. Suppose that $\inf_x E(x, x) > \bar{E}$. There exists $L > 0$ such that, given any weak KAM solution u and two points $x_0 > y_0 + L$, there cannot exist simultaneously an increasing u-calibrated configuration ending at x_0 and a decreasing u-calibrated configuration ending at y_0.*

Proof. Suppose that $(x_{-m}, \ldots, x_0)$ is increasing, $(y_{-n}, \ldots, y_0)$ is decreasing, with $x_0 > y_0$, and that m and n have been chosen so that $y_0 - R \leq x_{-m} \leq y_0$ and $x_0 \leq y_{-n} \leq x_0 + R$. From item ii of proposition 26, we have

$$
\begin{aligned}
u(y_0) - u(y_{-n}) &= S(y_{-n}, y_0) \geq \gamma|y_{-n} - y_0| - \delta \\
&\geq \gamma|x_0 - x_{-m}| - 2\gamma R - \delta \geq \gamma(x_0 - y_0) - 2\gamma R - \delta.
\end{aligned}
$$

Recall any weak KAM solution is Lipschitz continuous, with Lipschitz constant bounded by some fixed value K (lemma 11). Hence, applying proposition 26 with $\alpha = \gamma/2$ we see that for some $\beta > 0$,

$$\begin{aligned} u(y_0) - u(y_{-n}) &\leq 2KR + u(x_{-m}) - u(x_0) = 2KR - S(x_{-m}, x_0) \\ &\leq 2KR + |S(x_{-m}, x_0)| \leq 2KR + \frac{\gamma}{2}|x_0 - x_{-m}| + \beta \\ &\leq 2KR + \frac{\gamma}{2}(x_0 - y_0) + \frac{\gamma}{2}R + \beta. \end{aligned}$$

Therefore,

$$x_0 - y_0 \leq \frac{2}{\gamma}\Big(2KR + \frac{\gamma}{2}R + \beta + 2\gamma R + \delta\Big).$$

It is then enough to take $L := \lceil 5R + (4KR + 2\beta + 2\delta)/\gamma \rceil + 1$. ☐

We introduce vocabulary to quickly refer to possible classes of solutions.

Definition 29. Let $u : \mathbb{R} \to \mathbb{R}$ be a function. We say that

(i) u is of type I (or u has *a sublinear variation*) if

$$\forall \alpha > 0,\ \exists \beta \geq 0,\ \forall x, y \in \mathbb{R},\ |u(x) - u(y)| \leq \alpha|x - y| + \beta; \tag{5.4}$$

(ii) u is of type II (or u is *linearly decreasing with respect to the preferred ordering*) if

$$\exists \gamma, \delta > 0,\ \forall x, y \in \mathbb{R},\ \big[x \text{ precedes } y \ \Rightarrow\ u(y) - u(x) \leq -\gamma|x - y| + \delta\big];$$

(iii) u is of type III (or u is of mixed type) it it is of type I on points that precede 0 and of type II on points that succeed 0 according to the preferred ordering.

Proposition 30. *Let $E(x, y)$ be a weakly twist interaction that is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω. Suppose that $\inf_x E(x, x) > \bar{E}$. Let u be a weak KAM solution. Then, the following properties hold.*

(i) If u is of type I, any u-calibrated configuration is compatible.
(ii) If u is of type II, any u-calibrated configuration is anti-compatible.
(iii) If u is of type III, there is $T \geq 0$ such that any u-calibrated configuration ending at $x \leq -T$ is increasing and ending at $x \geq T$ is decreasing.

An equivalent way to describe the role of T in the last item above is to say that, according to the preferred ordering, any u-calibrated configuration that ends at a point preceding 0 by at least T is compatible, as well as any u-calibrated configuration that ends at a point succeeding 0 by at least T is anti-compatible.

Proof. Assume the preferred ordering is the increasing order.

Item i. By contradiction, assume (x_{-1}, x_0) is u-calibrated and decreasing. We can extend it to an arbitrarily large decreasing u-calibrated configuration $(x_{-n}, \ldots, x_{-1}, x_0)$. From item ii of proposition 26, there are $\gamma_1 > 0$ and $\delta_1 \geq 0$ fulfilling

$$u(x_0) - u(x_{-n}) = S(x_{-n}, x_0) \geq \gamma_1|x_{-n} - x_0| - \delta_1.$$

However, as u is of type I, it has sublinear variation, so that for $\alpha = \gamma_1/2$ there is $\beta_1 \ge 0$ such that

$$|u(x_0) - u(x_{-n})| \le \frac{\gamma_1}{2}|x_{-n} - x_0| + \beta_1.$$

As $|x_{-n} - x_0| \to \infty$ as $n \to \infty$, we have reached a contradiction.

Item ii. Suppose, by reduction to the absurd, that (x_{-1}, x_0) is an increasing u-calibrated pair, and extend it to an arbitrarily large increasing u-calibrated configuration $(x_{-n}, \dots, x_{-1}, x_0)$. As u is of type II, there are $\gamma_2 > 0$ and $\delta_2 \ge 0$

$$-S(x_{-n}, x_0) = u(x_{-n}) - u(x_0) \ge \gamma_2|x_0 - x_{-n}| - \delta_2.$$

Since S is sublinear for positively ordered variables, for some $\beta_2 > 0$,

$$|S(x_{-n}, x_0)| \le \frac{\gamma_2}{2}|x_0 - x_{-n}| + \beta_2.$$

We obtain a contradiction similar to the first item.

Item iii. Let us consider T of the form $T := RN$, where $R > 0$ is the upper bounded for successive jumps of Mañé calibrated configurations obtained in proposition 16 and $N > 1$ will be chosen later. Assume that (x_{-1}, x_0) is a decreasing u-calibrated pair such that $x_0 \le -T$. Extend it to a decreasing u-calibrated configuration $(x_{-N}, \dots, x_{-1}, x_0)$. In particular, $x_{-N} \le 0$. As u is of type I on $\mathbb{R}^-$, the computations in the first part imply that

$$Nr \le |x_{-N} - x_0| \le \frac{2}{\gamma_1}(\beta_1 + \delta_1).$$

Denote then $N_1 := \lceil \frac{2}{\gamma_1 r}(\beta_1 + \delta_1) \rceil + 1$. Similarly, if $(x_{-N}, \dots, x_0)$ is increasing u-calibrated configuration such that $x_0 \ge T$, one obtains $x_{-N} \ge 0$. As u is of type II on $\mathbb{R}^+$, the computations of the second part lead us to take into account $N_2 := \lceil \frac{2}{\gamma_2 r}(\beta_2 + \delta_2) \rceil + 1$. It is enough to choose $N := \max\{N_1, N_2\}$. □

Proposition 31. *Let $E(x, y)$ be a weakly twist interaction that is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω. Suppose that $\inf_x E(x, x) > \bar{E}$. Then*

- *(i) there are examples of weak KAM solutions of each of the three types.*
- *(ii) every weak KAM solution u is of one of three types:*
  - *(a) u is of type I if there exists a bi-infinite u-calibrated compatible configuration;*
  - *(b) u is of type II if there exists a bi-infinite u-calibrated anti-compatible configuration;*
  - *(c) u is of type III if there is no bi-infinite u-calibrated configuration.*

Proof. Assume for instance the preferred ordering is the increasing order. The case of decreasing order is similar.

Part 1. We prove the existence of weak KAM solutions of type I and II. Lemma 18 shows that one can always ensure the existence of an increasing as well as a decreasing calibrated configuration as required to apply theorem 12. We claim that the resulting weak KAM solutions are of type I and II, respectively. Recall from lemma 11 that any weak KAM solution is Lipschitz continuous, with a Lipschitz constant that only depends on the interaction model E. Recall also that the Mañé potential is Lipschitz continuous as stated in corollary 17.

The Lipschitz weak KAM solution u_I obtained from the increasing calibrated configuration inherits the sublinear growth for positively ordered variables of the Mañé potential, and it is thus of type I. More concretely, for such a solution u_I we have

$$\forall\, k < \ell, \quad u_I(x_\ell) - u_I(x_k) = S(x_k, x_\ell)$$

along the increasing Mañé calibrated configuration $(x_n)_{n\in\mathbb{Z}}$. Let $R > 0$ denote an upper bound for the successive jumps of a calibrated configuration. Given $y - x > 2R$, we may choose k and ℓ with $x \in [x_k, x_{k+1})$ and $y \in (x_{\ell-1}, x_\ell]$ so that

$$\begin{aligned} |u_I(x) - u_I(y)| &\leq |S(x_k, x_\ell)| + 2\mathrm{Lip}(u_I)R \\ &\leq |S(x, y)| + 2\big(\mathrm{Lip}(u_I) + \mathrm{Lip}(S)\big)R. \end{aligned}$$

Item i of proposition 26 shows that the Mañé potential has sublinear growth in this situation, we immediately conclude that u_I has sublinear variation.

By its turn, the solution u_{II} obtained from the decreasing calibrated configuration is linearly decreasing as a consequence of the behavior of the Mañé potential for negatively ordered variables. In more precise terms, u_{II} satisfies

$$\forall\, k < \ell, \quad u_{II}(y_\ell) - u_{II}(y_k) = S(y_k, y_\ell),$$

where $(y_n)_{n\in\mathbb{Z}}$ is a particular decreasing Mañé calibrated configuration. Similarly as above, for $y - x > 2R$ one may find $\ell > k$ such that

$$S(y, x) - 2\big(\mathrm{Lip}(u_{II}) + \mathrm{Lip}(S)\big)R \leq S(y_k, y_\ell) - 2\mathrm{Lip}(u_{II})R \leq u_{II}(x) - u_{II}(y).$$

The fact that u_{II} is of type II follows thus from item ii of proposition 26.

Part 2. We prove the existence of a weak KAM solution of type III. Let v_I and v_{II} be weak KAM solutions of type I and II, respectively. We may assume $v_I(0) = v_{II}(0) = 0$. Define $v_{III} := \min\{v_I, v_{II}\}$. Then v_{III} is again a weak KAM solution. Let $\gamma_{II}, \delta_{II} > 0$ be constants used to describe v_{II} as of type II. Since v_I is of type I, let $\beta_I > 0$ be the corresponding constant associated with $\alpha_I := \gamma_{II}/2$. For every $x \geq 2(\beta_I + \delta_{II})/\gamma_{II}$, we have the following inequalities

$$v_I(x) = v_I(x) - v_I(0) \geq -\alpha_I x - \beta_I,$$
$$v_{II}(x) = v_{II}(x) - v_{II}(0) \leq -\gamma_{II} x + \delta_{II},$$

which yield $v_{II}(x) \leq v_I(x) - \frac{\gamma_{II}}{2}x + \beta_I + \delta_{II} \leq v_I(x)$. Note that whenever $x \leq -2(\beta_I + \delta_{II})/\gamma_{II}$, we get

$$v_I(x) \leq -\alpha_I x + \beta_I, \quad v_{II}(x) \geq -\gamma_{II} x - \delta_{II},$$

so that $v_I(x) \leq v_{II}(x) + \frac{\gamma_{II}}{2}x + \beta_I + \delta_{II} \leq v_{II}(x)$. To simplify, denote then $T := 2(\beta_I + \delta_{II})/\gamma_{II}$. Let $K \geq 0$ be the Lipschitz constant of all weak KAM solutions (see lemma 11). If $0 \leq x \leq y$, let $x_T = \max\{T, x\}$, $y_T = \max\{T, y\}$. Then $T \leq x_T \leq y_T$ and

$$\begin{aligned}
v_{III}(y) - v_{III}(x) &\leq v_{III}(y_T) - v_{III}(x_T) + 2KT \\
&= v_{II}(y_T) - v_{II}(x_T) + 2KT \\
&\leq -\gamma_{II}|y_T - x_T| + \delta_{II} + 2KT \\
&\leq -\gamma_{II}|y - x| + \delta_{II} + 2T(K + \gamma_{II}).
\end{aligned}$$

We have proved that v_{III} is of type II on $\mathbb{R}^+$. Given any $\alpha > 0$, there is $\beta \geq 0$ such that

$$\forall\, x, y \in \mathbb{R}, \ |v_I(x) - v_I(y)| \leq \alpha|x - y| + \beta.$$

If $x \leq y \leq 0$, let $x_T = \min\{x, -T\}$, $y_T = \min\{y, -T\}$. Then $x_T \leq y_T \leq -T$ and

$$\begin{aligned}
|v_{III}(x) - v_{III}(y)| &\leq |v_{III}(x_T) - v_{III}(y_T)| + 2KT \\
&= |v_I(x_T) - v_I(y_T)| + 2KT \\
&\leq \alpha|x_T - y_T| + \beta + 2KT \leq \alpha|x - y| + \beta_{III},
\end{aligned}$$

with $\beta_{III} = \beta + 2T(K + \alpha)$. We have proved that v_{III} is of type I on $\mathbb{R}^-$.

Part 3. Conversely, we prove that every weak KAM solution u is one of the three types. Let $\mathcal{B}_u^-$ be the set of points $x \in \mathbb{R}$ such that all u-calibrated configurations ending at x are increasing. Let $\mathcal{B}_u^+$ be the set of points x for which all u-calibrated configurations ending at x are decreasing. From lemma 28, note that $\sup(\mathbb{R} \setminus \mathcal{B}_u^+) \leq \inf \mathcal{B}_u^+ + L$. We discuss thus three possibilities.

Case $\mathcal{B}_u^+ = \mathbb{R}$. For every $x \in \mathbb{R}$, all u-calibrated configurations ending at x are decreasing. Given $A > 0$, one constructs a decreasing u-calibrated configuration ending at $-A$ of size n sufficiently large, $-A = x_0 < x_{-1} < \cdots < x_{-n}$, where $x_{-n} > A$ and $r \leq |x_{-k} - x_{-k+1}| \leq R$ for every k. By re-indexing, one has a family of sequences $x_{i_N}^N < \cdots < x_{-j_N}^N$ with $|x_0^N| \leq R$ and $i_N, j_N \to +\infty$. Using a diagonal extraction, one obtains a bi-infinite decreasing u-calibrated configuration $(x_k)_{k\in\mathbb{Z}}$ with bounded jumps and a minimal spacing. For every $x < y - R$, we choose $k \leq \ell$ so that $|x_\ell - x| \leq R$, $|y - x_k| \leq R$. Let $\gamma, \delta > 0$ be given as in item ii of proposition 26. Using the Lipschitz constant K of u and the fact that $(x_k)_{k\in\mathbb{Z}}$ is Mañé calibrated, it follows that

$$\begin{aligned} u(x) - u(y) &\geq u(x_\ell) - u(x_k) - 2KR = S(x_k, x_\ell) - 2KR \\ &\geq \gamma|x_k - x_\ell| - \delta - 2KR \geq \gamma|y - x| - \delta - 2R(\gamma + K). \end{aligned}$$

We have proved that u is of type II.

Case $\mathbb{R} \setminus \mathcal{B}_u^+ \neq \emptyset$ is unbounded from above. Note that $\mathbb{R} \setminus \mathcal{B}_u^+$ is the set of points x for which there exists an increasing u-calibrated configuration ending at x. When it is unbounded from above, lemma 28 ensures that $\mathcal{B}_u^- = \mathbb{R}$ and hence all u-calibrated configurations are increasing. One can construct as above a strictly increasing bi-infinite u-calibrated configuration $(x_k)_{k\in\mathbb{Z}}$ that is also Mañé calibrated. Let K denote the Lipschitz constant of u. For every $x < y - R$, if $k \leq \ell$ is chosen so that $|x - x_k| \leq R$ and $|y - x_\ell| \leq R$, for every $\alpha > 0$ and $\beta \geq 0$ given by proposition 26, one obtains

$$\begin{aligned} |u(y) - u(x)| &\leq |u(x_\ell) - u(x_k)| + 2KR = |S(x_k, x_\ell)| + 2KR \\ &\leq \alpha|x_\ell - x_k| + \beta + 2KR \leq \alpha|y - x| + \beta + 2R(\alpha + K). \end{aligned}$$

We have proved that u is type I.
Case $b := \sup(\mathbb{R} \setminus \mathcal{B}^+(u)) \in \mathbb{R}$. Let $a := b - L$, where $L > 0$ is the constant obtained in lemma 28. Then any u-calibrated configuration ending at $x_0 < a$ is increasing, and any u-calibrated configuration ending at $y_0 > b$ is decreasing. In particular, there is no bi-infinite u-calibrated configuration. Fix once for all an increasing u-calibrated configuration $(x_{-k})_{k\geq 0}$ ending at a point $x_0 < \min\{a, 0\}$ and a decreasing u-calibrated configuration $(y_k)_{k\geq 0}$ ending at a point $y_0 > \max\{b, 0\}$. Both are Mañé calibrated. For $x < y - R$ with $y \leq x_0$, making use of the increasing u-calibrated configuration as above, given $\alpha > 0$ we have that

$$|u(y) - u(x)| \leq \alpha|y - x| + \beta + 2R(\alpha + K),$$

where $\beta \geq 0$ is guaranteed by proposition 26. For $x < y - R$ with $x \geq y_0$, making use of the decreasing u-calibrated configuration as in the first case, we see that

$$u(x) - u(y) \geq \gamma|y - x| - \delta - 2R(\gamma + K),$$

where $\gamma, \delta > 0$ are the constants guaranteed by item ii of proposition 26. These facts show that u is of type III. □

We highlight an immediate corollary of the proof of the result above.

Corollary 32. *Let $E(x, y)$ be a weakly twist interaction that is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω. Suppose that $\inf_x E(x, x) > \bar{E}$. Given a weak KAM solution u, with respect to the preferred ordering,*

(i) u is of type I if, and only if, for any $y \in \mathbb{R}$, every point belonging to $\arg\min\{u(\cdot) + E(\cdot, y)\}$ precedes y,

(ii) *u is of type II if, and only if, for any $y \in \mathbb{R}$, every point belonging to* $\arg\min\{u(\cdot) + E(\cdot, y)\}$ *succeeds y,*

(iii) *if u is of type III, if and only if, there exists an interval I of length at most L, where $L > 0$ is given as in lemma 28, for which, for all y preceding I every element of* $\arg\min\{u(\cdot) + E(\cdot, y)\}$ *precedes y, and for all y succeeding I every element of* $\arg\min\{u(\cdot)+E(\cdot, y)\}$ *succeeds y.*

The next proposition shows that solutions of the same type not only have the same asymptotic behavior characteristics but actually lie at a uniform distance from each other.

Proposition 33. *Let $E(x, y)$ be a weakly twist interaction that is pattern equivariant with respect to a linearly repetitive quasi-periodic set ω. Assume* $\inf_x E(x, x) > \bar{E}$. *If u and v are two weak KAM solutions of the same type, then* $\sup_{x\in\mathbb{R}} |u(x) - v(x)| < +\infty$.

Proof. Assume the preferred ordering is the increasing one. There are three cases depending on the types of u, v.

Assume that u and v are of type I. Let $(x_k)_{k\in\mathbb{Z}}$ be an increasing bi-infinite u-calibrated configuration. We may assume $|x_0| \le R$, where R is given by proposition 16. We first show by induction that, if c is some constant, if $u(x_k) \ge v(x_k) + c$ for some k, then $u(x_{k+1}) \ge v(x_{k+1}) + c$. Indeed, using the fact that v is a sub-action, we see that

$$\begin{aligned} u(x_{k+1}) &= u(x_k) + E(x_k, x_{k+1}) - \bar{E} \\ &\ge v(x_k) + E(x_k, x_{k+1}) - \bar{E} + c \ge v(x_{k+1}) + c. \end{aligned}$$

Starting with $c = u(x_0) - v(x_0)$, one thus obtains

$$\forall\, k \ge 0,\ u(x_k) - v(x_k) \ge u(x_0) - v(x_0).$$

We then extend the above inequality to every $x > 0$. We choose $k \ge 0$ such that $|x - x_k| \le R$. Using the Lipschitz constant K of u and v, one has

$$\forall\, x \ge 0,\ u(x) - v(x) \ge u(x_k) - v(x_k) - 2KR \ge u(x_0) - v(x_0) - 2KR.$$

By permuting the role of u and v, one obtains

$$\sup_{x\ge 0} |u(x) - v(x)| \le \sup_{y\in[-R,R]} |u(y) - v(y)| + 2KR.$$

Similarly if $u(x_k) \le v(x_k) + c$ for some $k \le 0$, by induction one gets $u(x_{k-1}) \le v(x_{k-1}) + c$. One concludes as before that

$$\begin{aligned} &\forall\, x \le 0,\ u(x) - v(x) \le u(x_0) - v(x_0) + 2KR, \quad \text{and} \\ &\sup_{x\le 0} |u(x) - v(x)| \le \sup_{y\in[-R,R]} |u(y) - v(y)| + 2KR. \end{aligned}$$

If u and v are of type II, the proof is completely analogous using decreasing bi-infinite u-calibrated and v-calibrated configurations.

Assume that u and v are of type III. Let $T \geq 0$ be the maximum of the respective constants for u and v as given in proposition 30. Through decreasing u-calibrated and v-calibrated configurations ending at T, in a similar way as above, we get $\sup_{x \geq T} |u(x) - v(x)| \leq |u(T) - v(T)| + 2KR$. Then using increasing u-calibrated and v-calibrated configurations ending at $-T$, we obtain $\sup_{x \leq -T} |u(x) - v(x)| \leq |u(-T) - v(-T)| + 2KR$. □

6. Periodic interaction models

Theorem 5 illustrates that, when the non-degeneracy condition is satisfied ($\inf_x E(x,x) > \bar{E}$), the asymptotic behaviour of a weak KAM solution is related to the one of calibrated configurations. A noticeable point in the proof is the existence of unbounded Mañé calibrated configurations, at least in one direction. We show the non-degeneracy hypothesis is necessary for these results by providing an example of a degenerate interaction for which any Mañé calibrated configuration is bounded but admitting a weak KAM solution with linear growth. In more precise words, we prove in this section theorem 7. Unexpectedly, the example falls in the classical framework of a periodic interaction mechanical model of Frenkel-Kontorova type.

Hypotheses 34. Let $E : \mathbb{R} \times \mathbb{R} \to \mathbb{E}$ be an interaction model satisfying all the assumptions (H1-4) of hypotheses 2. We will say that E is a *periodic interaction model* whenever

(H_{per}) E is periodic: $\forall\, x, y \in \mathbb{R}, \quad E(x, y) = E(x + 1, y + 1)$.

Periodicity suggests to introduce another interaction $E^{per} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$E^{\text{per}}(x, y) := \min_{k \in \mathbb{Z}} E(x, y + k).$$

Note that $E^{per} : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ may be seen as a function on the torus, that is, E^{per} is doubly periodic. But we will see it is important to keep the real line orientation and consider configurations on $\mathbb{R}$ and not on $\mathbb{T}$. It is easy to verify that the two ground actions coincide,

$$\bar{E} = \bar{E}^{\text{per}}.$$

As in definition 9 and in (2.1), we may similarly introduce periodic versions of the Mañé potential and Lax-Oleinik operator (now acting on continuous periodic functions):

$$\forall\, x, y \in \mathbb{R}, \quad S^{\text{per}}(x, y) = \inf_{n \geq 1} \inf_{x = x_0, \ldots, x_n = y} \sum_{k=0}^{n-1} \big(E^{per}(x_k, x_{k+1}) - \bar{E}\big),$$

$$\forall\, u \in C^0(\mathbb{T}),\ \forall\, y \in \mathbb{R}, \quad T^{per}[u](y) := \inf_{x \in \mathbb{R}} \big\{u(x) + E^{per}(x, y)\big\}.$$

It is easy to check the following properties.

Lemma 35. *Let E be an interaction model satisfying the hypotheses (34).*

(i) The periodic Mañé potential for the interaction E^{per} is related to the Mañé potential by

$$\forall\, x, y \in \mathbb{R}, \quad S^{per}(x,y) = \inf_{k\in\mathbb{Z}} S(x, y+k).$$

(ii) If $u \in C^0(\mathbb{R})$ is 1-periodic, then

$$T^{per}[u] = u + \bar{E} \quad \Leftrightarrow \quad T[u] = u + \bar{E}.$$

(iii) If u is a 1-periodic weak KAM solution (for either E or E^{per}), the optimal points in the two definitions of T and T^{per} coincide modulo $\mathbb{Z}$: for every $y \in \mathbb{R}$, we have

$$\arg\min_{x\in\mathbb{R}}\{u(x) + E^{per}(x,y) - \bar{E}\} = \arg\min_{x\in\mathbb{R}}\{u(x) + E(x,y) - \bar{E}\} + \mathbb{Z}.$$

By taking into account the periodic potential

$$V(x) := 1 - \cos(2\pi x),$$

we will focus on the following version of the standard Frenkel-Kontorova model

$$E_\lambda(x,y) := \frac{1}{2}|y - x - \lambda|^2 + KV(x), \qquad \lambda \in \mathbb{R},\ K \in \mathbb{R}_+. \tag{6.1}$$

Note that E_λ satisfies assumptions (H1-4) of hypotheses 2, but it may be degenerate. Let $\bar{E}_\lambda$ denote the corresponding ground action.

We now fix $K > 0$ and discuss the properties on the periodic interaction model (6.1) with respect to λ. Denote by $S_\lambda(x,y)$ the respective Mañé potential. Since the interaction E_0 may be written as a sum of even functions, the Mañé potential and the ground action preserve some symmetries:

$$\begin{aligned} S_0(0,y) &= S_0(0,-y), \\ S_0(x,y) + KV(y) &= S_0(y,x) + KV(x), \\ E_{\lambda+1}(x,y) = E_\lambda(x,y), &\qquad E_{-\lambda}(x,y) = E_\lambda(-x,-y), \\ E_\lambda(x,y) - \frac{1}{2}\lambda^2 &= E_0(x,y) - \lambda(y-x), \\ \bar{E}_{\lambda+1} = \bar{E}_\lambda, &\qquad \bar{E}_{-\lambda} = \bar{E}_\lambda. \end{aligned}$$

We resume in the next proposition the main properties of the Mañé potential and of the Mañé calibrated configurations that will give theorem 7. More precisely, for suitable parameters K and λ, this model is degenerate (item iii), the Mañé potential defines a weak KAM solution (item vi) of linear growth (items i and ii) but all Mañé calibrated configurations are bounded (item vii).

Proposition 36. *Let $K > 0$ and $c_0(K) := \min\{\sqrt{1+2K} - 1, \frac{1}{40}\}$. Then for every $|\lambda| < c_0(K)$, we have*

(i) $\forall\, y \in \mathbb{R},\ S_0(0,y) \geq c_0(K)(|y| - \frac{1}{2})$,
(ii) $\exists\, c_1(K) > c_0(K),\ \forall\, x, y \in \mathbb{R},\ S_0(x,y) \leq c_1(K)(|y-x| + 1)$,
(iii) $\bar{E}_\lambda = \inf_{x\in\mathbb{R}} E_\lambda(x,x) = \frac{1}{2}\lambda^2,\ \bar{E}_\lambda = E_\lambda(x,x) \ \Leftrightarrow\ x \in \mathbb{Z}$

(iv) $S_\lambda(x,y) = S_0(x,y) - \lambda(y-x)$, $S_\lambda(0,y) \geq (c_0(K) - |\lambda|)|y| - \frac{1}{2}c_0(K)$,
(v) $\forall\, i<j<k,\ i,j,k \in \mathbb{Z},\quad S_\lambda(i,k) = S_\lambda(i,j) + S_\lambda(j,k)$,
(vi) $S_\lambda(0,x)$ *is a uniformly Lipschitz weak KAM solution,*
(vii) *if* $(x_k)_{k\in\mathbb{Z}}$ *is a Mañé calibrated configuration for the interaction* E_λ*, then there is an integer* n *such that* $x_k \in [n, n+1]$, $\forall\, k \in \mathbb{Z}$.

Note that this model is non-degenerate for some parameters. Actually, it will be shown in proposition 37 of appendix A (for $\rho = 1$) that this periodic Frenkel-Kontorova model is non-degenerate for large $|\lambda|$ compared to K, namely:

$$\forall\, 0 \leq K < \frac{\lambda^2}{8}, \qquad \bar{E}_\lambda < \inf_{x\in\mathbb{R}} E_\lambda(x,x).$$

Proof of proposition 36.

Item i. By the symmetries of S_0, we may suppose that $y > 0$. Note that $\bar{E}_0 = 0$. Thanks to lemma 14, for the computation of $S_0(0,y)$, it is enough to consider monotone configurations $0 = x_0 < x_1 < \ldots < x_n = y$. Denote

$$\{i_\alpha < i_{\alpha+1} < \ldots < i_\beta\} := \{0 \leq i < n : [x_i, x_{i+1}) \cap (\mathbb{Z} + \tfrac{1}{2}) \neq \emptyset\}.$$

Clearly, $0 \leq x_{i_\alpha} < \frac{1}{2}$, $x_{i_{k+1}} - x_{i_k+1} < 1$, and $y - x_{i_\beta} < 1$. By positivity of E_0, we have

$$\sum_{i=0}^{n-1} E_0(x_i, x_{i+1}) \geq \sum_{k=\alpha}^{\beta} E_0(x_{i_k}, x_{i_k+1}).$$

We claim that there exists a constant $c \in (0, \frac{1}{4})$ such that, for any such a subconfiguration, $E_0(x_{i_k}, x_{i_k+1}) > c\,(x_{i_k+1} - x_{i_k} + 1)$, $\forall\, k = \alpha, \ldots, \beta$. Using $0 \leq x_{i_{k+1}} - x_{i_k+1} \leq 1$, the claim will imply

$$\sum_{k=\alpha}^{\beta} E_0(x_{i_k}, x_{i_k+1}) \geq c\,(y - x_{i_\alpha}). \tag{6.2}$$

Let us prove the claim. Denote by $q_k + \frac{1}{2}$ the smallest element of $\mathbb{Z} + \frac{1}{2}$ belonging to $[x_{i_k}, x_{i_k+1})$. If we write $u_k = q_k + \frac{1}{2} - x_{i_k}$ and $v_k = x_{i_k+1} - (q_k + \frac{1}{2})$, to show our claim it is enough to assure the existence of a constant $c \in (0, \frac{1}{4})$ such that, for every $u, v \geq 0$,

$$F(u,v) := \frac{1}{2}(u+v)^2 + K(1 + \cos 2\pi u) - c\,(u+v+1) \geq 0.$$

The minimum of $F(u,v)$ over $u, v \geq 0$ takes place on the boundary of the domain, since there is no critical points in the interior of the domain. Hence, we have to consider the following three cases.

Case 1. On the border $u = 0$, note that

$$F(0,v) = \frac{1}{2}v^2 - cv - c + 2K \geq 2K - \frac{1}{2}(c+1)^2 + \frac{1}{2} \geq 0,$$

whenever $c \in [0, \sqrt{1+4K} - 1]$.

Case 2a. On the border $v = 0$ and $u \geq \frac{1}{4}$, we have

$$F(u, 0) \geq \frac{1}{2}u^2 - c\,(u + 1) \geq \frac{1}{32} - c\frac{5}{4} \geq 0,$$

whenever $c \in [0, \frac{1}{40}]$.

Case 2b. Finally, on the border $v = 0$ and $0 \leq u \leq \frac{1}{4}$, observe that

$$F(u, 0) \geq \frac{1}{2}u^2 + K - c\,(u + 1) \geq K - \frac{1}{2}(c + 1)^2 + \frac{1}{2} \geq 0,$$

whenever $c \in [0, \sqrt{1 + 2K} - 1]$.

To summarize, we have shown that the inequality (6.2) holds for the constant $c = c_0(K) = \min\{\sqrt{1 + 2K} - 1, \frac{1}{40}\}$. With the obvious estimate

$$\sum_{i=0}^{i_\alpha - 1} E_0(x_i, x_{i+1}) \geq c_0(K)(x_{i_\alpha} - \tfrac{1}{2}),$$

we conclude that $S_0(0, y) \geq c_0(K)(y - \frac{1}{2})$ for all $y > 0$.

Item ii. The existence of $c_1(K)$ follows from lemma 13.

Item iii. On the one hand, as $\bar{E}_\lambda = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n} \frac{1}{n} E_\lambda(x_0, \ldots, x_n)$, from the configuration $x_k = 0$ for all k, one obtains $\bar{E}_\lambda \leq \frac{1}{2}\lambda^2$.

On the other hand, let us introduce the function

$$u_\lambda(x) = \inf_{n\geq 1} \inf_{x_0=0,\ldots,x_n=x} \Big[E_\lambda(x_0, \ldots, x_n) - n\frac{\lambda^2}{2}\Big].$$

Since $E_\lambda(x, y) - \frac{\lambda^2}{2} = E_0(x, y) - \lambda(y - x)$, clearly $u_\lambda(x) = S_0(0, x) - \lambda x$, and in particular, u is a well-defined function. Note that

$$u_\lambda(x) \geq (c_0(K) - |\lambda|)|x| - \tfrac{1}{2}c_0(K) \geq -\tfrac{1}{2}c_0(K), \quad \forall\, x \in \mathbb{R},$$

$$u_\lambda(x) + E_\lambda(x, y) \geq u_\lambda(y) + \tfrac{1}{2}\lambda^2, \quad \forall\, x, y \in \mathbb{R}. \tag{6.3}$$

Define then $u_\lambda^{\mathrm{per}}(x) := \inf_{p\in\mathbb{Z}} u_\lambda(x + p)$. Using the constant $c_1(K)$ of item ii, it follows that

$$-\tfrac{1}{2}c_0(K) \leq u_\lambda^{\mathrm{per}}(x) \leq \inf_{p\in\mathbb{Z}} \big[(c_1(K) + |\lambda|)(|x + p| + 1)\big] \leq 2(c_1(K) + |\lambda|),$$

namely, u_λ^{per} is bounded. From (6.3), we obtain for every $y \in \mathbb{R}$,

$$\begin{aligned} u_\lambda^{\mathrm{per}}(y) + \tfrac{1}{2}\lambda^2 &\leq \inf_{x\in\mathbb{R}} \inf_{p,q\in\mathbb{Z}} \big[u_\lambda(x + p) + E_\lambda(x, y + q)\big] \\ &= \inf_{x\in\mathbb{R}} \big[u_\lambda^{\mathrm{per}}(x) + E_\lambda^{\mathrm{per}}(x, y)\big]. \end{aligned}$$

Therefore, we clearly get

$$\bar{E}_\lambda = \lim_{n\to+\infty} \inf_{x_0,\ldots,x_n\in\mathbb{R}} \frac{1}{n}\big[E_\lambda(x_0, \ldots, x_n) + u_\lambda^{\mathrm{per}}(x_0) - u_\lambda^{\mathrm{per}}(x_n)\big] \geq \tfrac{1}{2}\lambda^2.$$

Item iv. It follows from $\bar{E}_\lambda = \frac{1}{2}\lambda^2$.

Item v. From item iv, it is enough to prove the relation for $\lambda = 0$. In addition, since $S_0(m, n) = S_0(0, n - m)$ for any integers $m < n$, one just needs to argue that $S_0(0,k) = kS_0(0,1)$ for all positive integers k. Such a fact follows from

$$S_0(0,k+1) = S_0(0,k) + S_0(0,1), \qquad \forall\, k \geq 1.$$

To see that the equality above holds, note first that clearly $S_0(0,k+1) \leq S_0(0,k) + S_0(k,k+1) = S_0(0,k) + S_0(0,1)$. Recall that we may consider only monotone configurations $(x_0 = 0, \dots, x_n = k+1)$ in the expression defining $S_0(0,k+1)$ (lemma 14). Observe that for $u, v \geq 0$ and $j \in \mathbb{Z}$,

$$E_0(j-u, j+v) - [E_0(j-u,j) + E_0(j,j+v)] = \frac{1}{2}(u+v)^2 - \frac{1}{2}u^2 - \frac{1}{2}v^2 \geq 0.$$

For $m = \max\{0 \leq i \leq n : x_i < k\}$, this inequality implies the configuration obtained by concatainnaiting the configurations $(x_0 = 0, \dots, x_m, k)$ and $(k, x_{m+1}, \dots, x_n = k+1)$ does not increase the total energy. This ensures the opposite inequality $S_0(0,k+1) \geq S_0(0,k) + S_0(0,1)$.

Item (vi). Thanks to item (iv), it is enough to argue that $S_0(0,\cdot)$ is a weak KAM solution. Note first that $x \mapsto S_0(0,x)$ is a continuous function. Obviously, for all x and y

$$S_0(0,y) - S_0(0,x) \leq S_0(x,y) \leq E_0(x,y) - \bar{E}_0 = E_0(x,y).$$

On the other hand, for each positive integer k, there exists a subconfiguration $(x_1^k, \dots, x_{n_k}^k)$ such that

$$S_0(0,y) + \frac{1}{k} \geq E_0(0, x_1^k, \dots, x_{n_k}^k, y) \geq 0.$$

In particular, as

$$0 \leq E_0(x_{n_k}^k, y) \leq S_0(0,y) + 1 \qquad \forall\, k \geq 1,$$

the superlinearity implies the sequence $\{x_{n_k}^k\}$ is at a bounded distance from y, and therefore admits an accumulation point. Passing to a subsequence if necessary, we then assume that $x_{n_k}^k \to \bar{x}$. We have for all k

$$S_0(0,y) + \frac{1}{k} \geq S_0(0, x_{n_k}^k) + E_0(x_{n_k}^k, y),$$

which thanks to the continuity yields

$$S_0(0,y) \geq S_0(0,\bar{x}) + E_0(\bar{x}, y).$$

We next show that the minimum in

$$S_\lambda(0,y) = \min_{x \in \mathbb{R}} \big[S_\lambda(0,x) + E_\lambda(x,y) - \bar{E}_\lambda\big],$$

is attained at some x which satisfies $|y-x| \leq R$ for some $R > 0$ independent of y. The infimum is realized at some x thanks to the superlinearity of E_λ. Using item ii, one obtains

$$S_\lambda(x,y) \leq (c_1(K) + |\lambda|)(|y-x| + 1), \quad \forall\, x, y \in \mathbb{R}.$$

On the one hand,

$$S_\lambda(0,y) \leq S_\lambda(0,x) + S_\lambda(x,y) \leq S_\lambda(0,x) + (c_1(K) + |\lambda|)(|y-x| + 1).$$

On the other hand, thanks to the superlinearity of E_λ, there exists a constant $c_2(K,\lambda)$ such that

$$E_\lambda(x,y) - \bar{E}_\lambda \geq (c_1(K) + |\lambda| + 1)|y-x| - c_2(K,\lambda), \quad \forall\, x,y \in \mathbb{R}.$$

Using the two previous estimates and $S_\lambda(0,y) = S_\lambda(0,x) + E_\lambda(x,y) - \bar{E}_\lambda$, one gets

$$|y-x| \leq c_1(K) + |\lambda| + c_2(K,\lambda) := R.$$

The Lipschitz constant of $y \mapsto S_\lambda(0,y)$ depends on the Lipschitz constant of $y \mapsto E_\lambda(x,y)$ uniformly on $|y-x| \leq R$.

Item vii. Observe that by item iv, a Mañé calibrated configuration for E_λ is also Mañé calibrated for E_0. So without lost of generality, we can assume $\lambda = 0$.

Part 1. We show that (x_k, x_{k+1}) cannot contain an integer. By contradiction, if $n \in (x_k, x_{k+1})$, the twist property (see lemma 22 of [22]) implies

$$\begin{aligned} S_0(x_k, x_{k+1}) &= S_0(x_k, x_{k+1}) + E_0(n,n), \\ &= E_0(x_k, x_{k+1}) + E_0(n,n), \\ &> E_0(x_k, n) + E_0(n, x_{k+1}) \geq S_0(x_k, x_{k+1}), \end{aligned}$$

and we reach an absurd.

Part 2. We show there cannot exist an integer $p \in \mathbb{Z}$ such that for some index $k \in \mathbb{Z}$ and some integer $l \geq 0$,

$$x_{k-1} < x_k = x_{k+1} = \cdots = x_{k+l} < x_{k+l+1}, \quad x_k = p \in \mathbb{Z}.$$

The other case $x_{k-1} > x_k > x_{k+l+1}$ and $x_k \in \mathbb{Z}$ is done similarly. By contradiction, on the one hand, the function

$$x \mapsto E_0(x_{k-1}, x, \cdots, x, x_{k+l+1}) \qquad (x \text{ repeated } l+1 \text{ times})$$

reaches its minimum at $x = x_k$ and for small $\epsilon > 0$ one has

$$E_0(x_{k-1}, x_k + \epsilon, \cdots, x_{k+l} + \epsilon, x_{k+l+1}) = S_0(x_{k-1}, x_{k+l+1}) + O(\epsilon^2).$$

On the other hand, the twist property implies there exists $\alpha > 0$ (independent of ϵ) such that

$$\begin{aligned} E_0(x_{k-1}, x_k + \epsilon) &= E_0(x_{k-1}, x_k + \epsilon) + E_0(x_k, x_k) \\ &\geq E_0(x_{k-1}, x_k) + E_0(x_k, x_k + \epsilon) + \alpha\epsilon. \end{aligned}$$

Thus

$$\begin{aligned} & S_0(x_{k-1}, x_{k+l+1}) + O(\epsilon^2) \geq \\ & \qquad \geq E_0(x_{k-1}, x_k, x_k + \epsilon, \cdots, x_{k+l} + \epsilon, x_{k+l+1}) + \alpha\epsilon \\ & \qquad \geq S_0(x_{k-1}, x_{k+l+1}) + \alpha\epsilon, \end{aligned}$$

and we obtain a contradiction.

The two previous parts show that $(x_k)_{k \in \mathbb{Z}}$ cannot overlap an interval of the form $[n, n+1]$, $n \in \mathbb{Z}$. ☐

As suggested by one of the referees, the way order emerges and the relationship with the non-degeneracy condition may be illustrated through the study of one-parameter periodic families. Proposition 8 summarizes this study and we conclude this section with its proof. Recall that E_0 is a given periodic interaction model and $E_\lambda(x,y) = E_0(x,y) - \lambda(y-x)$ for $\lambda \in \mathbb{R}$.

Proof of proposition 8.

Item i. Clearly, $\inf_x E_\lambda(x,x) = \inf_x E_0(x,x)$. Assume

$$\lambda > \inf_{x\in\mathbb{R}} E_0(x,x+1) - \inf_{x\in\mathbb{R}} E_0(x,x).$$

Then by taking configurations of the form $(x, x+1, \ldots, x+n)$ where $x \in \arg\min_x E_0(x,x+1)$, one obtains

$$\inf_{x_0,\ldots,x_n} E_\lambda(x_0,\ldots,x_n) \le \sum_{k=1}^{n} E_\lambda(x+k-1, x+k) = n\big(E_0(x,x+1) - \lambda\big),$$
$$\bar{E}_\lambda \le \inf_{x\in\mathbb{R}} E_0(x,x+1) - \lambda < \inf_{x\in\mathbb{R}} E_0(x,x) = \inf_{x\in\mathbb{R}} E_\lambda(x,x).$$

We have proved that E_λ is non-degenerate. Suppose now

$$\lambda < \inf_{x\in\mathbb{R}} E_0(x,x) - \inf_{x\in\mathbb{R}} E_0(x+1,x).$$

Then by taking configurations of the form $(x+n, \ldots, x+1, x)$ where $x \in \arg\min_x E_0(x+1,x)$, one has

$$\inf_{x_0,\ldots,x_n} E_\lambda(x_0,\ldots,x_n) \le \sum_{k=1}^{n} E_\lambda(x+k, x+k-1) = n\big(E_0(x+1,x) + \lambda\big),$$
$$\bar{E}_\lambda \le \inf_{x\in\mathbb{R}} E_0(x+1,x) + \lambda < \inf_{x\in\mathbb{R}} E_0(x,x) = \inf_{x\in\mathbb{R}} E_\lambda(x,x).$$

We have again showed that E_λ is non-degenerate.

Item ii. Let

$$\lambda > \inf_{x\in\mathbb{R}} E_0(x,x+1) - \inf_{x,y\in\mathbb{R}} E_0(x,y).$$

Suppose by contradiction there are fundamental configurations $(x_0^n, \ldots, x_n^n)$ of arbitrarily large size n for E_λ that are decreasing. Then, one has

$$E_\lambda(x_0^n, \ldots, x_0^n) \ge n \inf_{x,y\in\mathbb{R}} E_0(x,y) + \lambda(x_0^n - x_n^n) \ge n \inf_{x,y\in\mathbb{R}} E_0(x,y),$$
$$\bar{E}_\lambda \ge \inf_{x,y\in\mathbb{R}} E_0(x,y).$$

However, the first part shows that

$$\bar{E}_\lambda \le \inf_{x\in\mathbb{R}} E_0(x,x+1) - \lambda.$$

We thus obtain a contradiction. Therefore, a preferred ordering exists and is the increasing one.

Item iii. We change the ordering by taking the new interaction model

$$\tilde{E}_\lambda(x, y) := E_\lambda(-x, -y) = \tilde{E}_0(x, y) - (-\lambda)(y - x).$$

Item iv. Note that by its very definition the function

$$\lambda \mapsto \bar{E}_\lambda = \lim_{n \to +\infty} \frac{1}{n} \inf_{x_0, \ldots, x_n} [E_0(x_0, \ldots, x_n) - \lambda(x_n - x_0)]$$

is concave. Therefore, the parameters λ for which E_λ is degenerated correspond to the convex and closed set $D := \{\lambda \in \mathbb{R} : \bar{E}_\lambda = \inf_x E_0(x, x)\}$, which is an interval eventually empty. According to proposition 26, a parameter λ whose potential E_λ is non-degenerated belongs to either Λ_+ or Λ_-. Hence the real line is partitioned into $\mathbb{R} = \Lambda_- \sqcup D \sqcup \Lambda_+$. Moreover, the previous items show that both sets $\Lambda_+ \cap \mathbb{R}_+$ and $\Lambda_- \cap \mathbb{R}_-$ are not empty.

We claim that:

(a) for all $\lambda \in \Lambda_+$, there is $\epsilon > 0$ such that $(\lambda - \epsilon, +\infty) \subset \Lambda_+$;
(b) for all $\lambda \in \Lambda_-$, there is $\epsilon > 0$ such that $(-\infty, \lambda + \epsilon) \subset \Lambda_-$;

From claims (a) and (b), we conclude the sets Λ_+ and Λ_- are both open and stable by positive and negative rays, respectively. It follows, by the connexity of $\mathbb{R}$, the set D cannot be empty, and $D = [\lambda_-, \lambda_+]$ where $\lambda_- = \sup \Lambda_-$ and $\lambda_+ = \inf \Lambda_+$. These facts prove item iv.

We only show the first claim, since the reasoning to obtain the other is analogous. Let $\lambda \in \Lambda_+$. There exists $\epsilon > 0$ such that for any parameter $\mu \in (\lambda - \epsilon, +\infty)$, the interaction E_μ is non-degenerated. As a matter of fact, if $\big(z_0^n(\lambda), \ldots, z_n^n(\lambda)\big)$ is an increasing fundamental configuration for the interaction E_λ, recalling that

$$E_0\big(z_0^n(\lambda), \ldots, z_n^n(\lambda)\big) - \lambda\big(z_n^n(\lambda) - z_0^n(\lambda)\big) = E_\lambda\big(z_0^n(\lambda), \ldots, z_n^n(\lambda)\big) \le n\bar{E}_\lambda,$$

one has for $\mu \in \mathbb{R}$,

$$\begin{aligned} \bar{E}_\mu &\le \lim_{n \to +\infty} \frac{1}{n} E_\mu\big(z_0^n(\lambda), \ldots, z_n^n(\lambda)\big) \\ &= \lim_{n \to +\infty} \frac{1}{n} \big[E_\lambda\big(z_0^n(\lambda), \ldots, z_n^n(\lambda)\big) + (\lambda - \mu)\big(z_n^n(\lambda) - z_0^n(\lambda)\big)\big] \\ &\le \bar{E}_\lambda + (\lambda - \mu)^+ R(\lambda), \end{aligned}$$

where $R(\lambda) > 0$ is the uniform interdistance established by lemma 25. It follows that the potential E_μ is non-degenerated for any $\mu > \lambda - \epsilon$ where $0 < \epsilon \le \big(\inf_x E_0(x, x) - \bar{E}_\lambda\big) / R(\lambda)$.

It remains to show the preferred ordering for E_μ is increasing. For contradiction, let us assume the preferred ordering for E_μ is decreasing. Let $(z_0^n(\mu), \ldots, z_n^n(\mu))$ be a decreasing fundamental configuration for E_μ.

Note that

$$\begin{aligned} S_\lambda(z_0^n(\mu), z_n^n(\mu)) + &n\bar{E}_\lambda + (\lambda-\mu)(z_n^n(\mu) - z_0^n(\mu)) \leq \\ &\leq E_\lambda(z_0^n(\mu), \ldots, z_n^n(\mu)) + (\lambda-\mu)(z_n^n(\mu) - z_0^n(\mu)) \\ &= E_\mu(z_0^n(\mu), \ldots, z_n^n(\mu)) \leq n\bar{E}_\mu. \end{aligned}$$

Thanks to proposition 26, there are $\gamma(\lambda) > 0$ and $\delta(\gamma) \geq 0$ such that

$$(\gamma(\lambda) + \mu - \lambda)|z_n^n(\mu) - z_0^n(\mu)| - \delta(\lambda) \leq n(\bar{E}_\mu - \bar{E}_\lambda) \leq n(\lambda-\mu)^+ R(\lambda).$$

This inequality does not hold for n large enough whenever $\mu > \lambda$. To obtain a contradiction also in the case $\mu < \lambda$ with μ near to λ, we will need to look closely at the rotational properties of the fundamental configurations. More precisely, up to consider a smaller ϵ, we may assume that $\gamma(\lambda)/2 \geq \epsilon$. For $\mu \geq \lambda - \epsilon$, by passing to the limit as n tends to infinity in the above inequality, we get

$$\frac{\gamma(\lambda)}{2}\phi(\mu) \leq (\gamma(\lambda) - \epsilon)\phi(\mu) \leq \epsilon R(\lambda).$$

where $\phi(\mu) > 0$ is the uniform lower bound for a rotation number of any fundamental configuration for the non-degenerated interaction E_μ as established in lemma 25. If such an inequality were valid, $\lim_{\mu\to\lambda^-}\phi(\mu) = 0$ would occur. However, we actually have $\liminf_{\mu\to\lambda}\phi(\mu) > 0$, from which we conclude our proof. To see that, recall from (4.1), lemma 15, and lemma 25, that $\phi(\mu)$ is computed by the formula

$$\phi(\mu) = \frac{\eta(\mu)^2}{\bar{E}_\mu - \inf_{x,y\in\mathbb{R}} E_\mu(x,y) + \eta(\mu)}, \tag{6.4}$$

where $\eta(\mu) > 0$ is any number fulfilling

$$\forall\, x, y \in \mathbb{R},\ |y - x| < \eta(\mu) \ \Rightarrow\ E_\mu(x,y) - \bar{E}_\mu > \eta(\mu), \tag{6.5}$$

We first show that (6.5) may be satisfied with an independent $\eta_* := \min\left\{R(\lambda), \frac{\eta(\lambda)}{2}\right\}$. Indeed, if $|x-y| < \eta_*$ and $|\mu - \lambda| \leq \frac{\eta(\lambda)}{4R(\lambda)}$, then

$$\begin{aligned} E_\mu(x,y) - \bar{E}_\mu &\geq E_\lambda(x,y) - \bar{E}_\lambda - |\mu-\lambda|R(\lambda) - |\mu-\lambda||x-y| \\ &> \eta(\lambda) - 2|\mu-\lambda|R(\lambda) \geq \frac{\eta(\lambda)}{2} \geq \eta_*. \end{aligned}$$

We now bound from below $\phi(\mu)$ in (6.4). The superlinearity of E_λ implies that, given $\epsilon_* := \frac{\eta(\lambda)}{4R(\lambda)} > 0$, there is a constant $B_\lambda > 0$ such that

$$\forall x, y \in \mathbb{R},\ E_\lambda(x,y) - \bar{E}_\lambda \geq \epsilon_*|x-y| - B_\lambda.$$

In particular, for every $|\mu - \lambda| \leq \epsilon_*$, one obtains for all $x, y \in \mathbb{R}$,

$$E_\mu(x,y) \geq E_\lambda(x,y) - \epsilon_*|x-y| \geq \bar{E}_\lambda - B_\lambda \geq \bar{E}_\mu - \epsilon_* R(\lambda) - B_\lambda,$$

so that $\bar{E}_\mu - \inf_{x,y\in\mathbb{R}} E_\mu(x,y) \le \epsilon_* R(\lambda) + B_\lambda$. Therefore, we have proved that

$$\phi(\mu) \ge \frac{\eta_*^2}{\epsilon_* R(\lambda) + B_\lambda + \eta_*} \qquad \text{if } |\mu - \lambda| \le \epsilon_* = \frac{\eta(\lambda)}{4R(\lambda)}.$$

(If precision is required, the reader may easily check from the previous arguments that

$$\epsilon := \min\left\{ \frac{\inf_x E_0(x,x) - \bar{E}_\lambda}{R(\lambda)}, \frac{\gamma(\lambda)}{2}, \frac{\eta(\lambda)}{4R(\lambda)}, \frac{\gamma(\lambda)\eta_*^2}{4(\epsilon_* R(\lambda) + B_\lambda + \eta_*)R(\lambda)} \right\}$$

could be proposed as an explicit definition to be considered in claim (a).)

□

Appendices

Appendix A. Non-degenerate almost crystalline models

Our aim is to provide examples of pattern equivariant interactions that fulfill hypothesis $\inf_x E(x,x) > \bar{E}$. For this purpose, we focus on one-dimensional quasicrystals studied in [20]. More concretely, given $\alpha \in (0, 1/2)$ and $\rho > 0$, we will consider a quasi-periodic set $\omega(\alpha,\rho) = \omega = \{q_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ defined by

$$\forall\, k \in \mathbb{Z}, \quad q_k = k + (\rho - 1)\lfloor k\alpha \rfloor.$$

Note that $q_k - q_{k-1} = 1 + (\rho - 1)a_k$, where

$$a_k = \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor.$$

Since $\alpha < 1$, we have $(a_k)_{k\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ and $q_k - q_{k-1}$ equals 1 or ρ whenever a_k equals 0 or 1, respectively. As $\alpha < 1/2$, a_k and a_{k+1} cannot be both equal to 1. In fact, $(a_k)_{k\in\mathbb{Z}}$ is periodic if α is rational, and is called a Sturmian sequence when α is irrational [29].

Since ω is uniformly discrete, it satisfies the (finite local complexity) property. The fact that ω obeys (repetitivity) can be assured essentially because irrational rotations on the circle are minimal. Moreover, since these rotations are also uniquely ergodic, they satisfy an additional property: each type of pattern occurs with a positive density (see [18] for a modern presentation). In precise terms, they fulfill

uniform pattern distribution: for any pattern $\mathtt{P}$, there is a positive number $\nu(\mathtt{P}) > 0$ such that for any nested sequence of bounded open intervals $I_1 \subset I_2 \subset \ldots \subset I_k \subset \ldots$ with unbounded sequence of lengths $(|I_k|)_k$, the quantity

$$\frac{N_{\mathtt{P}}(I_k)}{|I_k|}$$

converges to $\nu(\mathtt{P})$, where $N_{\mathtt{P}}(I)$ denotes the number of patterns in $\omega \cap I$ equivalent to $\mathtt{P}$.

The uniform pattern distribution is a consequence of the uniform limits

$$\mu_{\mathtt{b}} = \lim_{n-m\to+\infty} \frac{\#_{\mathtt{b}} a_{[m,n)}}{n-m}, \tag{A.1}$$

where $\#_{\mathtt{b}} a_{[m,n)}$ is the number of times the subword $\mathtt{b} = b_1 \cdots b_r \in \{0,1\}^r$, $r \geq 1$, appears in the word $a_{[m,n)} = a_m a_{m+1} \cdots a_{n-1}$ for $n > m$, $m, n \in \mathbb{Z}$.

A discrete set that satisfies (finite local complexity), (repetitivity) and (uniform pattern distribution) is said to be a *quasicrystal*. See [24] for details.

Let us explain how to define a C^2 interaction model $E(x,y)$ that is twist and pattern equivariant with respect to ω. Regarding the interaction with the substrate, we introduce a pattern equivariant V obtained by translating two functions V_1 and V_ρ according to the patterns of two consecutive points of ω. Concretely, for $V_1 : [0,1] \to \mathbb{R}$ and $V_\rho : [0,\rho] \to \mathbb{R}$ defined as $V_1(x) = V_\rho(\rho x)/\rho^2 = 1 - \cos 2\pi x$, consider for every $k \in \mathbb{Z}$ and $x \in [q_k, q_{k+1})$,

$$V(x) = \begin{cases} V_1(x - q_k), & \text{if } q_{k+1} - q_k = 1, \\ V_\rho(x - q_k), & \text{if } q_{k+1} - q_k = \rho. \end{cases}$$

See the figure below for the graph of an example. Let the interaction be defined by

$$E_\lambda(x,y) := \frac{1}{2}|y - x - \lambda|^2 + KV(x), \qquad \lambda \in \mathbb{R},\ K \in \mathbb{R}_+.$$

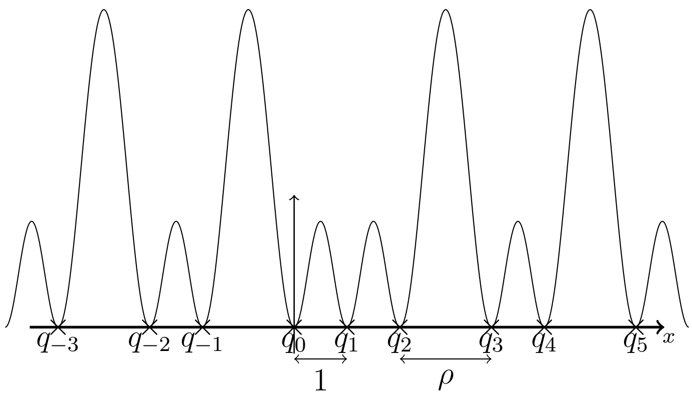

FIGURE 7. Graph of the potential V for $\alpha = 1/\sqrt{5}$ and $\rho = \sqrt{3}$

The next proposition provides examples of non-degenerate interactions for which theorem 5 does apply. Remark that for $\rho = 1$ the quasicrystal is nothing but the lattice $\mathbb{Z}$. For $\rho \neq 1$, note that the quasicrystal $\omega(\alpha,\rho)$ is linearly repetitive when the coefficients of the continued fraction of α are bounded [1] (*e.g.* when α is quadratic by the Lagrange's continued fraction theorem). It is known that the set of α satisfying this condition is a Baire meager set, of zero Lebesgue measure but with Hausdorff dimension 1.

Proposition 37. *For* $0 < \rho < 1 + \alpha^{-1/2}$*, denote* $c_1(\lambda) := \frac{\lambda^2}{8}\frac{1-\alpha(1-\rho)^2}{1+\alpha(\rho^2-1)}$*. Then for every* $K \in (0, c_1(\lambda))$*,*

$$\inf_{x\in\mathbb{R}} E_\lambda(x,x) > \bar{E}_\lambda.$$

In the periodic case, that is, when $\rho = 1$, the proposition gives a simple estimate between λ and K for the non-degeneracy. For $K < \frac{\lambda^2}{8}$, the periodic Frenkel-Kontorova model is non-degenerate.

Proof. Obviously $\inf_x E_\lambda(x,x) = \lambda^2/2$. Since

$$\bar{E}_\lambda = \lim_{n\to\infty} \inf_{x_0,\ldots,x_n} \frac{1}{n} E_\lambda(x_0,\ldots,x_n),$$

the strategy consists in finding a long configuration by concatenating short configurations located in each cell $[q_k, q_{k+1}]$ so that the mean of the successive interactions is strictly smaller than $\lambda^2/2$. Let $\ell \geq 1$. For cells (q_k, q_{k+1}) of length $q_{k+1} - q_k = 1$, we use a translate $q_k + \mathbf{y}^\ell$ of the subconfiguration $\mathbf{y}^\ell$ in $(0,1)$

$$\mathbf{y}^\ell = \big(y_0^\ell, \ldots, y_{2^\ell-1}^\ell\big) := \Big(\frac{1}{2^{\ell+1}} + \frac{j}{2^\ell}\Big)_{j=0}^{2^\ell-1}.$$

For cells of length $q_{k+1} - q_k = \rho$, we use a translate of the homothetic subconfiguration $\rho\mathbf{y}^\ell = (\rho y_0^\ell, \ldots, \rho y_{2^\ell-1}^\ell)$. More precisely, we define for every $k \geq 1$ a configuration $\mathbf{x}_k^\ell$ of $k2^\ell + 1$ points obtained by concatenating k translates of some homothetic $\mathbf{y}^\ell$ and by adding an extra translate of y_0^ℓ

$$\mathbf{x}_k^\ell := \big((q_1 - q_0)\mathbf{y}^\ell, q_1 + (q_2 - q_1)\mathbf{y}^\ell, \ldots, q_{k-1} + (q_k - q_{k-1})\mathbf{y}^\ell,$$
$$q_k + (q_{k+1} - q_k)y_0^\ell\big) \in \mathbb{R}^{k2^\ell+1}.$$

Note that $\mathbf{x}_k^\ell$ has 2^ℓ entries belonging to each interval $(q_0, q_1), \ldots, (q_{k-1}, q_k)$, and an extra point of (q_k, q_{k+1}). The total energy $E_\lambda(\mathbf{x}_k^\ell)$ consists in two terms. The term coming from the external potential is a sum of partial sums, of the form either $\sum_{x\in\mathbf{y}^\ell+q} V_1(x)$ or $\sum_{x\in\rho\mathbf{y}^\ell+q} V_\rho(x)$. The main observation is that in both cases each partial sum boils down to

$$c\big(2^\ell - \sum_{j=0}^{2^\ell-1} \cos(2\pi y_j^\ell + d)\big), \qquad \text{with } c, d \text{ constants, } c \in \{1, \rho^2\}.$$

Since $\sum_{j=0}^{2^\ell-1} \cos(2\pi y_j^\ell) = \sum_{j=0}^{2^\ell-1} \sin(2\pi y_j^\ell) = 0$, we thus have

$$\sum_{x \text{ entry into } \mathbf{y}^\ell+q} V_1(x) = 2^\ell \quad \text{and} \quad \sum_{x \text{ entry into } \rho\mathbf{y}^\ell+q} V_\rho(x) = \rho^2 2^\ell.$$

The term coming from the mutual interaction between neighboring atoms can be calculated according to the cases of entries belonging to possible cells, and presents three values. In fact, let $x < y$ be consecutive entries into $\mathbf{x}_k^\ell$. The energy $\frac{1}{2}|y - x - \lambda|^2$ takes one of the following values:

$$\begin{aligned} &\frac{\lambda^2}{2} + \frac{\rho}{2^\ell}\Big(\frac{\rho}{2^{\ell+1}} - \lambda\Big) && \text{if } x, y \in [q_{k-1}, q_k], \text{ with } a_k = 1, \\ &\frac{\lambda^2}{2} + \frac{1}{2^\ell}\Big(\frac{1}{2^{\ell+1}} - \lambda\Big) && \text{if } x, y \in [q_{k-1}, q_{k+1}], \text{ with } a_k = a_{k+1} = 0, \\ &\frac{\lambda^2}{2} + \frac{\rho+1}{2^{\ell+1}}\Big(\frac{\rho+1}{2^{\ell+2}} - \lambda\Big) && \text{if } x \in [q_{k-1}, q_k] \text{ and } y \in [q_k, q_{k+1}], \\ & && \text{with } a_k \neq a_{k+1} \end{aligned}$$

For $k \geq 1$, let $\#_i^k$ denote the number of times the subword $i \in \{0, 1\}$ appears in the word $a_1 \cdots a_{k+1}$. Only three types of subwords of length 2 appear in $(a_k)_{k\in\mathbb{Z}}$: 00, 01 and 10. Introduce similarly $\#_{ij}^k$ as the number of times the subword ij, $i, j \in \{0, 1\}$, appears in the word $a_1 \cdots a_{k+1}$. Clearly $\#_{11}^k = 0$. Then

$$\begin{aligned} E_\lambda(\mathbf{x}_k^\ell) := {}& K2^\ell(\#_0^k + \rho^2\#_1^k) + \#_{00}^k 2^\ell\Big[\frac{\lambda^2}{2} + \frac{1}{2^\ell}\Big(\frac{1}{2^{\ell+1}} - \lambda\Big)\Big] \\ &+ \#_{01}^k(2^\ell - 1)\Big[\frac{\lambda^2}{2} + \frac{1}{2^\ell}\Big(\frac{1}{2^{\ell+1}} - \lambda\Big)\Big] + \#_{01}^k\Big[\frac{\lambda^2}{2} + \frac{\rho+1}{2^{\ell+1}}\Big(\frac{\rho+1}{2^{\ell+2}} - \lambda\Big)\Big] \\ &+ \#_{10}^k(2^\ell - 1)\Big[\frac{\lambda^2}{2} + \frac{\rho}{2^\ell}\Big(\frac{\rho}{2^{\ell+1}} - \lambda\Big)\Big] + \#_{10}^k\Big[\frac{\lambda^2}{2} + \frac{\rho+1}{2^{\ell+1}}\Big(\frac{\rho+1}{2^{\ell+2}} - \lambda\Big)\Big]. \end{aligned}$$

We divide by $k2^\ell$ and let $k \to +\infty$. From (A.1), we have $\#_0^k/k \to \mu_0$ and a similar result for $\#_1^k$, $\#_{00}^k$, $\#_{01}^k$ and $\#_{10}^k$. Since the word 1 is always preceded and followed by 0 in $(a_k)_k$, note then that $\mu_1 = \mu_{01} = \mu_{10} = (1 - \mu_{00})/2$. We thus obtain

$$\begin{aligned} \lim_{k\to+\infty} \frac{E_\lambda(\mathbf{x}_k^\ell)}{k2^\ell} = \frac{\lambda^2}{2} + K(1+\mu_1(\rho^2 - 1)) + \frac{1+\mu_{00}}{2}\frac{1}{2^\ell}\Big(\frac{1}{2^{\ell+1}} - \lambda\Big) \\ + \frac{1-\mu_{00}}{2}\frac{\rho}{2^\ell}\Big(\frac{\rho}{2^{\ell+1}} - \lambda\Big) - \frac{1-\mu_{00}}{2^\ell}\frac{(1-\rho)^2}{2^{2\ell+3}}. \end{aligned}$$

We want to find a condition on $\lambda > 0$ so that $\bar{E}_\lambda < \frac{\lambda^2}{2}$. Ignoring the nonpositive term, it is enough to choose λ so that

$$2^\ell K(1 + \mu_1(\rho^2 - 1)) + \frac{1+\mu_{00}}{2}\Big(\frac{1}{2^{\ell+1}} - \lambda\Big) + \frac{1-\mu_{00}}{2}\rho\Big(\frac{\rho}{2^{\ell+1}} - \lambda\Big) < 0,$$

or equivalently

$$2^\ell K(1+\mu_1(\rho^2-1)) + \frac{1+\mu_{00}}{2}\frac{1}{2^{\ell+1}} + \frac{1-\mu_{00}}{2}\frac{\rho^2}{2^{\ell+1}} < \lambda\Big(\frac{1+\mu_{00}}{2} + \rho\frac{1-\mu_{00}}{2}\Big).$$

Assume ℓ has been chosen so that $\frac{1}{2^\ell} < \lambda \leq \frac{1}{2^{\ell-1}}$. It suffices to rewrite the inequality above with $\frac{1}{2^\ell}$ instead of λ. Then K must satisfy

$$2^\ell K(1 + \mu_1(\rho^2 - 1)) < \frac{1+\mu_{00}}{2}\frac{1}{2^{\ell+1}} + \frac{1-\mu_{00}}{2}\frac{\rho(2-\rho)}{2^{\ell+1}}.$$

As $\lambda \leq \frac{1}{2^{\ell-1}}$, we have $\frac{\lambda^2}{8} \leq \frac{1}{2^{2\ell+1}}$. Hence, it is enough to choose K so that

$$K < \frac{\lambda^2}{8}\Big(\frac{1+\mu_{00}}{2} + \frac{1-\mu_{00}}{2}\rho(2-\rho)\Big)\frac{1}{1+\mu_1(\rho^2-1)}.$$

Recalling that $(1-\mu_{00})/2 = \mu_1$, we have

$$\frac{1+\mu_{00}}{2} + \frac{1-\mu_{00}}{2}\rho(2-\rho) = 1-\mu_1+\mu_1\rho(2-\rho) = 1-\mu_1(1-\rho)^2.$$

We thus have proved that $\bar{E}_\lambda < \frac{\lambda^2}{2}$ whenever $K < \frac{\lambda^2}{8}\frac{1-\mu_1(1-\rho)^2}{1+\mu_1(\rho^2-1)}$. A direct computation [29] shows that $\mu_1 = \alpha$. This proves the proposition. $\square$

Appendix B. The calibration level of the sublinearity

Here we show that, within the framework delimited by hypotheses 2 and definition 4, the only calibration level associated with sublinear weak KAM solutions is exactly that given by the ground action $\bar{E}$. This result is nothing unexpected. We present the argument, however, for the convenience of the reader. We obviously make use of the Lax-Oleinik operator T as introduced in (2.1). Furthermore, by sublinearity we mean the notion described by (5.4). The precise statement is the following one.

Proposition 38. *Let E be a weakly twist interaction that is pattern equivariant with respect to a quasi-periodic set. Suppose that $\inf_x E(x,x) > \bar{E}$. Let u be a sublinear function such that $T[u] = u + c$ for some constant c. Then $c = \bar{E}$.*

The fact that u is a fixed point for the Lax-Oleinik operator associated with the interaction $E - c$ determines its regularity.

Lemma 39. *Let E be an interaction model satisfying assumptions (H1-3) of hypotheses 2. If u is a function satisfying $T[u] = u+c$ for some c, then there are $K_c, R_c \geq 0$ (depending only on the interaction model E and on the constant c) such that*

(i) u is Lipschitz continuous with $\mathrm{Lip}(u) \leq K_c$,
(ii) $\forall\, y \in \mathbb{R},\ \arg\min\{u(\cdot) + E(\cdot, y)\} \subset [y - R_c, y + R_c]$.

Proof. Just repeat *verbatim* the proof of lemma 11 by taking into account $\tilde{K} = \sup_{|y-x|\leq 1} E(x,y) - c$. $\square$

Proof of proposition 38. We first show the inequality $c \geq \bar{E}$. To that end, let $(y_0, \ldots, y_n)$ be a configuration such that

$$u(y_n) - u(y_0) = E(y_0, \ldots, y_n) - nc.$$

By the sublinearity of u, given $\alpha > 0$, there is β with respect to which $|u(y_n) - u(y_0)| \leq \alpha|y_n - y_0| + \beta$. Therefore, thanks to the previous lemma, we obtain

$$c \geq -\frac{\alpha n R_c + \beta}{n} + \frac{1}{n}\inf_{(x_0,\ldots,x_n)} E(x_0, \ldots, x_n),$$

and the claim follows by passing to limit as n tends to infinity and then by considering α arbitrarily close to zero.

We now show that $\bar{E} \geq c$. Let $(z_0, \ldots, z_n)$ be a fundamental configuration of size n (recall definition 21). We thus have

$$E(z_0, \ldots, z_n) \leq n\bar{E}, \quad \text{and}$$
$$u(z_n) - u(z_0) \leq E(z_0, \ldots, z_n) - nc.$$

We then apply lemma 25 and again the sublinearity of u to get

$$c \leq \bar{E} + \frac{\alpha n R + \beta}{n},$$

from which the conclusion follows as for the converse inequality. □

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