# A Ruelle-Perron-Frobenius theorem for expanding circle maps with an indifferent fixed point

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**Abstract**

We establish an original result for the thermodynamic formalism in the context of expanding circle transformations with an indifferent fixed point. For an observable whose modulus of continuity is linked to the dynamics near such a fixed point, by identifying an appropriate linear space to evaluate the action of the transfer operator, we show that there is a strictly positive eigenfunction associated with the maximal eigenvalue given as the exponential of the topological pressure. Taking into account also the corresponding eigenmeasure, the invariant probability thus obtained is proved to be the unique Gibbs-equilibrium state of the system.

**Keywords:** non-uniformly expanding dynamics, intermittent maps, thermodynamic formalism, equilibrium states, modulus of continuity.

**Mathematical subject classification:** 37D25, 37E05, 37D35, 26A12, 26A15.

## 1 Introduction and Statement of Results

### 1.1 Contextualization

The pioneering works of Sinai, Ruelle and Bowen provided the impetus for the development of a fruitful area in ergodic theory of differentiable systems, which made available a wide range of techniques to construct invariant measures with significant statistical properties. Thermodynamic formalism for uniformly hyperbolic systems and Hölder continuous potentials has today a well-established theoretical ground and contributions to extend it consider different settings and approaches.

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In this work, we focus on expanding maps on the circle that have an indifferent fixed point and we take into account potentials with a modulus of continuity whose only imposition is dictated by the behavior of the dynamics in a neighborhood of this fixed point. The study of this class of maps is linked to mathematical models of intermittency (see [Man80, PM80]) and the initial mathematical results on this topic were obtained in [Tha80].

An essential issue that this article addresses is the determination of conditions to be observed by a given class of potentials to ensure that there is a single equilibrium state of the system. For frameworks that are similar to ours, Ruelle-Perron-Frobenius type theorems and theorems about the existence and uniqueness of equilibrium states, central results of this article, are recorded in the literature. The main lines of research focus on potentials obeying properties related to fundamental entities of thermodynamic formalism, such as the topological pressure and the transfer operator, and/or belonging to certain specific classes of regularity.

Without any intention of being exhaustive, it is worth mentioning some works to illustrate advances in this current. For piecewise monotone interval transformations, Hofbauer and Keller [HK82] have studied equilibrium states associated with potentials of bounded variation whose oscillation is strictly smaller than the topological entropy. In the same dynamic context, Denker, Keller and Urbański [DKU90] proved that, on each topologically transitive component, there is at most one equilibrium state associated with a potential either of bounded variation or with bounded distortion under the transformation, which, besides one of these regularity conditions, has as its supremum a value strictly smaller than its topological pressure. Existence and uniqueness of equilibrium states were obtained by Liverani, Saussol and Vaienti [LSV98] for a class of piecewise monotone transformations on a totally ordered, compact set and for observables named as contracting potentials. Among the properties requested to be a contracting potential [LSV98, Definition 3.4], there is the demand that the supremum of one of its Birkhoff sums is strictly smaller than the logarithm of the infimum of the corresponding iterate of the transfer operator applied to the function identically equal one. For smooth interval maps, the condition on the potential introduced by Hofbauer and Keller was used by Bruin and Todd [BT08] to propose a proof for existence and uniqueness of equilibrium states by means of an inducing scheme. In [LR14], for a sufficiently regular one-dimensional map satisfying a weak form of hyperbolicity, Li and Rivera-Letelier showed that, given a Hölder continuous potential, the supremum of one of its Birkhoff averages is strictly smaller than its topological pressure, a condition that guarantees in particular the existence and uniqueness of equilibrium states. Generalizing an optimal transportation method successfully applied for the thermodynamic formalism in the uniformly expanding context [KLS15], recently Kloeckner [Klo20] was able to prove a Ruelle-Perron-Frobenius theorem and to study equilibrium states for non-uniformly expanding maps and observables named as flat potentials. Flatness is a property that requires a uniform regularity control on all Birkhoff sums taken along pairing trajectories following a common transition kernel (see [Klo20, Definition 2.13] for technical details).

One of the main contributions of our work is the identification of an easily verifiable property relating, in a neighborhood of the indifferent fixed point, the joint behavior of the dynamics and a pair of moduli of continuity (see condition (1) below). Here one modulus describes the regularity of a potential and the other one indicates the space on which the transfer operator's action should be considered in order to obtain relevant spectral information. This is a sufficient condition to ensure key results of the thermodynamic formalism from known methods and techniques, specially from potential theory or harmonic analysis. In practical terms, it is possible, for example, to fix a dynamics on the circle among the members of the analyzed family and without difficulty to determine possible classes of regularity of potentials for which a Ruelle-Perron-Frobenius theorem and the existence and uniqueness of Gibbs-equilibrium states can be proved. In section 1.4 of [GI20] there are illustrations of this simple compatibility procedure for a particular situation. We provide more general examples along this paper.

Roughly speaking, the condition of coherence between the dynamics around the indifferent fixed point and both classes of regularity guarantees that these moduli of continuity are nicely related along backward orbits (see Definition 1 and Propostion 7). This feature thus allows establishing a direct Ruelle-Peron-Frobenius theorem in a non-uniformly hyperbolic setting without inducing: we obtain a strictly positive eigenfunction of the transfer operator when looking at its action on the space of the functions with the suggested regularity. Existence and uniqueness of equilibrium state are discussed taking advantage of Rokhlin formula, being useful to have the eigenequation to eliminate the possibility of the Dirac delta at the indifferent fixed point being an aspiring to equilibrium state.

In a preliminary version of this article, we did not emphasize the actual extent of our results. In particular, it was pointed out that there is an overlap between our results and those obtained by Kloeckner [Klo20]. It should first be noted that, in cases where both works apply, the proofs are independent, and our strategy is dissociated from the approach in [Klo20], which is focused on determining a contraction rate of the dual of the transfer operator through couplings, a method introduced in [Sta17]. Here the proposal of a comprehensive scenario by means of a compatibility between dynamics and moduli of continuity allows us to go further: the generality of the maps makes it very easy to present examples and to work with classes of regularity far beyond the usual Hölder modulus environment. Corollaries 4 and 5, for instance, illustrate the fact that our results cover a range of examples and how simple can it be to ensure their application in wider situations. Furthermore, from a mainly theoretical perspective, it is relevant to have a more accurate understanding of how the specific form of the indifferent fixed point influences thermodynamic formalism. In significant cases, one cannot deal with the Hölder regularity class, being necessary to consider different moduli to ensure a Ruelle-Perron-Frobenius theorem. The general formulation developed here seeks to contribute to a global understanding of these aspects, as clearly is the purpose of Kloeckner's work as well.

### 1.2 Dynamics and Regularity Classes

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1)$ denote the circle endowed with the standard metric

$$d(x, y) = \min\{|x - y|, |x - y \pm 1|\}, \quad x, y \in [0, 1).$$

We consider a family $\mathscr{F}$ of continuous maps $T : \mathbb{T} \to \mathbb{T}$ which are non-uniformly expanding with an indifferent fixed point. More precisely, we suppose that $T$ is of the form $T(x) := x(1 + V(x)) \mod 1$, for all $x \in [0, 1)$, where the continuous and increasing function $V : [0, +\infty) \to [0, +\infty)$ satisfies that $V(1)$ is a positive integer and for some $\sigma \geq 0$

$$\lim_{x\to 0} \frac{V(tx)}{V(x)} = t^{\sigma}, \text{ for all } t > 0.$$

When $\sigma > 0$, $V$ is called *regularly varying with index* $\sigma$, and when $\sigma = 0$, $V$ is called *slowly varying*. For the main properties of these families of functions, we refer the reader to [Sen76]. We remark that any map $T$ in $\mathscr{F}$ is topologically mixing. As a matter of fact, $T$ is topologically exact in the sense that, for every open nonempty set $U \subset \mathbb{T}$, there is $M \geq 1$ for which $T^M(U) = \mathbb{T}$.

By a *modulus of continuity*, we mean a continuous and non-decreasing function $\omega : [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$. Let $\mathcal{M}$ be the set of all concave modulus of continuity. For $\omega \in \mathcal{M}$, we denote by $\mathscr{C}_\omega(\mathbb{T})$ the linear space of functions $\varphi : \mathbb{T} \to \mathbb{R}$ with a multiple of $\omega$ as modulus of continuity: $|\varphi(x) - \varphi(y)| \leq C\omega(d(x, y))$ for some constant $C > 0$, for all $x, y \in [0, 1)$. For $\varphi \in \mathscr{C}_\omega(\mathbb{T})$, we denote the smallest constant that guarantees this condition of regularity by

$$|\varphi|_\omega := \sup_{x\neq y} \frac{|\varphi(x) - \varphi(y)|}{\omega(d(x, y))}.$$

A central notion in this work will be the $T$-compatibility of a concave modulus of continuity with respect to another one.

**Definition 1.** *For a map $T \in \mathscr{F}$, given $\omega, \Omega \in \mathcal{M}$, we say that $\Omega$ is $T$-compatible with respect to $\omega$ whenever the following property holds.*

*There are constants $\varrho_1 > 0$ and $C_1 > 0$ such that, given any sequence $\{x_k\}_{k\geq 0}$ in $\mathbb{T}$ with $T(x_{k+1}) = x_k$ for $k \geq 0$, and a point $y_0 \in \mathbb{T}$ with $d(x_0, y_0) < \varrho_1$, there is a unique pre-orbit $\{y_k\}_{k\geq 1}$ of $y_0$ (that is, $T(y_{k+1}) = y_k$ for $k \geq 0$) fulfilling*

$$d(x_k, y_k) \leq d(x_0, y_0) < \varrho_1 \qquad \textit{and}$$

$$\Omega\big(d(x_k, y_k)\big) + C_1 \sum_{j=1}^{k} \omega\big(d(x_j, y_j)\big) \leq \Omega\big(d(x_0, y_0)\big) \qquad \forall\, k \geq 1.$$

*Moreover, this correspondence between pre-orbits of $x_0$ and $y_0$ is one-to-one.*

We will show (see Proposition 7) that a sufficient condition for $T$-compatibility of $\Omega$ with respect to $\omega$ is

$$\liminf_{x\to 0} \frac{V(x)}{\omega(x)} \Big(\Omega\big((1 + c)x\big) - \Omega(x)\Big) > 0 \tag{1}$$

for all sufficiently small constant $c > 0$. Note, in particular, that such a property implies $\liminf_{x\to 0} \frac{\omega(x)}{V(x)} = 0$.

This condition allows us to easily provide families of examples of $T$-compatibility. By way of illustration, suppose that $V$ and $V^2$ are concave near to the origin. If we consider $\omega = V^2$ and $\Omega = V$ in this neighborhood, then it is clear that condition (1) holds whenever $V$ has index $\sigma > 0$.

For a more concrete situation, remember that a prototypical example in $\mathscr{F}$ is the Manneville-Pomeau map defined, for a fixed $s \in (0,1)$, as $T_s(x) := x(1 + x^s)$ mod 1. Consider the class of modulus of continuity $\omega_{\alpha,\beta} : [0, +\infty) \to [0, +\infty)$, defined for $0 \le \alpha < 1$ and $\beta \ge 0$ with $\alpha + \beta > 0$ as

$$\omega_{\alpha,\beta}(x) := \begin{cases} x^\alpha(-\log x)^{-\beta}, & 0 < x < x_0, \\ x_0^\alpha(-\log x_0)^{-\beta}, & x \ge x_0, \end{cases}$$

where $x_0 = x_0(\alpha, \beta)$ is taken small enough so that $\omega_{\alpha,\beta}$ is concave. This class was taken into account in the work of Kloeckner [Klo20]. While $\omega_{\alpha,0}$ is reduced to the Hölder continuity, $\omega_{0,\beta}$ determines a class that is larger than local Hölder continuity. Note that for $x$ sufficiently small,

$$\omega_{\alpha,\beta}\big((1 + c)x\big) \ge (1 + c)^\alpha \omega_{\alpha,\beta}(x). \tag{2}$$

From this inequality, it follows that, for all $s \in (0, \alpha)$, the modulus $\omega_{\alpha-s,\beta}$ is $T_s$-compatible with respect to $\omega_{\alpha,\beta}$ since condition (1) is immediately checked.

To provide an application example for the slowly varying scenario, consider the family of maps $S_k(x) = x(1 + W_k(x))$ mod 1, with $k$ a positive integer, where in a neighborhood of the origin $W_k(x)$ is of the form $A_k(\log^k 1/x)^{-1}$ for some constant $A_k > 0$. (Here $\log^k$ stands for the $k$-times composition of the logarithm function.) Suppose that $\omega(x)$ and $\Omega(x)$ are defined, respectively, as $(\log^k 1/x)^{-1}(\log 1/x)^{-1}(\log^2 1/x)^{-2}$ and $(\log^2 1/x)^{-1}$ in a small neighborhood of the origin so that both are concave. From the calculus exercise

$$\lim_{x\to 0} \log 1/x \, \log\Big(\frac{\log 1/x}{\log 1/(1 + c)x}\Big) = \log(1 + c),$$

one may verify that condition (1) holds, and therefore the $S_k$-compatibility of (up to some convenient truncation on the right) the modulus $(\log^2 1/x)^{-1}$ with respect to $(\log^k 1/x)^{-1}(\log 1/x)^{-1}(\log^2 1/x)^{-2}$.

## 1.3 Thermodynamics and Main Results

Let $f$ be a real continuous map with modulus of continuity $\omega \in \mathcal{M}$. We define the *transfer operator* associated with $f$ as

$$\mathscr{L}_f \phi(x) := \sum_{y \in T^{-1}(x)} e^{f(y)} \phi(y), \qquad \forall\, \phi \in C^0(\mathbb{T}),$$

where $C^0(\mathbb{T})$ denotes the linear space of continuous functions endowed with the uniform norm $||\cdot||_\infty$. We have that $\mathscr{L}_f$ is a bounded linear operator. For every $n \ge 1$ and $x \in \mathbb{T}$, consider the Birkhoff sum $S_n f(x) := \sum_{j=0}^{n-1} f \circ T^j(x)$. Then, clearly

$$\mathscr{L}_f^n \phi(x) = \sum_{y \in T^{-n}(x)} e^{S_n f(y)} \phi(y).$$

Let $\mathscr{L}_f^*$ denote the operator on finite signed Borel measures defined by

$$\int \phi \, d(\mathscr{L}_f^* m) = \int \mathscr{L}_f \phi \, dm, \quad \forall \, \phi \in C^0(\mathbb{T}).$$

In other terms, $\mathscr{L}_f^*$ is the *dual operator* of $\mathscr{L}_f$. Here we focus on its restriction on the convex subset Prob($\mathbb{T}$) of Borel probability measures on $\mathbb{T}$. Let us represent by $M(\mathbb{T}, T)$ the space of all $T$-invariant probability measures on $\mathbb{T}$. For a probability $m \in M(\mathbb{T}, T)$, we denote by $h_m(T)$ its metric entropy. For a continuous potential $f : \mathbb{T} \to \mathbb{R}$, we introduce by means of the variational principle the topological pressure as

$$P(T, f) := \sup_{m \in M(\mathbb{T}, T)} \left\{ h_m(T) + \int f \, dm \right\}. \tag{3}$$

Our central result states that, whenever $T$-compatibility can be verified, a Ruelle-Perron-Frobenius theorem holds.

**Theorem 2.** *Let $T : \mathbb{T} \to \mathbb{T}$ be a map in $\mathscr{F}$ such that $T(x) = x(1 + V(x)) \mod 1$. Let $\Omega \in \mathcal{M}$ be a $T$-compatible modulus of continuity with respect to $\omega \in \mathcal{M}$. If $f \in \mathscr{C}_\omega(\mathbb{T})$, there exists a probability measure $\nu \in$ Prob($\mathbb{T}$) and a positive constant $\chi$ such that*

$$\mathscr{L}_f^* \nu = \chi \nu.$$

*The number $\chi$ is a simple eigenvalue of the operator $\mathscr{L}_f$ and there is a positive function $h \in \mathscr{C}_\Omega(\mathbb{T})$ such that*

$$\mathscr{L}_f h = \chi h.$$

*The constant $\chi$ is a maximal eigenvalue in the sense that the $\mathscr{L}_f$ acting on complex-valued continuous functions does not admit as eigenvalue another constant of absolute value greater than or equal to $\chi$. Moreover, supposing that $\int h \, d\nu = 1$, for every continuous function $\phi$, the sequence $\{\chi^{-n} \mathscr{L}_f^n \phi\}$ converges uniformly on $\mathbb{T}$ to $h \int \phi \, d\nu$ as $n$ goes to infinity.*

We are also able to establish the existence and uniqueness of Gibbs-equilibrium states.

**Theorem 3.** *In the context of Theorem 2, the measure $\mu := h\nu$ is a $T$-invariant probability such that*

$$h_\mu(T) + \int f \, d\mu = \log \chi = P(T, f).$$

*For any $m \in M(\mathbb{T}, T)$ with $m \neq \mu$, one has $P(T, f) > h_m(T) + \int f \, dm$. In particular, the probability $\mu$ is the unique equilibrium measure associated with $f$. Furthermore, $\mu$ is a Gibbs measure in the sense that, for every sufficiently small $r > 0$, there is a constant $K_r > 0$ such that, for $x \in \mathbb{T}$ and $n \geq 1$,*

$$K_r^{-1} \leq \frac{\mu\big(\{y : d(T^j(x), T^j(y)) < r, \ 0 \leq j \leq n\}\big)}{e^{S_n f(x) - nP(T,f)}} \leq K_r.$$

As one may expect, already known results can be seen as particular cases of the above theorems. By way of illustration, we recover as a corollary a result for the Manneville-Pomeau family, studied, for instance, by [Klo20, Theorem A] and [LR14, Corollary 2.5], who considered for a potential $f \in C_{\omega_{\alpha,0}}(\mathbb{T})$ (Hölder modulus of continuity), the transfer operator $\mathscr{L}_f$ acting on $\mathscr{C}_{\omega_{\alpha-s},0}(\mathbb{T})$.

**Corollary 4.** *For $s \in (0,1)$, consider the Manneville-Pomeau map $T_s(x) = x + x^{s+1}$ mod 1. Whenever $0 < s < \alpha < 1$ and $\beta \geq 0$, for any potential $f \in \mathscr{C}_{\omega_{\alpha,\beta}}(\mathbb{T})$, the transfer operator $\mathscr{L}_f$ acting on $\mathscr{C}_{\omega_{\alpha-s,\beta}}(\mathbb{T})$ satisfies the Ruelle-Perron-Frobenius theorem. Furthermore, the invariant probability arising from the corresponding eigenfunction and eigenmeasure is the unique Gibbs-equilibrium state associated with $f$.*

Nor is it surprising that, due to the malleability of the framework considered, it is not difficult to present families of new examples. As far as we know, the result below is not registered in the literature. The reader may produce several others from condition (1).

**Corollary 5.** *Given a positive integer $k$, let $S_k(x) = x(1 + W_k(x))$ mod 1 be an element of $\mathscr{F}$ for which $W_k(x) = A_k(\log^k 1/x)^{-1}$, with $A_k > 0$, for any $x$ small enough. Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous potential with a positive multiple of $(\log^k 1/x)^{-1}(\log 1/x)^{-1}(\log^2 1/x)^{-2}$ as modulus. Then a Ruelle-Perron-Frobenius theorem holds when one considers the action of the associated transfer operator $\mathscr{L}_f$ on the linear space of the continuous functions that admit a positive multiple of $(\log^2 1/x)^{-1}$ as a modulus of continuity. In addition, existence and uniqueness of Gibbs-equilibrium state associated with $f$ are also guaranteed.*

In section 2 we show that condition (1) is sufficient to guarantee $T$-compatibility and we provide examples of associated moduli of continuity that satisfy this condition. Sections 3 and 4 are devoted to the proofs of the theorems above. For both results, the proof strategies fall on argumentative lines already present in the literature. Among key references are [Bal00, Bow75, PP90, VO16, Rue04]. A major contribution here is the identification of the linear space $\mathscr{C}_\Omega(\mathbb{T})$ as an appropriate set for the search of eigenfunctions of the transfer operator $\mathscr{L}_f$ when $f \in \mathscr{C}_\omega(\mathbb{T})$.

## 2 A Sufficient Condition for $T$-compatibility

By its very definition, it follows that $T \in \mathscr{F}$ is expanding outside any half-closed arc that does not contain the origin, or without loss of generality outside any subset of the form $[0, \epsilon)$, $0 < \epsilon < 1$. Indeed, as $T$ has exactly $N_V := 1 + V(1)$ inverse branches, let $\varrho_V \in (0, 1/2)$ be such that $|x - y| < \varrho_V$ implies $|x-y|N_V + |V(x) - V(y)| < 1/2$. It is thus easy to show that, for all $x, y \in [\epsilon, 1)$ with $d(x, y) < \varrho_V$,

$$d(T(x), T(y)) \geq \lambda(\epsilon)\, d(x, y),$$

where $\lambda(\epsilon) := 1 + V(\epsilon) \to 1$ as $\epsilon \to 0$. A quantitative version of its non-uniformly expanding property on the whole circle is provided by the following lemma. This result is analogous to Lemma 4 of [GI20] and its proof is included by convenience of the reader.

**Lemma 6.** *There exists a constant $\varrho_0 > 0$ such that for $x, y \in \mathbb{T}$ with $d(x,y) < \varrho_0$,*

$$d(T(x), T(y)) \geq d(x,y) \left(1 + \frac{1}{2^{\sigma+2}} V(d(x,y))\right).$$

*Proof.* Let $x, y \in \mathbb{T}$ be such that $d(x,y) < \varrho_V$, where $\varrho_V$ is defined as above. We consider two situations.

*Either the (smallest) open arc from $y$ to $x$ does not contain the origin.* We may suppose then $0 \leq y \leq x \leq 1$. Hence $x(1+V(x)) - y(1+V(y)) \leq |x-y| N_V + |V(x) - V(y)| < 1/2$ and the fact that $V$ is increasing imply that

$$d(T(x), T(y)) = (x-y)\big(1 + V(x)\big) + y\big(V(x) - V(y)\big) \geq d(x,y) \left(1 + V(d(x,y))\right).$$

*Or the open arc from $y$ to $x$ contains the origin.* By the previous case, we have

$$d(T(x), 0) \geq d(x, 0) \left(1 + V(d(x,0))\right) \quad \text{and} \quad d(0, T(y)) \geq d(0, y) \left(1 + V(d(y,0))\right),$$

which by adding yields

$$d(T(x), T(y)) \geq d(x,y) + d(x,0)\, V(d(x,0)) + d(y,0)\, V(d(0,y)).$$

However, as $V$ is increasing,

$$\begin{aligned} d(x,0)\, V(d(x,0)) + d(y,0)\, V(d(0,y)) &\geq \\ &\geq \max\{d(x,0), d(y,0)\}\, V\big(\max\{d(x,0), d(y,0)\}\big) \\ &\geq \frac{1}{2} d(x,y)\, V\Big(\frac{1}{2} d(x,y)\Big). \end{aligned}$$

Using the fact that $V$ has a varying property, let $\varrho_0 \in (0, \varrho_V)$ be such that $V\big(\frac{1}{2}\gamma\big) \geq \frac{1}{2^{\sigma+1}} V(\gamma)$ for $0 \leq \gamma < \varrho_0$. We have thus shown that, whenever $d(x,y) < \varrho_0$,

$$d(T(x), T(y)) \geq d(x,y) \left(1 + \frac{1}{2^{\sigma+2}} V(d(x,y))\right).$$

□

The next result allows to check $T$-compatibility in concrete examples.

**Proposition 7.** *For a map $T(x) = x(1+V(x)) \mod 1$ in $\mathscr{F}$, suppose that the moduli of continuity $\omega, \Omega \in \mathcal{M}$ fulfill*

$$\liminf_{x \to 0} \frac{V(x)}{\omega(x)} \Big(\Omega\big((1+c)x\big) - \Omega(x)\Big) > 0$$

*for $c > 0$ small enough. Then $\Omega$ is $T$-compatible with respect to $\omega$.*

*Proof.* We shall show that there are $\varrho_1 \in (0, \varrho_0)$ and $C_1 > 0$ such that, for any $x_1, x_0$ in $\mathbb{T}$ with $T(x_1) = x_0$ and all $y_0 \in \mathbb{T}$ with $d(x_0, y_0) < \varrho_1$, one has a unique $y_1 \in \mathbb{T}$, with $T(y_1) = y_0$ and $d(x_1, y_1) \leq d(x_0, y_0) < \varrho_1$, satisfying

$$\Omega\big(d(x_1, y_1)\big) + C_1\, \omega\big(d(x_1, y_1)\big) \leq \Omega\big(d(x_0, y_0)\big). \tag{4}$$

Moreover, we shall argue that the correspondence $x_1 \mapsto y_1$ is injective.

For a fixed $c \in (0, \frac{1}{2^{\sigma+2}}]$ such that the above limit inferior is positive, define $C_1 := \frac{1}{2} \liminf \frac{V(x)}{\omega(x)}\big(\Omega((1+c)x) - \Omega(x)\big)$. Let $\varrho_1 \leq \varrho_0/2$ be such that $V(x) \in [0,1]$ and $V(x)\big(\Omega((1+c)x) - \Omega(x)\big) \geq C_1\omega(x)$ whenever $0 < x < \varrho_1$, where $\varrho_0$ is as in the statement of Lemma 6. For $x_0, x_1, y_0 \in \mathbb{T}$ with $T(x_1) = x_0$ and $d(x_0, y_0) < \varrho_1$, we can choose $y_1 \in T^{-1}(y_0)$ with $d(x_1, y_1) \leq d(x_0, y_0) < \varrho_1$. Then from Lemma 6,

$$d(x_0, y_0) = d(T(x_1), T(y_1)) \geq d(x_1, y_1)\,\big(1 + c\,V(d(x_1, y_1))\big).$$

Since $\Omega$ is non-decreasing, we have $\Omega\big(d(x_0, y_0)\big) \geq \Omega\big(d(x_1, y_1)\,\big(1 + c\,V(d(x_1, y_1))\big)\big)$. For $\gamma = d(x_1, y_1)$, we can write $\gamma(1 + c\,V(\gamma)) = (1 - V(\gamma))\,\gamma + V(\gamma)\,(1+c)\,\gamma$. As $\Omega$ is concave, we see that

$$\begin{aligned}
\Omega\big(\gamma\,(1 + c\,V(\gamma))\big) &\geq (1 - V(\gamma))\,\Omega(\gamma) + V(\gamma)\,\Omega\big((1+c)\,\gamma\big) \\
&= \Omega(\gamma) + V(\gamma)\,\Big(\Omega((1+c)\,\gamma) - \Omega(\gamma)\Big) \\
&\geq \Omega(\gamma) + C_1\omega(\gamma).
\end{aligned}$$

Thus, we have shown that, for $x_0, x_1, y_0 \in [0,1)$ with $T(x_1) = x_0$ and $d(x_0, y_0) < \varrho_1$, there is $y_1 \in T^{-1}(y_0)$ with $d(x_1, y_1) < \varrho_1$ for which inequality (4) holds. It remains to argue that $x_1 \mapsto y_1$ is a well-defined one-to-one map. But Lemma 6 implies that two pre-images $\bar{z}_1 \neq z_1$ of a point $z_0$ must satisfy $d(\bar{z}_1, z_1) \geq \varrho_0$. Hence, we have $\min\{d(x_1, \bar{y}_1), d(\bar{x}_1, y_1)\} \geq \varrho_0 - d(x_1, y_1) > \varrho_1$ whenever $\bar{x}_1 \neq x_1$ and $\bar{y}_1 \neq y_1$ are pre-images, respectively, of $x_0$ and $y_0$, which completes the proof of the lemma. $\square$

**Associated moduli fulfilling condition** (1)

Condition (1) is so flexible that it is not surprising there may be different moduli $T$-compatible with a given modulus $\omega$. (Actually, inequality (2) allows to show, via condition (1), that for any $\bar{\beta} \in [0, \beta]$ the modulus $\omega_{\alpha-s,\bar{\beta}}$ is $T_s$-compatible with $\omega_{\alpha,\beta}$.) The question of whether there is a possible, say, canonical choice naturally arises. Far from providing an answer to this interesting question, we would like to describe how a specific modulus of continuity $\Omega$ can be determined in certain situations. This construction is based on a case considered in [GI20].

For a map $T$ in $\mathscr{F}$ described as $T(x) = x(1 + V(x)) \mod 1$, suppose there exists $\omega \in \mathcal{M}$ for which there are constants $\xi_0 > 1$ and $\eta_0 \in (0,1)$ and a function $c : (1, \xi_0] \to (1, +\infty)$ such that

$$\frac{\omega(\xi x)}{V(\xi x)} \geq c(\xi) \frac{\omega(x)}{V(x)}, \qquad \forall\, x \in (0, \eta_0),\, \forall\, \xi \in (1, \xi_0]. \tag{5}$$

Note, for instance, that for the Manneville-Pomeau map $T_s$ (for which $V(x) = x^s$), and the modulus of continuity $\omega_{\alpha,\beta}$, the above hypothesis follows immediately with $c(\xi) = \xi^{\alpha-s}$: for $\gamma$ sufficiently small,

$$\frac{\omega_{\alpha,\beta}(\xi x)}{(\xi x)^s} \geq \xi^{\alpha-s} \frac{x^\alpha(-\log x)^{-\beta}}{x^s} = \xi^{\alpha-s} \frac{\omega_{\alpha,\beta}(x)}{x^s}.$$

We can define a new modulus of continuity $\Omega$ in $\mathcal{M}$ by means of (5). In order to do that, fix a parameter $\tau > 0$. First, let $\vartheta_0 : [0,\infty) \to [0,\infty)$ be the continuous function defined as

$$\vartheta_0(x) := \begin{cases} \frac{\omega(x)}{V(x)}, & x > 0, \\ 0, & x = 0, \end{cases}$$

and let $\vartheta_1 : [0,\infty) \to [0,\infty)$ be the continuous non-decreasing function given as

$$\vartheta_1(x) = \begin{cases} \max\limits_{0\le y\le x} \vartheta_0(y), & 0 \le x \le \tau, \\ \max\limits_{[0,\tau]} \vartheta_0, & x \ge \tau, \end{cases}$$

Denote then $\vartheta_1^*$ the *concave conjugate Legendre transform* of $\vartheta_1$:

$$\vartheta_1^*(x) = \min_{y\in[0,\infty)} [xy - \vartheta_1(y)], \quad \forall\, x \ge 0.$$

It is easy to see that $\vartheta_1^*$ is concave, non-decreasing and continuous on $[0,\infty)$. Moreover, since for any $x > 0$ and $\epsilon > 0$, $\vartheta_1^*(x) \le \epsilon - \vartheta_1(\epsilon/x) \le \epsilon$, we conclude that $\vartheta_1^*$ is bounded and non-positive. Its *concave conjugate Legendre transform*,

$$\vartheta_1^{**}(x) = \min_{y\in[0,\infty)} [xy - \vartheta_1^*(y)], \quad \forall\, x \ge 0,$$

is also a continuous concave non-decreasing function. Moreover $\vartheta_0(x) \le \vartheta_1(x) \le \vartheta_1^{**}(x)$ for all $x \in [0,\tau]$. Actually, $\vartheta_1^{**}$ is the *smallest* concave function that lies above $\vartheta_1$ on $[0,\tau]$. Note that $\vartheta_1^{**}(0) = -\max\vartheta_1^*$. We have thus obtained a function $\Omega := \vartheta_1^{**} + \max\vartheta_1^*$ that belongs to $\mathcal{M}$.

As an illustration, note that for the Manneville-Pomeau map $T_s$ and the modulus of continuity $\omega_{\alpha,\beta}$, whenever $0 < s < \alpha < 1$ and $\beta \ge 0$, we have $\vartheta_0 = \vartheta_1 = \vartheta_1^{**} = \Omega = \omega_{\alpha-s,\beta}$ on $[0,\tau]$ if we take $\tau = \min\{x_0(\alpha,\beta), x_0(\alpha-s,\beta)\}$.

Perhaps it is (at least conceptually) meaningful to note that, when $\tau$ is taken small enough, the modulus $\Omega$ could be introduced, up to some truncation, as the concave hull of function $\frac{\omega}{V}$. Indeed, inequality (5) implies that $\frac{\omega}{V}$ is increasing on $(0,\eta_0)$, and therefore the step $\vartheta_1$ is unnecessary for $\tau < \eta_0$. In this case, instead of the double Legendre transform, one could prefer to define $\Omega$ using the infimum of all affine functions bounding $\frac{\omega}{V}$ from above.

To see that the modulus of continuity $\Omega$ so obtained is $T$-compatible with respect to the initial modulus $\omega$, the reader can easily adapt the arguments from the proof of Proposition 3 of [GI20], whose core strategy is exactly to show that, for this constructed modulus $\Omega$ and the pair $\omega$ and $V$ fulfilling (5), one always has the condition (1) checked. Even though an assumption like (5) may be interpreted as somewhat restrictive, its value here is in pointing out that there may be more appropriate or convenient choices of $T$-compatible moduli.

# 3 Proof of Theorem 2

The statements of Theorem 2 are obtained from Proposition 8, Proposition 11 and Proposition 10. We recall that $T : \mathbb{T} \to \mathbb{T}$ is a map in $\mathscr{F}$ such that $T(x) =$

$x(1+V(x)) \mod 1$, $\omega$ and $\Omega$ are moduli of continuity in $\mathcal{M}$ such that $\Omega$ is $T$-compatible with respect to $\omega$ (recall Definition 1), and $f$ is a potential that belongs to $\mathscr{C}_\omega(\mathbb{T})$.

**Eigenproperties**

**Proposition 8.** *The transfer operator $\mathscr{L}_f$ and its dual share a common positive eigenvalue $\chi$. This number is a simple eigenvalue for $\mathscr{L}_f$, associated with which there is a positive eigenfunction $h$ belonging to $\mathscr{C}_\Omega(\mathbb{T})$.*

*Proof.* Recall that $\mathrm{Prob}(\mathbb{T})$ denotes the space of Borel probability measures on $\mathbb{T}$ equipped with the weak-star topology and the dual operator $\mathscr{L}_f^*$ acts on it as follows

$$\int \phi \, d(\mathscr{L}_f^* \mu) = \int \mathscr{L}_f \phi \, d\mu, \qquad \forall \, \phi \in C^0(\mathbb{T}).$$

Let $\mathbb{1}$ denote the function identically equal to 1 on $\mathbb{T}$. The function $\Phi$ defined on $\mathrm{Prob}(\mathbb{T})$ as

$$\Phi(\mu) := \frac{\mathscr{L}_f^* \mu}{\int \mathscr{L}_f \mathbb{1} d\mu}$$

is clearly continuous. Since $\mathrm{Prob}(\mathbb{T})$ is a convex and compact set which is invariant by $\Phi$, then Schauder-Tyckhonov theorem guarantees $\Phi$ admits a fixed point of $\nu \in \mathrm{Prob}(\mathbb{T})$. Hence, for $\chi := \int \mathscr{L}_f \mathbb{1} \, d\nu > 0$, we have

$$\mathscr{L}_f^* \nu = \chi \, \nu.$$

Let $\varrho_1$ and $C_1$ be the constants that characterize the $T$-compatibility of $\Omega$ with respect to $\omega$ (see Definition 1). Given $f \in \mathscr{C}_\omega$, denote $\kappa_f := C_1^{-1} |f|_\omega$. Consider thus the following subset of $C^0(\mathbb{T})$:

$$\Lambda := \left\{ \phi \in C^0(\mathbb{T}) : \, \phi \geq 0, \, \int \phi \, d\nu = 1, \; \phi(x) \leq \phi(y) \, e^{\kappa_f \, \Omega(d(x,y))} \text{ if } d(x,y) < \varrho_1 \right\}.$$

We have that $\Lambda$ is a convex and closed nonempty subset of $C^0(\mathbb{T})$. We claim that $\Lambda$ is uniformly bounded. In fact, let $\{A_i\}_{i=1}^L$ be a finite cover of $\mathbb{T}$ by open arcs of length $\varrho_1$ and let $z_i \in \mathbb{T}$ denote the center of $A_i$. Note that we may always suppose that these points are positively oriented, that is, $z_1 < z_2 < \ldots < z_L$. Hence, given $x, y \in \mathbb{T}$, with $x < y$, consider indexes $i_x \leq i_y$ for which $x \in A_{i_x}$ and $y \in A_{i_y}$, so that $d(x, z_{i_x}) < \varrho_1/2$, $d(y, z_{i_y}) < \varrho_1/2$, and for every $i_x \leq i < i_y$, $d(z_i, z_{i+1}) < \varrho_1$. For an arbitrary $\phi$ in $\Lambda$, the local property in the definition of this set provides

$$\begin{aligned} \phi(x) \leq & \phi(z_{i_x}) e^{\kappa_f \Omega(d(x, z_{i_x}))} \\ \leq & \phi(y) \exp\left( \kappa_f \big( \Omega(d(x, z_{i_x})) + \sum_{i=i_x}^{i_y - 1} \Omega(d(z_i, z_{i+1})) + \Omega(d(z_{i_y}, y)) \big) \right) \\ \leq & \phi(y) e^{L \, \kappa_f \, \Omega(d(x,y))}. \end{aligned} \tag{6}$$

As $0 \le \min \phi \le \int \phi d\nu = 1$, in particular for $x, y \in \mathbb{T}$ such that $\phi(x) = \|\phi\|_\infty$ and $\phi(y) = \min \phi$, it follows

$$\|\phi\|_\infty \le \min \phi \; e^{L\,\kappa_f\,\Omega(1/2)} \le e^{L\,\kappa_f\,\Omega(1/2)}.$$

The above estimates also ensure that $\Lambda$ is equicontinuous. In fact, as $|e^a - 1| \le |a|\, e^{|a|}$, we have

$$|\phi(x) - \phi(y)| \le \|\phi\|_\infty \left| e^{L\,\kappa_f\,\Omega(d(x,y))} - 1 \right| \le L\kappa_f e^{2L\,\kappa_f\,\Omega(1/2)}\,\Omega(d(x,y)), \tag{7}$$

for $\phi \in \Lambda$ and $x, y \in \mathbb{T}$. Therefore, by Arzelà-Ascoli theorem, the set $\Lambda$ is compact.

For $\phi \in \Lambda$, define

$$\mathscr{T}(\phi) := \mathscr{L}_{f-\log\chi}\phi = \frac{1}{\chi}\mathscr{L}_f\phi = \frac{\mathscr{L}_f\phi}{\int \mathscr{L}_f \mathbb{1}\, d\nu} \ge 0.$$

The set $\Lambda$ is invariant under the operator $\mathscr{T}$, that is: $\mathscr{T}(\Lambda) \subseteq \Lambda$. Indeed, $\mathscr{T}$ is clearly a positive operator and we note that for $\phi \in \Lambda$,

$$\int \mathscr{T}(\phi)\; d\nu = \int \frac{1}{\chi}\mathscr{L}_f\phi\; d\nu = \int \frac{\phi}{\chi}\; d(\mathscr{L}_f^*\nu) = \int \phi\; d\nu = 1.$$

Recall that we denote $N_V = 1 + V(1)$. For a pair of points $x, y \in \mathbb{T}$ with $d(x, y) \le \varrho_1$, if for $1 \le i \le N_V$, $x_i$ denotes a preimage of $x$, let $y_i$ be the corresponding preimage of $y$ as stated in Definition 1. Then, the fact that $\mathscr{T}(\phi)(x) \le \mathscr{T}(\phi)(y)\, e^{\kappa_f\,\Omega(d(x,y))}$ is a consequence of

$$\begin{aligned}
\mathscr{L}_f\phi(x) &= \sum_{i=1}^{N_V} e^{f(x_i)}\phi(x_i) \le \sum_{i=1}^{N_V} e^{f(x_i)}\phi(y_i)\, e^{\kappa_f\,\Omega(d(x_i,y_i))} \\
&\le \sum_{i=1}^{N_V} e^{f(y_i)+|f|_\omega\,\omega(d(x_i,y_i))}\phi(y_i)\, e^{\kappa_f\,\Omega(d(x_i,y_i))} \\
&= \sum_{i=1}^{N_V} e^{f(y_i)}\phi(y_i)\, e^{\kappa_f\left(C_1\omega(d(x_i,y_i))+\Omega(d(x_i,y_i))\right)} \\
&\le \sum_{i=1}^{N_V} e^{f(y_i)}\phi(y_i)\, e^{\kappa_f\,\Omega(d(x,y))} = \mathscr{L}_f\phi(y)\, e^{\kappa_f\,\Omega(d(x,y))}.
\end{aligned}$$

(Note that for the last inequality we apply the $T$-compatibility of $\Omega$ with respect to $\omega$.) Applying the Schauder-Tychonoff theorem for $\mathscr{T} : \Lambda \to \Lambda$, there is $h \in \Lambda$ such that $\mathscr{L}_f h = \chi h$. From (7), $h \in \mathscr{C}_\Omega(\mathbb{T})$. To show that $h > 0$, we suppose by contradiction that $h(z) = 0$ for some $z \in \mathbb{T}$. Hence $\chi^{-n}\mathscr{L}_f^n h(z) = 0$ for every $n \ge 1$. Then for every $y \in T^{-n}(z)$ we have that $h(y) = 0$. Since $T$ is topologically mixing, the set $\bigcup_{n\ge0} T^{-n}(z)$ is dense, which implies that $h = 0$ in $\mathbb{T}$. But since $h \in \Lambda$, we have $\int h d\nu = 1$, which is a contradiction.

To prove that $\chi$ is a simple eigenvalue of the operator $\mathscr{L}_f$, let $\phi$ be a continuous function such that $\mathscr{L}_f\phi = \chi\,\phi$. Since $\mathbb{T}$ is compact, there is $z \in \mathbb{T}$ such that

$$\min_{x\in\mathbb{T}} \frac{\phi(x)}{h(x)} = \frac{\phi(z)}{h(z)}.$$

The function $\hat{\phi} := \phi - \frac{\phi(z)}{h(z)} h$ is continuous. Moreover, $\hat{\phi}$ verifies for every $n \geq 1$,

$$\mathscr{L}_f^n \hat{\phi}(z) = \mathscr{L}_f^n \phi(z) - \chi^n \phi(z) = 0.$$

Hence, since $\hat{\phi}$ is nonnegative, as above the fact that $T$ is topologically mixing implies that $\hat{\phi} = 0$ on $\mathbb{T}$, i.e., $\phi = \frac{\phi(z)}{h(z)} h$. Therefore, every eigenfunction for $\chi$ is a multiple of $h$. $\square$

**Iterates of the transfer operator**

We focus on the behavior of iterates of the transfer operator $\mathscr{L}_f$ and we derive in particular the maximal character of the eigenvalue $\chi$. Henceforward, eigenfunction $h$ and eigenprobability $\nu$ are supposed to fulfill $\int h \, d\nu = 1$.

**Lemma 9.** *For $\phi \in C^0(\mathbb{T})$, the sequence $\left\{\frac{1}{\chi^n} \mathscr{L}_f^n \phi\right\}_{n \geq 1}$ is uniformly equicontinuous and uniformly bounded.*

*Proof.* Let $x, y \in \mathbb{T}$ be such that $d(x, y) < \varrho_1$. Given $\{x_k\}_{k \geq 1}$ a pre-orbit of $x$, the $T$-compatibility of $\Omega$ with respect to $\omega$ ensures that there exists a unique pre-orbit $\{y_k\}_{k \geq 1}$ of $y$ such that $d(x_k, y_k) \leq d(x, y) < \varrho_1$ and $\Omega\big(d(x_k, y_k)\big) + C_1 \sum_{j=1}^{k} \omega\big(d(x_j, y_j)\big) \leq \Omega\big(d(x, y)\big)$, for all $k \geq 1$. In particular, for $f \in \mathscr{C}_\omega(\mathbb{T})$, we have the following estimates for the corresponding Birkhoff sums

$$\begin{aligned} |S_n f(x_n) - S_n f(y_n)| \leq & |f|_\omega \sum_{j=0}^{n-1} \omega(d(T^j(x_n), T^j(y_n))) \\ \leq & \kappa_f \Big( \Omega(d(x, y)) - \Omega(d(x_n, y_n)) \Big) \leq \kappa_f \, \Omega(d(x, y)), \end{aligned} \tag{8}$$

where, as before, $\kappa_f = C_1^{-1} |f|_\omega$.

Keeping the notation of pairs of pre-images $(x_n, y_n)$ associated by the correspondence established by $T$-compatibility of $\Omega$ with respect to $\omega$, we can write for every $\phi \in C^0(\mathbb{T})$

$$\Big| \mathscr{L}_f^n \phi(x) - \mathscr{L}_f^n \phi(y) \Big| \leq \sum_{(x_n, y_n)} \Big| e^{S_n f(x_n)} \phi(x_n) - e^{S_n f(y_n)} \phi(y_n) \Big|$$

If we denote $\omega_\phi(t) := \sup\{|\phi(x) - \phi(y)| : x, y \in \mathbb{T}, \, d(x, y) \leq t\}$, we easily obtain

$$\begin{aligned} \Big| e^{S_n f(x_n)} \phi(x_n) - & e^{S_n f(y_n)} \phi(y_n) \Big| \leq \\ & \leq e^{S_n f(x_n)} \, \omega_\phi(d(x_n, y_n)) + ||\phi||_\infty \Big| e^{S_n f(x_n)} - e^{S_n f(y_n)} \Big| \\ & \leq e^{S_n f(x_n)} \, \omega_\phi(d(x, y)) + ||\phi||_\infty \, e^{S_n f(y_n)} \Big| e^{S_n f(x_n) - S_n f(y_n)} - 1 \Big|. \end{aligned}$$

Using (8) we get

$$\Big| e^{S_n f(x_n) - S_n f(y_n)} - 1 \Big| \leq \kappa_f e^{\kappa_f \, \Omega(1/2)} \, \Omega(d(x, y)).$$

Therefore, we have shown that

$$\Big|\mathscr{L}_f^n\phi(x)-\mathscr{L}_f^n\phi(y)\Big|\le$$
$$\le \omega_\phi(d(x,y))\,\mathscr{L}_f^n\mathbb{1}(x)+||\phi||_\infty\kappa_f e^{\kappa_f\,\Omega(1/2)}\Omega(d(x,y))\,\mathscr{L}_f^n\mathbb{1}(y). \quad (9)$$

Note now that, from (6), the positive eigenfunction $h$ (as any other element of the set $\Lambda$) satisfies $\frac{h(T^n(w))}{h(w)}\le e^{L\,\kappa_f\,\Omega(1/2)}$, for all $n\ge 1$ and $w\in\mathbb{T}$. Hence, we see that

$$\frac{1}{\chi^n}\mathscr{L}_f^n\mathbb{1}(z)=\frac{1}{\chi^n}\sum_{w\in T^{-n}(z)}e^{S_nf(w)}\le\frac{1}{\chi^n}\sum_{w\in T^{-n}(z)}e^{S_nf(w)}\frac{h}{h\circ T^n}(w)e^{L\,\kappa_f\,\Omega(1/2)}$$
$$=e^{L\,\kappa_f\,\Omega(1/2)}\frac{1}{\chi^n h(z)}\mathscr{L}_f^n h(z)=e^{L\,\kappa_f\,\Omega(1/2)}.$$

From the above discussion, we deduce that

$$\Big|\frac{1}{\chi^n}\mathscr{L}_f^n\phi(x)-\frac{1}{\chi^n}\mathscr{L}_f^n\phi(y)\Big|\le$$
$$\le e^{L\,\kappa_f\,\Omega(1/2)}\Big(\omega_\phi(d(x,y))+||\phi||_\infty\kappa_f e^{\kappa_f\,\Omega(1/2)}\Omega(d(x,y))\Big),$$

from which we conclude that $\big\{\frac{1}{\chi^n}\mathscr{L}_f^n\phi\big\}$ is uniformly equicontinuous. Moreover, since $\Big|\frac{1}{\chi^n}\mathscr{L}_f^n\phi(x)\Big|\le||\phi||_\infty\frac{1}{\chi^n}\mathscr{L}_f^n\mathbb{1}(x)\le||\phi||_\infty e^{L\,\kappa_f\,\Omega(1/2)}$, the sequence $\big\{\frac{1}{\chi^n}\mathscr{L}_f^n\phi\big\}$ is also uniformly bounded. $\square$

**Proposition 10.** *The sequence $\big\{\frac{1}{\chi^n}\mathscr{L}_f^n\phi\big\}_{n\ge1}$ converges uniformly to $h\int\phi\,d\nu$.*

*Proof.* Thanks to the previous lemma and Arzelà-Ascoli theorem, we need to argue that any convergent subsequence $\big\{\frac{1}{\chi^{n_j}}\mathscr{L}_f^{n_j}\phi\big\}$ has as uniform limit $h\int\phi\,d\nu$. Suppose that $\big\{\frac{1}{\chi^{n_j}}\mathscr{L}_f^{n_j}\phi\big\}$ converges uniformly to $\phi_\infty$.

Consider the normalized potential $\tilde f=f+\log h-\log h\circ T-\log\chi$ and note that for all $n\ge1$ and $\psi\in C^0(\mathbb{T})$, $\frac{1}{\chi^n}\mathscr{L}_f^n\psi=h\mathscr{L}_{\tilde f}^n\big(\frac{\psi}{h}\big)$. Since $\mathscr{L}_{\tilde f}\psi\le\max\psi\,\mathscr{L}_{\tilde f}\mathbb{1}=\max\psi$, note also that

$$\cdots\le\max\mathscr{L}_{\tilde f}^n\psi\le\cdots\le\max\mathscr{L}_{\tilde f}^2\psi\le\max\mathscr{L}_{\tilde f}\psi\le\max\psi. \quad (10)$$

We thus have that $\Big\{\max\mathscr{L}_{\tilde f}^n\big(\frac{\phi}{h}\big)\Big\}_{n\ge1}$ is non-increasing and $\Big\{\mathscr{L}_{\tilde f}^{n_j}\big(\frac{\phi}{h}\big)\Big\}_{j\ge1}$ converges uniformly $\frac{\phi_\infty}{h}$, so that, given $\epsilon>0$, for $j$ sufficiently large,

$$\max\mathscr{L}_{\tilde f}^k\Big(\frac{\phi_\infty}{h}\Big)\ge\max\mathscr{L}_{\tilde f}^k\Big(\mathscr{L}_{\tilde f}^{n_j}\big(\frac{\phi}{h}\big)\Big)+\epsilon\ge\max\mathscr{L}_{\tilde f}^{n_{k+j}}\big(\frac{\phi}{h}\big)+\epsilon,$$

for any fixed $k$. By passing to the limit as $j$ tends to infinity and then considering $\epsilon>0$ arbitrarily small, from (10) we conclude that

$$\max\mathscr{L}_{\tilde f}^k\big(\frac{\phi_\infty}{h}\big)=\max\frac{\phi_\infty}{h}\qquad\forall\,k\ge1. \quad (11)$$

As $\mathscr{L}_{\tilde f}^k \mathbb{1} = \mathbb{1}$, it follows that

$$T^{-k}\Big(\arg\max \mathscr{L}_{\tilde f}^k\big(\frac{\phi_\infty}{h}\big)\Big) \subset \arg\max \frac{\phi_\infty}{h} \qquad \forall\, k \geq 1.$$

Since $T$ is topologically mixing, we thus obtain that $\frac{\phi_\infty}{h}$ attains its maximum value in any nonempty open set of $\mathbb{T}$. Hence, by continuity, $\frac{\phi_\infty}{h}$ is identically constant.

Note now that by the dominated convergence theorem

$$\lim_{j\to\infty} \int \mathscr{L}_{\tilde f}^{n_j}\big(\frac{\phi}{h}\big)\, d\mu = \int \frac{\phi_\infty}{h}\, d\mu = \frac{\phi_\infty}{h}.$$

Since $\int \mathscr{L}_{\tilde f}^n \psi\, d\mu = \int \psi\, d\mu$ for all $n \geq 1$ and $\psi \in C^0(\mathbb{T})$, we have shown that $\phi_\infty = h \int \frac{\phi}{h} d\mu = h \int \phi\, d\nu$. $\square$

We can now complete the proof of Theorem 2 by discussing the maximality of the eigenvalue $\chi$.

**Proposition 11.** *When acting on complex-valued continuous functions on $\mathbb{T}$, the transfer operator $\mathscr{L}_f$ does not possess another eigenvalue with an absolute value strictly greater than or equal to $\chi$.*

*Proof.* Note that $\mathscr{L}_f \phi = c\phi$ for a (non null) complex-valued continuous function $\phi$ and a constant $|c| \geq \chi$ if, and only if, $\mathscr{L}_{\tilde f}\left(\frac{\phi}{h}\right) = \frac{c}{\chi}\frac{\phi}{h}$, where as before $\tilde f = f + \log h - \log h \circ T - \log \chi$. Thus, it suffices to show that $\mathscr{L}_{\tilde f}$ acting on complex-valued continuous functions admits only 1 as eigenvalue outside the open unit disc. Suppose then $\mathscr{L}_{\tilde f}\phi = c\phi$ with $|c| \geq 1$. Clearly, $|\phi| \leq \mathscr{L}_{\tilde f}^k |\phi|$ for all $k \geq 1$, so that $\max|\phi| \leq \max \mathscr{L}_{\tilde f}^k |\phi| \leq \max|\phi| \max \mathscr{L}_{\tilde f}^k \mathbb{1} = \max|\phi|$. We are exactly in the same situation as (11). Therefore, we obtain that $|\phi|$ is constant, which we may assume equal to 1. Hence, we write $\phi(x) = e^{2\pi i\theta(x)}$ and $c = be^{2\pi i\gamma}$ with $b \geq 1$ and $\gamma \in \mathbb{R}$. Since $\mathscr{L}_{\tilde f} e^{2\pi i\theta} = be^{2\pi i(\theta+\gamma)}$ represents a convex combination of extremal points of the unit disc, we conclude that $b = 1$ and $\theta(y) = \theta(x) + \gamma \mod 1$ for all $x \in \mathbb{T}$ and $y \in T^{-1}(x)$. In particular, for $x = y = 0$ we see that $\gamma \in \mathbb{Z}$, and therefore $c = 1$. $\square$

# 4 Proof of Theorem 3

In this section, we discuss a succession of intermediate results, from which we will derive Theorem 3. Throughout the entire section, we will assume without mentioning the hypotheses of Theorem 2. Moreover, the positive eigenfunction $h$ and the eigenprobability $\nu$, both obtained in Theorem 2, are from now on supposed to be related as $\int h\, d\nu = 1$. The statements of Theorem 3 can be recovered from the statements of Lemma 13, Proposition 15 and Proposition 16.

**Equilibrium states from Rokhlin formula**

For topological dynamical systems, whenever the measure entropy of $T$ is upper semi-continuous with respect to the measure, one may guarantee the existence of an invariant probability attaining the supremum in the variational expression (3) of the topological pressure, namely, the existence of an *equilibrium state*. Moreover, if the topological entropy of the system is finite, then the extreme points of the convex set of equilibrium states are exactly the ergodic members of this set. See, for instance, [Wal82, Theorem 9.13]. The fact that, for the maps we are dealing with, the measure entropy, regarded as a function of the measure, is upper semi-continuous follows from a general result for piecewise monotone mappings of the circle [MS80, Corollary 2'].

Our aim now is to show that $\mu = h\nu$ is the unique equilibrium state associated with $f$. A key element in our argument will be Rokhlin formula for the measure entropy. We briefly recall the main ingredients.

Let $m$ denote a Borel probability measure on $\mathbb{T}$. Let $\{\mathscr{A}_n\}_{n\geq 1}$ be a sequence of measurable (countable) partitions of $\mathbb{T}$, with finite $m$-entropy, such that $\mathscr{A}_n \preceq \mathscr{A}_{n+1}$ for all $n$. We say that $\{\mathscr{A}_n\}_{n\geq 1}$ is *$m$-generating* if $\bigcup_{n\geq 1} \mathscr{A}_n$ generates the Borel $\sigma$-algebra, up to $m$-measure zero. For $m$-almost every $x \in \mathbb{T}$, we denote $\mathscr{A}_n(x)$ the element of the partition $\mathscr{A}_n$ to which $x$ belongs. A sufficient condition for $\{\mathscr{A}_n\}_{n\geq 1}$ to be $m$-generating is to satisfy

$$\operatorname{diam}(\mathscr{A}_n(x)) \to 0 \quad \text{as } n \to \infty, \qquad \text{for } m\text{-a.e. } x \in \mathbb{T}. \tag{12}$$

(For a proof of this fact, see, for instance, the proof of Corollary 9.2.8 in [VO16].)

Given a map $T$ in $\mathscr{F}$, a measurable function $J_m(T) : \mathbb{T} \to [0, \infty)$ is a *Jacobian* of $T$ with respect to $m$ if for any measurable set $A$ such that $T|_A$ is injective,

$$m(T(A)) = \int_A J_m(T)\, dm.$$

Whenever $m$ is a $T$-invariant probability measure, existence and uniqueness (up to $m$-measure zero) of a Jacobian of $T$ with respect to $m$ are well known. (For a more general result, see [VO16, Proposition 9.7.2].) For a $T$-invariant probability $m$, it is easy to see that $J_m(T) > 0$ $m$-almost everywhere. Moreover,

$$\sum_{y \in T^{-1}(x)} \frac{1}{J_m(T)(y)} = 1 \quad \text{for } m\text{-almost every } x \in \mathbb{T}. \tag{13}$$

The following formula, due to V. Rokhlin, allows us to compute the entropy from the Jacobian. For a proof of this classical result, see, for instance, [Par69].

**Theorem** (Rokhlin formula)**.** *Let $T$ be a locally invertible measurable transformation and $m$ be a $T$-invariant probability measure. Suppose that domains of invertibility of $T$ provide a partition $\mathscr{A}_0$ such that the sequence $\left\{ \vee_{j=1}^{n} T^{-j}(\mathscr{A}_0)\right\}_{n\geq 1}$ is $m$-generating. Then*

$$h_m(T) = \int \log J_m(T)\, dm.$$

An application of the previous facts will be summarized in the following lemma.

**Lemma 12.** *For $T$ a map in $\mathscr{F}$, let $m$ be a $T$-invariant probability that does not charge $0$. If the set of pre-images $\{a_i\}_{i=0}^{N_V-1}$ of $a_0 := 0 =: a_{N_V}$ is supposed to be positively oriented, let $A_i$ be the positively oriented open arc from $a_i$ to $a_{i+1}$. Denote $\mathscr{A} := \{A_i\}_{i=0}^{N_V-1}$ and $\mathscr{A}_n := \bigvee_{j=0}^{n-1} T^{-j}(\mathscr{A})$. Then $\{\mathscr{A}_n\}_{n\geq 1}$ is $m$-generating. In particular, the measure $m$ satisfies the Rokhlin formula for the entropy $h_m(T)$ and the Jacobian $J_m(T)$.*

*Proof.* Obviously by invariance $m$ does not charge any point $a_i$, and therefore $\{A_i\}_{i=0}^{N_V-1}$ is a partition of $\mathbb{T}$ with respect to $m$. To prove that the monotone sequence $\{\mathscr{A}_n\}$ is $m$-generating we will show that (12) holds. Actually, this property follows easily from the fact that $T$ is topologically exact. Indeed, an element $\mathscr{A}_n(x)$ of $\mathscr{A}_n$ is of the form

$$\mathscr{A}_n(x) = \bigcap_{j=0}^{n-1} T^{-j}(A_{i_j}),$$

where $A_{i_j}$ is the open arc from $a_{i_j}$ to $a_{i_j+1}$, $i_j \in \{0, 1, \cdots, N_V - 1\}$. Now if the diameters would not shrink for a particular $x \in \mathbb{T} \setminus \bigcap_{j\geq 0} T^{-j}\big(\{a_0, \ldots, a_{N_V-1}\}\big)$, then there would exist $\kappa > 0$ and a sequence $\{n_j\}_{j\geq 0}$, with $n_j \to +\infty$ as $j \to +\infty$, such that for every $j \geq 0$,

$$\operatorname{diam}(\mathscr{A}_{n_j}(x)) \geq \kappa.$$

Hence, for an open nonempty subset $U \subset \displaystyle\bigcap_{j=0}^{+\infty} \mathscr{A}_{n_j}(x)$, we would have

$$T^k(U) \subset T^k\big(\bigcap_{j=0}^{+\infty} \mathscr{A}_{n_j}(x)\big) \subset A_{i_k}, \quad \forall\, k \geq 1.$$

However, there exists $M > 0$ such that

$$\mathbb{T} = T^M(U) \subset A_{i_M},$$

which is a contradiction. Thus, property (12) holds and the Rokhlin formula can be applied to the probability $m$. □

Our first goal is to show that the measure obtained from the eigenfunction of the transfer operator and the eigenprobability of the dual operator satisfies the conditions of the preceding lemma.

**Lemma 13.** *For a map $T$ in $\mathscr{F}$, let $\{a_i\}_{i=1}^{N_V-1}$ denote the points of $\mathbb{T} \setminus \{0\}$ such that $T(a_i) = 0$. Then, $\mu = h\nu$ is a $T$-invariant probability that does not charge either $0$ or any $a_i$, $i = 1, \cdots, N_V - 1$. Furthermore, the Jacobian of $T$ with respect to $\mu$ is given as $J_\mu(T) = \chi\, \frac{h\circ T}{h}\, e^{-f}$.*

*Proof.* Consider once more the normalized potential $\tilde{f} = f + \log h - \log h \circ T - \log \chi$ and the associated transfer operator $\mathscr{L}_{\tilde{f}}$. The invariant property of $\mu$ follows thus immediately: for all $\psi$ in $C^0(\mathbb{T})$,

$$\int \psi \circ T \, d\mu = \int \psi \circ T \, d(\mathscr{L}_{\tilde{f}}^* \mu) = \int \mathscr{L}_{\tilde{f}}(\psi \circ T) \, d\mu = \int \psi \mathscr{L}_{\tilde{f}} \mathbb{1} \, d\mu = \int \psi \, d\mu.$$

Note now that, by this invariant property,

$$\mu(\{0\}) = \mu(T^{-1}(0)) = \mu(\{0\}) + \sum_{i=1}^{N_V - 1} \mu(\{a_i\}),$$

which implies $\mu(\{a_i\}) = 0$ for $i = 1, \cdots, N_V - 1$. We also note that $\mu(\{0\}) = 0$. Otherwise, if we suppose $\mu(\{0\}) > 0$, we would have for $\psi$ in $C^0(\mathbb{T})$,

$$\begin{aligned} \frac{1}{\chi h(0)} \mathscr{L}_f(h\psi)(0) \, \mu(\{0\}) &= \int_{\{0\}} \mathscr{L}_{\tilde{f}} \psi \, d\mu = \int \mathscr{L}_{\tilde{f}}(\mathbb{1}_{T^{-1}(0)} \psi) \, d\mu \\ &= \int_{T^{-1}(0)} \psi \, d\mu = \psi(0) \, \mu(\{0\}). \end{aligned}$$

(Here $\mathbb{1}_{T^{-1}(0)}$ represents the indicator function on the set of pre-images of 0.) Hence, the following (linear) equation would hold for every $\psi$ in $C^0(\mathbb{T})$,

$$\big(e^{f(0)} - \chi\big) h(0) \, \psi(0) + \sum_{i=1}^{N_V - 1} e^{f(a_i)} h(a_i) \, \psi(a_i) = 0,$$

which is clearly impossible.

With respect to the Jacobian, let $A$ be a measurable set such that $T|_A$ is injective. For a sequence $\{\psi_n\} \subset C^0(\mathbb{T})$ converging to the indicator function on $A$ $\nu$-almost every point, by the dominated convergence theorem,

$$\begin{aligned} \int_A \chi \frac{h \circ T}{h} e^{-f} \, d\mu &= \lim_{n \to \infty} \int \chi \, h \circ T \, e^{-f} \, \psi_n \, d\nu = \lim_{n \to \infty} \int \mathscr{L}_f(h \circ T \, e^{-f} \, \psi_n) \, d\nu \\ &= \lim_{n \to \infty} \int \mathscr{L}_f(e^{-f} \, \psi_n) \, d\mu = \mu(T(A)), \end{aligned}$$

since $\mathscr{L}_f\big(e^{-f}\psi_n\big)(x) = \sum_{y \in T^{-1}(x)} \psi_n(y) \to \mathbb{1}_{T(A)}(x)$, $\nu$-almost every $x \in \mathbb{T}$. $\square$

It is well known that the topological pressure may be introduced by means of open coverings. We recall the main aspects of this formulation here and we refer the reader to [Wal82] for more details. Given an open cover $\mathscr{A}$ of $\mathbb{T}$, consider

$$p_n(T, f, \mathscr{A}) := \inf_{\mathscr{B}} \sum_{B \in \mathscr{B}} \exp\big(\sup_{x \in B} S_n f(x)\big),$$

where $\mathscr{B}$ is a finite subcover of $\mathbb{T}$ contained in $\mathscr{A} \vee T^{-1}\mathscr{A} \vee \cdots \vee T^{-(n-1)}\mathscr{A}$. Then the topological pressure may be defined as

$$P(T, f) := \lim_{\epsilon \to 0} \sup_{\operatorname{diam}(\mathscr{A}) \leq \epsilon} \lim_{n \to \infty} \frac{1}{n} \log p_n(T, f, \mathscr{A}).$$

**Lemma 14.** *The following inequality holds:* $\log\chi \le P(T,f)$.

*Proof.* Let $\mathscr{A}$ be an open cover of $\mathbb{T}$ with diameter less than $\varrho_0$, the positive constant described in Lemma 6. If $\mathscr{B}$ is a finite subcover of $\mathbb{T}$ contained in $\vee_{j=0}^{n-1}T^{-j}(\mathscr{A})$, by the very definition of $\varrho_0$, for all $x \in \mathbb{T}$ any two distinct points of $T^{-n}(x)$ belong to distinct elements of $\mathscr{B}$. Then

$$\chi^n = \chi^n \nu(\mathbb{T}) = \int \mathscr{L}_f^n \mathbb{1}\, d\nu \le \int \sum_{B\in\mathscr{B}} \exp(\sup_B S_n f)\, d\nu = \sum_{B\in\mathscr{B}} \exp(\sup_B S_n f).$$

Taking the infimum among all finite subcovers contained in $\vee_{j=0}^{n-1}T^{-j}(\mathscr{A})$, we obtain $\log\chi \le \frac{1}{n}\log p_n(T,f,\mathscr{A})$, which yields $\log\chi \le P(T,f)$. □

Given $m \in M(\mathbb{T},T)$ and a measurable function $\phi : \mathbb{T}\to\mathbb{R}$, keeping in mind (13), consider now for $m$-almost every $x \in \mathbb{T}$

$$\mathscr{J}_m(\phi)(x) := \sum_{y\in T^{-1}(x)} \frac{1}{J_m(T)(y)}\phi(y).$$

We highlight two well-known main properties:

$$\int \phi\, dm = \int \mathscr{J}_m(\phi)\, dm, \tag{14}$$

$$\int \mathscr{J}_m(\log\psi)\, dm \le \log \int \mathscr{J}_m(\psi)\, dm, \tag{15}$$

for every measurable functions $\phi,\psi : \mathbb{T}\to\mathbb{R}$ fulfilling integrability conditions. For details, see [VO16, Section 9.7].

For the next proposition, we also remark a basic fact: the eigenequation $\mathscr{L}_f h = \chi h$ considered at the fixed point gives us $\big(e^{f(0)} - \chi\big)h(0) + \sum_i e^{f(a_i)}h(a_i) = 0$, from which we conclude that

$$f(0) < \log\chi. \tag{16}$$

**Proposition 15.** *The $T$-invariant probability $\mu = h\nu$ is the unique equilibrium measure associated with $f$, and*

$$h_\mu(T) + \int f\, d\mu = \log\chi = P(T,f).$$

*Proof.* Let $m$ be an equilibrium state associated with $f$. We claim that $m$ does not charge the indifferent fixed point. Indeed, replacing $m$ by one of its ergodic components if necessary, we can assume the $m$ is ergodic. Suppose by contradiction that $m(\{0\}) > 0$. Then for any measurable set $B$ with $m(B) = 0$, we have $0 \notin B$, so that the Dirac measure $\delta_0$ supported at the fixed point 0 satisfies $\delta_0(B) = 0$, which means that $\delta_0$ is absolutely continuous with respect to $m$. Birkhoff's ergodic theorem ensures that for a bounded measurable function $\phi$,

$$\lim_{n\to\infty} \frac{1}{n} S_n(\phi)(x) = \tilde{\phi}(x), \qquad m\text{-almost every } x \in \mathbb{T}, \tag{17}$$

where $\tilde{\phi}$ is $m$-almost everywhere constant and equals to $\int \phi\, dm$. Since $\delta_0$ is absolutely continuous with respect to $m$, equality (17) holds for $\delta_0$-almost every $x \in \mathbb{T}$. In particular, $\int \phi\, d\delta_0 = \int \tilde{\phi}\, d\delta_0 = \int \phi\, dm$, and we conclude that $m = \delta_0$. Inequality (16) and Lemma 14 guarantee that $\int f\, d\delta_0 < P(T, f)$. Hence $m = \delta_0$ is not an equilibrium measure, which is a contradiction.

Hence by the $T$-invariance of $m$, this probability does not give any mass to the pre-images of 0. Applying thus Lemma 12, $m$ admits a Jacobian $J_m(T)$ that satisfies $h_m(T) = \int \log J_m(T)\, dm$. We use (14) and (15) to see that

$$\begin{aligned} P(T,f) - \log\chi &= \int \log J_m(T)\, dm + \int f\, dm - \log\chi \\ &= \int \log\big(\chi^{-1} \frac{h}{h\circ T}\, e^f J_m(T)\big)\, dm \\ &= \int \mathscr{J}_m\Big(\log\big(\chi^{-1}\frac{h}{h\circ T}\, e^f J_m(T)\big)\Big)\, dm \\ &\le \log \int \mathscr{J}_m\big(\chi^{-1}\frac{h}{h\circ T}\, e^f J_m(T)\big)\, dm \\ &= \log \int \frac{1}{\chi h}\, \mathscr{L}_f(h)\, dm = 0. \end{aligned}$$

Therefore, together with Lemma 14, we get $P(T, f) = \log\chi$.

By Lemma 13, the Jacobian of $T$ with respect to $\mu = h\nu$ is $J_\mu(T) = \chi\, \frac{h\circ T}{h} e^{-f}$. Note then that

$$\begin{aligned} h_\mu(T) &= \int \log J_\mu(T)\, d\mu = \int \log\big(\chi\, \frac{h}{h\circ T}\, e^{-f}\big)\, d\mu \\ &= -\int f\, d\mu + \log\chi = -\int f\, d\mu + P(T,f), \end{aligned}$$

which shows that $\mu$ is an equilibirum state.

Concerning uniqueness, note first that when $h_m(T) = \int \log J_m(T)\, dm$ (which we already showed to be necessarily the case for an equilibrium state), by (14) we have

$$h_m(T) + \int f\, dm - P(T,f) = \int \mathscr{J}_m\Big(\log\big(\chi^{-1}\frac{h}{h\circ T}\, e^f\big) + \log J_m(T)\Big)\, dm.$$

It is well known that if $p_1, \cdots, p_n$ are nonnegative real numbers such that $\sum_i p_i = 1$ and $b_1, \cdots, b_n$ are arbitrary real numbers, then

$$\sum_{i=1}^n \big(p_i b_i - p_i \log p_i\big) \le \log\big(\sum_{i=1}^n e^{b_i}\big)$$

with equality only when $p_i = \frac{e^{b_i}}{\sum_j e^{b_j}}$. Therefore, thanks to (13), we get

$$\mathscr{J}_m\Big(\log\big(\chi^{-1}\frac{h}{h\circ T}\, e^f\big) + \log J_m(T)\Big) \le \log\big(\frac{1}{\chi h}\mathscr{L}_f h\big) = 0,$$

with equality if, and only if, $J_m(T) = \chi \frac{h \circ T}{h} e^{-f} = J_\mu(T)$, $m$-almost everywhere. In particular, to obtain that $\mu$ is the unique equilibrium state associated with $f$, it suffices to argue that $\mu$ is the only invariant probability that admits $\chi \frac{h \circ T}{h} e^{-f}$ as its (almost everywhere) Jacobian. By definition $\mathscr{J}_m^n(\phi) = \mathscr{J}_\mu^n(\phi)$ $m$-almost everywhere, for every continuous function $\phi$ and all $n \geq 1$. Besides, since $\prod_{j=0}^{n-1} J_\mu(T) \circ T^j = \chi^n \frac{h \circ T^n}{h} e^{-S_n f}$, we see that

$$\mathscr{J}_\mu^n(\phi) = \frac{1}{\chi^n h} \mathscr{L}_f^n(\phi h).$$

Thus, from Proposition 10, $\mathscr{J}_\mu^n(\phi) \to \int \phi \, d\mu$ uniformly as $n$ tends to infinity. However, by (14) $\int \phi \, dm = \int \mathscr{J}_m^n(\phi) \, dm$. Hence, applying the dominated convergence theorem, we have, for any continuous function $\phi$, $\int \phi \, dm = \int \phi \, d\mu$, so that $m = \mu$. $\square$

**Gibbs measure**

In order to show the Gibbs' property obeyed by the probability $\mu = h\nu$, for every $x \in \mathbb{T}$, $r > 0$ and $n \geq 0$, define the corresponding dynamic ball

$$B(x,n,r) := \{y \in \mathbb{T} : d(T^j(x), T^j(y)) < r, \; j = 0, 1, \cdots, n\}.$$

**Proposition 16.** *The equilibrium state $\mu$ is a Gibbs measure: given $r \in (0, \varrho_1)$ (where $\varrho_1$ is the constant from Definition 1), there exists a constant $K_r > 0$ such that for $x \in \mathbb{T}$ and $n \geq 1$*

$$K_r^{-1} \leq \frac{\mu\big(B(x,n,r)\big)}{e^{S_n f(x) - nP(T,f)}} \leq K_r.$$

*Proof.* It is well known that the Jacobian of $T^n$, $n \geq 1$, with respect to $\mu$ may be described (almost everywhere) as $\prod_{j=0}^{n-1} J_\mu(T) \circ T^j$. Therefore, thanks to Lemma 13 and Proposition 15, we have

$$J_\mu(T^n) = e^{nP(T,f)} \frac{h \circ T^n}{h} e^{-S_n f}. \tag{18}$$

Let $r \in (0, \varrho_1)$, $x \in \mathbb{T}$ and $n \geq 1$. Then for $z \in B(x,n,r)$, since $d(T^n(z), T^n(x)) < \varrho_1$, there is a unique pre-orbit $\{z_n\}$ of $T^n(z)$ satisfying the properties in Definition 1. In particular, $z_{n-k} = T^k(z)$ for $k = 0, 1, \ldots, n$. Hence, from (8), we conclude that

$$|S_n f(x) - S_n f(z)| \leq \kappa_f \, \Omega(1/2), \qquad \forall \, z \in B(x,n,r). \tag{19}$$

By Lemma 6, $T^n|_{B(x,n,r)}$ is injective. Moreover, being an element of $\Lambda$, it follows from inequality (6) that the eigenfunction $h$ satisfies $e^{-L\kappa_f \Omega(1/2)} \leq \frac{h(T^n(z))}{h(z)} \leq$

$e^{L\kappa_f \Omega(1/2)}$ for any $z \in \mathbb{T}$ and $n \geq 1$. Hence, using (18) and (19), we have

$$\begin{aligned}
\mu\big(T^n(B(x,n,r))\big) &= \int_{B(x,n,r)} J_\mu(T^n)(z)\, d\mu(z) \\
&= \int_{B(x,n,r)} e^{nP(T,f)-S_n f(z)} \frac{h \circ T^n(z)}{h(z)}\, d\mu(z). \\
&\geq e^{-L\kappa_f \Omega(1/2)} e^{nP(T,f)-S_n f(x)} \int_{B(x,n,r)} e^{S_n f(x) - S_n f(z)}\, d\mu(z) \\
&\geq K^{-1} \frac{\mu(B(x,n,r))}{e^{S_n f(x) - nP(T,f)}}, \qquad (20)
\end{aligned}$$

where $K = e^{(L+1)\kappa_f \Omega(1/2)}$. Similarly

$$\begin{aligned}
\mu\big(T^n(B(x,n,r))\big) &\leq e^{L\kappa_f \Omega(1/2)} e^{nP(T,f)-S_n f(x)} \int_{B(x,n,r)} e^{S_n f(x) - S_n f(z)}\, d\mu(z) \\
&\leq K \frac{\mu(B(x,n,r))}{e^{S_n f(x) - nP(T,f)}}. \qquad (21)
\end{aligned}$$

Since the uniqueness of pre-orbits in Definition 1 ensures that $T^n(B(x,n,r)) = T(B(T^{n-1}(x),1,r))$, a particular application of (20) and (21) shows that the value $\mu\big(T^n(B(x,n,r))\big)$ belongs to the interval

$$\Big(\frac{K^{-1}}{e^{\max f - P(T,f)}} \inf_{y\in\mathbb{T}} \mu(B(y,1,r)),\ \frac{K}{e^{\min f - P(T,f)}} \sup_{y\in\mathbb{T}} \mu(B(y,1,r))\Big).$$

Hence to complete the proof, it remains to argue that $\inf_{y\in\mathbb{T}} \mu(B(y,1,r)) > 0$. In fact, as the dynamics is topologically exact, for each $y \in \mathbb{T}$, there exists a positive integer $M_y$ such that the restriction of $T^{M_y-1}$ on $B(y,1,r)$ is injective and has image strictly contained in $\mathbb{T}$, and also that $T^{M_y}(B(y,1,r)) = \mathbb{T}$. By continuity, $M_y$ is locally constant: for any $\hat{y}$ sufficiently close to $y$, we have $M_{\hat{y}} = M_y$. By compactness, one may find a finite cover $\{B(y_i,1,r)\}$ of $\mathbb{T}$ with $M_i := M_{y_i}$ constant on each $B(y_i,1,r)$. Now, for an arbitrary $y \in \mathbb{T}$, consider $i$ such that $y \in B(y_i,1,r)$ as well as the corresponding $M_i$. Fix then a half-open arc $A_y \subset B(y,1,r)$ for which $T^{M_i} : A_y \to \mathbb{T}$ is bijective. Once again taking advantage of Jacobians, we see that

$$\begin{aligned}
1 = \mu(\mathbb{T}) &= \mu\big(T^{M_i}(A_y)\big) \\
&\leq e^{L\kappa_f \Omega(1/2)} e^{M_i P(T,f)} \int_{A_y} e^{-S_{M_i} f(z)}\, d\mu(z) \\
&\leq e^{L\kappa_f \Omega(1/2)} e^{M_i (P(T,f) - \min f)} \mu(A_y),
\end{aligned}$$

which yields

$$\inf_{y\in\mathbb{T}} \mu(B(y,1,r)) \geq e^{-L\kappa_f \Omega(1/2)} \min_i e^{M_i(\min f - P(T,f))} > 0.$$

$\square$

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