An ergodic description of ground states

Eduardo Garibaldi*
Departamento de Matemática
Universidade Estadual de Campinas
13083-859 Campinas - SP, Brasil
garibaldi@ime.unicamp.br

Philippe Thieullen†
Institut de Mathématiques
Université Bordeaux 1, CNRS, UMR 5251
F-33405 Talence, France
Philippe.Thieullen@math.u-bordeaux1.fr

July 8, 2014

Abstract

Given a translation-invariant Hamiltonian H, a ground state on the lattice $\mathbb{Z}^d$ is a configuration whose energy, calculated with respect to H, cannot be lowered by altering its states on a finite number of sites. The set formed by these configurations is translation-invariant. Given an observable Ψ defined on the space of configurations, a minimizing measure is a translation-invariant probability which minimizes the average of Ψ. If Ψ_0 is the mean contribution of all interactions to the site 0, we show that any configuration of the support of a minimizing measure is necessarily a ground state.

Keywords: bounded-spin lattice system, ergodic optimization, ground state, minimizing probability

2010 Mathematics Subject Classification: 37A60, 37J50, 82B20

1 Introduction

A dynamical system, in a very broad sense, is given by a compact metric space Ω and a $\mathbb{Z}^d$-action $\theta^j : \Omega \to \Omega$, $j \in \mathbb{Z}^d$, acting continuously on Ω. A probability measure μ is said to be translation invariant if $\mu(\theta^j(B)) = \mu(B)$ for any Borel set B of Ω. The central matter of ergodic optimization is to understand the set of the probability measures which minimize the average of a given continuous observable $\Psi : \Omega \to \mathbb{R}$. We call minimizing ergodic value of Ψ the quantity

$$\bar{\Psi} = \inf \Big\{ \int \Psi \, d\mu : \mu \text{ is a translation-invariant probability} \Big\}.$$

A minimizing measure is a translation-invariant probability which attains the above infimum. For $\mathbb{Z}$-actions, the subject has been extensively studied (for details, see Jenkinson's notes [5]). Apparently no general result is known for $\mathbb{Z}^d$-actions.

*Supported by FAPESP 2009/17075-8 and Brazilian-French Network in Mathematics.
†Supported by FAPESP 2009/17075-8. Funding from the European Union's Seventh Framework Program (FP7/2007-2013) under grant agreement No. 318999 BREUDS.

A similar problem exists in the theory of Gibbs measures on bounded-spin lattice systems, as described, for instance, in [1, 2, 3]. The configuration space Ω is then given by the lattice $\mathbb{Z}^d$ of sites whose states belong to a fixed compact space Ω_0. Formally $\Omega = \Omega_0^{\mathbb{Z}^d}$ is equipped with the product topology. The energy of a configuration $\omega \in \Omega$ is usually understood as an infinite-volume limit, to be properly defined, of energies calculated on finite volumes. A finite volume is simply a finite subset Λ of $\mathbb{Z}^d$ and the energy of a configuration restricted to a finite volume Λ is computed using a single continuous function $H_\Lambda : \Omega \to \mathbb{R}$. Usually H_Λ describes all admissible interaction energies (internal to Λ or representing some coupling with the exterior), and has the form $H_\Lambda = \sum_{A\cap\Lambda\neq\emptyset} \Phi_A$ for some family of continuous functions $\Phi_A : \Omega \to \mathbb{R}$ indexed by finite subsets $A \subset \mathbb{Z}^d$. A ground-state configuration ω is a configuration whose energy $H_\Lambda(\omega)$ on ea ch fixed volume Λ cannot be lowered by changing the states of the sites restricted to Λ. We denote by $\Omega_{GS}(H)$ the set of ground-state configurations. Although $\Omega_{GS}(H)$ is defined without mentioning the temperature, it may be seen as a set containing the support of limit Gibbs states obtained as the absolute temperature tends to zero. More informations can be found in [2], appendix B.2. We shall not say anything on this issue. Our goal will consist in showing that $\Omega_{GS}(H)$ contains the support of any minimizing measure of a particular observable $\Psi_0 = \sum_{A\ni 0} \frac{1}{\#A}\Phi_A$ that summarizes the total mean energy contribution to the site 0.

2 Framework and Main Results

The spin values are described here by a compact metrizable space Ω_0. We introduce then the configuration space $\Omega = \Omega_0^{\mathbb{Z}^d}$. A configuration $\omega \in \Omega$ is described by giving the states $\omega = \{\omega_j\}_{j\in\mathbb{Z}^d}$ at all sites $j \in \mathbb{Z}^d$. Endowed with the product topology, Ω is a compact metrizable space. If d_0 is a metric compatible with the topology of Ω_0 and if $\|j\| := |j_1| + \ldots + |j_d|$ denotes the L^1 norm of an index $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, one may define a metric on Ω compatible with the product topology by $\mathsf{d}(\omega, \bar\omega) = \sum_{j\in\mathbb{Z}^d} 2^{-\|j\|}\mathsf{d}_0(\omega_j, \bar\omega_j)$.

Notice that $\mathbb{Z}^d$ acts on Ω by translation. Let $\{e_1, \ldots, e_d\}$ be the canonical basis for the lattice $\mathbb{Z}^d$. For each $i \in \{1, \ldots, d\}$, we consider the shift transformation $\theta_i : \Omega \to \Omega$ given by $\theta_i(\omega) = \{\omega_{j+e_i}\}_{j\in\mathbb{Z}^d}$. Given $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, we define $\theta^j := \theta_1^{j_1} \circ \theta_2^{j_2} \circ \cdots \circ \theta_d^{j_d}$.

Let $\mathcal{F}$ denote the collection of finite subsets of $\mathbb{Z}^d$. We call diameter of a set $A \in \mathcal{F}$ the real number $\mathrm{diam}(A) = \max\{\|j - k\| : j, k \in A\}$. For $r > 0$, we call inner r-boundary $\partial_r^-(A)$ and outer r-boundary $\partial_r^+(A)$ the two sets adjacent to A

$$\partial_r^-(A) := \{j \in A : \|j - k\| \leq r \text{ for some } k \in \mathbb{Z}^d \setminus A\},$$
$$\partial_r^+(A) := \{j \in \mathbb{Z}^d \setminus A : \|j - k\| \leq r \text{ for some } k \in A\}.$$

We call r-boundary $\partial_r(A) := \partial_r^-(A) \cup \partial_r^+(A)$. For $A \in \mathcal{F}$ and $\omega \in \Omega$, the notation $\omega|_A$ or simply ω_A will denote the restriction of the configuration ω to the set A. The cardinality of a subset $\Lambda \subset \mathbb{Z}^d$ is denoted by $\#A$, and the complement by $\Lambda^{\complement}$.

The Hamiltonian of the system will be defined through a family of local interactions $\Phi_A : \Omega \to \mathbb{R}$, where each $\Phi_A(\omega)$ takes into account the local interaction energy of the configuration ω_A. More precisely, we recall a standard definition.

Definition 2.1. *We call* translation-invariant interaction family *any collection of continuous maps $\Phi_A : \Omega \to \mathbb{R}$, indexed by $A \in \mathcal{F}$ such that*

i. $\Phi_{j+A}(\omega) = \Phi_A(\theta^j(\omega))$, *for all $j \in \mathbb{Z}^d$ and $\omega \in \Omega$;*

ii. $\omega_A = \bar{\omega}_A$ *implies* $\Phi_A(\omega) = \Phi_A(\bar{\omega})$.

In addition, we say that $\{\Phi_A\}_{A\in\mathcal{F}}$ is absolutely summable *if*

$$\sum_{A:0\in A} \|\Phi_A\|_\infty < \infty.$$

We also recall the following related notions.

Definition 2.2. *A translation-invariant interaction family $\{\Phi_A\}_{A\in\mathcal{F}}$ is said to have* finite-range *if there exists an integer $r > 0$ such that $\Phi_A \equiv 0$ whenever $diam(A) > r$. In this case, we also say that the translation-invariant interaction family has range $r > 0$. In particular for $r = 1$, the interaction takes into account the* nearest neighbors *only. (Notice that any translation-invariant interaction family with finite-range is absolutely summable.) We say that $\{\Phi_A\}_{A\in\mathcal{F}}$ is a* long-range interaction family *if $\Phi_A \not\equiv 0$ for sets A with arbitrarily large diameter.*

Given a translation-invariant absolutely summable interaction family $\{\Phi_A\}_{A\in\mathcal{F}}$, the associated *Hamiltonian* $H : \mathcal{F} \times \Omega \to \mathbb{R}$ is defined by

$$H(\Lambda, \cdot) = H_\Lambda := \sum_{A\,:\,A\cap\Lambda\neq\emptyset} \Phi_A, \quad \forall\, \Lambda \in \mathcal{F}. \tag{2.1}$$

It follows from the absolute summability condition of the interaction family that the associated Hamiltonian is a well defined function. For each $\Lambda \in \mathcal{F}$, H_Λ is actually a continuous function obtained as a uniform limit of continuous functions on Ω. Moreover, from its uniform continuity, it follows that H_Λ is quasi-local in the sense that

$$\lim_{\bar{\Lambda}\uparrow\mathbb{Z}^d} \sup_{\substack{\omega,\bar{\omega}\in\Omega \\ \omega_{\bar{\Lambda}}=\bar{\omega}_{\bar{\Lambda}}}} \big(H_\Lambda(\omega) - H_\Lambda(\bar{\omega})\big) = 0.$$

Notice also that the Hamiltonian inherits the invariance of the interaction family

$$H_\Lambda \circ \theta^j = H_{j+\Lambda} \quad \forall\, j \in \mathbb{Z}^d,\ \forall\, \Lambda \in \mathcal{F}.$$

The literature is filled with examples of such a formalism. For the Ising model, for instance, the energy of a configuration $\omega \in \{-1,1\}^{\mathbb{Z}^d}$ is formally given by the Hamiltonian $H(\omega) = -J\sum_{<i\,j>} \omega_i\omega_j - h\sum_j \omega_j$, where the first sum is over pairs of adjacent spins, J is a coupling constant which describes (according to its sign) ferromagnetic or antiferromagnetic phenomena, and h represents an external magnetic field. In terms of translation-invariant interactions, the Ising model highlights the nearest-neighbor interaction family given by the functions $\Phi_{\{0\}} = -h\omega_0$,

$\Phi_{\{0,e_k\}} = -J\omega_0\omega_{e_k}$, $k = 1,\dots,d$, and their translations. The classical Heisenberg model is another example of finite-range interaction model. In this case, the local state space is the unit Euclidean sphere, $\Omega_0 = \{\omega_0 \in \mathbb{R}^3 : \|\omega_0\| = 1\}$, and the formal Hamiltonian is $H(\omega) = -J\sum_{<ij>}\omega_i \cdot \omega_j$. By its turn, Dyson model exhibits long-range interactions with the introduction of a pair-wise coupling which decreases with the distance between the spins. In this model, the lattice is one-dimensional and, for $\omega \in \{-1,1\}^{\mathbb{Z}}$, the Hamiltonian takes the form $H(\omega) = -J\sum_{i>j}\omega_i\omega_j/(i-j)^\alpha$ with $\alpha \in (1,2)$. The non-null functions of the absolutely summable interaction family are thus $\Phi_{\{0,k\}}(\omega) = -Jk^{-\alpha}\omega_0\omega_k$, $k \ge 1$, as well as their translations. All these examples are considered in the results that follow.

In order to be able to apply Birkhoff's ergodic theorem, we shall consider a unique function $\Psi_0 : \Omega \to \mathbb{R}$, which corresponds to the normalized contribution of all interaction energies at the site 0, and is defined by

$$\Psi_0 := \sum_{A\,:\,0\in A}\frac{1}{\#A}\Phi_A. \tag{2.2}$$

Notice that Ψ_0 is a continuous real valued function thanks to the absolutely summability condition. Moreover, Ψ_0 is also quasi-local:

$$\lim_{\bar\Lambda\uparrow\mathbb{Z}^d}\ \sup_{\substack{\omega,\bar\omega\in\Omega\\ \omega_{\bar\Lambda}=\bar\omega_{\bar\Lambda}}}\big(\Psi_0(\omega)-\Psi_0(\bar\omega)\big) = 0.$$

Notice also that we could have introduced a notion of energy of a configuration ω restricted to $\Lambda \in \mathcal{F}$ by using the Birkhoff's sum

$$S_\Lambda\Psi_0(\omega) := \sum_{k\in\Lambda}\Psi_0\circ\theta^k(\omega).$$

For $\omega,\bar\omega \in \Omega$ and $\Lambda \in \mathcal{F}$, we denote by $\bar\omega_\Lambda\omega_{\mathbb{Z}^d\setminus\Lambda}$ the configuration of Ω that coincides with $\bar\omega$ on Λ and with ω on $\mathbb{Z}^d\setminus\Lambda$. Let $\mathcal{M}(\Omega)$ be the set of Borel probability measures equipped with the weak* topology. Let $\mathcal{M}(\Omega,\theta)$ be the subset of translation-invariant probability measures

$$\mathcal{M}(\Omega,\theta) := \left\{\mu\in\mathcal{M}(\Omega) : \mu\circ\theta^j = \mu,\ \ \forall\, j\in\mathbb{Z}^d\right\}.$$

We shall use Birkhoff's ergodic theorem for sequences of square boxes

$$\Lambda_n := [-n,n]^d\cap\mathbb{Z}^d, \quad \forall\, n\in\mathbb{N}.$$

Our main goal is to describe the set of ground-state configurations. We choose two possible definitions. The first one is more general since no hypothesis of invariance needs to be assumed. The second one is closer to notions that one finds in ergodic optimization.

Definition 2.3. *We say that $\omega\in\Omega$ is a* ground-state configuration *with respect to the Hamiltonian H if*

$$H_\Lambda(\omega) \le H_\Lambda(\bar\omega_\Lambda\omega_{\mathbb{Z}^d\setminus\Lambda}), \quad \forall\,\bar\omega\in\Omega,\ \forall\,\Lambda\in\mathcal{F}.$$

Let $\Omega_{GS}(H)$ be the set of ground-state configurations.

We say that $\mu \in \mathcal{M}(\Omega, \theta)$ is a minimizing measure *for the function Ψ_0 if*

$$\int_\Omega \Psi_0 \, d\mu \le \int_\Omega \Psi_0 \, d\nu, \quad \forall \, \nu \in \mathcal{M}(\Omega, \theta).$$

Let $\mathcal{M}_{min}(\Omega, \theta, H)$ be the set of minimizing measures.

We call ground-state energy *of the Hamiltonian H the constant*

$$\bar{H} := \inf_{\omega \in \Omega} \liminf_{n \to \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega).$$

It is easy to show that ground-state configurations do exist for a translation-invariant absolutely summable interaction family. We shall give a short proof for completeness (see proposition 3.1). The existence of minimizing measures is clearly guaranteed by the weak* compactness of $\mathcal{M}(\Omega, \theta)$. Moreover, by the ergodic decomposition theorem (see, for instance, [3], section 7.3), there always exist ergodic minimizing measures.

Our theorem states that a translation-invariant probability measure is minimizing if, and only if, its support lies on the set of ground-state configurations. The necessity part is more difficult to prove and uses a kind of maximal lemma for ground-state configurations.

Theorem 2.4. *Let H be an Hamiltonian defined by a translation-invariant absolutely summable interaction family. Then, one has*

i. $\bar{H} = \lim_{n \to \infty} \frac{1}{\#\Lambda_n} \inf_{\omega \in \Omega} H_{\Lambda_n}(\omega)$*;*

ii. $\bar{H} = \int_\Omega \Psi_0 \, d\mu, \quad \forall \, \mu \in \mathcal{M}_{min}(\Omega, \theta, H)$*;*

iii. $\bar{H} = \lim_{n \to \infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega), \quad \forall \, \omega \in \Omega_{GS}(H)$*;*

iv. (Subordination Principle) if μ is a translation-invariant probability, then

$$\mathrm{supp}(\mu) \subseteq \Omega_{GS}(H) \implies \mu \in \mathcal{M}_{min}(\Omega, \theta, H);$$

v. (Main Result) if μ is a translation-invariant probability, then

$$\mu \in \mathcal{M}_{min}(\Omega, \theta, H) \implies \mathrm{supp}(\mu) \subseteq \Omega_{GS}(H).$$

The previous theorem extends several results in Schrader's article [7]. In his work, Schrader first considers a configuration space of the form $\{0,1\}^{\mathbb{Z}^d}$, which enables him to identify configurations to subsets of $\mathbb{Z}^d$. We do not restrict our analysis to a finite state space Ω_0. In order to define the ground-state energy $\bar{H}$, he only considers Hamiltonians with free boundary conditions $H_{\Lambda,Free} := \sum_{A \subseteq \Lambda} \Phi_A$. We use the more natural Hamiltonian $H_\Lambda = \sum_{A \cap \Lambda \neq \emptyset} \Phi_A$, which takes into account the interactions across the boundary of Λ. Moreover, Schrader does not introduce the normalized contribution of all interaction energies Ψ_0 and makes no connection between the present notion of minimizing measures (as found in ergodic optimization theory) and his notion of ground-state translation-invariant measures. We have changed a little bit the terminology: we use the expression *ground-state configuration* for configurations $\omega \in \Omega_{GS}(H)$, no name is given in [7]; we use the expression *minimizing measure* for translation-invariant probability measures μ which minimize $\int \Psi_0 \, d\mu$ as in ergodic optimization theory, a similar notion is used in [7] and is called ground state without asking the shift invariance. Part *iv* of theorem 2.4 is similar to theorem 4.6 in [7], part *v* is similar to theorem 4.8 there. Our motivation to extend Schrader's article from the simple state space $\Omega_0 = \{0, 1\}$ to any general compact state space is to make clear the connection between two notions: a notion of configuration with the lowest possible energy where no average is computed, and a notion of ground-state energy which uses mean values of a unique energy function Ψ_0.

3 Proof of Theorem 2.4

From now on, without being restated each time, we assume that $\{\Phi_A\}_{A \in \mathcal{F}}$ is a translation-invariant absolutely summable interaction family. We begin by showing that ground-state configurations do exist. We give a short proof for completeness.

Proposition 3.1. *The set of ground-state configurations with respect to H is a non-empty, closed and translation-invariant set.*

Proof. For $\Lambda \in \mathcal{F}$, let $\Omega_{GS,\Lambda}$ be the set of the configurations $\omega \in \Omega$ such that

$$\forall \, \omega' \in \Omega, \quad \omega'_{\mathbb{Z}^d \setminus \Lambda} = \omega_{\mathbb{Z}^d \setminus \Lambda} \quad \Longrightarrow \quad H_\Lambda(\omega) \le H_\Lambda(\omega').$$

We choose a reference configuration $\bar{\omega} \in \Omega$. Notice that $\Omega_{GS,\Lambda}$ clearly contains the minimum points of the continuous map $\omega_\Lambda \in \Omega_0^\Lambda \mapsto H_\Lambda(\omega_\Lambda \bar{\omega}_{\mathbb{Z}^d \setminus \Lambda}) \in \mathbb{R}$. Moreover, $\Omega_{GS,\Lambda}$ is closed and monotone with respect to Λ in the following sense: if $\Lambda \subset \Lambda'$, then $\Omega_{GS,\Lambda'} \subset \Omega_{GS,\Lambda}$. (We use the fact that $H_{\Lambda'} = H_\Lambda + \sum_{A \cap \Lambda = \emptyset, \ A \cap \Lambda' \neq \emptyset} \Phi_A$.) In particular, the family $\{\Omega_{GS,\Lambda}\}_{\Lambda \in \mathcal{F}}$ has the finite intersection property, and by compactness, $\Omega_{GS}(H) = \bigcap_{\Lambda \in \mathcal{F}} \Omega_{GS,\Lambda}$ is non-empty and translation invariant. $\square$

The proof of the main result of theorem 2.4 will be given at the end of this section. We shall first need the following two lemmas which can be found in the literature. We have nevertheless included short proofs for convenience of the reader and for clarification of the notations. We show in the first lemma how one can neglect the long-range interaction.

Lemma 3.2. $\displaystyle\lim_{n \to \infty} \frac{1}{\#\Lambda_n} \sum_{j \in \Lambda_n} \sum_{\substack{A \ni j \\ A \not\subset \Lambda_n}} \|\Phi_A\|_\infty = 0.$

Proof. For $\Lambda_m \subset \Lambda_n$, we have that

$$\begin{aligned}\sum_{j\in\Lambda_n}\sum_{\substack{A\ni j\\ A\not\subset\Lambda_n}}\|\Phi_A\|_\infty &\le \sum_{\substack{j\in\Lambda_n\\ j+\Lambda_m\subset\Lambda_n}}\sum_{\substack{A\ni j\\ A\not\subset j+\Lambda_m}}\|\Phi_A\|_\infty + \sum_{\substack{j\in\Lambda_n\\ j+\Lambda_m\not\subset\Lambda_n}}\sum_{A\ni j}\|\Phi_A\|_\infty \\ &\le \sum_{j\in\Lambda_n}\sum_{\substack{A\ni j\\ A\not\subset j+\Lambda_m}}\|\Phi_A\|_\infty + \#\{j\in\Lambda_n : j+\Lambda_m\not\subset\Lambda_n\}\sum_{A\ni 0}\|\Phi_A\|_\infty \\ &= \#\Lambda_n \sum_{\substack{A\ni 0\\ A\not\subset\Lambda_m}}\|\Phi_A\|_\infty + \#\big(\Lambda_n\setminus\Lambda_{n-m}\big)\sum_{A\ni 0}\|\Phi_A\|_\infty.\end{aligned}$$

By the absolute summability, given $\epsilon > 0$, there is $m \in \mathbb{N}$ with $\sum_{\substack{A\ni 0\\ A\not\subset\Lambda_m}} \|\Phi_A\|_\infty < \frac{\epsilon}{2}$. One may find $n_0 > m$ such that $\frac{\#(\Lambda_n\setminus\Lambda_{n-m})}{\#\Lambda_n}\sum_{A\ni 0}\|\Phi_A\|_\infty < \frac{\epsilon}{2}$ for all $n \ge n_0$. Therefore, whenever $n \ge n_0$, we obtain

$$\sum_{j\in\Lambda_n}\sum_{\substack{A\ni j\\ A\not\subset\Lambda_n}}\|\Phi_A\|_\infty \le \epsilon\#\Lambda_n,$$

which finishes the proof. □

We show in the second lemma that the two average energies $\frac{1}{\#\Lambda_n}H_{\Lambda_n}(\omega)$ and $\frac{1}{\#\Lambda_n}S_{\Lambda_n}\Psi_0(\omega)$ are comparable and have uniformly controlled oscillations.

Lemma 3.3.

i. $\displaystyle\lim_{n\to\infty}\frac{1}{\#\Lambda_n}\sup_{\omega_{\Lambda_n}=\bar\omega_{\Lambda_n}}\big(S_{\Lambda_n}\Psi_0(\omega) - S_{\Lambda_n}\Psi_0(\bar\omega)\big) = 0;$

ii. $\displaystyle\lim_{n\to\infty}\frac{1}{\#\Lambda_n}\sup_{\omega_{\Lambda_n}=\bar\omega_{\Lambda_n}}\big(H_{\Lambda_n}(\omega) - H_{\Lambda_n}(\bar\omega)\big) = 0;$

iii. $\displaystyle\lim_{n\to\infty}\frac{1}{\#\Lambda_n}\|H_{\Lambda_n} - S_{\Lambda_n}\Psi_0\|_\infty = 0.$

Proof. Notice first that

$$S_\Lambda\Psi_0 = \sum_{j\in\Lambda}\sum_{\substack{A\ni j\\ A\subset\Lambda}}\frac{1}{\#A}\Phi_A + \sum_{j\in\Lambda}\sum_{\substack{A\ni j\\ A\not\subset\Lambda}}\frac{1}{\#A}\Phi_A = \sum_{A\subset\Lambda}\Phi_A + \sum_{\substack{A\cap\Lambda\neq\emptyset\\ A\cap\Lambda^{\complement}\neq\emptyset}}\frac{\#(A\cap\Lambda)}{\#A}\Phi_A. \quad (3.1)$$

So for $\omega_\Lambda = \bar\omega_\Lambda$, we have

$$\begin{aligned}S_\Lambda\Psi_0(\omega_\Lambda) - S_\Lambda\Psi_0(\bar\omega_\Lambda) &= \sum_{j\in\Lambda}\sum_{\substack{A\ni j\\ A\not\subset\Lambda}}\frac{1}{\#A}\big(\Phi_A(\omega_\Lambda) - \Phi_A(\bar\omega_\Lambda)\big) \\ &\le 2\sum_{j\in\Lambda}\sum_{\substack{A\ni j\\ A\not\subset\Lambda}}\frac{1}{\#A}\|\Phi_A\|_\infty \le \sum_{j\in\Lambda}\sum_{\substack{A\ni j\\ A\not\subset\Lambda}}\|\Phi_A\|_\infty,\end{aligned}$$

and item i follows then from lemma 3.2.

In order to prove item iii, we decompose H_Λ similarly

$$H_\Lambda = \sum_{A\subset\Lambda} \Phi_A + \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \Phi_A = \sum_{j\in\Lambda} \sum_{\substack{A\ni j \\ A\subset\Lambda}} \frac{1}{\#A}\Phi_A + \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \Phi_A. \tag{3.2}$$

From the above equality and from (3.1), we have

$$S_\Lambda\Psi_0 - H_\Lambda = \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \Big(\frac{\#(A\cap\Lambda)}{\#A} - 1\Big)\Phi_A,$$

$$\begin{aligned}\|S_\Lambda\Psi_0 - H_\Lambda\|_\infty &\leq \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \|\Phi_A\|_\infty = \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \sum_{j\in A\cap\Lambda} \frac{1}{\#(A\cap\Lambda)}\|\Phi_A\|_\infty \\ &= \sum_{j\in\Lambda} \sum_{\substack{A\ni j \\ A\not\subset\Lambda}} \frac{1}{\#(A\cap\Lambda)}\|\Phi_A\|_\infty \leq \sum_{j\in\Lambda} \sum_{\substack{A\ni j \\ A\not\subset\Lambda}} \|\Phi_A\|_\infty. \end{aligned} \tag{3.3}$$

Then item iii follows from lemma 3.2 and item ii follows from i and iii. □

Remark 3.4. *For a translation-invariant interaction family with finite range r, the properties pointed out in the above lemma may be more precisely stated as follows:*

i. $\displaystyle\sup_{\Lambda\in\mathcal{F}} \sup_{\omega_\Lambda=\bar\omega_\Lambda} \frac{1}{\#\partial_1^+\Lambda}\left(S_\Lambda\Psi_0(\omega) - S_\Lambda\Psi_0(\bar\omega)\right) < \infty;$

ii. $\displaystyle\sup_{\Lambda\in\mathcal{F}} \sup_{\omega_\Lambda=\bar\omega_\Lambda} \frac{1}{\#\partial_1^+\Lambda}\left(H_\Lambda(\omega) - H_\Lambda(\bar\omega)\right) < \infty;$

iii. $\displaystyle\|H_\Lambda - S_\Lambda\Psi_0\|_\infty \leq \#\partial_r^-(\Lambda)\sum_{A\ni 0}\|\Phi_A\|_\infty.$

From lemma 3.3, item iii, we immediately obtain the following corollary which shows that $\bar H$ is a minimizing ergodic value of a unique observable in the context of ergodic optimization (see [5]). We shall show soon in proposition 3.7 that we can actually permute $\inf_{\omega\in\Omega}$ and $\liminf_{n\to+\infty}$ and that $\bar H$ is obtained by minimizing $\int\Psi_0\,d\mu$ over all translation-invariant probabilities μ.

Corollary 3.5. *The minimizing ergodic value of H is given by*

$$\bar H = \inf_{\omega\in\Omega}\liminf_{n\to\infty}\frac{1}{\#\Lambda_n}S_{\Lambda_n}\Psi_0(\omega).$$

In the following proposition, we first show that the limit in theorem 2.4, item iii, does exist. The proof is similar to Birkhoff's ergodic proof for uniquely ergodic systems. The identification of the limit to $\bar H$ will be done in proposition 3.8.

Proposition 3.6. *If ω is a ground-state configuration with respect to H, then both limits $\displaystyle\lim_{n\to\infty}\frac{1}{\#\Lambda_n}H_{\Lambda_n}(\omega) = \lim_{n\to\infty}\frac{1}{\#\Lambda_n}S_{\Lambda_n}\Psi_0(\omega)$ do exist.*

Proof. Let ω be a ground-state configuration for H. By lemma 3.3, item *iii*, it is enough to show that the second limit exists. Set

$$L = \liminf_{n\to\infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n}\Psi_0(\omega).$$

Given $\epsilon > 0$, consider a positive integer N large enough in such a way that

$$\frac{1}{\#\Lambda_N} S_{\Lambda_N}\Psi_0(\omega) < L + \epsilon,$$

$$\frac{1}{\#\Lambda_N} \|S_{\Lambda_N}\Psi_0 \circ \theta^j - H_{j+\Lambda_N}\|_\infty = \frac{1}{\#\Lambda_N} \|S_{\Lambda_N}\Psi_0 - H_{\Lambda_N}\|_\infty \le \epsilon, \tag{3.4}$$

$$\text{and} \quad \frac{1}{\#\Lambda_N} \sup_{\omega'_{\Lambda_N} = \omega''_{\Lambda_N}} \big(S_{\Lambda_N}\Psi_0(\omega') - S_{\Lambda_N}\Psi_0(\omega'')\big) \le \epsilon. \tag{3.5}$$

Suppose now that the integers $m, n \ge 1$ and $\ell \in \{0, 1, \dots, 2N\}$ are such that $2n + 1 = m(2N + 1) + \ell$. We choose a subset of indices J of Λ_n so that the translates of Λ_N, $\{j + \Lambda_N\}_{j\in J}$, are pairwise disjoint, are contained inside Λ_n, and essentially cover Λ_n in the sense that

$$\#(\Lambda_n \setminus \cup_{j\in J}(j + \Lambda_N)) \le C_{d,N} m^{d-1},$$

where $C_{d,N}$ is a constant that depends only on d and N. We then decompose the Birkhoff's sum

$$S_{\Lambda_n}\Psi_0 = \sum_{j\in J} S_{\Lambda_N}\Psi_0 \circ \theta^j + \sum_{k\in\Lambda_n\setminus\cup_{j\in J}(j+\Lambda_N)} \Psi_0 \circ \theta^k.$$

We define for each index $j \in J$ a configuration ω^j which coincides with $\theta^{-j}(\omega)$ on $j + \Lambda_N$ and with ω on $\mathbb{Z}^d \setminus (j + \Lambda_N)$. Since ω is minimizing, we have

$$\begin{aligned} S_{\Lambda_N}\Psi_0 \circ \theta^j(\omega) &= S_{j+\Lambda_N}\Psi_0(\omega) \le H_{j+\Lambda_N}(\omega) + \epsilon\#\Lambda_N \\ &\le H_{j+\Lambda_N}(\omega^j) + \epsilon\#\Lambda_N = H_{\Lambda_N}(\theta^j(\omega^j)) + \epsilon\#\Lambda_N \\ &\le S_{\Lambda_N}\Psi_0(\theta^j(\omega^j)) + 2\epsilon\#\Lambda_N \le S_{\Lambda_N}\Psi_0(\omega) + 3\epsilon\#\Lambda_N. \end{aligned}$$

By adding these inequalities over $j \in J$, one obtains

$$S_{\Lambda_n}\Psi_0(\omega) \le \#J(S_{\Lambda_N}\Psi_0(\omega) + 3\epsilon\#\Lambda_N) + C_{d,N} m^{d-1}\|\Psi_0\|_\infty.$$

Since $\#J \le \#\Lambda_n/\#\Lambda_N$, it follows that

$$\limsup_{n\to\infty} \frac{1}{\#\Lambda_n} S_{\Lambda_n}\Psi_0(\omega) \le L + 4\epsilon.$$

Since $\epsilon > 0$ can be chosen as close as one wants to zero, the proof is complete. $\square$

We have then the following characterization of the minimizing ergodic value $\bar{H}$.

Proposition 3.7 (Items *i* and *ii* of Theorem 2.4)**.**

$$\bar{H} = \min_{\mu\in\mathcal{M}(\Omega,\theta)} \int_\Omega \Psi_0\, d\mu = \lim_{n\to\infty} \frac{1}{\#\Lambda_n} \inf_{\omega\in\Omega} S_{\Lambda_n}\Psi_0(\omega) = \lim_{n\to\infty} \frac{1}{\#\Lambda_n} \inf_{\omega\in\Omega} H_{\Lambda_n}(\omega).$$

Proof. The last equality will follow from lemma 3.3, item *iii*. Besides, by standard superaditivity argument (see, for instance, Proposition 4.10 in [4]), one can ensure that the limit $\lim_{n\to\infty}\frac{1}{\#\Lambda_n}\inf_{\omega\in\Omega} S_{\Lambda_n}\Psi_0(\omega)$ exists. Therefore, we notice that

$$\min_{\mu}\int \Psi_0\, d\mu \geq \bar{H} \geq \lim_{n\to+\infty}\frac{1}{\#\Lambda_n}\inf_{\omega\in\Omega} S_{\Lambda_n}\Psi_0(\omega) \tag{3.6}$$

The second inequality of (3.6) comes from corollary 3.5. To prove the first one, we use Birkhoff's ergodic theorem (see, for example, Theorem 2.1.5 in [6]). Suppose that $\mu \in \mathcal{M}(\Omega,\theta)$ is ergodic. By Birkhoff's ergodic theorem, μ-almost every configuration ω satisfies

$$\int \Psi_0\, d\mu = \lim_{n\to\infty}\frac{1}{\#\Lambda_n} S_{\Lambda_n}\Psi_0(\omega) \geq \bar{H}.$$

Hence, the first inequality of (3.6) follows from the existence of a minimizing ergodic probability.

To conclude the proof, it thus suffices to show that

$$\lim_{n\to\infty}\frac{1}{\#\Lambda_n}\inf_{\omega} S_{\Lambda_n}\Psi_0(\omega) \geq \min_{\mu}\int \Psi_0\, d\mu.$$

For each n, consider a configuration $\omega^n \in \Omega$ such that $S_{\Lambda_n}\Psi_0(\omega^n) = \inf_\omega S_{\Lambda_n}\Psi_0(\omega)$ and define a Borel probability measure

$$\mu_n := \frac{1}{\#\Lambda_n}\sum_{j\in\Lambda_n}\delta_{\theta^j(\omega^n)} \in \mathcal{M}(\Omega).$$

Let $\mu \in \mathcal{M}(\Omega)$ be any weak* limit for a subsequence $\{\mu_{n_k}\}$. Clearly by construction,

$$\lim_{n\to\infty}\inf_{\omega\in\Omega}\frac{S_{\Lambda_n}\Psi_0(\omega)}{\#\Lambda_n} = \lim_{k\to\infty}\frac{S_{\Lambda_{n_k}}\Psi_0(\omega^{n_k})}{\#\Lambda_{n_k}} = \lim_{k\to\infty}\int \Psi_0\, d\mu_{n_k} = \int \Psi_0\, d\mu.$$

Moreover, μ is translation-invariant: for any continuous function f, one has

$$\forall\, i = 1,\ldots,d, \qquad \left|\int (f\circ\theta_i - f)\ d\mu_{n_k}\right| \leq 2\frac{\#\partial_1^+\Lambda_{n_k}}{\#\Lambda_{n_k}}\,\|f\|_\infty \overset{k\to\infty}{\longrightarrow} 0,$$

which indeed shows the invariance of μ. □

The next proposition also contains items of our theorem.

Proposition 3.8 (Items *iii* and *iv* of Theorem 2.4)**.**

$$\forall\, \omega \in \Omega_{GS}(H), \quad \lim_{n\to\infty}\frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) = \bar{H},$$

$$\forall\, \mu \in \mathcal{M}(\Omega,\theta), \quad \operatorname{supp}(\mu) \subset \Omega_{GS}(H) \quad \Longrightarrow \quad \bar{H} = \int \Psi_0\, d\mu.$$

Proof. Let $\omega \in \Omega$ be a ground-state configuration. From proposition 3.6, we know that the limit $\lim_{n\to\infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega)$ exists. Besides, from the very definition of $\bar{H}$, we get $\bar{H} \le \lim_{n\to\infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega)$. Let ω' be a configuration of Ω and let Λ be a finite subset of $\mathbb{Z}^d$. Define $\bar{\omega}$ equal to ω on $\mathbb{Z}^d \setminus \Lambda$ and equal to ω' on Λ. Using equation (3.2) and the upper bound given by (3.3), we have

$$\begin{aligned} H_\Lambda(\omega) \le H_\Lambda(\bar{\omega}) &= \sum_{A \subset \Lambda} \Phi_A(\bar{\omega}) + \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \Phi_A(\bar{\omega}) \\ &\le H_\Lambda(\omega') + 2 \sum_{\substack{A\cap\Lambda\neq\emptyset \\ A\cap\Lambda^{\complement}\neq\emptyset}} \|\Phi_A\|_\infty \le H_\Lambda(\omega') + 2 \sum_{j\in\Lambda} \sum_{\substack{A\ni j \\ A\not\subset\Lambda}} \|\Phi_A\|_\infty. \end{aligned}$$

We thus obtain, thanks to lemma 3.2 and for any configuration ω',

$$\lim_{n\to\infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) \le \liminf_{n\to\infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega'),$$

which yields $\lim_{n\to\infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) \le \bar{H}$.

If μ is translation invariant and has support included in $\Omega_{GS}(H)$, by Birkhoff's ergodic theorem, the support of μ contains a generic configuration ω, which implies that μ is minimizing thanks to lemma 3.3, item *iii*, and the first part of the proof. □

Before proving item v of Theorem 2.4, we will need the following estimate.

Lemma 3.9. *For any $\Lambda \in \mathcal{F}$ and $M \in \mathbb{N}$,*

$$\Big\| H_\Lambda - \sum_{\substack{A\cap\Lambda\neq\emptyset \\ diam(A)\le 2dM}} \Phi_A \Big\|_\infty \le \#\Lambda \sum_{\substack{A\ni 0 \\ A\not\subset\Lambda_M}} \|\Phi_A\|_\infty.$$

Proof. Indeed

$$H_\Lambda - \sum_{\substack{A\cap\Lambda\neq\emptyset \\ \mathrm{diam}(A)\le 2dM}} \Phi_A = \sum_{\substack{A\cap\Lambda\neq\emptyset \\ \mathrm{diam}(A)> 2dM}} \Phi_A = \sum_{k\in\Lambda} \sum_{\substack{A\ni k \\ \mathrm{diam}(A)> 2dM}} \frac{1}{\#(A\cap\Lambda)} \Phi_A.$$

The result follows then from the translation invariance of the family $\{\Phi_A\}_{A\in\mathcal{F}}$. □

We conclude by the proving the main result of this paper. The proof uses an estimate which is similar to the one that we find in the proof of the ergodic maximal lemma.

Proposition 3.10 (Item v of Theorem 2.4)**.** *The support of any minimizing probability is included in the set of ground-state configurations $\Omega_{GS}(H)$.*

Proof. The proof is done by contradiction. Let μ be an ergodic minimizing probability whose support is not included in $\Omega_{GS}(H)$. The open set $U = \Omega \setminus \Omega_{GS}(H)$ intersects $\mathrm{supp}(\mu)$ and satisfies $\mu(U) > 0$. We choose a generic configuration $\omega \in U$

in the following sense. Let $\{V_\ell\}_{\ell\geq 0}$ be a countable basis of open sets for the product topology of Ω. For each $\ell \geq 0$, consider the characteristic function $\chi_\ell : \Omega \to \{0,1\}$ of V_ℓ. By Birkhoff's ergodic theorem, there exists a Borel set $B \subset U$ such that $\mu(B) = \mu(U)$ and, for any $\omega \in B$ and $\ell \geq 0$,

$$\lim_{n\to\infty} \frac{S_{\Lambda_n}\Psi_0(\omega)}{\#\Lambda_n} = \int \Psi_0 \, d\mu \quad \text{and} \quad \lim_{n\to\infty} \frac{S_{\Lambda_n}\chi_\ell(\omega)}{\#\Lambda_n} = \mu(V_\ell). \tag{3.7}$$

We choose once for all $\omega \in B \cap \operatorname{supp}(\mu)$.

Since ω is not a ground-state configuration, there exist $\tilde\omega \in \Omega$, $\tilde N \in \mathbb{N}$ and $\tilde\eta > 0$ such that

$$\tilde\omega_{\mathbb{Z}^d\setminus\Lambda_{\tilde N}} = \omega_{\mathbb{Z}^d\setminus\Lambda_{\tilde N}} \quad \text{and} \quad H_{\Lambda_{\tilde N}}(\tilde\omega) < H_{\Lambda_{\tilde N}}(\omega) - \tilde\eta. \tag{3.8}$$

Notice that $\tilde\omega$ is not any more generic. Thanks to lemma 3.9 and the summability condition of $\{\Phi_A\}_{A\in\mathcal{F}}$, we may choose a positive integer M such that

$$\Big\| H_{\Lambda_{\tilde N}} - \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \operatorname{diam}(A)\leq 2dM}} \Phi_A \Big\|_\infty = \Big\| \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \operatorname{diam}(A)> 2dM}} \Phi_A \Big\|_\infty \leq \frac{\tilde\eta}{8}. \tag{3.9}$$

Set $\tilde M = \tilde N + 2dM$. Note that by the triangular inequality

$$\operatorname{diam}(A) \leq 2dM \ \text{ and } \ A \cap \Lambda_{\tilde N} \neq \emptyset \quad \Longrightarrow \quad A \subset \Lambda_{\tilde M}.$$

We now choose an open neighborhood V_ℓ of ω in the following way. The map

$$\omega' \in \Omega \mapsto P(\omega') := \omega'_{\Lambda_{\tilde M}\setminus\Lambda_{\tilde N}} \tilde\omega_{\mathbb{Z}^d\setminus(\Lambda_{\tilde M}\setminus\Lambda_{\tilde N})} \in \Omega$$

is continuous. Then, there is an open set V_ℓ containing ω such that, for all $\omega' \in V_\ell$,

$$\Big| \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \operatorname{diam}(A)\leq 2dM}} \Phi_A \circ P(\omega') - \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \operatorname{diam}(A)\leq 2dM}} \Phi_A \circ P(\omega) \Big| \leq \frac{\tilde\eta}{8}, \tag{3.10}$$

$$\big| H_{\Lambda_{\tilde N}}(\omega') - H_{\Lambda_{\tilde N}}(\omega) \big| \leq \frac{\tilde\eta}{8}. \tag{3.11}$$

We choose once for all such a neighborhood V_ℓ.

Notice that $V_\ell \cap \operatorname{supp}(\mu) \neq \emptyset$ and in particular $\mu(V_\ell) > 0$. Since ω is generic in the sense of the two equalities in (3.7), we can choose n large enough so that

$$\frac{S_{\Lambda_n}\chi_\ell(\omega)}{\#\Lambda_{n+\tilde N}} \geq \frac{\mu(V_\ell)}{2} \quad \text{and} \quad \frac{H_{\Lambda_{n+\tilde N}}(\omega)}{\#\Lambda_{n+\tilde N}} < \bar H + \frac{\tilde\eta}{8}\frac{\mu(V_\ell)}{\#\Lambda_{2\tilde M}}. \tag{3.12}$$

Denote $A_n := \{j \in \Lambda_n : \theta^j(\omega) \in V_\ell\}$. Let then $B_n \subset A_n$ be a maximal subcollection of indices such that $(j + \Lambda_{\tilde M}) \cap (k + \Lambda_{\tilde M}) = \emptyset$ whenever $j, k \in B_n$ are distinct. Since for all $j \in A_n$ there must exist $k \in B_n$ such that $(j + \Lambda_{\tilde M}) \cap (k + \Lambda_{\tilde M}) \neq \emptyset$, we have

$$\begin{aligned} \#A_n &\leq \#\big\{(j,k) \in A_n \times B_n : (j + \Lambda_{\tilde M}) \cap (k + \Lambda_{\tilde M}) \neq \emptyset\big\} \\ &\leq \sum_{k\in B_n} \#(A_n \cap (k + \Lambda_{\tilde M} - \Lambda_{\tilde M})) \leq \#B_n \, \#\Lambda_{2\tilde M}, \end{aligned}$$

which together with (3.12) yields

$$\frac{\#B_n}{\#\Lambda_{n+\tilde N}} > \frac{\mu(V_\ell)}{2\#\Lambda_{2\tilde M}}. \tag{3.13}$$

Let $\omega^n \in \Omega$ be the configuration which coincides with $\theta^{-j}(\tilde\omega)$ on $j+\Lambda_{\tilde N}$ for every $j \in B_n$ and with ω on the complement $\mathbb{Z}^d \setminus \sqcup_{j\in B_n}(j+\Lambda_{\tilde N})$:

$$\forall\, j \in B_n, \quad \omega^n_{j+\Lambda_{\tilde N}} = \big(\theta^{-j}(\tilde\omega)\big)_{j+\Lambda_{\tilde N}} \quad \text{and} \quad \omega^n_{\mathbb{Z}^d\setminus\sqcup_{j\in B_n}(\Lambda_{\tilde N}+j)} = \omega_{\mathbb{Z}^d\setminus\sqcup_{j\in B_n}(\Lambda_{\tilde N}+j)}.$$

Notice that $\theta^j(\omega^n)$ coincides with $\tilde\omega$ on $\Lambda_{\tilde N}$ and with $\theta^j(\omega)$ on $\Lambda_{\tilde M}\setminus\Lambda_{\tilde N}$. From the definition of the map P, we obtain that

$$\big(\theta^j(\omega^n)\big)_{\Lambda_{\tilde M}} = \big(P(\theta^j(\omega^n))\big)_{\Lambda_{\tilde M}} \text{ and } P(\theta^j(\omega^n)) = P(\theta^j(\omega)), \quad \forall\, j \in B_n.$$

Hence, since $\theta^j(\omega) \in V_\ell$ for any $j \in B_n$, thanks to (3.10) we get

$$\sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset\\ \operatorname{diam}(A)\le 2dM}} \Phi_A(\theta^j(\omega^n)) \le \frac{\tilde\eta}{8} + \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset\\ \operatorname{diam}(A)\le 2dM}} \Phi_A(\tilde\omega). \tag{3.14}$$

We decompose $H_{\Lambda_{n+\tilde N}}(\omega^n) = H^{\mathrm{I}}(\omega^n)+H^{\mathrm{II}}(\omega^n)+H^{\mathrm{III}}(\omega^n)$ in three terms, where for a configuration $\bar\omega$ of Ω

$$\begin{aligned}
H^{\mathrm{I}}(\bar\omega) &:= \sum_{\substack{A\cap(\bigsqcup_{j\in B_n}(j+\Lambda_{\tilde N}))\neq\emptyset\\ \operatorname{diam}(A)\le 2dM}} \Phi_A(\bar\omega),\\
H^{\mathrm{II}}(\bar\omega) &:= \sum_{\substack{A\cap\Lambda_{n+\tilde N}\neq\emptyset\\ A\cap(\bigsqcup_{j\in B_n}(j+\Lambda_{\tilde N}))=\emptyset}} \Phi_A(\bar\omega),\\
H^{\mathrm{III}}(\bar\omega) &:= \sum_{\substack{A\cap(\bigsqcup_{j\in B_n}(j+\Lambda_{\tilde N}))\neq\emptyset\\ \operatorname{diam}(A)>2dM}} \Phi_A(\bar\omega).
\end{aligned}$$

For the first term, we obtain

$$
\begin{aligned}
H^{\mathrm{I}}(\omega^n) &= \sum_{j\in B_n} \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \mathrm{diam}(A)\le 2dM}} \Phi_A(\theta^j(\omega^n)) && (\{j+\Lambda_{\tilde M}\}_{j\in B_n} \text{ are disjoint}) \\
&\le \#B_n\Big(\sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \mathrm{diam}(A)\le 2dM}} \Phi_A(\tilde\omega) + \frac{\tilde\eta}{8}\Big) && (\text{inequality (3.14)}) \\
&\le \#B_n\Big(H_{\Lambda_{\tilde N}}(\tilde\omega) + \frac{\tilde\eta}{4}\Big) && (\text{inequality (3.9)}) \\
&\le \#B_n\Big(H_{\Lambda_{\tilde N}}(\omega) - \frac{3\tilde\eta}{4}\Big) && (\text{inequality (3.8)}) \\
&\le \sum_{j\in B_n} \Big(H_{\Lambda_{\tilde N}}(\theta^j(\omega)) - \frac{5\tilde\eta}{8}\Big) && (\text{inequality (3.11)}) \\
&\le \sum_{j\in B_n} \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \mathrm{diam}(A)\le 2dM}} \Big(\Phi_A(\theta^j(\omega)) - \frac{\tilde\eta}{2}\Big) && (\text{inequality (3.9)}) \\
&= \sum_{\substack{A\cap(\bigsqcup_{j\in B_n}(j+\Lambda_{\tilde N}))\neq\emptyset \\ \mathrm{diam}(A)\le 2dM}} \Phi_A(\omega) - \frac{\tilde\eta}{2}\#B_n && (\{j+\Lambda_{\tilde M}\}_{j\in B_n} \text{ are disjoint}) \\
&= H^{\mathrm{I}}(\omega) - \frac{\tilde\eta}{2}\#B_n
\end{aligned}
$$

For the second term, we use the fact that ω^n and ω coincide on $\mathbb{Z}^d \setminus \sqcup_{j\in B_n}(j+\Lambda_{\tilde N})$ so that

$$
H^{\mathrm{II}}(\omega^n) = \sum_{\substack{A\cap\Lambda_{n+\tilde N}\neq\emptyset \\ A\cap(\bigsqcup_{j\in B_n}(j+\Lambda_{\tilde N}))=\emptyset}} \Phi_A(\omega) \;=\; H^{\mathrm{II}}(\omega).
$$

For the third term, we use again twice inequality (3.9). Here A may intersect several $j+\Lambda_{\tilde N}$, $j\in B_n$, but

$$
\begin{aligned}
H^{\mathrm{III}}(\omega^n) - H^{\mathrm{III}}(\omega) &= \sum_{\substack{A\cap(\bigsqcup_{j\in B_n}(j+\Lambda_{\tilde N}))\neq\emptyset \\ \mathrm{diam}(A)>2dM}} \Phi_A(\omega^n) - \Phi_A(\omega) \\
&\le \sum_{j\in B_n} \sum_{\substack{A\cap\Lambda_{\tilde N}\neq\emptyset \\ \mathrm{diam}(A)>2dM}} |\Phi_A(\theta^j(\omega^n)) - \Phi_A(\theta^j(\omega))| \;\le\; \frac{\tilde\eta}{4}\#B_n.
\end{aligned}
$$

We finally collect the three terms and obtain for n sufficiently large

$$
\inf_{\bar\omega\in\Omega} \frac{H_{\Lambda_{n+\tilde N}}(\bar\omega)}{\#\Lambda_{n+\tilde N}} \le \frac{H_{\Lambda_{n+\tilde N}}(\omega^n)}{\#\Lambda_{n+\tilde N}} \le \frac{H_{\Lambda_{n+\tilde N}}(\omega)}{\#\Lambda_{n+\tilde N}} - \frac{\tilde\eta}{4}\frac{\#B_n}{\#\Lambda_{n+\tilde N}} \le \bar H - \frac{\tilde\eta}{8}\frac{\mu(V_\ell)}{\#\Lambda_{2\tilde M}},
$$

which contradicts the characterization of the constant $\bar H$ in proposition 3.7. □

Since any periodic configuration gives rise to a translation-invariant probability measure supported on its orbit, we point out an immediate corollary.

Corollary 3.11. *A periodic configuration $\omega \in \Omega$ that verifies*

$$\lim_{n\to\infty} \frac{1}{\#\Lambda_n} H_{\Lambda_n}(\omega) = \bar{H}$$

is a ground-state configuration with respect to H.

References

[1] A. Bovier, *Statistical mechanics of disordered systems: A mathematical perspective*, Cambridge Series in Statistical and Probabilistic Mathematics **18**, Cambridge University Press, Cambridge, 2006.

[2] A. C. D. van Enter, R. Fernández and A. D. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, *Journal of Statistical Physics* **72** (1993), 879-1167.

[3] H. O. Georgii, *Gibbs measures and phase transitions*, 2nd edition, De Gruyter Studies in Mathematics **9**, Walter de Gruyter, Berlin, 2011.

[4] N. M. Hugenholtz, C^*-albegras and statistical mechanics, in *Proceedings of Symposia in Pure Mathematics*, *Volume* 38, *Part* 2 (American Mathematical Society, Providence, Rhode Island, 1982), pp. 407-465.

[5] O. Jenkinson, Ergodic optimization, *Discrete and Continuous Dynamical Systems, Series A* **15** (2006), 197-224.

[6] G. Keller, *Equilibrium states in ergodic theory*, London Mathematical Society Students Texts **42**, Cambridge University Press, Cambridge, 1998.

[7] R. Schrader, Ground states in classical lattice systems with hard core, *Communications in Mathematical Physics* **16** (1970), 247-260.