Counting numerical semigroups of a given genus via even gaps

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Introduction

Let $S$ be a numerical semigroup.

- $G(S) := \mathbb{N}_0 \setminus S$ - set of gaps of $S$;
- $g(S) := \#G(S)$ - genus of $S$;
- $n_g := \#\{S : g(S) = g\}$.

Examples

- $n_0 = 1 \quad \mathbb{N}_0$
- $n_1 = 1 \quad \mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2 \quad \mathbb{N}_0 \setminus \{1, 2\}$ and $\mathbb{N}_0 \setminus \{1, 3\}$
- $n_3 = 4 \quad \mathbb{N}_0 \setminus \{1, 2, 3\}, \mathbb{N}_0 \setminus \{1, 2, 4\}, \mathbb{N}_0 \setminus \{1, 2, 5\}$ and $\mathbb{N}_0 \setminus \{1, 3, 5\}$
Interest

Studying the behavior of $n_g$.

Main Goal (but still not solved)

\[ n_g \leq n_{g+1}, \text{ for all } g. \]
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A brief survey

First Bound

If \( g(S) = g \), then \( 2g + \mathbb{N}_0 \subset S \). Hence,

\[
 n_g \leq \binom{2g - 1}{g}.
\]

M. Bras-Amorós and A. de Mier - 2007

\[
 n_g \leq C_g = \frac{1}{g + 1} \binom{2g}{g}.
\]
A brief survey

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$$n_g \leq C_g = \frac{1}{g + 1} \binom{2g}{g}.$$
From now on, let \( \varphi = \frac{1 + \sqrt{5}}{2} \).

M. Bras-Amorós - 2006/2008 (Conjecture)

1. \( n_g + n_{g+1} \leq n_{g+2}, \text{ for all } g; \)
2. \( \lim_{g \to \infty} \frac{n_{g+1}}{n_g} = \varphi; \)
3. \( \lim_{g \to \infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1. \)
Let \((F_n)_{n \geq 0} = (1, 1, 2, 3, 5, 8, 13, \ldots)\) be the Fibonacci sequence. Then

\[2F_g \leq n_g \leq 1 + 3 \cdot 2^{g-3}, \forall g \geq 3.\]

where \(a_g\) and \(c_g\) are coefficients of some explicit generating functions.

\[a_g \leq n_g \leq c_g, \forall g \geq 1\]
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S. Elizalde - 2010

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A. Zhai - 2011/2013

1. \[ \lim_{g \to \infty} \frac{n_{g+1}}{n_g} = \varphi; \]

2. \[ \lim_{g \to \infty} \frac{n_{g+1} + n_g}{n_{g+2}} = 1. \]

Remark

- Zhai’s first item implies that \( n_g < n_{g+1} \), for \( g \gg 0 \).
- Checking if \( n_g \leq n_{g+1} \) for all \( g \) is still an open problem (weaker conjecture).
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- Zhai’s first item implies that \( n_g < n_{g+1} \), for \( g \gg 0 \).
- Checking if \( n_g \leq n_{g+1} \) for all \( g \) is still an open problem (weaker conjecture).
\[ m(S) := \min\{s \in S : s \neq 0\} \] - multiplicity of \( S \);

\[ N(m, g) := \#\{S : g(S) = g \text{ and } m(S) = m\} \]
| g \ m | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | N(g) |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|-----|
| 0     | 1  |     |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| 1     | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| 2     | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 1   |
| 3     | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 2   |
| 4     | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 4   |
| 5     | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 7   |
| 6     | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 12  |
| 7     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 23  |
| 8     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 39  |
| 9     | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    | 67  |
| 10    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    | 118 |
| 11    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    | 204 |
| 12    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    |    | 343 |
| 13    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    |    | 592 |
| 14    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    |    | 1001|
| 15    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    |    |    |    | 1693|
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| 19    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    |    |    |    |    |    | 13467|
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| 24    | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |    | 170863|
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|       |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    | 467224|
Ordinarization transform of a semigroup:
- Remove the multiplicity (smallest non-zero non-gap)
- Add the largest gap (the Frobenius number).

The result is another numerical semigroup.
The genus is kept constant in all the transforms.
Repeating several times (:= ordinarization number) we obtain an ordinary semigroup.

$r(S)$ - ordinarization number of $S$;
$n_{g,r} := \#\{S : g(S) = g \text{ and } r(S) = r\}$. 
$r(S)$ - ordinarization number of $S$;

$n_{g,r} := \#\{S : g(S) = g \text{ and } r(S) = r\}$. 
If $r > \max\{\frac{g}{3} + 1, \left\lfloor \frac{g+1}{2} \right\rfloor - 14\}$, then $n_{g,r} \leq n_{g+1,r}$.

If $r > \frac{g}{3}$, then $n_{g,r} \leq n_{g+1,r}$. 
If \( r > \max\{\frac{g}{3} + 1, \left\lfloor \frac{g+1}{2} \right\rfloor - 14\} \), then \( n_{g,r} \leq n_{g+1,r} \).

If \( r > \frac{g}{3} \), then \( n_{g,r} \leq n_{g+1,r} \).
Our approach

- \( \gamma(S) \): number of even gaps of \( S \) - \( \# [G(S) \cap 2\mathbb{Z}] \);
- \( \gamma \)-hyperelliptic semigroup: numerical semigroup with \( \gamma \) even gaps;
- \( N_\gamma(g) := \# \{ S : g(S) = g \text{ and } \gamma(S) = \gamma \} \).

\[ n_g = \sum_{\gamma=0}^{g} N_\gamma(g) \]
Our approach

- $\gamma(S)$: number of even gaps of $S - \#[G(S) \cap 2\mathbb{Z}]$;
- $\gamma$-hyperelliptic semigroup: numerical semigroup with $\gamma$ even gaps;
- $N_\gamma(g) := \#\{S : g(S) = g \text{ and } \gamma(S) = \gamma\}$.

\[
n_g = \sum_{\gamma=0}^{g} N_\gamma(g)\]
Examples

- \( n_0 = 1 \ (\mathbb{N}_0) \) and
  \[
  N_\gamma(0) = \begin{cases} 
  1, & \text{if } \gamma = 0 \\
  0, & \text{if } \gamma \geq 1.
  \end{cases}
  \]

- \( n_1 = 1 \ (\mathbb{N}_0 \setminus \{1\}) \) and
  \[
  N_\gamma(1) = \begin{cases} 
  1, & \text{if } \gamma = 0 \\
  0, & \text{if } \gamma \geq 1.
  \end{cases}
  \]

- \( n_2 = 2 \ (\mathbb{N}_0 \setminus \{1, 2\} \text{ and } \mathbb{N}_0 \setminus \{1, 3\}) \) and
  \[
  N_\gamma(2) = \begin{cases} 
  1, & \text{if } \gamma = 0 \\
  1, & \text{if } \gamma = 1 \\
  0, & \text{if } \gamma \geq 2.
  \end{cases}
  \]
If $\gamma$ and $g$ are the number of even gaps and the genus of a numerical semigroup $S$, respectively, then $3\gamma \leq 2g$.

**Remark**

If $\gamma$ is even, then

$$\mathbb{N}_0 \setminus (\{2, 4, \ldots, 2\gamma\} \cup \{1, 3, \ldots, \gamma - 1\})$$

is a numerical semigroup with genus $g = \frac{3\gamma}{2}$.

$$n_g = \sum_{\gamma=0}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_{\gamma}(g)$$
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$$n_g = \sum_{\gamma=0}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_{\gamma}(g)$$
Theorem 1

Let $\gamma$ be a positive integer and $g \geq 3\gamma$. Then

$$N_\gamma(g) = N_\gamma(3\gamma).$$

Thus, $N_\gamma(g) = N_\gamma(g + 1)$, for all $g \geq 3\gamma$. 
Notice that

\[
  n_g = \sum_{\gamma=0}^{\lfloor g/3 \rfloor} N_{\gamma}(g) + \sum_{\gamma=\lfloor g/3 \rfloor+1}^{2g/3} N_{\gamma}(g).
\]

\[
  n_{g+1} = \sum_{\gamma=0}^{\lfloor g/3 \rfloor} N_{\gamma}(g+1) + \sum_{\gamma=\lfloor g/3 \rfloor+1}^{2(g+1)/3} N_{\gamma}(g+1).
\]

Theorem 1 states that \( N_{\gamma}(g) = N_{\gamma}(g+1) \), for \( \gamma \leq \frac{g}{3} \).
Notice that

\[ n_g = \sum_{\gamma=0}^{\left\lfloor \frac{g}{3} \right\rfloor} N_\gamma(g) + \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_\gamma(g). \]

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Theorem 1 states that \( N_\gamma(g) = N_\gamma(g + 1) \), for \( \gamma \leq \frac{g}{3} \).
Notice that

\[ n_g = \sum_{\gamma=0}^{\left\lfloor \frac{g}{3} \right\rfloor} N_{\gamma}(g) + \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_{\gamma}(g). \]

\[ n_{g+1} = \sum_{\gamma=0}^{\left\lfloor \frac{g}{3} \right\rfloor} N_{\gamma}(g+1) + \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor+1}^{\left\lfloor \frac{2(g+1)}{3} \right\rfloor} N_{\gamma}(g+1). \]

Theorem 1 states that \( N_{\gamma}(g) = N_{\gamma}(g + 1) \), for \( \gamma \leq \frac{g}{3} \).
Corollary

\[ n_g \leq n_{g+1} \]

if, and only if,

\[ \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor +1}^{\left\lfloor \frac{2g}{3} \right\rfloor} N_\gamma(g) \leq \sum_{\gamma=\left\lfloor \frac{g}{3} \right\rfloor +1}^{\left\lfloor \frac{2(g+1)}{3} \right\rfloor} N_\gamma(g+1). \]
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Conjecture

Let $\gamma$ be a non-negative integer. Then

$$N_\gamma(g) \leq N_\gamma(g + 1), \forall g.$$

Remark

If this Conjecture holds, then $n_g \leq n_{g+1}$ for all $g$. 
Conjecture

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Remark

If this Conjecture holds, then $n_g \leq n_{g+1}$ for all $g$. 
Construction of a $\gamma$-hyperelliptic semigroup of genus $g$

- $\gamma$ - positive integer
- $T = \mathbb{N}_0 \setminus \{q_1, \ldots, q_\gamma\}$ numerical semigroup

$$S = 2 \cdot T \cup (2 \cdot \mathbb{N}_0 + 1) \setminus \{“suitable\ choice”\ of\ g - \gamma\ odd\ numbers\}$$
“suitable choice” ensures that the final set is closed under addition.

- \( S \) is a \( \gamma \)-hyperelliptic semigroup: even gaps and even non-gaps are determined by \( T \).
- \( S \) has genus \( g \) if, and only if, the number of green points chosen as gaps is \( g - \gamma - k \).

**Lemma**

Let \( S \) be a \( \gamma \)-hyperelliptic semigroup of genus \( g \) and \( O \) the first odd number in \( S \). Then

\[
2g - 4\gamma + 1 \leq O \leq 2g - 2\gamma + 1.
\]
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- $S$ has genus $g$ if, and only if, the number of green points chosen as gaps is $g - \gamma - k$.

**Lemma**

Let $S$ be a $\gamma$-hyperelliptic semigroup of genus $g$ and $O$ the first odd number in $S$. Then

$$2g - 4\gamma + 1 \leq O \leq 2g - 2\gamma + 1.$$
“suitable choice” ensures that the final set is closed under addition.

- $S$ is a $\gamma$-hyperelliptic semigroup: even gaps and even non-gaps are determined by $T$.
- $S$ has genus $g$ if, and only if, the number of green points choosen as gaps is $g - \gamma - k$.

**Lemma**

Let $S$ be a $\gamma$-hyperelliptic semigroup of genus $g$ and $O$ the first odd number in $S$. Then

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**Proof (Thm 1)**

- Odd $q_i$ must be gaps! (otherwise, $S \ni q_i + q_i = 2q_i \notin S$)
- $g \geq 3\gamma \implies O \geq 2\gamma + 1 > q_\gamma \geq q_i$, for all $i$. 

\[O \geq 2g - 4\gamma + 1\]
Proof (Thm 1)

Odd \( q_i \) must be gaps! (otherwise, \( S \ni q_i + q_i = 2q_i \notin S \))

\[ g \geq 3\gamma \quad \Rightarrow \quad O \geq 2\gamma + 1 > q_\gamma \geq q_i, \text{ for all } i. \]
Proof (Thm 1)

- \( S_\gamma(g) := \{ S : g(S) = g \text{ and } \gamma(S) = \gamma \} \)
- For a fixed \( g \geq 3\gamma \), we find a bijection between \( S_\gamma(g) \) and \( S_\gamma(3\gamma) \)
Proof (Thm 1)

• Given $S(g) \in S_{\gamma}(g)$, let $O(g)$ be the first odd number of $S(g)$

• Let $M := g - 3\gamma$. Making a translation by $-2M$ only on the odd numbers higher than or equal to $O(g)$, we obtain a NS $S$, such that $O(3\gamma) = O(g) - 2M \geq 2\gamma + 1$ (and this is the first odd number of $S$)

• The even gaps of $S(g)$ and $S$ are the same, as the odd gaps of $S(g)$ and $S$ lower than $O(3\gamma)$. The odd gaps of $S(g)$ and $S$ higher than $O(3\gamma)$ are translated by $-2M$

• Under this construction, we have $g(S) = g - M = 3\gamma$. Hence, $S \in S_{\gamma}(3\gamma)$ and $\#S_{\gamma}(g) \leq \#S_{\gamma}(3\gamma)$

• Similarly (by making a translation by $+2M$), we can verify the other inequality and the result follows.
A related problem

- \( \gamma \) non-negative integer
- For \( g \geq 3\gamma \), the sequence \( N_\gamma(g) \) is constant and equal to \( N_\gamma(3\gamma) \)
- A natural task is about the behavior of \( f_\gamma := N_\gamma(3\gamma) \)

**Lemma**

Let \( \gamma \) be a non-negative integer and \( M_\gamma := 2^\gamma \left( \frac{\gamma}{2} + 1 \right) - 1 \). Then

\[
M_\gamma + (n_\gamma - \gamma) \cdot (\gamma + 1) \leq f_\gamma \leq M_\gamma + (n_\gamma - \gamma) \cdot 2^\gamma.
\]
A related problem

- $\gamma$ non-negative integer
- For $g \geq 3\gamma$, the sequence $N_\gamma(g)$ is constant and equal to $N_\gamma(3\gamma)$
- A natural task is about the behavior of $f_\gamma := N_\gamma(3\gamma)$

**Lemma**

Let $\gamma$ be a non-negative integer and $M_\gamma := 2^\gamma \left(\frac{\gamma}{2} + 1\right) - 1$. Then

$$M_\gamma + (n_\gamma - \gamma) \cdot (\gamma + 1) \leq f_\gamma \leq M_\gamma + (n_\gamma - \gamma) \cdot 2^\gamma.$$
Theorem 2

Let $\epsilon > 0$. Then

$$\lim_{\gamma \to \infty} \frac{f_\gamma}{(2\varphi + \epsilon)^\gamma} = 0$$

and

$$\lim_{\gamma \to \infty} \frac{f_\gamma}{2^\gamma} = \infty.$$ 

It suggests that the asymptotic behavior of $f_\gamma$ is exponential of order $\beta^\gamma$, where $2 < \beta \leq 2\varphi$. 
Theorem 2

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Conjecture

\[ \lim_{\gamma \to \infty} \frac{f_\gamma}{f_{\gamma-1}} = \phi^2 \approx 2.618 \]

and

\[ \lim_{\gamma \to \infty} \frac{f_\gamma}{n_{2\gamma}} = C, \]

where \( C \) is a constant.

Remark

There is a relation between the sequence \( f_\gamma \) and the conjecture proposed by M. Bras-Amorós (11). In fact, if \( f_\gamma \) is an increasing sequence, then the conjecture is also true.
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M. Bernardini and F. Torres, Counting numerical semigroups of a given genus via even gaps, *preprint.*


Thank You!
Feliz Cumpleaños, Fernando!
(22/06/2016)