AN ALPHABETICAL APPROACH TO NIVAT'S CONJECTURE

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Abstract. Since techniques used to address the Nivat's conjecture usually relies on Morse-Hedlund Theorem, an improved version of this classical result may mean a new step towards a proof for the conjecture. In this paper, considering an alphabetical version of the Morse-Hedlund Theorem, we show that, for a configuration $\eta \in A^{\mathbb{Z}^2}$ that contains all letters of a given finite alphabet A, if its complexity with respect to a quasi-regular set $\mathcal{U} \subset \mathbb{Z}^2$ (a finite set whose convex hull on $\mathbb{R}^2$ is described by pairs of edges with identical size) is bounded from above by $\frac{1}{2}|\mathcal{U}| + |A| - 1$, then η is periodic.

1. Introduction

Fixed a finite alphabet A (with at least two elements), for $n \in \mathbb{N}$, the *n-complexity* of an infinite sequence $\xi = (\xi_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}}$, denoted by $P_\xi(n)$, is defined to be the number of distinct words of the form $\xi_j \xi_{j+1} \cdots \xi_{j+n-1}$ appearing in ξ. In 1938, Morse and Hedlund [10, 11] proved one of the most famous results in symbolic dynamics which establishes a connection between periodic sequences (sequences for which there is an integer $m \geq 1$ such that $\xi_{i+m} = \xi_i$ for all $i \in \mathbb{Z}$) and complexity. More specifically, they proved that $\xi \in A^{\mathbb{Z}}$ is periodic if, and only if, there exists $n \in \mathbb{N}$ such that $P_\xi(n) \leq n$.

A natural extension of the complexity function to higher dimensions is obtained when we consider, instead of words, blocks of symbols. More precisely, the $n_1 \times \cdots \times n_d$*-complexity* of a configuration $\eta = (\eta_g)_{g \in \mathbb{Z}^d} \in A^{\mathbb{Z}^d}$, denoted by $P_\eta(n_1, \ldots, n_d)$, is defined to be the number of distinct $n_1 \times \cdots \times n_d$ blocks of symbols appearing in η. Of course periodicity also has a natural higher dimensional generalization: $\eta \in A^{\mathbb{Z}^d}$ is *periodic* if there exists a vector $h \in (\mathbb{Z}^d)^*$, called period of η, such that $\eta_{g+h} = \eta_g$ for all $g \in \mathbb{Z}^d$. If $\eta \in A^{\mathbb{Z}^2}$ has two linearly independent periods, it is said to be *doubly periodic*. A configuration that is not periodic is said to be *aperiodic*.

The Nivat's Conjecture [12] is the natural generalization of the Morse-Hedlund Theorem for the two-dimensional case.

Conjecture (Nivat). *For a configuration $\eta \in A^{\mathbb{Z}^2}$, if there exist integers $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, then η is periodic.*

The first step towards the conjecture was given by Sander and Tijdeman [14]: they showed that if $P_\eta(n, 2) \leq 2n$ (or if $P_\eta(2, n) \leq 2n$) for some integer $n \in \mathbb{N}$, then $\eta \in A^{\mathbb{Z}^2}$ is periodic. Other weak forms of the Nivat's Conjecture were obtained in [6, 13, 5, 3, 9, 8, 2]. Moreover, Sander and Tijdeman [15] found counter-examples to the analogue of Nivat's Conjecture in higher dimensions, that is, they showed that,

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for $d \geq 3$, there exist aperiodic configurations $\eta \in \{0,1\}^{\mathbb{Z}^d}$ such that $P_\eta(n,\ldots,n) = 2n^{d-1}+1$.

The best result known so far was obtained by Bryna Kra and Van Cyr [4]. Using the notion of expansive subspaces of $\mathbb{R}^2$ introduced by Boyle and Lind, they shed a new light towards a proof for Nivat's Conjecture by relating expansive subspaces to periodicity. In particular, they proved that if there exist integers $n,k \in \mathbb{N}$ such that $P_\eta(n,k) \leq \frac{1}{2}nk$, then $\eta \in A^{\mathbb{Z}^2}$ is periodic.

Our main result (Theorem 2.2) is an "alphabetical" version of Cyr and Kra's bound, which in fact provides a slight improvement in estimation. Moreover, we consider the complexity function with respect to a more general class of sets, called quasi-regular sets (see Definition 2.1). In the particular case of blocks, we show that, for a configuration $\eta \in A^{\mathbb{Z}^2}$ that contains all letters of A, if there exist integers $n,k \in \mathbb{N}$ such that

$$P_\eta(n,k) \leq \frac{1}{2}nk + |A| - 1 = \left(\frac{1}{2} + \frac{|A|-1}{nk}\right)nk, \tag{1.1}$$

where $|A|$ denotes the cardinality of the alphabet A, then η is periodic.

Here is an example of configuration that satisfies (1.1) but does not satisfy the condition of Cyr and Kra's Theorem. Let A be the alphabet formed by the colours "white" and "black" and define $\eta \in A^{\mathbb{Z}^2}$ as $\eta_g :=$ "black" if $g = (a,a) + b(\sum_{i=6}^{c} i, 0)$, where $a \in \mathbb{Z}$, $b \in \{-1,0,1\}$ and $c \geq 6$, and $\eta_g :=$ "white" otherwise (see Figure 1). Note that $P_\eta(n,k) = n+k$ when $n+k \leq 7$ and that from the symmetries of such configuration

$$P_\eta(n,k) = n+k+\frac{1}{2}(n+k-7)(n+k-6)$$

when $n+k > 7$. It is easy to see that there are no integers $n,k \in \mathbb{N}$ such that $P_\eta(n,k) \leq \frac{1}{2}nk$. However, one has $P_\eta(3,4) = 7 = \frac{1}{2}\cdot 12 + |A| - 1$.

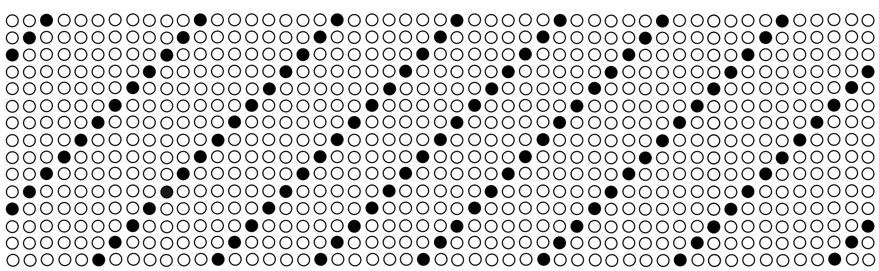


FIGURE 1. Representation of the configuration $\eta \in A^{\mathbb{Z}^2}$.

2. Initial concepts and main result

Let A be endowed with the discrete topology. It is well known that the configuration space $A^{\mathbb{Z}^d}$ equipped with the product topology is a metrizable compact space. For each $u \in \mathbb{Z}^d$, let $T^u : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ be the shift application, i.e., for $\eta = (\eta_g)_{g\in\mathbb{Z}^d} \in A^{\mathbb{Z}^d}$, the configuration $T^u\eta$ is defined by $(T^u\eta)_g := \eta_{g+u}$ for all $g \in \mathbb{Z}^d$. It is easy to see that, with respect to this topology, the $\mathbb{Z}^d$-action by shift applications $(T^u : u \in \mathbb{Z}^d)$ is continuous. Let $Orb\,(\eta) := \{T^u\eta : u \in \mathbb{Z}^d\}$ denote the $\mathbb{Z}^d$-orbit of $\eta \in A^{\mathbb{Z}^d}$ and set $X_\eta := \overline{Orb\,(\eta)}$, where the bar denotes the closure.

Following Sander and Tijdeman [15], for a nonempty set $\mathcal{S} \subset \mathbb{Z}^d$, the $\mathcal{S}$-complexity of $\eta \in A^{\mathbb{Z}^d}$, denoted by $P_\eta(\mathcal{S})$, is defined to be the number of distinct $\mathcal{S}$-configurations of the form $(T^u\eta)|\mathcal{S} \in A^{\mathcal{S}}$, where $u \in \mathbb{Z}^d$ and $\cdot|\mathcal{S}$ means the restriction to the set $\mathcal{S}$. The set of all $\mathcal{S}$-configurations of $\eta \in A^{\mathbb{Z}^d}$ is denoted by

$$L(\mathcal{S}, \eta) := \left\{(T^u\eta)|\mathcal{S} \in A^{\mathcal{S}} : u \in \mathbb{Z}^d\right\}.$$

Clearly $\mathcal{T} \subset \mathcal{S}$ implies $P_\eta(\mathcal{T}) \leq P_\eta(\mathcal{S})$. If $\mathcal{S} \subset \mathbb{Z}^d$, then $P_\eta(\mathcal{S}) = P_{T^u\eta}(\mathcal{S}+g)$ for all $u, g \in \mathbb{Z}^d$ and $P_x(\mathcal{S}) \leq P_\eta(\mathcal{S})$ for any $x \in X_\eta$. We remark that for a $n_1 \times \cdots \times n_d$ block based at the origin, i.e.,

$$R_{n_1,\ldots,n_d} := \left\{(t_1, \ldots, t_d) \in \mathbb{Z}^d : 0 \leq t_i < n_i \text{ for every } 1 \leq i \leq d\right\},$$

the previous notion $P_\eta(n_1, \ldots, n_d)$ coincides with $P_\eta(R_{n_1,\ldots,n_d})$.

A set $\mathcal{S} \subset \mathbb{Z}^2$ is called *convex* if its convex hull in $\mathbb{R}^2$, denoted by $\operatorname{conv}(\mathcal{S})$, is closed and $\mathcal{S} = \operatorname{conv}(\mathcal{S}) \cap \mathbb{Z}^2$. If $\mathcal{S} \subset \mathbb{Z}^2$ is a convex set, a point $g \in \mathcal{S}$ is a *vertex* of $\mathcal{S}$ when $\mathcal{S}\backslash\{g\}$ is a convex subset, and a line segment w contained at the boundary of $\operatorname{conv}(\mathcal{S})$ is an *edge* of $\mathcal{S}$ if it is an edge of the convex polygon $\operatorname{conv}(\mathcal{S}) \subset \mathbb{R}^2$. Let $V(\mathcal{S})$ and $E(\mathcal{S})$ denote, respectively, the sets of vertices and edges of $\mathcal{S}$.

Let $\mathcal{F}_C$ denote the family of convex, finite and nonempty subsets of $\mathbb{Z}^2$, and $\mathcal{F}_C^{Vol}$ denote the subfamily of $\mathcal{F}_C$ whose convex hull has positive area.

If $\mathcal{S} \subset \mathbb{Z}^2$ is a convex set (possibly infinite) such that $\operatorname{conv}(\mathcal{S})$ has non-null area, our standard convention is that the boundary of $\operatorname{conv}(\mathcal{S})$ is positively oriented. With this convention, each edge $w \in E(\mathcal{S})$ inherits a natural orientation from the boundary of $\operatorname{conv}(\mathcal{S})$.

In the sequel, by an oriented object we mean an oriented line, an oriented line segment or a vector. Remember that two vectors are parallel if they have the same direction and antiparallel if they have opposite directions. Two oriented objects in $\mathbb{R}^2$ are said to be *(anti)parallel* if the adjacent vectors to their respective orientations are (anti)parallel.

Definition 2.1. *We say that $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is a quasi-regular set when, for every edge $w \in E(\mathcal{U})$, there is an edge $w' \in E(\mathcal{U})$ antiparallel to w satisfying $|w' \cap \mathcal{U}| = |w \cap \mathcal{U}|$.*

We may now state our main result.

Theorem 2.2. *If $\eta \in A^{\mathbb{Z}^2}$ contains all letters of the alphabet A and there exists a quasi-regular set $\mathcal{U} \in \mathcal{F}_C^{Vol}$ such that $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |A| - 1$, then η is periodic.*

The proof of Theorem 2.2 will be done by contradiction. The global strategy consists basically in showing the existence of an aperiodic accumulation point which is doubly periodic on an unbounded region, and then in arguing that the cardinality of subconfigurations arising in the boundary of this region would be greater than possible. We will first apply the Alphabetical Morse-Hedlund Theorem to certain strips, defined from special generating sets, called balanced sets (see Definitions 3.8 and 4.6), to get periodic configurations, which will allow us, by an inductive argument, to construct such an accumulation point.

The rest of the paper is organized as follows. The next section reviews key concepts and results. In Section 4, with an alphabetical viewpoint, we obtain a wide range of propositions which will be useful for the proof of the main theorem. Roughly speaking, these results connect one-sided nonexpansive directions (which will be precisely defined in Section 3) and periodicity. Following methods highlighted by Cyr and Kra [4], in the Section 6 we provide bounds for the periods of any aperiodic configuration with an unbounded doubly periodic region (see Definition 5.1) and in the last section we prove Theorem 2.2. The results of this article are based on the PhD thesis of the first author written under the guidance of the second author.

3. Fundamental notions and facts

From now on, we will assume that the configuration $\eta \in A^{\mathbb{Z}^2}$ always contains all letters of the alphabet A, i.e., $P_\eta(\{g\}) = |A|$ for all $g \in \mathbb{Z}^2$.

3.1. **Alphabetical Morse-Hedlund Theorem.**

We reproduce here an alphabetical version of the celebrated result of Morse and Hedlund (see Theorem 7.4 in [10]).

Given a sequence $\xi = (\xi_t)_{t\in U} \in A^U$, where $U = \{a, a+1, \ldots\}$ and $a \in \mathbb{Z}$, the *n-complexity* of ξ, denoted by $P_\xi(n)$, is defined to be the number of distinct words of the form $\xi_t\xi_{t+1}\cdots\xi_{t+n-1}$, where $\{t, t+1, \ldots, t+n-1\} \subset U$. Such sequence is said to be *periodic* if there exists an integer $m \geq 1$ (called period) that satisfies $\xi_{i+m} = \xi_i$ for all $i \in U$.

Theorem 3.1 (Alphabetical Morse-Hedlund Theorem)**.** *Let $\xi = (\xi_i)_{i\in U} \in A^U$ be a sequence that contains all letters of A, where $U = \mathbb{Z}$ or $U = \{a, a+1, \ldots\}$ for some $a \in \mathbb{Z}$. Suppose there exists $n_0 \in \mathbb{N}$ such that $P_\xi(n_0) \leq n_0'$, where $n_0' := n_0 + |A| - 2$.*

- *(i) If $U = \{a, a+1, \ldots\}$, then the sequence $(\xi_t)_{t\in U+n_0'} \in A^{U+n_0'}$ is periodic of period at most n_0';*
- *(ii) If $U = \mathbb{Z}$, then the sequence $\xi \in A^{\mathbb{Z}}$ is periodic of period at most n_0'.*

As an immediate consequence, we have the next result.

Corollary 3.2. *Let $\eta \in A^{\mathbb{Z}^2}$ and suppose $P_\eta(\mathcal{S}) \leq |\mathcal{S}| + |A| - 2$ for some set $\mathcal{S} \in \mathcal{F}_C$. If $\mathrm{conv}(\mathcal{S})$ has null area, then η is periodic.*

3.2. **Expansive subdynamics.**

Let F be a subspace of $\mathbb{R}^d$. For each $g \in \mathbb{Z}^d$, denote $\mathrm{dist}(g, F) := \inf\{\|g - u\| : u \in F\}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. Given $t > 0$, the t-neighbourhood of F is defined as

$$F^t := \left\{g \in \mathbb{Z}^d : \mathrm{dist}(g, F) \leq t\right\}.$$

Let $X \subset A^{\mathbb{Z}^d}$ be a subshift (a closed subset which is invariant for shift applications). Following Boyle and Lind [1], we say that a subspace $F \subset \mathbb{R}^d$ is *expansive* on X if there exists $t > 0$ such that $x|F^t = y|F^t$ implies $x = y$ for any $x, y \in X$. If a subspace fails to meet this condition, it is called a *nonexpansive* subspace on X. Boyle and Lind [1, Theorem 3.7] showed that if $X \subset A^{\mathbb{Z}^d}$ is an infinite subshift, then, for $0 \leq k < d$, there exists a k-dimensional subspace of $\mathbb{R}^d$ that is nonexpansive on X.

As an immediate corollary from Boyle and Lind's Theorem, we highlight the following result.

Corollary 3.3. *For $\eta \in A^{\mathbb{Z}^2}$, if every one-dimensional subspace of $\mathbb{R}^2$ is expansive on X_η, then η is doubly periodic.*

The next result allows us to conclude that every configuration with at least two nonexpansive one-dimensional subspaces on X_η is not periodic.

Lemma 3.4. *If $\eta \in A^{\mathbb{Z}^2}$ is periodic of period $h \in (\mathbb{Z}^2)^*$, then every one-dimensional subspace $F \subset \mathbb{R}^2$ that does not contain h is expansive on X_η.*

Proof. Let $t > 0$ be such that $\{-h, h\} \subset F^t$. For each $g \in \mathbb{Z}^2$, there exists an integer $m \in \mathbb{Z}$ such that $g + mh \in F^t$. Since all configurations $x, y \in X_\eta$ are periodic of period $h \in (\mathbb{Z}^2)^*$, if $x|F^t = y|F^t$, then $x_g = x_{g+mh} = y_{g+mh} = y_g$. □

In the above lemma, note that there is no assumption about the expansiveness or nonexpansiveness of the one-dimensional subspace which contains the period h, so that doubly periodic configurations naturally satisfy its statement. Actually, if a configuration is doubly periodic, by applying Lemma 3.4 to linearly independent periods, we see that all its one-dimensional subspaces are expansive.

For a line $\ell \subset \mathbb{R}^2$, we also use ℓ to denote this line endowed of a given orientation. We believe that, according to the context, the reader will easily realize if we refer to a line or to an oriented line.

Notation 3.5. *For an oriented line $\ell \subset \mathbb{R}^2$, let $\rotatebox[origin=c]{180}{ℓ} \subset \mathbb{R}^2$ denote the oriented line antiparallel to ℓ that intersects it. Obviously $\rotatebox[origin=c]{180}{ℓ}$ determines the same points in $\mathbb{R}^2$ that ℓ, but is endowed of the opposite orientation.*

A convex set $\mathcal{H} \subset \mathbb{Z}^2$ is said to be a *half plane* if $\mathrm{conv}(\mathcal{H})$ has non-null area and $E(\mathcal{H})$ has only a single edge. In this case, the single edge $\ell \in E(\mathcal{H})$ is a line in $\mathbb{R}^2$. For an oriented line $\ell \subset \mathbb{R}^2$ that contains at least one point of $\mathbb{Z}^2$, let $\mathcal{H}(\ell) \subset \mathbb{Z}^2$ denote the unique half plane for which ℓ is its single (positively) oriented edge (see Figure 2). This means not only that $\ell \in E(\mathcal{H}(\ell))$ but also the orientation of the edge of the half plane $\mathcal{H}(\ell)$ agrees with the orientation of ℓ.

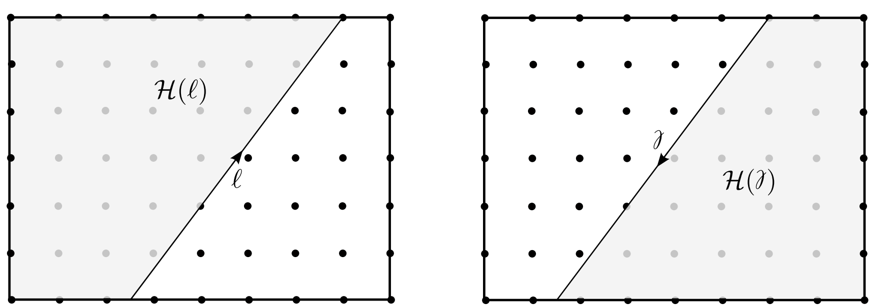


FIGURE 2. The oriented lines ℓ and $\rotatebox[origin=c]{180}{ℓ}$ and the half planes $\mathcal{H}(\ell)$ and $\mathcal{H}(\rotatebox[origin=c]{180}{ℓ})$.

Notation 3.6. *If ℓ is a rational oriented line (i.e., an oriented line with rational angular coefficient) that contains at least one point of $\mathbb{Z}^2$, let $\ell^{(-)} \subset \mathbb{R}^2$ denote the oriented line parallel to ℓ such that the half plane $\mathcal{H}(\ell^{(-)})$ is minimal (with respect to partial ordering by inclusion) among all half planes that strictly contains $\mathcal{H}(\ell)$. Likewise, let $\ell^{(+)} \subset \mathbb{R}^2$ denote the oriented line parallel to ℓ such that the half plane $\mathcal{H}(\ell^{(+)})$ is maximal among all half planes that are strictly contained in $\mathcal{H}(\ell)$ (see Figure 3).*

We use $\mathbb{G}_1$ to denote the set of all lines through the origin in $\mathbb{R}^2$, i.e., the set of one-dimensional subspaces. In a slight abuse of notation, we also say that oriented lines belong to $\mathbb{G}_1$. If $\ell \in \mathbb{G}_1$ is a rational oriented line, let $\vec{v}_\ell \in (\mathbb{Z}^2)^*$ denote the non-null vector parallel to ℓ of minimum norm.

In the sequel, we restate a refined version of the classical notion of expansiveness called one-sided nonexpansiveness, and introduced by Cyr and Kra in [4].

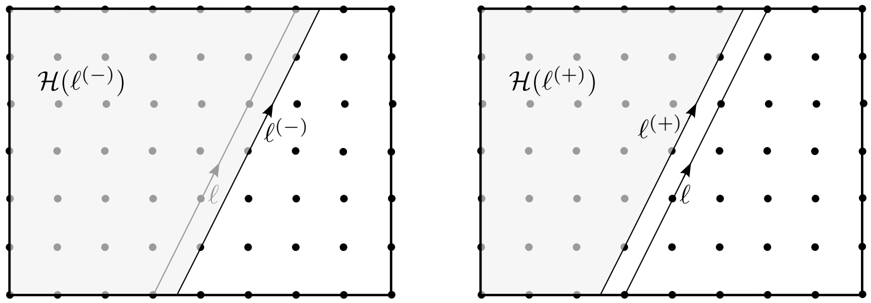


FIGURE 3. The oriented lines $\ell^{(-)}$ and $\ell^{(+)}$ and the half planes $\mathcal{H}(\ell^{(-)})$ and $\mathcal{H}(\ell^{(+)})$.

Definition 3.7. *Given $\eta \in A^{\mathbb{Z}^2}$, we say that an oriented line $\ell \in \mathbb{G}_1$ is a one-sided expansive direction on X_η if $x|\mathcal{H}(\ell) = y|\mathcal{H}(\ell)$ implies $x = y$ for any $x, y \in X_\eta$. If an oriented line $\ell \in \mathbb{G}_1$ fails to meet this condition, it is called a one-sided nonexpansive direction on X_η.*

As X_η is a compact subshift of $A^{\mathbb{Z}^2}$, it is easy to see that $\ell \in \mathbb{G}_1$ is an expansive line on X_η if, and only if, $\ell, \reflectbox{$\ell$} \in \mathbb{G}_1$ are one-sided expansive directions on X_η. Moreover, in all the text, whenever we consider a nonexpansive line, this actually means a nonexpansive rational line, since any irrational line is expansive on X_η if the configuration η satisfies $P_\eta(\mathcal{U}) \leq |\mathcal{U}| + |A| - 2$ for some $\mathcal{U} \in \mathcal{F}_C$ (see Remark 3.12).

3.3. **Generating sets.** The notion of generating set, deeply developed in [4], underlines configurations that admit a unique extension on extreme points of a given convex set.

Definition 3.8. *Let $\eta \in A^{\mathbb{Z}^2}$ and suppose $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set. A point $g \in \mathcal{S}$ is said to be η-generated by $\mathcal{S}$ if $P_\eta(\mathcal{S}) = P_\eta(\mathcal{S}\backslash\{g\})$. A set $\mathcal{S} \in \mathcal{F}_C$ for which each vertex is η-generated is called an η-generating set.*

Note that $g \in \mathcal{S}$ is η-generated by $\mathcal{S}$ if, and only if, for every $\gamma \in L(\mathcal{S}\backslash\{g\}, \eta)$, there exists a unique $\gamma' \in L(\mathcal{S}, \eta)$ such that $\gamma'|\mathcal{S}\backslash\{g\} = \gamma$.

Remark 3.9. *If $P_\eta(\mathcal{U}) \leq |\mathcal{U}| + |A| - 2$ for some set $\mathcal{U} \in \mathcal{F}_C$, then any convex set $\mathcal{S} \subset \mathcal{U}$ that is minimal among all convex sets $\mathcal{T} \subset \mathcal{U}$ fulfilling $P_\eta(\mathcal{T}) \leq |\mathcal{T}| + |A| - 2$ is an η-generating set. The fact that $1 + |A| - 2 < |A| = P_\eta(\{g\})$ for all $g \in \mathbb{Z}^2$ ensures that $\mathcal{S}$ has at least two points. In particular, if $w \in E(\mathcal{S})$, by minimality, one has $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S}\backslash w) \leq |w \cap \mathcal{S}| - 1$.*

Notation 3.10. *For an oriented line $\ell \subset \mathbb{R}^2$ and a convex set $\mathcal{S} \subset \mathbb{Z}^2$, we use $\ell_{\mathcal{S}}$ to denote the oriented line $\ell' \subset \mathbb{R}^2$ parallel to ℓ such that $\mathcal{S} \subset \mathcal{H}(\ell')$ and $\ell' \cap \mathcal{S} \neq \emptyset$. Note that either $\ell_{\mathcal{S}} \cap \mathcal{S}$ is a vertex of $\mathcal{S}$ or $\mathrm{conv}(\ell_{\mathcal{S}} \cap \mathcal{S}) \subset \mathbb{R}^2$ is an edge of $\mathcal{S}$ (see Figure 4).*

The next lemma is an immediate consequence of the existence of generating sets. Its proof is straightforward from definitions.

Lemma 3.11. *If $\ell \in \mathbb{G}_1$ is an oriented line and there is a set $\mathcal{S} \in \mathcal{F}_C$ such that $\ell_{\mathcal{S}} \cap \mathcal{S} = \{g_0\}$ is η-generated by $\mathcal{S}$, then ℓ is a one-sided expansive direction on X_η.*

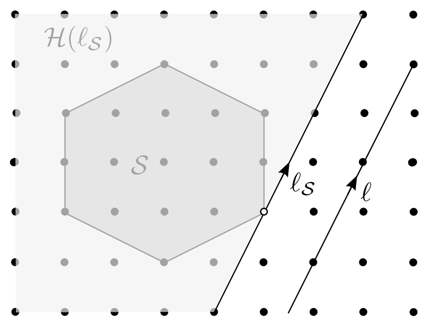


FIGURE 4. The set $\mathcal{S}$ and the oriented lines ℓ and $\ell_{\mathcal{S}}$.

Remark 3.12. *Given $\eta \in A^{\mathbb{Z}^2}$ with $P_\eta(\mathcal{U}) \leq |\mathcal{U}| + |A| - 2$ for some $\mathcal{U} \in \mathcal{F}_C$, if $\ell \in \mathbb{G}_1$ is an irrational oriented line, then there exists an η-generating set $\mathcal{S} \in \mathcal{F}_C$ such that $\ell_{\mathcal{S}} \cap \mathcal{S} = \{g_0\}$ is η-generated by $\mathcal{S}$ and so, from Lemma 3.11, it follows that ℓ is a one-sided expansive direction on X_η. Furthermore, if $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η, the above lemma also ensures that every η-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ has an edge parallel to ℓ, i.e., $|\ell_{\mathcal{S}} \cap \mathcal{S}| \geq 2$.*

The next lemma is an immediate consequence of the compactness of the subshift.

Lemma 3.13. *Let $\ell \in \mathbb{G}_1$ be a rational oriented line. If ℓ is a one-sided expansive direction on X_η, then there is a set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ such that $\ell_{\mathcal{S}} \cap \mathcal{S} = \{g_0\}$ is η-generated by $\mathcal{S}$.*

For $\ell \in \mathbb{G}_1$, we call *ℓ-strip* any lattice translation of $\ell^t = \{g \in \mathbb{Z}^2 : \text{dist}(g, \ell) \leq t\}$, where $t > 0$.

For $\eta \in A^{\mathbb{Z}^d}$ and $\mathcal{U} \subset \mathbb{Z}^d$ nonempty, we say that $\eta|\mathcal{U} \in A^{\mathcal{U}}$ is *periodic* of period $h \in (\mathbb{Z}^d)^*$ if $\eta_{g+h} = \eta_g$ for every $g \in \mathcal{U} \cap (\mathcal{U} - h)$. Clearly, this notion of periodicity extend the classical one. Given $\ell \in \mathbb{G}_1$, to indicate that some period h belongs to $\ell \cap \mathbb{Z}^2$, we say that $\eta|\mathcal{U}$ is *ℓ-periodic*.

For $x \in X_\eta$, if $\mathcal{S}$ is a subset of $\mathbb{R}^2$, we make a slight abuse of the notation by denoting $x|\mathcal{S}$ instead of $x|\mathcal{S} \cap \mathbb{Z}^2$.

Proposition 3.14. *If $\eta \in A^{\mathbb{Z}^2}$ is periodic and $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η, then $\reflectbox{\ensuremath{\ell}}$ is also a one-sided nonexpansive direction on X_η.*

Proof. Suppose, by contradiction, that $\reflectbox{\ensuremath{\ell}}$ is a one-sided expansive direction on X_η. Thanks to Lemma 3.13 there is a set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ such that $\reflectbox{\ensuremath{\ell}}_{\mathcal{S}} \cap \mathcal{S} = \{g_0\}$ is η-generated by $\mathcal{S}$. Let $F \subset \mathbb{Z}^2$ be an ℓ-strip that contains a translation of $\mathcal{S}$ and consider a finite convex set $B \subset F$ such that, for any $g, g' \in \mathbb{Z}^2$, $(T^g\eta)|B = (T^{g'}\eta)|B$ implies $(T^g\eta)|F = (T^{g'}\eta)|F$. Such a subset exists because according to Lemma 3.4 the line ℓ contains all periods of η. Let $\hat{\ell} \in \mathbb{G}_1$ be the orthogonal line to ℓ and consider a non-null vector $v \in \hat{\ell} \cap \mathcal{H}(\ell)$. For any $\tau \in \mathbb{Z}$ there exist $\tau \leq t < t' \leq \tau + P_\eta(B)$ with $(T^{tv}\eta)|B = (T^{t'v}\eta)|B$ and hence with $(T^{tv}\eta)|F = (T^{t'v}\eta)|F$. Since $g_0 \in \mathcal{S} \subset F$ is η-generated by $\mathcal{S}$, it is easy to argue by induction that the restriction of η to the half plane $\mathcal{H}(\ell)$ is periodic of period $(t' - t)v$, where $t' - t \leq P_\eta(B)$. As τ is arbitrary, we conclude that η is $\hat{\ell}$-periodic, but this contradicts Lemma 3.4. □

We introduce a notion motivated by the existence of sets that are not necessary η-generating, but, with respect to a fixed direction, part of their vertices are η-generated.

Definition 3.15. *Given an oriented line $\ell \subset \mathbb{R}^2$, a set $\mathcal{U} \in \mathcal{F}_C$ is said to be an (η, ℓ)-generating set if each point of $\ell_{\mathcal{U}} \cap V(\mathcal{U})$ is η-generated by $\mathcal{U}$.*

Of course, every η-generating set $\mathcal{U} \in \mathcal{F}_C$ is, in particular, an (η, ℓ)-generating set for every oriented line $\ell \subset \mathbb{R}^2$.

Remark 3.16. *Given $\eta \in A^{\mathbb{Z}^2}$, suppose $P_\eta(\mathcal{U}) \leq |\mathcal{U}| + |A| - 2$ for some set $\mathcal{U} \in \mathcal{F}_C$ and let $\ell \in \mathbb{G}_1$ be a rational oriented line. Set $\mathcal{S}_1 := \mathcal{U}$ and define $\mathcal{S}_{i+1} := \mathcal{S}_i \backslash \ell_{\mathcal{S}_i}$ for all $i \geq 1$. Let $I \geq 1$ be the greatest integer such that $P_\eta(\mathcal{S}_I) \leq |\mathcal{S}_I| + |A| - 2$. Note that any convex set $\mathcal{S} \subset \mathcal{S}_I$ that is minimal among all convex sets $\mathcal{T} \subset \mathcal{S}_I$ that satisfy $\mathcal{S}_I \backslash \ell_{\mathcal{S}_I} \subset \mathcal{T}$ and $P_\eta(\mathcal{T}) \leq |\mathcal{T}| + |A| - 2$ is (η, ℓ)-generating. Again, $1 + |A| - 2 < |A|$ implies that $\mathcal{S}$ has at least two points. Moreover, if $\mathcal{S} \backslash \ell_{\mathcal{S}}$ is not empty, then*

- *(i) $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \backslash \ell_{\mathcal{S}}) \leq |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$,*
- *(ii) there is a half plane $\mathcal{H} \subset \mathbb{Z}^2$ (whose edge is parallel to ℓ) such that $\mathcal{S} \backslash \ell_{\mathcal{S}} = \mathcal{U} \cap \mathcal{H}$.*

Note that, if we also suppose in the previous remark that $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{S} \in \mathcal{F}_C^{Vol}$, then Remark 3.12 provides the additional property $|\ell_{\mathcal{S}} \cap \mathcal{S}| \geq 2$.

4. Periodicity and balanced sets

Balanced sets are in particular generating sets for which additional hypotheses are imposed on their geometry and on the bounds of their complexity. Such properties fit well in the context of the Alphabetical Morse-Hedlund Theorem.

For a set $\mathcal{U} \in \mathcal{F}_C^{Vol}$, an oriented line $\ell \in \mathbb{G}_1$ and $\gamma \in L(\mathcal{U} \backslash \ell_{\mathcal{U}}, \eta)$, the number of distinct $\mathcal{U}$-configurations of η that extend γ is denoted by

$$N_{\mathcal{U}}(\ell, \gamma) := \left| \{\gamma' \in L(\mathcal{U}, \eta) : \gamma' | \mathcal{U} \backslash \ell_{\mathcal{U}} = \gamma\} \right|. \tag{4.1}$$

Note that $N_{\mathcal{U}}(\ell, \gamma) = 1$ means that $\gamma' | \mathcal{U} \backslash \ell_{\mathcal{U}} = \gamma = \gamma'' | \mathcal{U} \backslash \ell_{\mathcal{U}}$ implies $\gamma' = \gamma''$ for any $\mathcal{U}$-configurations $\gamma', \gamma'' \in L(\mathcal{U}, \eta)$, i.e., the $\mathcal{U} \backslash \ell_{\mathcal{U}}$-configuration γ admits exactly one extension to an $\mathcal{U}$-configuration of η. It is relevant to consider how many extensions to a specific edge of a convex set a given configuration possesses because such a value is closely related to nonexpansiveness and hence to the number of configurations that appear along a given direction. Conveniently, if $\ell \in \mathbb{G}_1$ is a rational oriented line, for a configuration $x \in X_\eta$, the set of $\mathcal{U}$-configurations of x along of ℓ is denoted by

$$L^\ell(\mathcal{U}, x) := \left\{ (T^{t\vec{v}_\ell} x) | \mathcal{U} \ : t \in \mathbb{Z} \right\}.$$

Lemma 4.1. *For $\eta \in A^{\mathbb{Z}^2}$ and a rational oriented line $\ell \in \mathbb{G}_1$, let $\mathcal{U} \in \mathcal{F}_C^{Vol}$ be an (η, ℓ)-generating set. If $x \in X_\eta$ and there is $\gamma \in L^\ell(\mathcal{U} \backslash \ell_{\mathcal{U}}, x)$ such that $N_{\mathcal{U}}(\ell, \gamma) = 1$, then, for every ℓ-strip $F \supset \mathcal{U} \backslash \ell_{\mathcal{U}}$ and any configuration $y \in X_\eta$, $x|F = y|F$ implies that $x|\ell_{\mathcal{U}} \cup F = y|\ell_{\mathcal{U}} \cup F$.*

Proof. Since $\gamma \in L^\ell(\mathcal{U} \backslash \ell_{\mathcal{U}}, x)$, there exists $\tau \in \mathbb{Z}$ such that $\gamma = (T^{\tau \vec{v}_\ell} x) | \mathcal{U} \backslash \ell_{\mathcal{U}}$. So, from $x|F = y|F$, we obtain $(T^{\tau \vec{v}_\ell} x) | \mathcal{U} \backslash \ell_{\mathcal{U}} = \gamma = (T^{\tau \vec{v}_\ell} y) | \mathcal{U} \backslash \ell_{\mathcal{U}}$. As $N_{\mathcal{U}}(\ell, \gamma) = 1$, we have then $(T^{\tau\vec{v}_\ell}x)|\mathcal{U} = (T^{\tau\vec{v}_\ell}y)|\mathcal{U}$, which implies

$$x|(\mathcal{U}+\tau\vec{v}_\ell)\cup F = y|(\mathcal{U}+\tau\vec{v}_\ell)\cup F. \tag{4.2}$$

By hypothesis, each vertex of $\mathcal{U}$ in $\ell_{\mathcal{U}}\cap\mathcal{U}$ is η-generated by $\mathcal{U}$. Hence, from (4.2) it follows by induction that $x|\ell_{\mathcal{U}}\cup F = y|\ell_{\mathcal{U}}\cup F$, which completes the proof. □

Clearly, if $\ell\in\mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{U}\in\mathcal{F}_C^{Vol}$ is an (η,ℓ)-generating set, then there are configurations $x\in X_\eta$ such that

$$N_{\mathcal{U}}(\ell,\gamma)>1 \quad \forall\ \mathcal{U}\backslash\ell_{\mathcal{U}}\text{-configuration } \gamma\in L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x). \tag{4.3}$$

We denote by $\mathcal{M}(\ell,\mathcal{U})$ the set formed by the configurations $x\in X_\eta$ that satisfy (4.3). The configurations that belong to $\mathcal{M}(\ell,\mathcal{U})$ are exactly the ones for which each $\mathcal{U}\backslash\ell_{\mathcal{U}}$-restriction admits multiple extensions to $\mathcal{U}$. It is not difficult to see that $P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}}) = \sum_{\gamma\in L(\mathcal{U}\backslash\ell_{\mathcal{U}},\eta)}(N_{\mathcal{U}}(\ell,\gamma)-1)$, which, for each $x\in\mathcal{M}(\ell,\mathcal{U})$, yields

$$P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}}) \geq \sum_{\gamma\in L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)}\big(N_{\mathcal{U}}(\ell,\gamma)-1\big) \geq \big|L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)\big|. \tag{4.4}$$

By applying Morse-Hedlund Theorem, Cyr and Kra showed that, under certain conditions, a suitable upper bound for $P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}})$ and so for $|L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)|$, imposes periodicity in some regions of specific configurations. In the sequel, we will employ a similar strategy.

Let $\ell\in\mathbb{G}_1$ be an oriented line. Given a set $\mathcal{U}\in\mathcal{F}_C^{Vol}$, for each oriented line $\ell'\subset\mathbb{R}^2$ parallel to ℓ such that $\ell'\cap\mathcal{U}\neq\emptyset$, we denote $i_{\mathcal{U}}(\ell')$ and $f_{\mathcal{U}}(\ell')$, respectively, the initial and the final points on $\ell'\cap\mathcal{U}$ according to the orientation of ℓ'. If $|\ell'\cap\mathcal{U}|$ is equal to 1, then $i_{\mathcal{U}}(\ell')$ and $f_{\mathcal{U}}(\ell')$ are the same point. For $p\in\mathbb{N}$, we define (see Figure 5)

$$\mathcal{I}^{\ell,p}(\mathcal{U}) := \left\{i_{\mathcal{U}}(\ell')\in\mathbb{Z}^2 : \ell' \text{ is parallel to } \ell,\ \ell'\neq\ell_{\mathcal{U}},\ |\ell'\cap\mathcal{U}|\geq p\right\}$$

and

$$\mathcal{F}^{\ell,p}(\mathcal{U}) := \left\{f_{\mathcal{U}}(\ell')\in\mathbb{Z}^2 : \ell' \text{ is parallel to } \ell,\ \ell'\neq\ell_{\mathcal{U}},\ |\ell'\cap\mathcal{U}|\geq p\right\}.$$

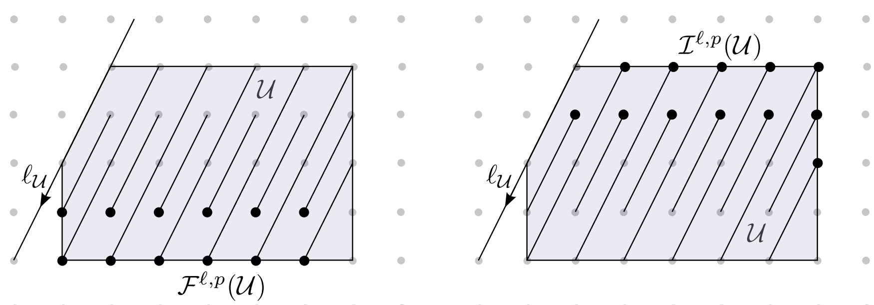


FIGURE 5. The oriented line $\ell_{\mathcal{U}}$ and for $p=2$ the sets $\mathcal{I}^{\ell,p}(\mathcal{U})$ and $\mathcal{F}^{\ell,p}(\mathcal{U})$.

For $x\in X_\eta$, we denote by

$$A^{\ell,p}(\mathcal{U},x) := \left\{(T^{t\vec{v}_\ell}x)|\mathcal{I}^{\ell,p}(\mathcal{U}) : t\in\mathbb{Z}\right\}$$

the finite alphabet induced by the sequence $\xi_t = (T^{t\vec{v}_\ell}x)|\mathcal{I}^{\ell,p}(\mathcal{U})$, where $t \in \mathbb{Z}$. This sequence is closely related to the restriction of the configuration x to the set $\bigcup_{t\in\mathbb{Z}}(\mathcal{I}^{\ell,p}(\mathcal{U}) + t\vec{v}_\ell)$ and this viewpoint will be essential in several arguments. This motivates the following definition.

Definition 4.2. *Let $\ell \in \mathbb{G}_1$ be a rational oriented line and let $\mathcal{U} \in \mathcal{F}_C^{Vol}$ and $p \in \mathbb{N}$. The set $\bigcup_{t\in\mathbb{Z}}(\mathcal{I}^{\ell,p}(\mathcal{U}) + t\vec{v}_\ell)$ is called an $(\ell,\mathcal{U},p)$-strip.*

Note that an $(\ell,\mathcal{U},p)$-strip can be seen as the intersection of $\mathbb{Z}^2$ with the union of finitely many appropriated lines, i.e., an $(\ell,\mathcal{U},p)$-strip can be seen as a set of the form $\mathbb{Z}^2 \cap (\ell_1 \cup \cdots \cup \ell_n)$, where each ℓ_i is parallel to ℓ, $\ell_i \neq \ell_{\mathcal{U}}$ and $|\ell_i \cap \mathcal{U}| \geq p$.

Similarly to Lemma 2.24 of [4], the following result shows how nonexpansiveness is connected to periodicity in certain regions of some configurations.

Lemma 4.3. *Given $\eta \in A^{\mathbb{Z}^2}$, suppose $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is an (η,ℓ)-generating set. If $x \in \mathcal{M}(\ell,\mathcal{U})$ and there is $p \in \mathbb{N}$ such that $P_\eta(\mathcal{U}) - P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}}) \leq p + |A^{\ell,p}(\mathcal{U},x)| - 2$, then the restriction of x to the $(\ell,\mathcal{U},p)$-strip is periodic of period $t\vec{v}_\ell$ for some $t \leq p + |A^{\ell,p}(\mathcal{U},x)| - 2$.*

Proof. Note that, from (4.4) and by hypothesis, it follows that $1 \leq |L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)| \leq p + |A^{\ell,p}(\mathcal{U},x)| - 2$. Set

$$\mathcal{R} := \bigcup_{t=0}^{p-1}(\mathcal{I}^{\ell,p}(\mathcal{U}) + t\vec{v}_\ell) \subset \mathcal{U}$$

and let $\xi = (\xi_t)_{t\in\mathbb{Z}}$ be the sequence defined by $\xi_t = (T^{t\vec{v}_\ell}x)|\mathcal{I}^{\ell,p}(\mathcal{U})$ for all $t \in \mathbb{Z}$. Note that, for every $\tau \in \mathbb{Z}$, the word $\xi_\tau\xi_{\tau+1}\cdots\xi_{\tau+p-1}$ is naturally identified with the $\mathcal{R}$-configuration $(T^{\tau\vec{v}_\ell}x)|\mathcal{R} \in L^\ell(\mathcal{R},x)$. If $|A^{\ell,p}(\mathcal{U},x)| = 1$, then there is nothing to argue. Otherwise, since

$$P_\xi(p) = |L^\ell(\mathcal{R},x)| \leq |L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)| \leq p + |A^{\ell,p}(\mathcal{U},x)| - 2,$$

the Alphabetical Morse-Hedlund Theorem ensures that the sequence ξ is periodic of period at most $p+|A^{\ell,p}(\mathcal{U},x)|-2$. Thus, the restriction of x to the $(\ell,\mathcal{U},p)$-strip is periodic of period $t\vec{v}_\ell$ for some $t \leq p + |A^{\ell,p}(\mathcal{U},x)| - 2$. ☐

We will need a version of the above lemma for half-strips. So let $\ell \in \mathbb{G}_1$ be a rational oriented line. If $\mathcal{U} \in \mathcal{F}_C^{Vol}$ and $a \in \mathbb{Z}$, for each $x \in X_\eta$, the sets of $\mathcal{U}$-configurations of x from the level a along of the directions $\vec{v}_\ell$ or $-\vec{v}_\ell$ are denoted, respectively, by

$$L^\ell_{a+}(\mathcal{U},x) := \left\{(T^{t\vec{v}_\ell}x)|\mathcal{U}\ : t \geq a\right\} \quad \text{and} \quad L^\ell_{a-}(\mathcal{U},x) := \left\{(T^{-t\vec{v}_\ell}x)|\mathcal{U}\ : t \geq a\right\}.$$

Recall that $-\vec{v}_\ell$ is parallel to $\rotatebox[origin=c]{180}{ℓ} \in \mathbb{G}_1$. Naturally, we are led to consider configurations $x \in X_\eta$ such that

$$N_{\mathcal{U}}(\ell,\gamma) > 1 \quad \forall\ \mathcal{U}\backslash\ell_{\mathcal{U}}\text{-configuration } \gamma \in L^\ell_{a+}(\mathcal{U}\backslash\ell_{\mathcal{U}},x) \tag{4.5}$$

or

$$N_{\mathcal{U}}(\ell,\gamma) > 1 \quad \forall\ \mathcal{U}\backslash\ell_{\mathcal{U}}\text{-configuration } \gamma \in L^\ell_{a-}(\mathcal{U}\backslash\ell_{\mathcal{U}},x). \tag{4.6}$$

We denote by $\mathcal{M}_{a+}(\ell,\mathcal{U})$ and $\mathcal{M}_{a-}(\ell,\mathcal{U})$ the sets formed by the configurations $x \in X_\eta$ that satisfy, respectively, (4.5) and (4.6). Note that $\mathcal{M}(\ell,\mathcal{U}) \subset \mathcal{M}_{a+}(\ell,\mathcal{U}) \cap \mathcal{M}_{a-}(\ell,\mathcal{U})$. It is clear that analogous inequalities as (4.4) also hold in this context, i.e., for $x \in \mathcal{M}_{a\pm}(\ell,\mathcal{U})$, one has

$$P_\eta(\mathcal{U}) - P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}}) \geq \left|L^\ell_{a\pm}(\mathcal{U}\backslash\ell_{\mathcal{U}},x)\right|. \tag{4.7}$$

For a configuration $x \in X_\eta$, the finite alphabets induced, respectively, by the sequences $\xi_t = (T^{t\vec{v}_\ell}x)|\mathcal{I}^{\ell,p}(\mathcal{U})$, where $t \geq a$, and $\xi'_t = (T^{-t\vec{v}_\ell}x)|\mathcal{F}^{\ell,p}(\mathcal{U})$, where $t \geq a$, are denote by

$$A^{\ell,p}_{a+}(\mathcal{U},x) := \Big\{(T^{t\vec{v}_\ell}x)|\mathcal{I}^{\ell,p}(\mathcal{U}) : t \geq a\Big\}$$

and

$$A^{\ell,p}_{a-}(\mathcal{U},x) := \Big\{(T^{-t\vec{v}_\ell}x)|\mathcal{F}^{\ell,p}(\mathcal{U}) : t \geq a\Big\}.$$

The sequences ξ_t and ξ'_t are closely related to the restriction of the configuration x to the sets $\bigcup_{t\geq a}(\mathcal{I}^{\ell,p}(\mathcal{U})+t\vec{v}_\ell)$ and $\bigcup_{t\geq a}(\mathcal{F}^{\ell,p}(\mathcal{U})-t\vec{v}_\ell)$, respectively, which motivates the following definition.

Definition 4.4. *Let $\ell \in \mathbb{G}_1$ be a rational oriented line and let $\mathcal{U} \in \mathcal{F}^{Vol}_C$, $p \in \mathbb{N}$ and $a \in \mathbb{Z}$. The sets*

$$F^+(a) := \bigcup_{t\geq a}(\mathcal{I}^{\ell,p}(\mathcal{U}) + t\vec{v}_\ell) \qquad \textit{and} \qquad F^-(a) := \bigcup_{t\geq a}(\mathcal{F}^{\ell,p}(\mathcal{U}) - t\vec{v}_\ell)$$

are called $(\ell,\mathcal{U},p)$-half-strips.

Note that $(\ell,\mathcal{U},p)$-half-strips can be seen as the intersection of $\mathbb{Z}^2$ with the union of finitely many appropriated semi-infinite lines (see Figure 6).

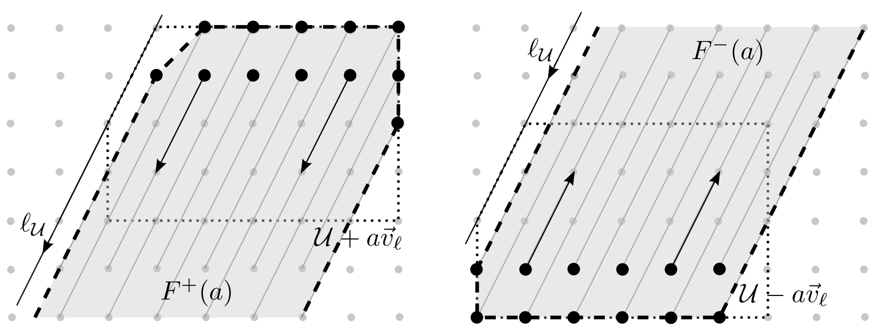


FIGURE 6. The oriented line $\ell_\mathcal{U}$ and for $p = 2$ the $(\ell,\mathcal{U},p)$-half-strips $F^+(a)$ and $F^-(a)$.

Lemma 4.5. *Given $\eta \in A^{\mathbb{Z}^2}$, suppose $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and that $\mathcal{U} \in \mathcal{F}^{Vol}_C$ is an (η,ℓ)-generator set. If $x \in \mathcal{M}_{a\pm}(\ell,\mathcal{U})$ and there exists $p \in \mathbb{N}$ such that $P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_\mathcal{U}) \leq p+|A^{\ell,p}_{a\pm}(\mathcal{U},x)|-2$, then the restriction of x to the $(\ell,\mathcal{U},p)$-half-strip $F^\pm(a+m_x)$ is periodic of period $\pm t\vec{v}_\ell$ for some $t \leq m_x := p + |A^{\ell,p}_{a\pm}(\mathcal{U},x)| - 2$.*

Proof. The proof is identical to that of the previous lemma, by taking (4.7) instead of (4.4) and using condition (ii) instead of condition (iii) in the Alphabetical Morse-Hedlund Theorem. □

If the set $\mathcal{U} \in \mathcal{F}^{Vol}_C$ does not have suitable geometrical properties, an $(\ell,\mathcal{U},p)$-(half-)strip might be a "nonconnected" set, in the sense that its convex hull contains further points of $\mathcal{U}$ than the (half-)strip itself. In such a situation, it could, for example, not be possible to extend to a half plane the periodicity obtained for an $(\ell,\mathcal{U},p)$-strip in Lemma 4.3. We will prevent this inconvenience by imposing that lines parallel to $\ell_{\mathcal{U}}$ intersect $\mathcal{U}$ in a sufficient number of points (see condition (i) in the next definition). This leads us to consider balanced sets, a class of generating sets also obeying the bounds on their complexity highlighted in the previous lemmas (see condition (ii) below).

Definition 4.6. *Given $\eta \in A^{\mathbb{Z}^2}$, let $\ell \in \mathbb{G}_1$ be a one-sided nonexpansive direction on X_η. An (η, ℓ)-generating set $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is said to be (ℓ, p)-balanced (on X_η), $p \in \mathbb{N}$, if the following conditions hold:*

(i) for every $\ell' \neq \ell_{\mathcal{U}}$ that contains at least one point of $\mathbb{Z}^2$ and is parallel to ℓ, $|\ell' \cap \mathcal{U}| \geq p$ whenever $\ell' \cap \operatorname{conv}(\mathcal{U}) \neq \emptyset$,

(ii) for each $x \in \mathcal{M}(\ell, \mathcal{U})$ with $|A^{\ell,p}(\mathcal{U}, x)| > 1$, we have that $P_\eta(\mathcal{U}) - P_\eta(\mathcal{U} \backslash \ell_{\mathcal{U}}) \leq p_x + |A^{\ell,p_x}(\mathcal{U}, x)| - 2$ for some positive integer $p_x \leq p$.

Under an appropriate bound on the complexity, Proposition 4.9 ensures the existence of balanced sets. Furthermore, as $x \in \mathcal{M}(\ell, \mathcal{U} + u)$ if, and only if, $T^u x \in \mathcal{M}(\ell, \mathcal{U})$, it is easy to argue that the property of being an (ℓ, p)-balanced set is invariant by translations. If $\mathcal{U} \in \mathcal{F}_C^{Vol}$ satisfies condition (i) of Definition 4.6, it is immediate that $A^{\ell,p}(\mathcal{U}, x) = A^{\ell,p_x}(\mathcal{U}, x)$ for every positive integer $p_x \leq p$ and $x \in X_\eta$.

Remark 4.7. *Let $\mathcal{U} \in \mathcal{F}_C^{Vol}$ be an (ℓ, p)-balanced set. If $x \in \mathcal{M}_{a\pm}(\ell, \mathcal{U})$ satisfies $|A_{s\pm}^{\ell,p}(\mathcal{U}, x)| > 1$ for infinitely many integer $s \geq a$, then it is easy to see that there exists a positive integer $p_x \leq p$ such that $P_\eta(\mathcal{U}) - P_\eta(\mathcal{U} \backslash \ell_{\mathcal{U}}) \leq p_x + |A_{a\pm}^{\ell,p_x}(\mathcal{U}, x)| - 2$.*

If $\varpi, \varpi' \in E(\mathcal{U})$ are antiparallel edges, where $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is a quasi-regular set (see Definition 2.1), let $R, S \subset \mathbb{R}^2$ denote the line segments connecting the initial and the final points of ϖ and ϖ'. Line segments such as $R, S \subset \mathbb{R}^2$ are called *axis of symmetry of $\mathcal{U}$*. It is easy to argue that each axis of symmetry $S \subset \mathbb{R}^2$ divides $\mathcal{U}$ in two subsets A_S and B_S with $A_S \cap B_S = S \cap \mathbb{Z}^2$, $A_S \cup B_S = \mathcal{U}$ and $|A_S| = |B_S|$.

Remark 4.8. *Let $\mathcal{U} \in \mathcal{F}_C^{Vol}$ and suppose $\varpi, \varpi' \in E(\mathcal{U})$ are antiparallel edges. For any oriented line $\ell \subset \mathbb{R}^2$ parallel to ϖ that intersects $\mathcal{U}$, if $|\varpi \cap \mathcal{U}| \leq |\varpi' \cap \mathcal{U}|$, since the length of ϖ is less or equal to the length of the line segment $\ell \cap \operatorname{conv}(\mathcal{U})$, then $|\ell \cap \mathcal{U}| \geq |\varpi \cap \mathcal{U}| - 1$.*

The following result shows how a strong complexity assumption ensures the existence of balanced sets for any one-sided nonexpansive direction.

Proposition 4.9. *Given $\eta \in A^{\mathbb{Z}^2}$ aperiodic, suppose there exists a quasi-regular set $\mathcal{U} \in \mathcal{F}_C^{Vol}$ for which $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |A| - 1$. If $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η, then there exists an (η, ℓ)-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ such that*

(i) $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\partial_{\mathcal{S}} \cap \mathcal{S}|$,

(ii) $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \backslash \ell_{\mathcal{S}}) \leq |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$.

In particular, it follows that $\mathcal{S} \in \mathcal{F}_C^{Vol}$ is an (ℓ, p)-balanced set with $p = |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$.

Proof. Initially, let $z \in \mathbb{R}^2$ be the intersection of two distinct axis of symmetry of $\mathcal{U}$. Since $\mathcal{U}$ is a quasi-regular set, the oriented line parallel to ℓ that passes through the point $z \in \mathbb{R}^2$ intersects antiparallel edges $\varpi, \varpi' \in E(\mathcal{U})$. Let $\partial' \subset \mathbb{R}^2$ be the oriented line antiparallel to ℓ for which the half plane $\mathcal{H}(\partial')$ is maximal among all half planes whose edge (antiparallel to ℓ) has nonempty intersection with both ϖ and ϖ'. By maximality, there exists a vertex on ϖ or ϖ', denoted by $u \in V(\mathcal{U})$, such that $u \in \ell' \cap \mathbb{Z}^2$. In particular, at most one of the initial and final points of

ϖ and ϖ' may not belong to $\mathcal{H}(\partial')$. So if $S \subset \mathbb{R}^2$ is the axis of symmetry with $u \in S$, then $\mathcal{H}(\partial')$ contains one of the sets $A_S, B_S \subset \mathcal{U}$. Without loss of generality, we assume that $B_S \subset \mathcal{H}(\partial')$ (see Figure 7). Then, since $|A_S| = |B_S|$, we conclude that

$$|\mathcal{U} \cap \mathcal{H}(\partial')| \geq |B_S| \geq |B_S| - \frac{1}{2}|A_S \cap B_S| + 1 = \frac{1}{2}|\mathcal{U}| + 1.$$

Denoting $\mathcal{T} := \mathcal{U} \cap \mathcal{H}(\partial')$, we have then

$$P_\eta(\mathcal{T}) - |\mathcal{T}| \leq P_\eta(\mathcal{U}) - |\mathcal{T}| \leq \frac{1}{2}|\mathcal{U}| + |A| - 1 - |\mathcal{T}| \leq \frac{1}{2}|\mathcal{U}| + |A| - 1 - \frac{1}{2}|\mathcal{U}| - 1,$$

so that $P_\eta(\mathcal{T}) \leq |\mathcal{T}| + |A| - 2$.

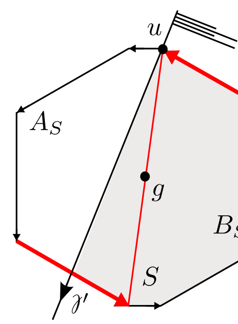


FIGURE 7. The line $\partial' \subset \mathbb{R}^2$ and the subsets $A_S, B_S \subset \mathcal{U}$.

Since (by construction) ℓ' has nonempty intersection with $\varpi, \varpi' \in E(\mathcal{U})$ and $\mathcal{U}$ is a convex set, the length of $\ell'' \cap \text{conv}(\mathcal{U})$ is less or equal to the length of $\ell' \cap \text{conv}(\mathcal{U})$ for all oriented line $\ell'' \subset \mathbb{R}^2$ parallel to ℓ. This means that

$$|\ell' \cap \mathcal{U}| = \max\{|\ell'' \cap \mathcal{U}| : \ell'' \text{ is parallel to } \ell\}. \tag{4.8}$$

According to Remark 3.16, there exists a half plane $\mathcal{H} \subset \mathbb{Z}^2$ (whose edge is parallel to ℓ) and an (η, ℓ)-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ such that

$$\mathcal{S} \backslash \ell_{\mathcal{S}} = (\mathcal{U} \cap \mathcal{H}(\partial')) \cap \mathcal{H} = \mathcal{T} \cap \mathcal{H}, \tag{4.9}$$

$$P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \backslash \ell_{\mathcal{S}}) \leq |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1. \tag{4.10}$$

Note that $\partial_{\mathcal{S}} = \partial_{\mathcal{S} \backslash \ell_{\mathcal{S}}} = \partial_{(\mathcal{U} \cap \mathcal{H}(\partial')) \cap \mathcal{H}} = \partial_{\mathcal{U} \cap \mathcal{H}(\partial')} = \partial'$, where the first equality holds for any set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ and the third follows because the edge of $\mathcal{H}$ is antiparallel to ∂'. Furthermore, from (4.9) we get $\partial_{\mathcal{S}} \cap \mathcal{S} = \partial_{\mathcal{S}} \cap (\mathcal{S} \backslash \ell_{\mathcal{S}}) = \partial' \cap \mathcal{U}$. So by (4.8) one has

$$|\partial_{\mathcal{S}} \cap \mathcal{S}| = |\ell' \cap \mathcal{U}| \geq |\ell_{\mathcal{S}} \cap \mathcal{U}| \geq |\ell_{\mathcal{S}} \cap \mathcal{S}|.$$

Finally, we claim that $\mathcal{S}$ is an (ℓ, p)-balanced set with $p := |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$. Indeed, Remark 4.8 ensures condition (i) of Definition 4.6, and from inequality (4.10), for each $x \in \mathcal{M}(\ell, \mathcal{S})$ with $|A^{\ell,p}(\mathcal{S}, x)| > 1$, we get $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \backslash \ell_{\mathcal{S}}) \leq p \leq p + |A^{\ell,p}(\mathcal{S}, x)| - 2$, i.e., condition (ii) of Definition 4.6 holds for $p_x = p$. ☐

Remark 4.10. *For $\eta \in A^{\mathbb{Z}^2}$ aperiodic, suppose that $P_\eta(\mathcal{U}) \leq |\mathcal{U}| + |A| - 2$ for some $\mathcal{U} \in \mathcal{F}_C$. If the antiparallel oriented lines $\ell, \partial \in \mathbb{G}_1$ are both one-sided nonexpansive directions on X_η, then the η-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ from Remark 3.9 has at least two antiparallel edges (see Remark 3.12), which are parallel to ℓ and ∂. Thus, if the edge parallel to ℓ is the smallest one, it is not difficult to see that $\mathcal{S}$ is (ℓ, p)-balanced with $p := |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$.*

The notion of balanced sets introduced here has some advantages. For example, if $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{S} \in \mathcal{F}_C^{Vol}$ is an (η, ℓ)-generating set where $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \backslash \ell_\mathcal{S}) \leq |\ell_\mathcal{S} \cap \mathcal{S}| - 1$, then $\mathcal{S}$ may be (ℓ, p)-balanced even when $|\ell_\mathcal{S} \cap \mathcal{S}| > |\partial_\mathcal{S} \cap \mathcal{S}|$. Indeed, if $|\partial_\mathcal{S} \cap \mathcal{S}| > 1$, it is enough to have

$$|\ell_\mathcal{S} \cap \mathcal{S}| \leq |\partial_\mathcal{S} \cap \mathcal{S}| + |A^{\ell,p}(\mathcal{S}, x)| - 2$$

for every configuration $x \in \mathcal{M}(\ell, \mathcal{S})$ with $|A^{\ell,p}(\mathcal{S}, x)| > 1$, where $p := |\partial_\mathcal{S} \cap \mathcal{S}| - 1$.

4.1. **Extending periodicity from strips.** For an (ℓ, p)-balanced set $\mathcal{T} \in \mathcal{F}_C^{Vol}$, we define

$$\varPhi_p(\ell, \mathcal{T}) := P_\eta(\mathcal{T}) - P_\eta(\mathcal{T} \backslash \ell_\mathcal{T})$$

if $|A^{\ell,p}(\mathcal{T}, x)| = 1$ for every $x \in \mathcal{M}(\ell, \mathcal{T})$ and

$$\varPhi_p(\ell, \mathcal{T}) := \max \left\{ p_x + |A^{\ell,p_x}(\mathcal{T}, x)| - 2 : x \in \mathcal{M}(\ell, \mathcal{T}) \text{ and } |A^{\ell,p_x}(\mathcal{T}, x)| > 1 \right\}$$

otherwise, where $p_x \leq p$ is the smallest integer fulfilling Definition 4.6. Although the definition of $\varPhi_p(\ell, \mathcal{T})$ may seem somewhat elaborate, we prefer to use it to make it clear that we can extend periodicity beyond strips even without explicitly using any hypothesis about complexity. When we consider our alphabetical upper bound for complexity, however, the function $\varPhi_p(\ell, \mathcal{T})$ may be replaced by a nice expression (see inequality (5.7)).

The next lemma will allow us to extend the periodicity to larger strips and then to half planes.

Lemma 4.11. *Given $\eta \in A^{\mathbb{Z}^2}$, suppose $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is an (ℓ, p)-balanced set. If the restriction of $x \in X_\eta$ to the $(\ell, \mathcal{U}, p)$-strip F is periodic of period $t'\vec{v}_\ell$ for some $t' \in \mathbb{N}$, then $x|\ell_\mathcal{U} \cup F$ is periodic of period $t\vec{v}_\ell$, where $t = t'$ if $x \notin \mathcal{M}(\ell, \mathcal{U})$ and $t \leq 2\varPhi_p(\ell, \mathcal{U})$ otherwise.*

Proof. Initially, if $x \notin \mathcal{M}(\ell, \mathcal{U})$, since $x|F$ is periodic of period $t'\vec{v}_\ell$, that is, $x|F = (T^{t'\vec{v}_\ell} x)|F$, from Lemma 4.1 it follows that $x|\ell_\mathcal{U} \cup F = (T^{t'\vec{v}_\ell} x)|\ell_\mathcal{U} \cup F$.

Suppose $x \in \mathcal{M}(\ell, \mathcal{U})$ and $|A^{\ell,p}(\mathcal{U}, x)| > 1$. In this case, from (4.4) we obtain that $|L^\ell(\mathcal{U} \backslash \ell_\mathcal{U}, x)| \leq p_x + \left|A^{\ell,p_x}(\mathcal{U}, x)\right| - 2 \leq \varPhi_p(\ell, \mathcal{U})$ where $p_x \leq p$ is the smallest integer fulfilling Definition 4.6. Therefore,

$$\begin{aligned} \left|L^\ell(\mathcal{U}, x)\right| - \varPhi_p(\ell, \mathcal{U}) &\leq \left(\sum_{\gamma \in L^\ell(\mathcal{U} \backslash \ell_\mathcal{U}, x)} N_\mathcal{U}(\ell, \gamma) \right) - \left|L^\ell(\mathcal{U} \backslash \ell_\mathcal{U}, x)\right| \\ &= \sum_{\gamma \in L^\ell(\mathcal{U} \backslash \ell_\mathcal{U}, x)} \Big(N_\mathcal{U}(\ell, \gamma) - 1 \Big) \\ &\leq \sum_{\gamma \in L(\mathcal{U} \backslash \ell_\mathcal{U}, \eta)} \Big(N_\mathcal{U}(\ell, \gamma) - 1 \Big) \\ &= P_\eta(\mathcal{U}) - P_\eta(\mathcal{U} \backslash \ell_\mathcal{U}) \leq p_x + \left|A^{\ell,p_x}(\mathcal{U}, x)\right| - 2, \end{aligned} \tag{4.11}$$

which yields

$$|L^\ell(\mathcal{U}, x)| \leq \varPhi_p(\ell, \mathcal{U}) + p_x + |A^{\ell,p_x}(\mathcal{U}, x)| - 2 \leq 2\varPhi_p(\ell, \mathcal{U}).$$

By the Pigeonhole Principle, we can assume, without loss of generality, that there exists a positive integer $t \leq 2\varPhi_p(\ell, \mathcal{U})$ such that $x|\mathcal{U} = (T^{t\vec{v}_\ell} x)|\mathcal{U}$. If we also have

$x|\mathcal{U}\cup F=(T^{t\vec{v}_\ell}x)|\mathcal{U}\cup F$, since $\mathcal{U}\in\mathcal{F}_C^{Vol}$ is an (η,ℓ)-generating set, by induction, we obtain

$$x|\ell_{\mathcal{U}}\cup F=(T^{t\vec{v}_\ell}x)|\ell_{\mathcal{U}}\cup F.$$

So to finish this case, it is enough to show that $x|F$ is periodic of period $t\vec{v}_\ell$. Indeed, let $\xi=(\xi_i)_{i\in\mathbb{Z}}$ be the sequence defined by $\xi_i:=(T^{i\,\vec{v}_\ell}x)|\mathcal{I}^{\ell,p_x}(\mathcal{U})$ for all $i\in\mathbb{Z}$. As $|A^{\ell,p_x}(\mathcal{U},x)|>1$ and

$$P_\xi(p_x)\leq|L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)|\leq p_x+|A^{\ell,p_x}(\mathcal{U},x)|-2,$$

let $1<p_0\leq p_x$ be the smallest integer such that $P_\xi(p_0)\leq p_0+|A^{\ell,p_0}(\mathcal{U},x)|-2$. It is easy to see that by minimality $P_\xi(p_0)=P_\xi(p_0-1)$, which means that a word of p_0-1 symbols admits exactly one extension to a word of p_0 symbols. Hence, since from $x|\mathcal{U}=(T^{t\vec{v}_\ell}x)|\mathcal{U}$ one has $\xi_0\xi_1\cdots\xi_{p_0-1}=\xi_t\xi_{t+1}\cdots\xi_{t+p_0-1}$, by induction it follows that the sequence ξ is periodic of period t. In other words, $x|F$ is periodic of period $t\vec{v}_\ell$.

Suppose $x\in\mathcal{M}(\ell,\mathcal{U})$ and $|A^{\ell,p}(\mathcal{U},x)|=1$. In this case, there exists a unique $\mathcal{U}\backslash\ell_{\mathcal{U}}$-configuration $\gamma\in L^\ell(\mathcal{U}\backslash\ell_{\mathcal{U}},x)$ and, therefore,

$$\left|L^\ell(\mathcal{U},x)\right|-1\leq N_{\mathcal{U}}(\ell,\gamma)-1\leq P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}}).$$

As before, we can assume that there is a positive integer $t\leq P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}})+1$ such that $x|\mathcal{U}=(T^{t\vec{v}_\ell}x)|\mathcal{U}$. Since $|A^{\ell,p}(\mathcal{U},x)|=1$, $x|F$ is in particular periodic of period $t\vec{v}_\ell$. The same argument as in the previous case allows us to conclude that the restriction of x to $\ell_{\mathcal{U}}\cup F$ is periodic of period $t\vec{v}_\ell$. Finally, as $P_\eta(\mathcal{U})>P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}})$, note that

$$P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}})+1\leq 2\big(P_\eta(\mathcal{U})-P_\eta(\mathcal{U}\backslash\ell_{\mathcal{U}})\big)=2\Phi_p(\ell,\mathcal{U}).$$

This completes the proof. □

A key element of the proof of the main result will be the possibility of extending periodicity to wider half-strips (see Lemma 4.13). It is necessary to avoid doubts regarding orientations and to introduce a precise notation for each situation. Therefore, for $(\ell,\mathcal{U},p)$-half-strips $F^+(a),F^-(a)\subset\mathbb{Z}^2$, we define (see Figure 8)

$$(\ell_{\mathcal{U}}\cup F^+)(a):=F^+(a)\cup\{i_{\mathcal{U}}(\ell_{\mathcal{U}})+t\vec{v}_\ell : t\geq a\}\tag{4.12}$$

and

$$(\ell_{\mathcal{U}}\cup F^-)(a):=F^-(a)\cup\{f_{\mathcal{U}}(\ell_{\mathcal{U}})-t\vec{v}_\ell : t\geq a\}.\tag{4.13}$$

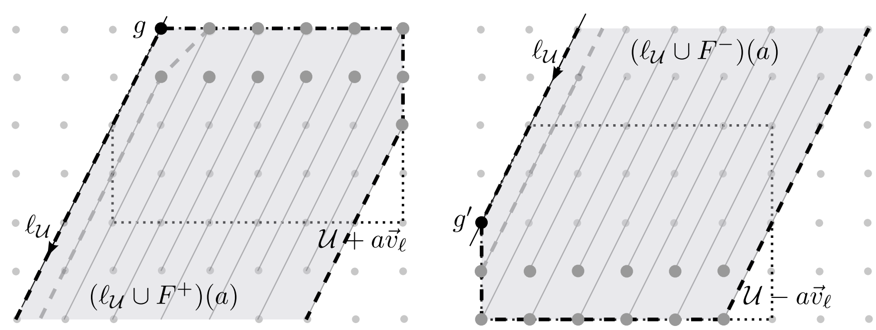


FIGURE 8. The points $g=i_{\mathcal{U}}(\ell_{\mathcal{U}})+a\vec{v}_\ell$ and $g'=f_{\mathcal{U}}(\ell_{\mathcal{U}})-a\vec{v}_\ell$ and the sets $(\ell_{\mathcal{U}}\cup F^+)(a)$ and $(\ell_{\mathcal{U}}\cup F^-)(a)$.

Lemma 4.12. *For $\eta \in A^{\mathbb{Z}^2}$ and a rational oriented line $\ell \in \mathbb{G}_1$, let $\mathcal{U} \in \mathcal{F}_C^{Vol}$ be an (η, ℓ)-generating set. If $x \notin \mathcal{M}_{a\pm}(\ell,\mathcal{U})$, then, for every $(\ell,\mathcal{U}, p)$-half-strip $F^{\pm}(a)$ with $\mathrm{conv}(F^{\pm}(a)) \cap \mathbb{Z}^2 = F^{\pm}(a)$ and any configuration $y \in X_\eta$, $x|F^{\pm}(a) = y|F^{\pm}(a)$ implies $x|(\ell_{\mathcal{U}} \cup F^{\pm})(a) = y|(\ell_{\mathcal{U}} \cup F^{\pm})(a)$.*

Proof. The proof is identical to that one of Lemma 4.1. □

Next lemma is the analogous of Lemma 4.11 for half-strips (recall that $-\vec{v}_\ell = \vec{v}_\partial$).

Lemma 4.13. *Given $\eta \in A^{\mathbb{Z}^2}$, suppose $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is an (ℓ, p)-balanced set. If the restriction of $x \in X_\eta$ to the $(\ell,\mathcal{U}, p)$-half-strip $F^{\pm}(a)$, $a \in \mathbb{Z}$, is periodic of period $\pm t'\vec{v}_\ell$ for some $t' \in \mathbb{N}$, then $x|(\ell_{\mathcal{U}} \cup F^{\pm})(a)$ is periodic of period $\pm t\vec{v}_\ell$, where $t = t'$ if $x \notin \mathcal{M}_{a\pm}(\ell,\mathcal{U})$ and $t \leq 2\Phi_p(\ell,\mathcal{U})$ otherwise.*

Proof. The proof is similar to that one of Lemma 4.11. If $x \notin \mathcal{M}_{a\pm}(\ell,\mathcal{U})$, we use Lemma 4.12 instead of Lemma 4.1. If $x \in \mathcal{M}_{a\pm}(\ell,\mathcal{U})$ and $|A^{\ell,p}_{a\pm}(\mathcal{U}, x)| > 1$, since the sequence $(T^{\pm t\vec{v}_\ell}x)_{t\geq a} \subset \mathcal{M}_{a\pm}(\ell,\mathcal{U})$ has an accumulation point $z \in \mathcal{M}(\ell,\mathcal{U})$ with $|A^{\ell,p}(\mathcal{U}, z)| > 1$, then there exists a smallest integer $p_z \leq p$ fulfilling Definition 4.6. Due to periodicity of $x|F^{\pm}(a)$, we have $A^{\ell,p_z}(\mathcal{U}, z) = A^{\ell,p_z}_{a\pm}(\mathcal{U}, x)$. So just use (4.7) instead of (4.4). □

Proposition 4.14. *Let $\eta \in A^{\mathbb{Z}^2}$ and suppose $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is an (ℓ, p)-balanced set. If the restriction of $x \in X_\eta$ to the $(\ell,\mathcal{U}, p)$-strip is periodic of period $t'\vec{v}_\ell$ for some $t' \in \mathbb{N}$, then, for any translation $\mathcal{U}'$ of $\mathcal{U}$ with $\mathcal{U}' \subset \mathcal{H}(\partial_{\mathcal{U}})$, the restriction of x to the $(\ell,\mathcal{U}', p)$-strip is periodic of period $t\vec{v}_\ell$, where $t \leq \max\{t', 2\Phi_p(\ell,\mathcal{U})\}$. In particular, the configuration $x|\mathcal{H}(\partial_{\mathcal{U}})$ is ℓ-periodic.*

Proof. Let $u \in (\mathbb{Z}^2)^*$ be such that $\ell + u = \ell^{(-)}$ (recall Notation 3.6). Let $P_{\mathcal{U}}$ denote the set of $\kappa \in \mathbb{Z}_+$ such that the restriction of $x \in X_\eta$ to the $(\ell,\mathcal{U} + \kappa u, p)$-strip is periodic of period $\tau'\vec{v}_\ell$ for some $\tau' \leq \max\{t', 2\Phi_{\ell,p}(\mathcal{U})\}$. Suppose, by contradiction, that $P_{\mathcal{U}}$ does not coincide with $\mathbb{Z}_+$. Let $\kappa' \in P_{\mathcal{U}}$ be the largest integer for which $i \in P_{\mathcal{U}}$ for every $0 \leq i \leq \kappa'$. Since the restriction of x to the $(\ell,\mathcal{U} + \kappa' u, p)$-strip $F_{\kappa'}$ is periodic of period $\tau'\vec{v}_\ell$ for some integer $\tau' \leq \max\{t', 2\Phi_p(\ell,\mathcal{U})\}$, according to Lemma 4.11, the configuration $x|\ell_{(\mathcal{U}+\kappa' u)} \cup F_{\kappa'}$ is periodic of period $\tau\vec{v}_\ell$, where $\tau = \tau'$ if $x \notin \mathcal{M}(\ell,\mathcal{U} + \kappa' u)$ and $\tau \leq 2\Phi_p(\ell,\mathcal{U} + \kappa' u)$ otherwise. The fact that $x \in \mathcal{M}(\ell,\mathcal{U}+\kappa' u)$ if, and only if, $T^{\kappa' u}x \in \mathcal{M}(\ell,\mathcal{U})$ implies $\Phi_p(\ell,\mathcal{U}+\kappa' u) = \Phi_p(\ell,\mathcal{U})$. So we have that the restriction of x to the $(\ell,\mathcal{U} + (\kappa' + 1)u, p)$-strip is periodic of period $\tau\vec{v}_\ell$ with $\tau \leq \max\{t', 2\Phi_p(\ell,\mathcal{U})\}$, which contradicts the maximality of $\kappa' \in \mathbb{Z}_+$. □

The thesis of corollary below was first obtained in Proposition 3.14 for periodic configurations. Here, as a consequence of the previous results this hypothesis can be replaced by the existence of a balanced set.

Corollary 4.15. *If $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is (ℓ, p)-balanced, then the oriented line ∂, antiparallel to $\ell \in \mathbb{G}_1$, is also a one-sided nonexpansive direction on X_η.*

Proof. Let $x, x' \in X_\eta$ be configurations where $x|\mathcal{H}(\ell) = x'|\mathcal{H}(\ell)$, but $x_g \neq x'_g$ for some $g \in \ell^{(-)} \cap \mathbb{Z}^2$. Let $\mathcal{U}' \in \mathcal{F}_C^{Vol}$ denote a translation of $\mathcal{U} \in \mathcal{F}_C^{Vol}$ such that $\ell_{\mathcal{U}'} = \ell^{(-)}$. It follows by Lemma 4.1 that $x, x' \in \mathcal{M}(\ell,\mathcal{U}')$. Thanks to Lemma 4.3, the restrictions of x and x' to the $(\ell,\mathcal{U}',p)$-strip are ℓ-periodic. Proposition 4.14 ensures then that the restrictions of x and x' to the half plane $\mathcal{H}(\partial_{\mathcal{U}'})$ are ℓ-periodic of periods $h,h' \in (\mathbb{Z}^2)^*$. Suppose, by contradiction, that $\partial \in \mathbb{G}_1$ is a one-sided expansive direction on X_η. Since $x|\mathcal{H}(\partial) = (T^h x)|\mathcal{H}(\partial)$ and $x'|\mathcal{H}(\partial) = (T^{h'}x')|\mathcal{H}(\partial)$, it follows by expansiveness that $x,x' \in X_\eta$ are ℓ-periodic. Thus, due to Proposition 3.14, the oriented lines $\ell,\partial \in \mathbb{G}_1$ are both one-sided expansive directions on both subshifts $\overline{Orb\,(x)}$ and $\overline{Orb\,(x')}$. With respect to the others lines, from Lemma 3.4 we get that they are also expansive on both subshifts. Hence, Corollary 3.3 implies that $x,x' \in X_\eta$ are doubly periodic. Therefore, as $x|\mathcal{H}(\ell) = x'|\mathcal{H}(\ell)$, we conclude that x and x' are equals, which is a contradiction. □

Note that, by the very definition, the vertices of an edge of an (ℓ,p)-balanced set parallel to ∂ are not necessarily generated. Thus, it is not a surprise that, to extend periodicity to the entire plane, one has to simultaneously consider (ℓ,p)-balanced and (∂,q)-balanced sets. The following result shows how balanced sets impose periodicity for some configurations.

Proposition 4.16. *For $\eta \in A^{\mathbb{Z}^2}$, suppose $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η and $\mathcal{U},\mathcal{T} \in \mathcal{F}_C^{Vol}$ are, respectively, (ℓ,p)-balanced and (∂,q)-balanced. If the restriction of $x \in X_\eta$ to the $(\ell,\mathcal{U},p)$-strip is periodic of period $t'\vec{v}_\ell$ for some $t' \in \mathbb{N}$, then, for any translation $\mathcal{U}'$ of $\mathcal{U}$, the restriction of x to the $(\ell,\mathcal{U}',p)$-strip is periodic of period $t\vec{v}_\ell$, where $t \leq \max\{t', 2\Phi_p(\ell,\mathcal{U})\}$. In particular, the configuration x is ℓ-periodic. Similarly, if the restriction of a configuration to the $(\partial,\mathcal{T},q)$-strip is periodic, then the configuration is ∂-periodic.*

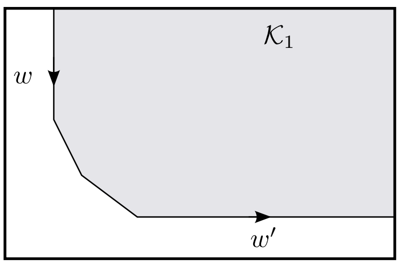
Proof. Initially, it follows from Proposition 4.14 applied to each one of the sets $\mathcal{U},\mathcal{T}' \in \mathcal{F}_C^{Vol}$ that x is ℓ-periodic, where $\mathcal{T}'$ is a translation of $\mathcal{T}$ with $\mathcal{T}' \subset \mathcal{H}(\partial_{\mathcal{U}})$. So let $u \in (\mathbb{Z}^2)^*$ be such that $\ell^{(-)} + u = \ell$. Let $Q_{\mathcal{U}}$ be the set of $\kappa \in \mathbb{Z}_+$ such that the restriction of $x \in X_\eta$ to the $(\ell,\mathcal{U}+\kappa u,p)$-strip is periodic of period $\tau\vec{v}_\ell$ for some $\tau \leq \max\{t', 2\Phi_p(\ell,\mathcal{U})\}$. Suppose, by contradiction, that $Q_{\mathcal{U}}$ does not coincide with $\mathbb{Z}_+$. Let $\kappa' \in Q_{\mathcal{U}}$ denote the largest integer for which $i \in Q_{\mathcal{U}}$ for every $0 \leq i \leq \kappa'$. Thus, Lemma 4.3 and Proposition 4.14 imply that $x \notin \mathcal{M}(\ell,\mathcal{U}+\kappa u)$ for all integer $\kappa > \kappa'$. Since x is ℓ-periodic, the Pigeonhole Principle ensures that there are integers $i > I \geq \kappa'$ such that $x|F + iu = x|F + Iu$, where $F \subset \mathbb{Z}^2$ denotes the $(\ell,\mathcal{U},p)$-strip. We can further assume that $I \geq \kappa'$ is the smallest integer with this property. Since $x|F + iu = x|F + Iu = (T^{(I-i)u}x)|F + iu$ and $x \notin \mathcal{M}(\ell,\mathcal{U}+iu)$, from Lemma 4.1, we get $x|(F+iu) \cup \ell_{(\mathcal{U}+iu)} = (T^{(I-i)u}x)|(F+iu) \cup \ell_{(\mathcal{U}+iu)}$. Since $F + (i-1)u \subset \ell_{(\mathcal{U}+iu)} \cup (F+iu)$, one has $x|F+(i-1)u = x|F+(I-1)u$ and so $I = \kappa'$ (by minimality of I). Hence, the restriction of x to the $(\ell,\mathcal{U}+iu,p)$-strip is periodic of period $\tau\vec{v}_\ell$ for some $\tau \leq \max\{t', 2\Phi_p(\ell,\mathcal{U})\}$. Therefore, by applying Proposition 4.14, one contradicts the maximality of $\kappa' \in Q_{\mathcal{U}}$. □

5. (ℓ,ℓ')-periodic maximal $\mathcal{K}$-configurations

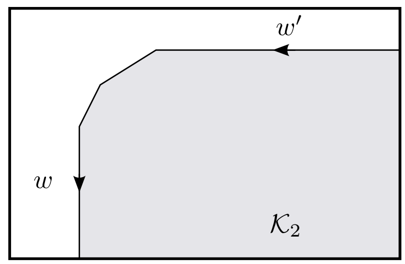
If $\eta \in A^{\mathbb{Z}^2}$ is an aperiodic configuration with $P_\eta(n,k) \leq \frac{1}{2}nk$ for some $n,k \in \mathbb{N}$, in [4] the authors proved that there always exists another aperiodic configuration $\varphi \in A^{\mathbb{Z}^2}$ whose restriction to a largest convex set $\mathcal{K}$ is doubly periodic, say, both ℓ-periodic and ℓ'-periodic. We will obtain the same result, but in a more general context (see Proposition 5.3). Such an aperiodic configuration φ is called an (ℓ,ℓ')-periodic maximal $\mathcal{K}$-configuration; see the next definition for accuracy.

Definition 5.1. *Let $\ell, \ell' \in \mathbb{G}_1$ be rational oriented lines, with $\ell \cap \ell' = \{0\}$, such that $\vec{v}_{\ell'} \in \mathcal{H}(\ell)$. Let $\mathcal{K} \subset \mathbb{Z}^2$ be a convex set with semi-infinite edges parallel to ℓ and ℓ'. An aperiodic configuration $\varphi \in A^{\mathbb{Z}^2}$ is said to be an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration if (i) $\varphi|\mathcal{K}$ is doubly periodic with periods in $\ell \cap (\mathbb{Z}^2)^*$ and $\ell' \cap (\mathbb{Z}^2)^*$, and (ii) $\mathcal{K}$ is maximal among all convex sets (positively oriented) with semi-infinite edges parallel to ℓ and ℓ' that satisfy condition (i).*

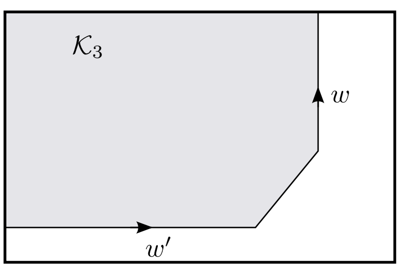
In the above definition, $\vec{v}_{\ell'} \in \mathcal{H}(\ell)$ means that (with respect to the orientation inhered from the boundary of conv $(\mathcal{K})$) the edge parallel to ℓ precedes the edge parallel to ℓ' (see Figure 9.A). In particular, only $(\ell, \mathcal{S}, p)$-half-strips of the form $F^-(a)$, $a \in \mathbb{Z}$, and $(\ell', \mathcal{S}', q')$-half-strips of the form $F^+(a')$, $a' \in \mathbb{Z}$, may be contained in $\mathcal{K}$ (see Definition 4.4 to recall the notion of half-strips).



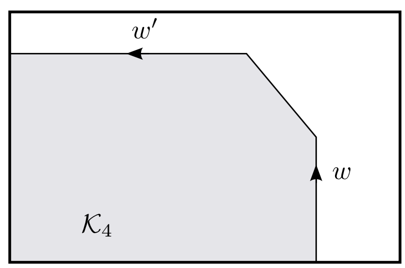
(A) φ is an (ℓ, ℓ')-periodic maximal $\mathcal{K}_1$-configuration.



(B) φ is an $(\reflectbox{$\ell$}', \ell)$-periodic maximal $\mathcal{K}_2$-configuration.



(C) φ is an $(\ell', \reflectbox{$\ell$})$-periodic maximal $\mathcal{K}_3$-configuration.



(D) φ is an $(\reflectbox{$\ell$}, \reflectbox{$\ell$}')$-periodic maximal $\mathcal{K}_4$-configuration.

FIGURE 9. Fixed two oriented lines, the four types of infinite convex regions over which periodic maximal configurations may be found are represented above. In each case, the semi-infinite edges $w, w' \in E(\mathcal{K}_i)$, $i = 1, 2, 3, 4$, are parallel to the corresponding oriented lines.

From now on, whenever we mention an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration, we are supposing that $\ell, \ell' \in \mathbb{G}_1$ are rational oriented lines with $\ell \cap \ell' = \{0\}$ and $\vec{v}_{\ell'} \in \mathcal{H}(\ell)$, as well as that $\mathcal{K}$ is a convex set with semi-infinite edges parallel to ℓ and ℓ'.

Note that, if $\eta \in A^{\mathbb{Z}^2}$ is an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration, the maximality of $\mathcal{K}$ and Lemma 3.13 imply that $\ell, \ell' \in \mathbb{G}_1$ are one-sided nonexpansive directions on X_η.

Definition 5.2. *For $\mathcal{U} \in \mathcal{F}_C^{Vol}$, a convex set $\mathcal{T} \subset \mathbb{Z}^2$ is said to be weakly $E(\mathcal{U})$-enveloped if, for every edge $\varpi \in E(\mathcal{T})$, there exists an edge $w \in E(\mathcal{U})$ parallel to ϖ with*

$|w \cap \mathcal{U}| \leq |\varpi \cap \mathcal{T}|$. *A set* $\mathcal{T} \in \mathcal{F}_C^{Vol}$ *weakly* $E(\mathcal{U})$*-enveloped verifying* $|E(\mathcal{T})| = |E(\mathcal{U})|$ *is said to be* $E(\mathcal{U})$*-enveloped.*

It is clear that every set $\mathcal{U} \in \mathcal{F}_C^{Vol}$ is $E(\mathcal{U})$-enveloped.

Proposition 5.3. *For* $\eta \in A^{\mathbb{Z}^2}$ *aperiodic, suppose there is a set* $\mathcal{U} \in \mathcal{F}_C$ *such that* $P_\eta(\mathcal{U}) \leq |\mathcal{U}| + |A| - 2$. *If there exist an* (ℓ, r)*-balanced set and an* (∂, q)*-balanced set, then one may always obtain a rational oriented line* $\ell' \in \mathbb{G}_1$ *and an infinite convex region* $\mathcal{K} \subset \mathbb{Z}^2$ *for which there exists an* (l, ℓ')*-periodic or an* (ℓ', l)*-periodic maximal* $\mathcal{K}$*-configuration* $\varphi \in X_\eta$*, where* $l \in \{\ell, \partial\}$[1].

Proof. Let $\mathcal{S} \in \mathcal{F}_C^{Vol}$ be an η-generating set as in Remark 3.9 and suppose, without loss of generality, $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\partial_{\mathcal{S}} \cap \mathcal{S}|$. (As effect of this choice, $l = \ell$ in this proof and we come across either the situation described in Figure 9.A or the one presented in Figure 9.B.) According to Remark 4.10, the set $\mathcal{S}$ fulfills all properties to be (ℓ, p)-balanced with $p := |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$. Since $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η, there must exist $x, y \in X_\eta$ such that $x|\mathcal{H}(\ell) = y|\mathcal{H}(\ell)$, but $x_g \neq y_g$ for some $g \in \ell^{(-)} \cap \mathbb{Z}^2$. Translating $\mathcal{S}$ if necessary, we can further assume that $(0,0) \in \mathcal{S}$ and that $\ell_{\mathcal{S}} = \ell^{(-)}$. Thanks to Lemma 4.1, $x, y \in \mathcal{M}(\ell, \mathcal{S})$. So Lemma 4.3 and Proposition 4.16 imply that x and y are ℓ-periodic, but the fact that $x_g \neq y_g$ for some $g \in \ell^{(-)} \cap \mathbb{Z}^2$ prevents both configurations $x|\mathcal{H}(\ell^{(-)})$ and $y|\mathcal{H}(\ell^{(-)})$ from being doubly periodic. Thus, we can assume the restriction of x to the half plane $\mathcal{H}(\ell^{(-)})$ is not doubly periodic.

Note that, for every ℓ-strip F, with $\mathcal{S} \backslash \ell_{\mathcal{S}} \subset F$, and any finite set $B \subset F$, there always exists $x' \in X_\eta$ such that $x|B = x'|B$, but $x|F \neq x'|F$. Indeed, otherwise, since $x|B = (T^g \eta)|B$ for some $g \in \mathbb{Z}^2$, the restriction of $T^g \eta$ to the $(\ell, \mathcal{S}, p)$-strip would be ℓ-periodic and then Proposition 4.16 would imply that η is periodic, contradicting the aperiodicity of η. This fact allows us to construct a sequence of $E(\mathcal{S})$-enveloped sets

$$A_0 \subset B_1 \subset A_1 \subset B_2 \subset A_2 \subset \cdots \subset B_i \subset A_i \subset \cdots,$$

with $A_0 \subset \mathcal{S}$ and $B_1 := \mathcal{S}$, and configurations $(\varphi_i)_{i \in \mathbb{N}} \in X_\eta$ such that, for each $i \geq 1$,

(i) $B_i \in \mathcal{F}_C^{Vol}$ is an $E(\mathcal{S})$-enveloped set with $\ell_{B_i} = \ell_{\mathcal{S}}$,
(ii) B_i contains both A_{i-1} and $[-i+1, i-1]^2 \cap \mathcal{H}(\ell^{(-)})$,
(iii) $x|B_i = \varphi_i|B_i$, but $x|F_i \neq \varphi_i|F_i$,
(iv) either A_i is a maximal set among all $E(\mathcal{S})$-enveloped sets $\mathcal{T} \in \mathcal{F}_C^{Vol}$ such that $B_i \subset \mathcal{T} \subset F_i^+$ and $x|\mathcal{T} = \varphi_i|\mathcal{T}$ or A_i is a maximal set among all $E(\mathcal{S})$-enveloped sets $\mathcal{T} \in \mathcal{F}_C^{Vol}$ such that $B_i \subset \mathcal{T} \subset F_i^-$ and $x|\mathcal{T} = \varphi_i|\mathcal{T}$,[2]

where

$$F_i := \bigcup_{j \in \mathbb{Z}} \big(B_i + j\vec{v}_\ell\big), \quad F_i^+ := \bigcup_{j \geq 0} \big(B_i + j\vec{v}_\ell\big) \text{ and } F_i^- := \bigcup_{j \geq 0} \big(B_i - j\vec{v}_\ell\big).$$

We need consider two cases separately. We will focus on the case where there exist infinitely many indexes i for which A_i is a maximal set among all $E(\mathcal{S})$-enveloped sets $\mathcal{T} \in \mathcal{F}_C^{Vol}$ such that $B_i \subset \mathcal{T} \subset F_i^+$ and $x|\mathcal{T} = \varphi_i|\mathcal{T}$ (see Figure 10).

[1]As ℓ' represents in this statement an arbitrary oriented line, note that all the situations present in Figure 9 are in fact contemplated.

[2]If there is not a maximal $E(\mathcal{S})$-enveloped set $B_i \subset A_i \subset F_i^+$, then there must exist a maximal $E(\mathcal{S})$-enveloped set $B_i \subset A_i \subset F_i^-$, since, otherwise, η would be periodic.

By passing to a subsequence, we can assume this holds for all i. For each integer $i \geq 1$, we have $\ell_{\mathcal{S}} \cap \mathcal{S} \subset \ell_{A_i} \cap A_i \subset \ell^{(-)}$. So let $g_1 := f_{A_1}(\ell^{(-)})$ be the final point of $\ell_{A_1} \cap A_1$ (with respect to the orientation of $\ell^{(-)}$) and, for each $i \geq 1$, let $k_i \in \mathbb{N}$ be such that $f_{A_i - k_i \vec{v}_\ell}(\ell^{(-)}) = g_1$. Setting $\hat{A}_i := A_i - k_i \vec{v}_\ell$, since

$$(T^{k_i \vec{v}_\ell} x)|\hat{A}_i = x|A_i = \varphi_i|A_i = (T^{k_i \vec{v}_\ell} \varphi_i)|\hat{A}_i,$$

then each $\hat{A}_i$ is a maximal set among all $E(\mathcal{S})$-enveloped sets $\mathcal{T} \in \mathcal{F}_C^{Vol}$ such that $B_i - k_i \vec{v}_\ell \subset \mathcal{T} \subset F_i^+ - k_i \vec{v}_\ell$ and $(T^{k_i \vec{v}_\ell} x)|\mathcal{T} = (T^{k_i \vec{v}_\ell} \varphi_i)|\mathcal{T}$. As $x \in X_\eta$ is ℓ-periodic, then there exists an integer $k \geq 0$ such that $T^{k \vec{v}_\ell} x = T^{k_i \vec{v}_\ell} x$ for infinitely many i. By passing to a subsequence, we can assume this holds for all i.

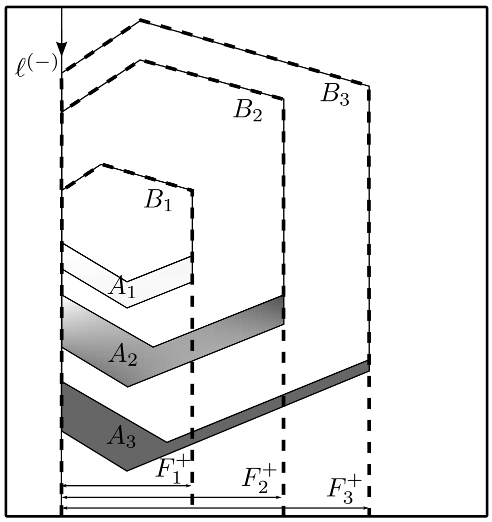


FIGURE 10. The sets $B_1 \subset A_1 \subset B_2 \subset A_2 \subset \cdots$.

Suppose $w_i(0), w_i(1), \ldots, w_i(K), w_i(K+1) \in E(\hat{A}_i)$ are enumerated according to the orientation inherited from the boundary of $\operatorname{conv}(\hat{A}_i)$, where $w_i(0)$ is parallel to ℓ and $w_i(K+1)$ is parallel to $\daleth$. Since $\bigcup_{i=1}^{\infty} A_i = \mathcal{H}(\ell^{(-)})$, let $1 \leq k_{min} \leq K$ be the smallest integer such that $|w_i(k_{min}) \cap \hat{A}_i| < |w_{i+1}(k_{min}) \cap \hat{A}_{i+1}|$ for infinity many i. By passing to a subsequence, we can assume that this holds for all i and, if $k_{min} > 1$, that $|w_i(k) \cap \hat{A}_i| = |w_{i+1}(k) \cap \hat{A}_{i+1}|$ for every $1 \leq k \leq k_{min} - 1$ and all $i \geq 1$. In particular, $\hat{A}_\infty := \bigcup_{i=1}^{\infty} \hat{A}_i$ is a convex weakly $E(\mathcal{S})$-enveloped set (see Figure 11), with two semi-infinite edges, one of which is parallel to $\ell \in \mathbb{G}_1$ and the other one is parallel to the edges $w_i(k_{min})$.

We define $\hat{\varphi}_i := T^{k \vec{v}_\ell} \varphi_i$ for all i and $\hat{x} := T^{k \vec{v}_\ell} x$. Since $\hat{\varphi}_{i'}|\hat{A}_i = \hat{\varphi}_i|\hat{A}_i$ for all $1 \leq i \leq i'$, by the compactness of X_η, the sequence $\{\hat{\varphi}_i\}_{i \in \mathbb{N}}$ has an accumulation point $\varphi \in X_\eta$ with $\varphi|\hat{A}_\infty = \hat{x}|\hat{A}_\infty$. In particular, it follows that $\varphi|\hat{A}_\infty$ is an ℓ-periodic configuration.

For each $0 \leq j \leq k_{min}$, let $l_j \subset \mathbb{R}^2$ denote the oriented line parallel to the edges $w_i(j) \in E(\hat{A}_i)$ such that $w_i(j) \subset l_j$ for all i, and let $\wp_{m+1} := \wp_m^{(-)}$ for all $m \geq 0$, where $\wp_0 := l_{k_{min}}$. Writing $\hat{A}_\infty^{(m)} := \mathcal{H}(l_{k_{min}-1}) \cap (\hat{A}_\infty \cup \wp_0 \cup \wp_1 \cup \cdots \cup \wp_m)$ for all $m \geq 0$, we claim that there exists $n \geq 0$ such that

$$\varphi|\hat{A}_\infty^{(n)} = \hat{x}|\hat{A}_\infty^{(n)}, \quad \text{but} \quad \varphi|\hat{A}_\infty^{(n+1)} \neq \hat{x}|\hat{A}_\infty^{(n+1)}. \tag{5.1}$$

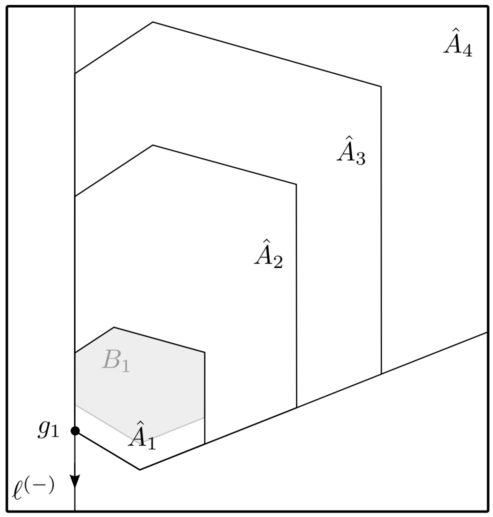


FIGURE 11. The sets $B_1 \subset \hat{A}_1 \subset \hat{A}_2 \subset \cdots \subset \hat{A}_\infty$.

Indeed, suppose, by contradiction, that $\varphi|\hat{A}_\infty^{(m)} = \hat{x}|\hat{A}_\infty^{(m)}$ for all integer $m \geq 1$. As each set $\hat{A}_i$ is $E(\mathcal{S})$-enveloped and $l_{k_{min}-1}$ and $l_{k_{min}}$ are fixed, then, for a sufficiently large $\imath$, there exists an integer $m \geq 1$ for which both $l_{k_{min}-1} \cap \wp_m$ and $l_{k_{min}+1} \cap \wp_m$ belong to $\mathbb{Z}^2$, with the distance from $l_{k_{min}-1} \cap \wp_m$ to $l_{k_{min}+1} \cap \wp_m$ greater than the length of $w_1(k_{min})$, where $l_{k_{min}+1} \subset \mathbb{R}^2$ denotes the oriented line parallel to $w_\imath(k_{min}+1) \in E(\hat{A}_\imath)$ such that $w_\imath(k_{min}+1) \subset l_{k_{min}+1}$. Hence, one has that

$$\hat{A}_\imath^{(m)} := \mathcal{H}(l_{k_{min}-1}) \cap \Big(\hat{A}_\imath \cup \wp_0 \cup \wp_1 \cup \cdots \cup \wp_m\Big) \cap \mathcal{H}(l_{k_{min}+1}) \subset (F_\imath^+ - k_\imath \vec{v}_\ell) \cap \hat{A}_\infty^{(m)}$$

is an $E(\mathcal{S})$-enveloped set. From our assumption it follows that $\varphi|\hat{A}_\imath^{(m)} = \hat{x}|\hat{A}_\imath^{(m)}$, which contradicts the maximality of $\hat{A}_\imath$.

We fix once for all $n \geq 0$ fulfilling (5.1). Let $\ell' \in \mathbb{G}_1$ be the oriented line parallel to the edges $w_i(k_{min})$. Suppose $\mathcal{T} \in \mathcal{F}_C^{Vol}$ is a translation of $\mathcal{S}$ such that $\ell'_\mathcal{T} = \wp_{n+1}$ and $a \in \mathbb{Z}$ satisfies $\mathcal{T} \backslash \ell'_\mathcal{T} + a\vec{v}_{\ell'} \subset \hat{A}_\infty^{(n)}$. We claim that $\varphi \in \mathcal{M}_{a+}(\ell', \mathcal{T})$. In fact, if there exists a $\mathcal{T}\backslash\ell'_\mathcal{T}$-configuration $\gamma \in L_{a+}^{\ell'}(\mathcal{T}\backslash\ell'_\mathcal{T}, \varphi)$ with $N_\mathcal{T}(\ell', \gamma) = 1$, from $\varphi|\hat{A}_\infty^{(n)} = \hat{x}|\hat{A}_\infty^{(n)}$, it follows that

$$\varphi|\hat{A}_\infty^{(n)} \cup (\mathcal{T} + i\vec{v}_{\ell'}) = \hat{x}|\hat{A}_\infty^{(n)} \cup (\mathcal{T} + i\vec{v}_{\ell'}) \tag{5.2}$$

for some $i \geq a$. Although $\hat{A}_\infty^{(n)}$ may be not weakly $E(\mathcal{S})$-enveloped, both $\hat{A}_\infty^{(n)}$ and $\mathcal{T}$ have suitable geometries to fully take advantage of (5.2). Then, as $\mathcal{T}$ is an η-generating set, from (5.2) it follows by induction that $\varphi|\hat{A}_\infty^{(n+1)} = \hat{x}|\hat{A}_\infty^{(n+1)}$, which contradicts (5.1).

Note that $\ell' \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η, since, otherwise, from Lemma 3.13 we could get (5.2) for some integer $i \geq a$ and, exactly as above, reach an absurd. Our goal now will be to construct a convex subset $\mathcal{K}$ of $\hat{A}_\infty^{(n)}$ (with two semi-infinite edges parallel to the oriented lines $\ell, \ell' \in \mathbb{G}_1$) such that $\varphi|\mathcal{K}$ is both ℓ-periodic and ℓ'-periodic.

Let $p' := |\ell'_\mathcal{T} \cap \mathcal{T}| - 1$. If $|A_{a+}^{\ell',p'}(\mathcal{T}, \varphi)| = 1$, then the restriction of φ to the $(\ell', \mathcal{T}, p')$-half-strip $F^+(a)$ is trivially periodic of period $\vec{v}_{\ell'}$. Otherwise, by minimality, one has $P_\eta(\mathcal{T}) - P_\eta(\mathcal{T}\backslash\ell'_{\mathcal{T}}) \leq p' \leq p' + |A^{\ell',p'}_{a+}(\mathcal{T},\varphi)| - 2$ (recall Remark 3.9). Hence, Lemma 4.5 implies, in particular, that, for some $b \geq a$, the restriction of φ to the $(\ell', \mathcal{T}, p')$-half-strip $F^+(b) \subset F^+(a)$ is periodic of period $h' := t'\vec{v}_{\ell'}$ for some $t' \geq 1$. Let $g'_0, g'_1 \in \ell'_{\mathcal{T}} \cap \mathcal{T}$ be the two distinct vertices of $\mathcal{T}$. Since $\mathcal{T}$ is convex and $|\ell_{\mathcal{T}} \cap \mathcal{T}| \leq |\partial_{\mathcal{T}} \cap \mathcal{T}|$, then the set $\mathcal{R} \in \mathcal{F}^{Vol}_C$ whose vertices are

$$g'_0, g'_0 - (|\ell_{\mathcal{T}} \cap \mathcal{T}| - 1)\vec{v}_\ell, g'_1 \;\text{ and }\; g'_1 - (|\ell_{\mathcal{T}} \cap \mathcal{T}| - 1)\vec{v}_\ell$$

is contained in $\mathcal{T}$. So any oriented line $\ell'' \subset \mathbb{R}^2$ parallel to ℓ' satisfying $\ell'' \cap \mathcal{R} \neq \emptyset$ also satisfies $|\ell'' \cap \mathcal{R}| \geq |\ell'_{\mathcal{R}} \cap \mathcal{R}| - 1 = |\ell'_{\mathcal{T}} \cap \mathcal{T}| - 1$. This means that the $(\ell', \mathcal{R}, p')$-half-strip $F^+_{\mathcal{R}}(b)$ lies in the $(\ell', \mathcal{T}, p')$-half-strip $F^+(b)$. Let $Q \subset \mathcal{H}(\ell_{\mathcal{R}+b\vec{v}_{\ell'}}) \cap \hat{A}^{(n)}_\infty$ be an $E(\mathcal{S})$-enveloped set large enough so that, for any integers $0 \leq r < s$, $\varphi|Q + rh' = \varphi|Q + sh'$ implies

$$\varphi|F_Q + rh' = \varphi|F_Q + sh' = T^{(s-r)h'}\varphi|F_Q + rh', \tag{5.3}$$

where F_Q is defined to be the set $\big(\bigcup_{i\in\mathbb{Z}}(Q - i\vec{v}_\ell)\big) \cap \hat{A}^{(n)}_\infty$. Of course this is possible because $\varphi|\hat{A}^{(n)}_\infty$ is ℓ-periodic. We claim that if $0 < r < s$ satisfy (5.3), then the restriction of φ to the set $\mathcal{H}(\ell_{\mathcal{R}+b\vec{v}_{\ell'}}) \cap \hat{A}^{(n)}_\infty \cap \mathcal{H}(\partial_{Q+rh'})$ is periodic of period $(s-r)h'$. In fact, since $\varphi|\hat{A}^{(n)}_\infty$ is periodic of period $h := t\vec{v}_\partial$ for some $t \geq 1$ (recall that $\vec{v}_\partial = -\vec{v}_\ell$), then

$$\varphi|(F^+_{\mathcal{R}}(b) + \iota h) \cup (F_Q + rh') = T^{(s-r)h'}\varphi|(F^+_{\mathcal{R}}(b) + \iota h) \cup (F_Q + rh'), \tag{5.4}$$

where $\iota \geq 1$. Let $\ell_0 := \ell_{Q+rh'}$ and set $\ell_{i+1} := \ell^{(-)}_i$ for all $i \geq 1$. Fix $\iota > |\mathcal{T}|$. Since any translation of $\mathcal{T}$ is η-generating and $|\ell_1 \cap (F^+_{\mathcal{R}}(b) + \iota h)| \geq |\ell_{\mathcal{T}} \cap \mathcal{T}| - 1$, from (5.4) we can, by induction, enlarge the set where φ and $T^{(s-r)h'}\varphi$ coincide by including a subset of $\ell_1 \cap \hat{A}^{(n)}_\infty$ which can be so large as we want (see Figure 12). The ℓ-periodicity of $\varphi|\ell_1 \cap \hat{A}^{(n)}_\infty$ and $T^{(s-r)h'}\varphi|\ell_1 \cap \hat{A}^{(n)}_\infty$ implies then $\varphi|\ell_1 \cap \hat{A}^{(n)}_\infty = T^{(s-r)h'}\varphi|\ell_1 \cap \hat{A}^{(n)}_\infty$. The same idea applied to the lines ℓ_i for $i = 2, \ldots, M$, where $\ell_M = \ell_{\mathcal{R}+b\vec{v}_{\ell'}}$, allows us to conclude that the restriction of φ to the set $\mathcal{H}(\ell_{\mathcal{R}+b\vec{v}_{\ell'}}) \cap \hat{A}^{(n)}_\infty \cap \mathcal{H}(\partial_{Q+rh'})$ is periodic of period $(s-r)h'$, which proves the claim. Since there exist $0 < t'_0 \leq P_\eta(Q)$ and infinitely many indexes $r > 0$ so that (5.3) holds for r and $s := r + t'_0$, from the previous discussion, we conclude that $\varphi|\mathcal{H}(\ell_{\mathcal{R}+b\vec{v}_{\ell'}}) \cap \hat{A}^{(n)}_\infty$ is periodic of period t'_0h'.

For $\mathcal{K}' := \mathcal{H}(\ell_{\mathcal{R}+b\vec{v}_{\ell'}}) \cap \hat{A}^{(n)}_\infty$, let $\mathcal{K} \subset \mathbb{Z}^2$ denote the maximal set among all convex sets $\mathcal{W} \supset \mathcal{K}'$ such that $\varphi|\mathcal{W}$ is both ℓ-periodic and ℓ'-periodic.

Since $\mathcal{K}$ is convex and $\mathcal{K}'$ has two semi-infinite edges parallel, respectively, to ℓ and ℓ' (recall that $\wp_n$ is parallel to ℓ'), the inclusions $\mathcal{K}' \subset \mathcal{K} \subset \mathcal{H}(\ell) \cap \mathcal{H}(\wp_n)$ ensure that $\mathcal{K}$ has two semi-infinite edges parallel, respectively, to ℓ and ℓ'. To conclude the proof, it is enough to show that $\mathcal{K} \subset \mathcal{H}(\ell) \cap \mathcal{H}(\wp_n)$. If $\mathcal{K} \not\subset \mathcal{H}(\wp_n)$, then $\wp_{n+1} \cap$ conv$(\mathcal{K})$ is a half line. As $\varphi|\mathcal{K}$ and $\hat{x}$ are ℓ-periodic and $\varphi|\hat{A}^{(n)}_\infty = \hat{x}|\hat{A}^{(n)}_\infty$, it follows that $\varphi|\hat{A}^{(n)}_\infty \cup (\wp_{n+1} \cap \mathcal{K}) = \hat{x}|\hat{A}^{(n)}_\infty \cup (\wp_{n+1} \cap \mathcal{K})$. But this equality is stronger than (5.2) and, by a similar reasoning that the one that succeeds it, one contradicts (5.1). Now, if $\mathcal{K} \not\subset \mathcal{H}(\ell)$, then $\ell'' \cap$ conv$(\mathcal{K})$ is a half line for every line $\ell'' \subset \mathcal{H}(\ell^{(-)})$ parallel to ℓ. This allows us to transfer the doubly periodicity of $\hat{x}|\mathcal{K} \cap \hat{A}^{(n)}_\infty \cap \mathcal{H}(\ell^{(-)})$ to $\hat{x}|\mathcal{H}(\ell^{(-)})$, which is an absurd, since we are assuming $x|\mathcal{H}(\ell^{(-)})$ is not doubly periodic.

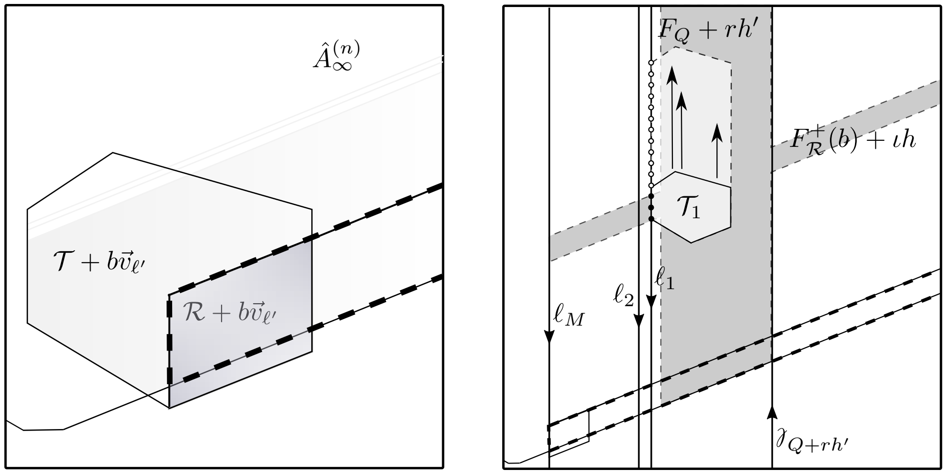


FIGURE 12. In the figure on the left, one has the sets $\mathcal{T} + b\vec{v}_\ell$ and $\mathcal{R} + b\vec{v}_\ell$. In the figure on the right, the grey region represents the set where the restrictions of φ and $T^{(s-r)h'}\varphi$ coincide. The set $\mathcal{T}_1$ denotes the translation of $\mathcal{T}$ such that $u_1 := f_{F^+_{\mathcal{R}}(b)+\iota h}(\ell_1)$ is the final point of $\ell_{\mathcal{T}_1} \cap \mathcal{T}_1$ (with respect to the orientation of $\ell_{\mathcal{T}_1}$). The white points denote the new set where the restrictions of φ and $T^{(s-r)h'}\varphi$ also coincide.

Finally, from Lemma 3.13 and maximality of $\mathcal{K}$, it follows that $\ell, \ell' \in \mathbb{G}_1$ are one-sided nonexpansive directions on $\overline{Orb\,(\varphi)}$, which means that $\varphi \in X_\eta$ is aperiodic and so an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration.

The proof of the case where there exist infinitely many i for which A_i is a maximal set among all $E(\mathcal{S})$-enveloped sets $\mathcal{T} \in \mathcal{F}_C^{Vol}$ such that $B_i \subset \mathcal{T} \subset F_i^-$ and $x|\mathcal{T} = \varphi_i|\mathcal{T}$ is analogous to the previous one by doing the obvious changes. Differently from the first case, here we get an (ℓ', ℓ)-periodic maximal $\mathcal{K}$-configuration $\varphi \in X_\eta$. $\square$

We highlight a class of η-generating sets whose existence comes naturally from a strong condition on the complexity. Although it is a concept derived immediately from our key hypothesis, whose fundamental properties could be absorbed in technical arguments, our main goal in introducing this notion is to simplify the statements of results and the expositions of proofs that will follow.

Definition 5.4. *Given $\eta \in A^{\mathbb{Z}^2}$, an η-generating set $\mathcal{S} \in \mathcal{F}_C$ is said to be a minimal lower complexity (mlc) η-generating set if*

(i) $P_\eta(\mathcal{S}) \leq \frac{1}{2}|\mathcal{S}| + |A| - 1$,

(ii) if $\mathcal{T} \subsetneq \mathcal{S}$ is convex and nonempty, then $P_\eta(\mathcal{T}) > \frac{1}{2}|\mathcal{T}| + |A| - 1$.

The existence of mlc η-generating sets is a straightforward consequence of our alphabetical bound assumption for complexity. As a matter of fact, if there exists a set $\mathcal{U} \in \mathcal{F}_C$ such that $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |A| - 1$, then any convex set $\mathcal{S} \subset \mathcal{U}$ that is minimal among all convex sets $\mathcal{T} \subset \mathcal{U}$ fulfilling $P_\eta(\mathcal{T}) \leq \frac{1}{2}|\mathcal{T}| + |A| - 1$ is an mlc η-generating set. The fact that $\frac{1}{2} + |A| - 1 < |A| = P_\eta(\{g\})$ for all $g \in \mathbb{Z}^2$ ensures that $\mathcal{S}$ has at least two points. In particular, if η is an aperiodic configuration, since $\frac{1}{2}|\mathcal{S}| + |A| - 1 \leq |\mathcal{S}| + |A| - 2$, from Corollary 3.2, we get $\mathcal{S} \in \mathcal{F}_C^{Vol}$.

Due to condition (ii) of Definition 5.4, if $\mathcal{T} \subsetneq \mathcal{S}$ is convex and nonempty, it is clear that $P_\eta(\mathcal{S}) - P_\eta(\mathcal{T}) < \frac{1}{2}|\mathcal{S}\backslash\mathcal{T}|$, which yields

$$P_\eta(\mathcal{S}) - P_\eta(\mathcal{T}) \leq \left\lceil \frac{1}{2}\left|\mathcal{S}\backslash\mathcal{T}\right| \right\rceil - 1, \tag{5.5}$$

where as usual $\lceil \cdot \rceil$ denotes the ceiling function.

Lemma 5.5. *For $\eta \in A^{\mathbb{Z}^2}$, if $\mathcal{S} \in \mathcal{F}_C^{Vol}$ is an mlc η-generating set, then $|\ell_{\mathcal{S}} \cap \mathcal{S}| \geq 3$ whenever $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η.*

Proof. From (4.4), one has $P_\eta(\mathcal{S}\backslash\ell_{\mathcal{S}}) < P_\eta(\mathcal{S})$ and so (5.5) applied to $\mathcal{T} = \mathcal{S}\backslash\ell_{\mathcal{S}}$ provides $|\ell_{\mathcal{S}} \cap \mathcal{S}| \geq 3$. □

Remark 5.6. *Given $\eta \in A^{\mathbb{Z}^2}$ aperiodic, suppose $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |A| - 1$ for some quasi-regular set $\mathcal{U} \in \mathcal{F}_C^{Vol}$. If $\ell \in \mathbb{G}_1$ is a nonexpansive line on X_η, Proposition 4.9 and Corollary 4.15 imply that the oriented lines $\ell, \reflectbox{$\ell$} \in \mathbb{G}_1$ are both one-sided nonexpansive directions on X_η. In particular, if $\mathcal{S} \in \mathcal{F}_C^{Vol}$ is an mlc η-generating set, then $\mathcal{S}$ is (ℓ, p)-balanced with $p = |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$ or $(\reflectbox{$\ell$}, q)$-balanced with $q = |\reflectbox{$\ell$}_{\mathcal{S}} \cap \mathcal{S}| - 1$, since, clearly, conditions (i) and (ii) of Proposition 4.9 hold in one of the two cases.*

Let $\ell \in \mathbb{G}_1$ be a one-sided nonexpansive direction on X_η and suppose $\mathcal{S} \in \mathcal{F}_C^{Vol}$ is an mlc η-generating set such that $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\reflectbox{ℓ}_{\mathcal{S}} \cap \mathcal{S}|$ – in particular, an (ℓ, p)-balanced set with $p := |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$. Since $1 \leq |A^{\ell,p}(\mathcal{S}, x)| \leq \lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}| \rceil - 1$ for every $x \in \mathcal{M}(\ell, \mathcal{S})$ (see (4.4)) and $\lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}| \rceil \leq p$, for each $x \in \mathcal{M}(\ell, \mathcal{S})$, there is a positive integer $p_x \leq p$ such that

$$\left\lceil \frac{1}{2}\left|\ell_{\mathcal{S}} \cap \mathcal{S}\right| \right\rceil - 1 = p_x + \left|A^{\ell,p}(\mathcal{S}, x)\right| - 2, \tag{5.6}$$

which yields $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S}\backslash\ell_{\mathcal{S}}) \leq p_x + |A^{\ell,p_x}(\mathcal{S}, x)| - 2$. Moreover, from (5.6) it follows that

$$2\Phi_p(\ell, \mathcal{S}) \leq 2\left\lceil \frac{1}{2}\left|\ell_{\mathcal{S}} \cap \mathcal{S}\right| \right\rceil - 2 \leq \left|\ell_{\mathcal{S}} \cap \mathcal{S}\right| - 1. \tag{5.7}$$

From the previous discussion, Lemmas 4.3, 4.11, 4.5 and 4.13 can be restated as follows. Recall that, for a rational oriented line ℓ, the vector $\vec{v}_{\reflectbox{$\ell$}} = -\vec{v}_\ell$ (of minimum norm) is antiparallel to ℓ and parallel to $\reflectbox{$\ell$}$.

Proposition 5.7. *Given $\eta \in A^{\mathbb{Z}^2}$, if there exists an mlc η-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ and $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on X_η with $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\reflectbox{ℓ}_{\mathcal{S}} \cap \mathcal{S}|$, then $\mathcal{S}$ is an (ℓ, p)-balanced set for $p = |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$ and satisfies the following conditions.*

- *(i) For $x \in \mathcal{M}(\ell, \mathcal{S})$, the restriction of x to the $(\ell, \mathcal{S}, p)$-strip F is periodic of period $t\vec{v}_\ell$ for some $t \leq \lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}| \rceil - 1$;*
- *(ii) If the restriction of $x \in X_\eta$ to the $(\ell, \mathcal{S}, p)$-strip F is periodic of period $t'\vec{v}_\ell$ for some $t' \geq 1$, then $x|\ell_{\mathcal{S}} \cup F$ is periodic of period $t\vec{v}_\ell$, where $t = t'$ if $x \notin \mathcal{M}(\ell, \mathcal{S})$ and $t \leq 2\lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}| \rceil - 2$ otherwise;*
- *(iii) For $x \in \mathcal{M}_{a\pm}(\ell, \mathcal{S})$, with $a \in \mathbb{Z}$, the restriction of x to the $(\ell, \mathcal{S}, p)$-half-strip $F^{\pm}(a + p)$ is periodic of period $\pm t\vec{v}_\ell$ for some $t \leq \lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}| \rceil - 1$;*
- *(iv) If the restriction of $x \in X_\eta$ to the $(\ell, \mathcal{S}, p)$-half-strip $F^{\pm}(a)$ is periodic of period $\pm t'\vec{v}_\ell$ for some $t' \geq 1$, then $x|(\ell_{\mathcal{S}} \cup F^{\pm})(a)$ is periodic of period $\pm t\vec{v}_\ell$, where $t = t'$ if $x \notin \mathcal{M}_{a\pm}(\ell, \mathcal{S})$ and $t \leq 2\lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}| \rceil - 2$ otherwise.*

For an mlc η-generating set, roughly speaking, its complexity is bounded from above by the sum of the complexity of any proper convex subset and the cardinality of the complement of this subset, as already seen in (5.5). The key point in the proof of Theorem 2.2 is to contradict such a property by taking into account a convenient proper convex subset. In order to estimate the complexity of this subset with respect to an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration φ, it is useful to relate it to the complexity of a suitable small part. This essentially can be done due to the fact that the periods of φ can be bounded by expressions depending on the cardinality of the edges of an mlc η-generating set (see Proposition 5.9).

If $\mathcal{T} \in \mathcal{F}_C$ and $\ell \in \mathbb{G}_1$, from now on, let $\operatorname{diam}_\ell(\mathcal{T})$ denote the number of distinct oriented lines parallel to ℓ that have nonempty intersection with $\mathcal{T}$.

Lemma 5.8. *Given $\eta \in A^{\mathbb{Z}^2}$, if $\mathcal{S} \in \mathcal{F}_C^{Vol}$ is mlc η-generating and $\ell, \partial \in \mathbb{G}_1$ are antiparallel one-sided nonexpansive directions on X_η, then there exists an η-generating set $\mathcal{T} \subset \mathcal{S}$ with $\operatorname{diam}_\ell(\mathcal{T}) \leq \lceil \frac{1}{2} \operatorname{diam}_\ell(\mathcal{S}) \rceil$.*

Proof. Initially, choose $\ell_1, \ell_2 \subset \mathbb{R}^2$ antiparallel oriented lines, with ℓ_1 parallel to ℓ, such that $\mathcal{U}_1 := \mathcal{S} \cap \mathcal{H}(\ell_1)$ and $\mathcal{U}_2 := \mathcal{S} \cap \mathcal{H}(\ell_2)$ satisfy $\operatorname{diam}_\ell(\mathcal{U}_1) = \lceil \frac{1}{2} \operatorname{diam}_\ell(\mathcal{S}) \rceil$ and $\operatorname{diam}_\ell(\mathcal{U}_2) = \lceil \frac{1}{2} \operatorname{diam}_\ell(\mathcal{S}) \rceil$. Since $\mathcal{U}_1 \subset \mathcal{S} \backslash \ell_{\mathcal{S}}$, $\mathcal{U}_2 \subset \mathcal{S} \backslash \partial_{\mathcal{S}}$ and $\ell, \partial \in \mathbb{G}_1$ are one-sided nonexpansive directions on X_η, one has $P_\eta(\mathcal{U}_1) < P_\eta(\mathcal{S})$ and $P_\eta(\mathcal{U}_2) < P_\eta(\mathcal{S})$. Hence, if $|\mathcal{U}_1| \geq \frac{1}{2}|\mathcal{S}|$, then

$$P_\eta(\mathcal{U}_1) - |\mathcal{U}_1| < P_\eta(\mathcal{S}) - |\mathcal{U}_1| \leq \frac{1}{2}|\mathcal{S}| + |A| - 1 - |\mathcal{U}_1| \leq \frac{1}{2}|\mathcal{S}| + |A| - 1 - \frac{1}{2}|\mathcal{S}| = |A| - 1,$$

that is, $P_\eta(\mathcal{U}_1) \leq |\mathcal{U}_1| + |A| - 2$. Thus, from Remark 3.9, there exists an η-generating set $\mathcal{T} \subset \mathcal{U}_1$ and, in particular, $\operatorname{diam}_\ell(\mathcal{T}) \leq \lceil \frac{1}{2} \operatorname{diam}_\ell(\mathcal{S}) \rceil$. If $|\mathcal{U}_1| < \frac{1}{2}|\mathcal{S}|$, it is clear that $|\mathcal{U}_2| \geq \frac{1}{2}|\mathcal{S}|$ and the same argument as before concludes the proof. □

We will make use in the next pages of a theorem that is a classic in the combinatorics of words and appears in the main textbooks. For the convenience of the reader, we recall its statement here.

Fine-Wilf Theorem ([7])**.** *Assuming the $\xi = (\xi_i)_{i \in \mathbb{N}}$ and $\xi' = (\xi'_i)_{i \in \mathbb{N}}$ are periodic sequences of periods q and q', respectively, if $\xi_i = \xi'_i$ for at least $q + q' - gcd(q, q')$ consecutive entries, then $\xi = \xi'$.*

The next result generalizes Claim 5.4 in [4]. We state it for an mlc η-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ with $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\partial_{\mathcal{S}} \cap \mathcal{S}|$ and an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration, but an analogous statement holds for (ℓ', ℓ)-periodic maximal $\mathcal{K}$-configurations by considering $F^+(a)$ instead of $F^-(a)$. The proposition shows that the existence of an mlc generating set provides upper bounds for the periods of restrictions to $\mathcal{K}$ of (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configurations, with one of these periods even remaining outside the convex region $\mathcal{K}$. The strategy of its proof consists in arguing that there exists a suitable translation of the mlc generating set such that each restriction to the associated half-strip admits multiple extensions, so that one may apply Proposition 5.7. Some notions used here can be recalled in Notations 3.6 and 3.10, Definition 4.4 and (4.13). Furthermore, recall that $\mathcal{M}_{a+}(\ell, \mathcal{U})$ and $\mathcal{M}_{a-}(\ell, \mathcal{U})$ are the sets formed by the configurations $x \in X_\eta$ that satisfy, respectively, (4.5) and (4.6).

Proposition 5.9. *For $\eta \in A^{\mathbb{Z}^2}$, suppose there exists an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration $\varphi \in X_\eta$. Let $\mathcal{R}' \in \mathcal{F}_C^{Vol}$ be an (ℓ', r')-balanced set and let $\mathcal{S} \in \mathcal{F}_C^{Vol}$*

be an mlc η-generating set such that $\ell_{\mathcal{S}} = \ell_{\mathcal{K}}^{(-)}$ and $|\ell_{\mathcal{S}} \cap \mathcal{S}| \le |\partial_{\mathcal{S}} \cap \mathcal{S}|$. Then the following conditions hold:

(i) $\varphi|\mathcal{K}$ is periodic of period $t_0\vec{v}_\partial$ for some $t_0 \le \lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}|\rceil - 1$ and periodic of period $t'_0\vec{v}_{\ell'}$ for some $t'_0 \le |\ell'_{\mathcal{S}} \cap \mathcal{S}| - 2$,

(ii) denoting $p := |\ell_{\mathcal{S}}\cap\mathcal{S}|-1$, for any $(\ell, \mathcal{S}, p)$-half-strip $F^-(a)$, $a \in \mathbb{Z}$, contained in $\mathcal{K}$, $\varphi|(\ell_{\mathcal{S}} \cup F^-)(a)$ is periodic of period $\tau_0\vec{v}_\partial$ for some $\tau_0 \le \lceil \frac{1}{2}|\ell_{\mathcal{S}}\cap\mathcal{S}|\rceil-1$.

Proof. Let $h = \kappa\vec{v}_\partial$ and $h' = \kappa'\vec{v}_{\ell'}$, with $\kappa, \kappa' \in \mathbb{N}$, denote periods of $\varphi|\mathcal{K}$. Recall that $\ell, \ell' \in \mathbb{G}_1$ are one-sided nonexpansive directions on $\overline{Orb\,(\varphi)}$ and so one-sided nonexpansive directions on X_η. Note that any accumulation point $\psi \in \overline{Orb\,(\varphi)}$ of $T^{j\vec{v}_{\ell'}}\varphi$, $j \ge 0$, is ℓ'-periodic. Indeed, as the restriction of ψ to the half plane $\mathcal{H}(\ell'_{\mathcal{K}})$ is ℓ'-periodic and $\mathcal{R}' \in \mathcal{F}_C^{Vol}$ is an (ℓ', r')-balanced set, Proposition 4.16 implies that ψ is ℓ'-periodic. Moreover, due to Lemma 3.13, the maximality of $\mathcal{K}$ prevents ℓ' from being a one-sided expansive direction on $\overline{Orb\,(\psi)}$. Hence, Proposition 3.14 implies that $\partial' \in \mathbb{G}_1$ is a one-sided nonexpansive direction on $\overline{Orb\,(\psi)}$ and then on X_η. In particular, Lemma 5.5 ensures that $|\ell'_{\mathcal{S}} \cap \mathcal{S}| \ge 3$ and $|\partial'_{\mathcal{S}} \cap \mathcal{S}| \ge 3$.

Let $a \in \mathbb{Z}$ be such that the $(\ell, \mathcal{S}, p)$-half-strip $F^-(a)$ is contained in $\mathcal{K}$. Since $\mathcal{S}$ is an (ℓ, p)-balanced set and $(T^h\varphi)|F^-(a) = \varphi|F^-(a) = (T^{h'}\varphi)|F^-(a)$, Lemma 4.12 and maximality of $\mathcal{K}$ provide $\varphi \in \mathcal{M}_{a-}(\ell, \mathcal{S})$. Thanks to condition (iii) of Proposition 5.7, the restriction of φ to the $(\ell, \mathcal{S}, p)$-half-strip $F^-(a+p)$ is periodic of period $t_0\vec{v}_\partial$ for some

$$t_0 \le \left\lceil \frac{1}{2}\left|\ell_{\mathcal{S}} \cap \mathcal{S}\right|\right\rceil - 1. \tag{5.8}$$

Moreover, since $\varphi|\mathcal{K}$ is ℓ-periodic and $F^-(a + p) \subset F^-(a) \subset \mathcal{K}$, it is clear that the restriction of φ to the $(\ell, \mathcal{S}, p)$-half-strip $F^-(a)$ is also periodic of period $t_0\vec{v}_\partial$.

To conclude the proof of the first condition it is enough to show that $\varphi|\mathcal{K}$ is periodic of period $t'_0\vec{v}_{\ell'}$ for some integer $t'_0 \le |\ell'_{\mathcal{S}} \cap \mathcal{S}| - 2$. Indeed, in this case, $F^-(a)\cap(F^-(a) + t'_0\vec{v}_{\ell'}) \ne \emptyset$ and, from the Fine-Wilf Theorem [7], we obtain that $\varphi|\mathcal{K}$ is periodic of period $t_0\vec{v}_\partial$.

We prove such a fact by considering two cases separately.

Case 1. Suppose $|\ell'_{\mathcal{S}} \cap \mathcal{S}| \le |\partial'_{\mathcal{S}} \cap \mathcal{S}|$ and define $q' := |\ell'_{\mathcal{S}} \cap \mathcal{S}| - 1$. Let $\mathcal{S}'$ be a translation of $\mathcal{S}$ with $\ell'_{\mathcal{S}'} = \ell'^{(-)}_{\mathcal{K}}$ and let $a' \in \mathbb{Z}$ be such that the $(\ell', \mathcal{S}', q')$-half-strip $F^+(a')$ is contained in $\mathcal{K}$. As before, Lemma 4.12 and maximality of $\mathcal{K}$ imply that $\varphi \in \mathcal{M}_{a'+}(\ell', \mathcal{S}')$. Thus, condition (iii) of Proposition 5.7 ensures that the restriction of φ to $F^+(a' + q')$ and therefore to $F^+(a')$ is periodic of period $t'_0\vec{v}_{\ell'}$ for some integer

$$t'_0 \le \left\lceil \frac{1}{2}\left|\ell'_{\mathcal{S}'} \cap \mathcal{S}'\right|\right\rceil - 1 = \left\lceil \frac{1}{2}\left|\ell'_{\mathcal{S}} \cap \mathcal{S}\right|\right\rceil - 1 \le \left|\ell'_{\mathcal{S}} \cap \mathcal{S}\right| - 2. \tag{5.9}$$

Clearly, for each $j \ge 0$, $\varphi|F^-(a) = (T^{jh'}\varphi)|F^-(a)$ and $\varphi|F^+(a') = (T^{jh}\varphi)|F^+(a')$. Let $m, m' \in \mathbb{N}$ be such that

$$\mathcal{W} := (F^-(a) + m'h') \cap (F^+(a') + mh)$$

contains a translation of $\mathcal{S}\backslash\{\ell_{\mathcal{S}}, \ell'_{\mathcal{S}}\}$. Let $\partial'_1 := \partial'^{(-)}_{\mathcal{W}}$ and define $\partial'_{i+1} := \partial'^{(-)}_i$ for all $i \ge 1$. For each $i \ge 1$, we write

$$(\mathbb{Z}^2\backslash\mathcal{H}(\partial_{\mathcal{W}})) \cap \partial'_i =: \{g_i + t\vec{v}_{\ell'} : t \ge 0\}.$$

Thanks to Lemma 5.8 there exists an η-generating set $\mathcal{T} \subset \mathcal{S}$ with $\mathrm{diam}_\ell(\mathcal{T}) \le \lceil \frac{1}{2}\,\mathrm{diam}_\ell(\mathcal{S})\rceil$. If $\mathfrak{T}_i$ is the translation of $\mathcal{T}$ where g_i is the initial point of $\partial'_{\mathfrak{T}_i} \cap \mathfrak{T}_i$

(with respect to the orientation of ∂'), by (5.9) we have $\mathfrak{T}_i - t_0'\vec{v}_{\ell'} \subset F^-(a)+m'h'$ (see Figure 13). Since $\varphi|F^-(a)+m'h'$ is periodic of period $t_0\vec{v}_\partial$, from (5.8) it follows

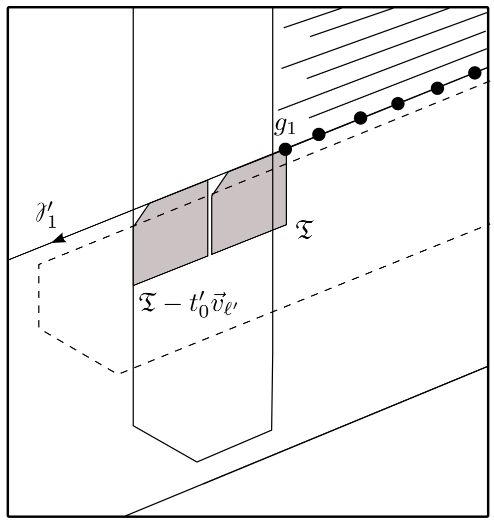


FIGURE 13. The sets $\mathfrak{T}_1$ and $\mathfrak{T}_1 - t_0'\vec{v}_{\ell'}$. The dashed region represents $F^+(a')+mh$.

that the restriction of φ to the set $(F^-(a)+m'h') \cup (F^+(a')+mh)$ is periodic of period $t_0'\vec{v}_{\ell'}$. In particular,

$$\varphi|\mathfrak{T}_1 \backslash \{g_1\} = (T^{-t_0'\vec{v}_{\ell'}}\varphi)|\mathfrak{T}_1 \backslash \{g_1\},$$

which yields $\varphi_{g_1} = (T^{-t_0'\vec{v}_{\ell'}}\varphi)_{g_1}$, because $\mathfrak{T}_1$ is an η-generating set. Applying an identical reasoning to the points $g_1+t\vec{v}_{\ell'}$, we obtain by induction that the restriction of φ to the set

$$(F^-(a)+m'h') \cup (F^+(a')+mh) \cup \{g_1+t\vec{v}_{\ell'} : t \geq 0\}$$

is periodic of period $t_0'\vec{v}_{\ell'}$. By repeating the reasoning for the others lines ∂_i', we conclude by induction that the restriction of φ to the set

$$(F^-(a)+m'h') \cup (F^+(a')+mh) \cup \{g_i+t\vec{v}_{\ell'} : i \geq 1,\ t \geq 0\}$$

is periodic of period $t_0'\vec{v}_{\ell'}$. Since by hypothesis $\varphi|\mathcal{K}$ is doubly periodic with periods in $\ell \cap (\mathbb{Z}^2)^*$ and $\ell' \cap (\mathbb{Z}^2)^*$, using the Fine-Wilf Theorem it is not difficult to conclude that $\varphi|\mathcal{K}$ is also periodic of period $t_0'\vec{v}_{\ell'}$, which completes this case.

Case 2. Suppose $|\ell'_{\mathcal{S}} \cap \mathcal{S}| > |\partial'_{\mathcal{S}} \cap \mathcal{S}|$ and define $q' := |\partial'_{\mathcal{S}} \cap \mathcal{S}| - 1$. If for some translation $\mathcal{S}'$ of $\mathcal{S}$ there is $a' \in \mathbb{Z}$ such that $\varphi \in \mathcal{M}_{a'-}(\partial', \mathcal{S}')$ and the $(\partial', \mathcal{S}', q')$-half-strip $F^-(a')$ lies in $\mathcal{K}$, then exactly as argued in Case 1 we have that $\varphi|\mathcal{K}$ is periodic of period $t_0'\vec{v}_{\ell'}$ for some $t_0' \leq \lceil \frac{1}{2}|\partial'_{\mathcal{S}} \cap \mathcal{S}| \rceil - 1 < |\partial'_{\mathcal{S}} \cap \mathcal{S}| - 1 \leq |\ell'_{\mathcal{S}} \cap \mathcal{S}| - 2$, which completes this case. From now on, we suppose $\varphi \notin \mathcal{M}_{a'-}(\partial', \mathcal{S}')$ for all translation $\mathcal{S}'$ of $\mathcal{S}$ and any $a' \in \mathbb{Z}$ such that the $(\partial', \mathcal{S}', q')$-half-strip $F^-(a')$ is contained in $\mathcal{K}$. Let $\psi \in \overline{Orb\,(\varphi)}$ be an accumulation point of the sequence $T^{j\vec{v}_{\ell'}}\varphi$, $j \geq 0$. Note that $\psi \notin \mathcal{M}(\partial', \mathcal{S}')$ for all translation $\mathcal{S}'$ of $\mathcal{S}$ such that $\mathcal{S}' \subset \mathcal{H}(\ell'_{\mathcal{K}})$. In fact, for $a' \in \mathbb{Z}$ with $\mathcal{S}'+a'\vec{v}_{\ell'} \subset \mathcal{K}$, let $\gamma \in L^{\partial'}_{a'-}(\mathcal{S}' \backslash \partial'_{\mathcal{S}'}, \varphi)$ be such that $N_{\mathcal{S}'}(\partial', \gamma) = 1$. The ℓ'-periodicity of $\varphi|\mathcal{K}$ ensures that $\gamma \in L^{\partial'}(\mathcal{S}'\backslash\partial'_{\mathcal{S}'}, \psi)$, which implies that $\psi \notin \mathcal{M}(\partial', \mathcal{S}')$. On the other hand, we claim that there is a translation $\mathcal{S}'_1$ of $\mathcal{S}$ such that $\psi \in \mathcal{M}(\partial', \mathcal{S}'_1)$. Indeed, suppose not. Let $\mathcal{S}'$ be a translation of $\mathcal{S}$ such that $\mathcal{S}' \not\subset \mathcal{H}(\ell'_{\mathcal{K}})$. For $t' \geq 1$, we define

$$Q := (\mathcal{S}' + \vec{v}_{\ell'}) \cup \cdots \cup (\mathcal{S}' + t'\vec{v}_{\ell'}).$$

By the Pigeonhole Principle, there exist $0 \leq i < j \leq P_\psi(Q)$ such that $\psi|Q + i\vec{v}_\ell = \psi|Q + j\vec{v}_\ell$ (see Figure 14). Since the accumulation point ψ is ℓ'-periodic, we may consider $t' \geq 1$ large enough so that

$$(T^{(i-j)\vec{v}_\ell}\psi)|F' + j\vec{v}_\ell = \psi|F' + i\vec{v}_\ell = \psi|F' + j\vec{v}_\ell, \tag{5.10}$$

where F' denotes an $(\partial', \mathcal{S}', q')$-strip. The contradiction hypothesis allows us to apply

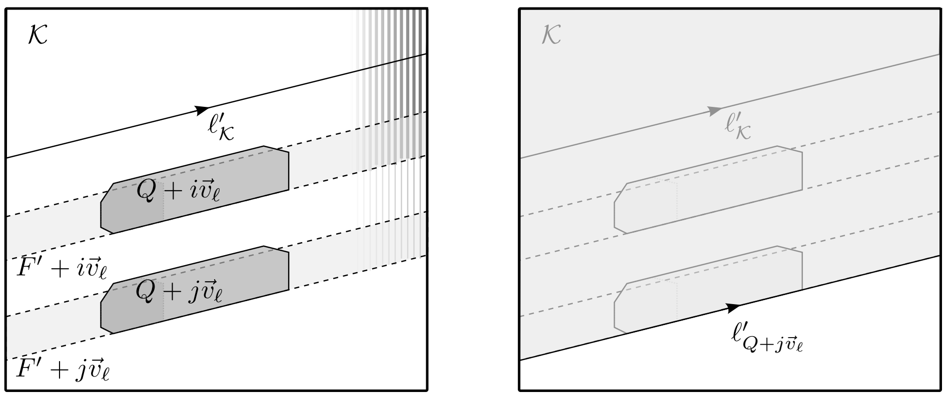


FIGURE 14. The sets $Q + i\vec{v}_\ell$ and $Q + j\vec{v}_\ell$ and the half plane $\mathcal{H}(\ell'_{Q+j\vec{v}_\ell})$.

Lemma 4.1 successively. Then, from (5.10), we obtain that the restriction of ψ to the half plane $\mathcal{H}(\ell'_{Q+j\vec{v}_\ell})$ is periodic of period $(i-j)\vec{v}_\ell$. Recall that any accumulation point of $T^{j\vec{v}_{\ell'}}\varphi$, $j \geq 0$, is ℓ'-periodic with a common bound for their periods (see Proposition 4.16). From this fact is not difficult to argue that there exists a half line $\varGamma \subset \ell'^{(-)}_{\mathcal{K}}$ such that $\varphi|\varGamma$ is ℓ'-periodic. Obviously any half line contained in $\varGamma$ fulfills the same property. So we can suppose $\varGamma$ is such that $\mathcal{K} \cup (\varGamma \cap \mathbb{Z}^2)$ is convex. As ψ and $T^{j\vec{v}_{\ell'}}\varphi$ coincide on arbitrarily large regions, this allows us to contradict the maximality of $\mathcal{K}$, since the set of doubly periodicity of φ would now include the half line $\varGamma$, which proves the claim.

So, let $\mathcal{S}'_1$ be a translation of $\mathcal{S}$ such that $\psi \in \mathcal{M}(\partial', \mathcal{S}'_1)$ and define $\mathcal{S}'_{j+1} := \mathcal{S}'_1 - ju$ for all $j \geq 1$, where $u \in \mathbb{Z}^2$ satisfies $\ell' + u = \ell'^{(-)}$. Let $J \geq 1$ be the largest integer such that $\psi \in \mathcal{M}(\partial', \mathcal{S}'_J)$. Thanks to condition (ii) of Proposition 5.7, the restriction of ψ to $\partial'_{\mathcal{S}'_J} \cup F'_J$ is periodic of period $t'_0\vec{v}_{\ell'}$ for some

$$t'_0 \leq 2\lceil \frac{1}{2}|\partial'_{\mathcal{S}} \cap \mathcal{S}|\rceil - 2 \leq |\partial'_{\mathcal{S}} \cap \mathcal{S}| - 1 \leq |\ell'_{\mathcal{S}} \cap \mathcal{S}| - 2,$$

where F'_J denotes the $(\partial', \mathcal{S}'_J, q')$-strip. Thus, since $\psi \notin \mathcal{M}(\partial', \mathcal{S}'_j)$ for all $j > J$, condition (ii) of Proposition 5.7 ensures that the restriction of ψ to the half plane $\mathcal{H}(\ell'_{\mathcal{S}'_J})$ is periodic of period $t'_0\vec{v}_{\ell'}$. Hence, as ψ and $T^{j\vec{v}_{\ell'}}\varphi$ coincide on arbitrarily large regions, it is easy to argue using the Fine-Wilf Theorem that $\varphi|\mathcal{K}$ is also periodic of period $t_0'\vec{v}_{\ell'}$. This concludes the proof of condition (i) in the statement of proposition.

We also prove condition (ii) by considering two cases separately. We begin by assuming that

$$(T^{i\,\vec{v}_\jmath}\varphi)|\mathcal{S} \neq (T^{j\,\vec{v}_\jmath+h'}\varphi)|\mathcal{S} \quad \forall\, i,j \geq a, \tag{5.11}$$

where $h' = \kappa'\vec{v}_{\ell'}$ is a period of $\varphi|\mathcal{K}$. Clearly,

$$(T^{t\vec{v}_\jmath}\varphi)|\mathcal{S}\backslash\ell_{\mathcal{S}} = (T^{t\vec{v}_\jmath+h'}\varphi)|\mathcal{S}\backslash\ell_{\mathcal{S}} \quad \forall\, t \geq a. \tag{5.12}$$

For $A := \{(T^{t\vec{v}_\jmath}\varphi)|\mathcal{S} : t \geq a\}$ and $B := \{(T^{t\vec{v}_\jmath+h'}\varphi)|\mathcal{S} : t \geq a\}$, from (5.11) and (5.12) one has $|A| + |B| = |A \cup B| \leq \sum_{\gamma\in C} N_{\mathcal{S}}(\ell,\gamma)$, where

$$C := \{(T^{t\vec{v}_\jmath+h'}\varphi)|\mathcal{S}\backslash\ell_{\mathcal{S}} : t \geq a\}.$$

In particular, as $|C| \leq |B|$, then

$$|A| \leq \left(\sum_{\gamma\in C} N_{\mathcal{S}}(\ell,\gamma)\right) - |C| \leq \sum_{\gamma\in C}\Big(N_{\mathcal{S}}(\ell,\gamma) - 1\Big) \leq P_\eta(\mathcal{S}) - P_\eta(\mathcal{S}\backslash\ell_{\mathcal{S}}).$$

Hence, (5.5) applied to $\mathcal{T} = \mathcal{S}\backslash\ell_{\mathcal{S}}$ provides $|A| \leq \lceil\frac{1}{2}|\ell_{\mathcal{S}}\cap\mathcal{S}|\rceil - 1$. Let $\xi = (\xi_t)_{t\geq a}$ be the sequence defined by $\xi_t := (T^{t\vec{v}_\jmath}\varphi)|\mathcal{F}^{\ell,p}(\mathcal{S}) \cup \{f_{\mathcal{S}}(\ell_{\mathcal{S}})\}$ for all integer $t \geq a$ and consider $n := \lceil\frac{1}{2}|\ell_{\mathcal{S}}\cap\mathcal{S}|\rceil - 1$. If $P_\xi(1) = 1$, then ξ is trivially periodic. Otherwise, from the fact that $n < |\ell_{\mathcal{S}}\cap\mathcal{S}| - 1$ we have that $P_\xi(n) \leq |A| \leq n \leq n + P_\xi(1) - 2$. The Alphabetical Morse-Hedlund Theorem implies that $(\xi_t)_{t\geq a+n}$ is periodic of period at most n. This means, even if $P_\xi(1) = 1$, that the restriction of φ to the set $(\ell_{\mathcal{S}}\cup F^-)(a+n)$ is periodic of period $\tau_0\vec{v}_\jmath$ for some $\tau_0 \leq \lceil\frac{1}{2}|\ell_{\mathcal{S}}\cap\mathcal{S}|\rceil - 1$. Being $\varphi|(\ell_{\mathcal{S}}\cup F^-)(a)$ ℓ-periodic (see Lemma 4.13), the inclusion $(\ell_{\mathcal{S}}\cup F^-)(a+n) \subset (\ell_{\mathcal{S}}\cup F^-)(a)$ ensures that $\varphi|(\ell_{\mathcal{S}}\cup F^-)(a)$ is also periodic of period $\tau_0\vec{v}_\jmath$, which concludes this case.

In the second case, we assume that there exist integers $i,j \geq a$ such that

$$\varphi|\mathcal{S} + i\vec{v}_\jmath = (T^{i\,\vec{v}_\jmath}\varphi)|\mathcal{S} = (T^{j\,\vec{v}_\jmath+h'}\varphi)|\mathcal{S} = (T^{h'}\varphi)|\mathcal{S} + j\vec{v}_\jmath. \tag{5.13}$$

As guaranteed by condition (i), $\varphi|F^-(i)$ and $(T^{h'}\varphi)|F^-(j)$ are periodic of period $t_0\vec{v}_\jmath$ for some $t_0 \leq \lceil\frac{1}{2}|\ell_{\mathcal{S}}\cap\mathcal{S}|\rceil - 1$. Hence, (5.13) and Remark 4.8 allow us to get

$$\varphi|F^-(i) \cup (\mathcal{S} + i\vec{v}_\jmath) = (T^{h'}\varphi)|F^-(j) \cup (\mathcal{S} + j\vec{v}_\jmath).$$

Being $\mathcal{S}$ an η-generating set, by induction, it follows that the restrictions of φ to the sets $(\ell_{\mathcal{S}}\cup F^-)(i)$ and $(\ell_{\mathcal{S}}\cup F^-)(j) + h' \subset \mathcal{K}$ coincide. Thus, since the restriction of φ to $(\ell_{\mathcal{S}}\cup F^-)(j) + h'$ is periodic of period $t_0\vec{v}_\jmath$, then $\varphi|(\ell_{\mathcal{S}}\cup F^-)(i)$ and therefore $\varphi|(\ell_{\mathcal{S}}\cup F^-)(a)$ is periodic of period $t_0\vec{v}_\jmath$. □

6. Proof of the main result

For the convenience of the reader, we recall the statement of the main result.

Theorem 2.2. *If $\eta \in A^{\mathbb{Z}^2}$ contains all letters of the alphabet A and there exists a quasi-regular set $\mathcal{U} \in \mathcal{F}_C^{Vol}$ such that $P_\eta(\mathcal{U}) \leq \frac{1}{2}|\mathcal{U}| + |A| - 1$, then η is periodic.*

In addition to the fundamental notion given in Definition 2.1 (*quasi-regular set*), the reader may find it beneficial to review Definitions 3.7 (*(non)expansive direction*), 3.8 (*η-generating set*), and 4.6 (*(ℓ,p)-balanced set*). For the proof, two concepts discussed in the preceding section will be crucial: it is useful to keep in mind the notions of (ℓ, ℓ')-*periodic maximal* $\mathcal{K}$-*configuration* (Definition 5.1) and of *mlc* η-*generating set* (Definition 5.4).

Regarding the proof strategy, supposing, by contradiction, that the thesis of Theorem 2.2 does not hold, following steps highlighted by Cyr and Kra, we will reach an absurd from the existence of an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration. The main idea is to argue that the number of configurations arising in the boundary of the doubly periodic maximal region would be greater than possible. We do not prove the Claims 6.1 and 6.2, but their proofs can be found, respectively, after Claim 5.7 and in Subsubsection 5.4.3 of [4]. The proof of the main theorem will be done by considering an mlc η-generating set $\mathcal{S} \in \mathcal{F}_C^{Vol}$ with $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\reflectbox{ℓ}_{\mathcal{S}} \cap \mathcal{S}|$ and by supposing the existence of an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration[3]. The alphabetical viewpoint will be not explicitly used here. Actually, our strong hypothesis on the complexity will be inherited by $\mathcal{S}$, providing thus a common bound (see (5.7)) for $p_x + |A^{\ell, p_x}(\mathcal{S}, x)|$ for all $x \in \mathcal{M}(\ell, \mathcal{S})$, where p_x is the smallest integer fulfilling Definition 4.6.

Proof of Theorem 2.2 By contradiction, suppose η is aperiodic. Let $\mathcal{S} \in \mathcal{F}_C^{Vol}$ be an mlc η-generating set. Corollary 3.3 ensures that there is at least one nonexpansive line on X_η, which is denoted by $\ell \in \mathbb{G}_1$. As observed in Remark 5.6, the antiparallel lines ℓ and $\reflectbox{$\ell$}$ are both one-sided nonexpansive directions on X_η. This allows us to assume, without loss of generality, $|\ell_{\mathcal{S}} \cap \mathcal{S}| \leq |\reflectbox{ℓ}_{\mathcal{S}} \cap \mathcal{S}|$. Hence, according to the proof of Proposition 5.3, there exists an (ℓ, ℓ')-periodic or an (ℓ', ℓ)-periodic maximal $\mathcal{K}$-configuration $\varphi \in X_\eta$. We suppose $\varphi \in X_\eta$ is an (ℓ, ℓ')-periodic maximal $\mathcal{K}$-configuration. Recall that $\ell' \in \mathbb{G}_1$ is a one-sided nonexpansive directions on $\overline{Orb\,(\varphi)}$ and so also on X_η which satisfies $\vec{v}_{\ell'} \in \mathcal{H}(\ell)$. We denote by $\psi \in A^{\mathbb{Z}^2}$ the unique doubly periodic configuration such that $\varphi|\mathcal{K} = \psi|\mathcal{K}$. Translating $\mathcal{S}$, we can assume that the $(\ell, \mathcal{S}, p)$-half-strip $F^-(0)$ lies in $\mathcal{K}$ and that $\ell_{\mathcal{S}} = \ell_{\mathcal{K}}^{(-)}$, where $p := |\ell_{\mathcal{S}} \cap \mathcal{S}| - 1$. As Proposition 4.9 guarantees the existence of a balanced set with respect to every given one-sided nonexpansive direction, by Proposition 5.9, $\varphi|\mathcal{K}$ and so ψ are doubly periodic configurations of periods $h := t_0 \vec{v}_{\reflectbox{ℓ}}$ and $h' := t_0' \vec{v}_{\ell'}$, where

$$t_0 \leq \left\lceil \frac{1}{2} |\ell_{\mathcal{S}} \cap \mathcal{S}| \right\rceil - 1 \leq |\ell_{\mathcal{S}} \cap \mathcal{S}| - 2, \quad t_0' \leq |\ell'_{\mathcal{S}} \cap \mathcal{S}| - 2. \tag{6.1}$$

Let $\reflectbox{$\ell$}_0 := \reflectbox{$\ell$}_{\mathcal{S}}$ and denote $\reflectbox{$\ell$}_{i+1} := \reflectbox{$\ell$}_i^{(+)}$ for all $i \geq 1$. For suitable integers $d \geq 1$, we define

$$\mathcal{S}(d) := \bigcup_{i=0}^{d-1} \left\{ f_{\mathcal{S}}(\reflectbox{ℓ}_i) + t \vec{v}_\ell : 0 \leq t \leq p - 2 \right\},$$

where $f_{\mathcal{S}}(\reflectbox{ℓ}_i) \in \mathbb{Z}^2$ is the final point of $\reflectbox{$\ell$}_i \cap \mathcal{S}$ (with respect to the orientation of $\reflectbox{$\ell$}_i$). Consider $\mathcal{S}^* := \mathcal{S} \backslash \{ i_{\mathcal{S}}(\reflectbox{ℓ}_i) : 0 \leq i \leq n \}$, where $n \geq 1$ is such that $\reflectbox{$\ell$}_n \cap \mathbb{Z}^2 = \ell_{\mathcal{S}} \cap \mathbb{Z}^2$ (See Figure 15). Recall that, thanks to Lemma 5.5, $p \geq 2$. Since $\mathcal{S}^*$ is a proper convex subset of $\mathcal{S}$ and $\mathcal{S}$ is an mlc η-generating set, by (5.5) one has

$$P_\eta(\mathcal{S}) - P_\eta(\mathcal{S}^*) \leq \left\lceil \frac{1}{2} |\mathcal{S} \backslash \mathcal{S}^*| \right\rceil - 1 < |\mathcal{S} \backslash \mathcal{S}^*| - 1. \tag{6.2}$$

Let $\wp_0 \subset \mathbb{R}^2$ be the oriented line parallel to ℓ such that $\wp_0 \cap (\mathcal{S} \backslash \mathcal{S}^*) = i_{\mathcal{S}}(\ell'_{\mathcal{S}})$ and denote $\wp_{i+1} = \wp_i^{(+)}$ for all $i \geq 0$. If $m' \geq 1$ is such that $\wp_{m'} \cap (\mathcal{S} \backslash \mathcal{S}^*) = f_{\mathcal{S}}(\ell'_{\mathcal{S}})$,

[3]For the case of an (ℓ', ℓ)-periodic maximal $\mathcal{K}$-configuration (which is the other possible situation according to the proof of of Proposition 5.3) the reasoning is analogous.

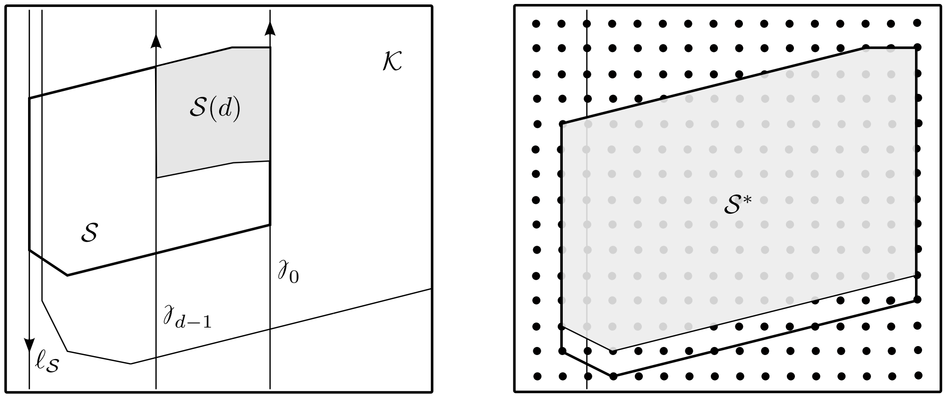


FIGURE 15. Representation of the sets $\mathcal{S}(d)$ and $\mathcal{S}^*$.

define $z_i := \wp_i \cap (\mathcal{S}\backslash\mathcal{S}^*)$ for each $0 \leq i \leq m'$. By (6.1) there exist integers $0 \leq i < j \leq m'$ such that $t'_0\vec{v}_{\ell'} = z_j - z_i$. Thus, we define

$$d' := \min\{j - i : 0 \leq i < j \leq m',\ \psi \text{ is periodic of period } z_j - z_i\}.$$

Identifying the vector $z_j - z_i$ to an oriented line segment $S_{ij} \subset \mathbb{R}^2$, note that $j - i + 1$ represents the number of distinct oriented lines $\ell'' \subset \mathbb{R}^2$ parallel to ℓ such that $\ell'' \cap \mathbb{Z}^2 \neq \emptyset$ and $\ell'' \cap S_{ij} \neq \emptyset$. Then, writing $\mathrm{diam}_\ell(\mathcal{S}) = (|\ell'_{\mathcal{S}} \cap \mathcal{S}| - 1)\mu + 1 + r$, where $\mu := j - i$ for any $0 \leq i < j \leq m'$ such that $\vec{v}_{\ell'} = z_j - z_i$ and $r \geq 0$, by (6.1) one has

$$\mathrm{diam}_\ell(\mathcal{S}) > (|\ell'_{\mathcal{S}} \cap \mathcal{S}| - 2)\mu + 1 + r \geq t'_0\mu + 1,$$

which yields $d' \leq t'_0\mu < \mathrm{diam}_\ell(\mathcal{S}) - 1$. In particular, $\ell_{\mathcal{S}} \cap \mathcal{S}(d') = \emptyset$. Defining $\bar{d} := |\mathcal{S}\backslash\mathcal{S}^*| - d' = \mathrm{diam}_\ell(\mathcal{S}) - d' > 1$, it is clear that $(\mathcal{S}(d') + lu) + i\vec{v}_\jmath \subset \mathcal{K}$ for every $i \geq 0$ and $0 \leq l \leq \bar{d} - 1$.

Let $u \in \mathbb{Z}^2$ be such that $\ell + u = \ell^{(-)}$. For each integer $l \geq 0$, we define $\mathcal{S}_l := \mathcal{S} + lu$ and $\mathcal{S}^*_l := \mathcal{S}^* + lu$. Since $\varphi \in X_\eta$ is aperiodic and $\ell \in \mathbb{G}_1$ is a one-sided nonexpansive direction on $\overline{Orb\,(\varphi)}$, from Proposition 4.9, Corollary 4.15 and Proposition 4.16, we conclude that the restriction of φ to every $(\ell, \mathcal{S}_l, p)$-strip is not ℓ-periodic. For each integer $0 \leq l \leq \bar{d} - 1$, let $a_l \in \mathbb{Z}$ be the smallest integer for which the restriction of φ to the set $(\ell_{\mathcal{S}_l} \cup F^-)(a_l)$ is ℓ-periodic, where $F^-(a_l)$ denotes as usual the corresponding $(\ell, \mathcal{S}_l, p)$-half-strip. We remark that the existence of a_l follows from Lemma 4.13 applied successively in order to get a larger region that contains $\mathcal{K}$ and such that the restriction of φ is ℓ-periodic (but not doubly periodic).

In the next two claims, we will count the number of configurations that arise in the boundary of the doubly periodic maximal region. The strategy is to show that there are so many such configurations that (6.2) may be contradicted.

Claim 6.1 (Claim 5.7 of [4]). *The following conditions hold.*

(i) $\left|\{\varphi|\mathcal{S}^*_l + (a_l - 1)\vec{v}_\jmath : 0 \leq l \leq \bar{d} - 1\}\right| = \bar{d}$.

(ii) *For each integer $0 \leq l \leq \bar{d} - 1$, there exist at least two distinct $\mathcal{S}_l$-configurations $\gamma', \gamma'' \in L(\mathcal{S}_l, \eta)$ such that $\gamma'|\mathcal{S}^*_l = (T^{(a_l - 1)\vec{v}_\jmath}\varphi)|\mathcal{S}^*_l = \gamma''|\mathcal{S}^*_l$.*

For each integer $0 \leq l \leq d' - 1$, let u_l denote the vector $z_{m'} - z_{m'-d'+l}$. Let $\mathcal{T}$ be a translation of $\mathcal{S}$ such that $\ell'_{\mathcal{T}} = \ell'^{(-)}_{\mathcal{K}}$ and $\mathcal{T}\backslash\ell'_{\mathcal{T}} \subset \mathcal{K}$, and let $\mathcal{T}^*$ be the corresponding translation of $\mathcal{S}^*$. Since z_i belongs to $\mathcal{S}\backslash\mathcal{S}^*$ and does not belong to

$\ell'_{\mathcal{S}^*}$ for every $0 \le i \le m'$, it is not difficult to check that $\mathcal{T}^* + u_l \subset \mathcal{K}$ for every integer $0 \le l \le d'-1$ (even when $z_{m'-d'+l} \in \ell'^{(-)}_{\mathcal{S}^*}$).

Claim 6.2 (Subsubsection 5.4.3 of [4])**.** *The following conditions hold.*

(iii) $\big|\{\varphi|\mathcal{T}^* + u_l : 0 \le l \le d'-1\}\big| = d'$.

(iv) For each integer $0 \le l \le d'-1$*, there exist at least two distinct* $\mathcal{T}$*-configurations* $\gamma', \gamma'' \in L(\mathcal{T}, \eta)$ *such that* $\gamma'|\mathcal{T}^* = (T^{u_l}\varphi)|\mathcal{T}^* = \gamma''|\mathcal{T}^*$.

To reach a contradiction and thus to conclude the proof, it is enough to show that the $\mathcal{S}^*$-configurations of condition (i) are different from $\mathcal{S}^*$-configurations of condition (iii). As a matter of fact, if this is the case, there exist at least $\bar{d}+d' = |\mathcal{S}\backslash\mathcal{S}^*|$ distinct $\mathcal{S}^*$-configurations $\gamma \in L(\mathcal{S}^*, \eta)$ such that $|\{\gamma' \in L(\mathcal{S}, \eta) : \gamma'|\mathcal{S}^* = \gamma\}| > 1$, which means that

$$P_\eta(\mathcal{S}) - P_\eta(\mathcal{S}^*) \ge \bar{d} + d' = |\mathcal{S}\backslash\mathcal{S}^*|,$$

contradicting (6.2). Note then that, since every configuration of condition (iii) belongs to $L(\mathcal{S}^*, \psi)$, it is enough to show that, for each integer $0 \le l \le \bar{d}-1$,

$$(T^{lu+i\,\vec{v}_\jmath}\varphi)|\mathcal{S}^* \notin L(\mathcal{S}^*, \psi), \quad \forall\ i \ge 0,$$

where $\ell + u = \ell^{(-1)}$. Suppose, by contradiction, that there exist $0 \le l \le \bar{d}-1$ and $i \ge 0$ such that

$$(T^{lu+i\,\vec{v}_\jmath}\varphi)|\mathcal{S}^* \in L(\mathcal{S}^*, \psi). \tag{6.3}$$

Since ψ is doubly periodic of periods $t_0\vec{v}_\jmath$ and $z_j - z_i$, with $0 \le i < j \le m'$ such that $d' = j - i$, the very definition of $\mathcal{S}(d') \subset \mathcal{S}^*$ ensures

$$\forall\ \gamma', \gamma'' \in L(\mathcal{S}^*, \psi), \quad \gamma'|\mathcal{S}(d') = \gamma''|\mathcal{S}(d') \ \text{ implies } \ \gamma' = \gamma''. \tag{6.4}$$

As $(\mathcal{S}(d') + lu) + i\vec{v}_\jmath \subset \mathcal{K}$ and $\varphi|\mathcal{K} = \psi|\mathcal{K}$, then $(T^{lu+i\,\vec{v}_\jmath}\varphi)|\mathcal{S}(d') \in L(\mathcal{S}(d'), \psi)$. Since $h' = t'_0\vec{v}_{\ell'}$ is a period of φ, $(T^{lu+i\,\vec{v}_\jmath}\varphi)|\mathcal{S}(d') = (T^{lu+i\,\vec{v}_\jmath+mh'}\varphi)|\mathcal{S}(d')$ for all $m \ge 1$. Note that, for m sufficiently large, we have $(\mathcal{S}^* + lu) + i\vec{v}_\jmath + mh' \subset \mathcal{K}$ and therefore $(T^{lu+i\,\vec{v}_\jmath+mh'}\varphi)|\mathcal{S}^* \in L(\mathcal{S}^*, \psi)$. Thus, by (6.3) and (6.4) it follows that

$$(T^{lu+i\,\vec{v}_\jmath}\varphi)|\mathcal{S}^* = (T^{lu+i\,\vec{v}_\jmath+mh'}\varphi)|\mathcal{S}^*.$$

For a half line $\Gamma := \ell_{\mathcal{S}} \cap (\ell_{\mathcal{S}} \cup F^-)(0)$, the above equality implies, in particular, that the restrictions given by $\varphi|\Gamma$ and $(T^{mh'}\varphi)|\Gamma$ coincide in at least $|\ell_{\mathcal{S}} \cap \mathcal{S}| - 2$ consecutive indexes. Since $\varphi|\Gamma$ and $(T^{mh'}\varphi)|\Gamma$ are, respectively, periodic of periods $k\vec{v}_\jmath$ and $t_0\vec{v}_\jmath$, with $k, t_0 \le \lceil \frac{1}{2}|\ell_{\mathcal{S}} \cap \mathcal{S}|\rceil - 1$ (condition (ii) of Proposition 5.9 and (6.1)), then, as $k + t_0 - \gcd(k, t_0) \le |\ell_{\mathcal{S}} \cap \mathcal{S}| - 2$, from Fine-Wilf Theorem, we obtain that $\varphi|\Gamma = (T^{mh'}\varphi)|\Gamma$, which contradicts the maximality of $\mathcal{K}$ and concludes the proof of the theorem. □

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