ReSEARCH ARtiCLE

On $\gamma$-hyperelliptic Numerical Semigroups

Fernando Torres*

Communicated by M. Mislove

Abstract

We extend results on Weierstrass semigroups at ramified points of double covering of curves to any numerical semigroup whose genus is large enough. As an application we strengthen the properties concerning Weierstrass weights stated in [23].

0. Introduction

Let $H$ be a numerical semigroup, that is, a subsemigroup of $(\mathbb{N}, +)$ whose complement is finite. Examples of such semigroups are the Weierstrass semigroups at non-singular points of algebraic curves.

In this paper we deal with the following type of semigroups:

Definition 0.1. Let $\gamma \geq 0$ be an integer. $H$ is called $\gamma$-hyperelliptic if the following conditions hold:

(E₁) $H$ has $\gamma$ even elements in $[2, 4\gamma]$.

(E₂) The $(\gamma + 1)$th positive element of $H$ is $4\gamma + 2$.

A 0-hyperelliptic semigroup is usually called hyperelliptic.

The motivation for study of such semigroups comes from the study of Weierstrass semigroups at ramified points of double coverings of curves. Let $\pi : X \to \bar{X}$ be a double covering of projective, irreducible, non-singular algebraic curves over an algebraically closed field $k$. Let $g$ and $\gamma$ be the genus of $X$ and $\bar{X}$ respectively. Assume that there exists $P \in X$ which is ramified for $\pi$, and denote by $m_i$ the $i$th non-gap at $P$. Then the Weierstrass semigroup $H(P)$ at $P$ satisfies the following properties (cf. [23], [24, Lemma 3.4]):

(P₁) $H(P)$ is $\gamma$-hyperelliptic, provided $g \geq 4\gamma + 1$ if $\text{char}(k) \neq 2$, and $g \geq 6\gamma - 3$ otherwise.

(P₂) $m_{2\gamma + 1} = 6\gamma + 2$, provided $g \geq 5\gamma + 1$.

(P₃) $m_{\frac{g}{2} - \gamma - 1} = g - 2$ or $m_{\frac{g}{2} - \gamma} = g - 1$, provided $g \geq 4\gamma + 2$.

(P₄) The weight $w(P)$ of $H(P)$ satisfies

$$\frac{g - 2\gamma}{2} \leq w(P) < \frac{g - 2\gamma + 2}{2}.$$  

* This work was realized while the author was in ICTP (Trieste - Italy) with a grant from the International Atomic Energy Agency and UNESCO.
Conversely if \( g \) is large enough and if any of the above properties is satisfied, then \( X \) is a double covering of a curve of genus \( \gamma \). A posteriori the four above properties become equivalent whenever \( g \) is large enough.

The goal of this paper is to extend these results for any semigroup \( H \) such that \( g(H) := \#(\mathbb{N} \setminus H) \) is large enough. We remark that there exist semigroups of genus large enough that cannot be realized as Weierstrass semigroups (see [3], [23, Scholium 3.5]).

The key tool used to prove these equivalences is Theorem 1.10 in Freiman's book [8] which have to do with addition of finite sets. From this theorem we deduce Corollary 2.4 which can be considered as analogous to Castelnuovo's genus bound for curves in projective spaces ([4], [1, p.116], [21, Corollary 2.8]). Castelnuovo's result is the key tool to deal with Weierstrass semigroups. Corollary 2.4 can also be proved by means of properties of addition of residue classes (see Remark 2.6).

In §2 we prove the equivalences \((P_1) \iff (P_2) \iff (P_3)\). The equivalence \((P_1) \iff (P_2)\) is proved under the hypothesis \( g(H) \geq 6\gamma + 4 \), while \((P_1) \iff (P_3)\) is proved under \( g(H) = 6\gamma + 5 \) or \( g(H) \geq 6\gamma + 8 \). In both cases the bounds on \( g(H) \) are sharp (Remark 2.9). We mention that the cases \( \gamma \in \{1, 2\} \) of \((P_1) \iff (P_3)\) were fixed by Kato [13, Lemmas 4.5,6,7].

In §3 we deal with the equivalence \((P_1) \iff (P_4)\). To this purpose we determine bounds for the weight \( w(H) \) of the semigroup \( H \), which is defined by

\[
w(H) := \sum_{i=1}^{g} (\ell_i - i),
\]

where \( g := g(H) \) and \( \mathbb{N} \setminus H = \{\ell_1, \ldots, \ell_g\} \). It is well known that \( 0 \leq w(H) \leq \binom{g}{2} \); clearly \( w(H) = 0 \iff H = \{g + i : i \in \mathbb{Z}^+\} \), and one can show that \( w(H) = \binom{g}{2} \iff H \) is \( \mathbb{N} \), or \( g(H) \geq 1 \) and \( H \) is hyperelliptic (see e.g. [7, Corollary III.5.7]). Associated to \( H \) we have the number

\[
\rho = \rho(H) := \#\{\ell \in \mathbb{N} \setminus H : \ell \text{ even}\}.
\]

In [23, Lemma 2.3] it has been shown that \( \rho(H) \) is the unique number \( \gamma \) satisfying condition \((E_1)\) of Definition 0.1, and

\[
(E_2') \ 4\gamma + 2 \in H.
\]

Thus we observe the following:

**Lemma 0.2.** Let \( H \) be a \( \gamma \)-hyperelliptic semigroup. Then

\[
\rho(H) = \gamma.
\]

We also observe that if \( g(H) \geq 1 \), then \( H \) is hyperelliptic if and only if \( \rho(H) = 0 \). In general, \( \rho(H) \) affects the values of \( w(H) \). Let us assume that \( \rho(H) \geq 1 \) (hence \( w(H) < \binom{g}{2} \)); then we find

\[
\left( g - 2\rho \right) \left( \frac{2}{g} \right) \leq w(H) \leq \begin{cases} 
\frac{(g - 2\rho)}{g(g - 1)} + 2\rho^2 & \text{if } g \geq 6\rho + 5 \\
\text{otherwise}
\end{cases}
\]

265
(see Lemmas 3.2 and 3.4). These bounds strengthen results of Kato [12, Thm.1] and Oliveira [19, p.435] (see Remark 3.7). From this result we prove \((P_1) \Leftrightarrow (P_4)\) (Theorem 3.6) under the hypothesis

\[
g(H) \geq \begin{cases} 
\max\{12\gamma - 1, 1\} & \text{if } \gamma \in \{0, 1, 2\}, \\
11\gamma + 1 & \text{if } \gamma \in \{3, 5\}, \\
\frac{2(\gamma - 4) + 88}{2} & \text{if } \gamma \in \{4, 6\}, \\
\gamma^2 + 4\gamma + 3 & \text{if } \gamma \geq 7.
\end{cases}
\tag{2}
\]

The cases \(\gamma \in \{1, 2\}\) of that equivalence was fixed by Garcia (see [9]). In this section we use ideas from Garcia’s [9, Proof of Lemma 8] and Kato’s [12, p. 144].

In §1 we recollect some arithmetical properties of numerical semigroups. We mainly remark the influence of \(\rho(H)\) on \(H\).

It is a pleasure to thank Pablo Azcue and Gustavo T. de A. Moreira for discussions about §2.

1. Preliminaries

Throughout this paper we use the following notation

- semigroup: numerical semigroup.
- Let \(H\) be a semigroup. The genus of \(H\) is the number \(g(H) := \#(\mathbb{N} \setminus H)\), which throughout this article will be assumed bigger than 0. The positive elements of \(H\) will be called the non-gaps of \(H\), and those of \(G(H) := \mathbb{N} \setminus H\) will be called the gaps of \(H\). We denote by \(m_i(H)\) the \(i\)th non-gap of \(H\). If a semigroup is generated by \(m, n, \ldots\) we denote \(H = (m, n, \ldots)\).
- \([x]\) stands for the integer part of \(x \in \mathbb{R}\).

In this section we recall some arithmetical properties of semigroups. Let \(H\) be a semigroup of genus \(g\). Set \(m_j := m_j(H)\) for each \(j\). If \(m_1 = 2\) then \(m_i = 2i\) for \(i = 1, \ldots, g\). Let \(m_i \geq 3\). By the semigroup property of \(H\) the first \(g\) non-gaps satisfy the following inequalities:

\[
m_i \geq 2i + 1 \quad \text{for } i = 1, \ldots, g - 2, \quad m_{g-1} \geq 2g - 2, \quad m_g = 2g
\tag{3}
\]

(see [2], [19, Thm.1.1]).

Let \(\rho\) be as in (1). From [24, Lemma 2.3] we have that

\[
\{4\rho + 2i : i \in \mathbb{N}\} \subseteq H.
\tag{4}
\]

From the definition of \(\rho\), \(H\) has \(\rho\) odd non-gaps in \([1, 2g - 1]\). Let denote by

\[u_\rho < \ldots < u_1\]

such non-gaps.

Lemma 1.1. Let \(H\) be a semigroup of genus \(g\), and \(\rho\) the number of even gaps of \(H\). Then

\[2g \geq 3\rho.\]
Proof. Let us assume that \( g \leq 2\rho - 1 \). From \( u_1 \leq 2g - 1 \) we have \( u_{2\rho-g+1} \leq 4g - 4\rho - 1 \). Let \( \ell \) be the biggest even gap of \( H \). Then \( \ell \leq 4g - 4\rho \). For suppose that \( \ell \geq 4g - 4\rho + 2 \). Thus \( \ell - u_{2\rho-g+1} \geq 3 \), and then \( H \) would have \( g - \rho + 1 \) odd gaps, namely \( 1, \ell - u_{2\rho-g+1}, \ldots, \ell - u_{\rho} \), a contradiction. Now since in \([2, 4g - 4\rho]\) there are \( 2g - 2\rho \) even numbers such that \( \rho \) of them are gaps, the lemma follows. 

Denote by \( f_i := f_i(H) \) the \( i \)th even non-gap of \( H \). Hence by (4) we have

\[
H = (f_1, \ldots, f_{\rho}, 4\rho + 2, u_{\rho}, \ldots, u_1).
\]

Observe that \( f_{\rho} = 4\rho \), and

\[
f_{g-\rho} = 2g.
\]

By [24, Lemma 2.1] and since \( g \geq 1 \) we have

\[
u_{\rho} \geq \max\{2g - 4\rho + 1, 3\}.
\]

In particular, if \( g \geq 4\rho \) we obtain

\[
m_1 = f_1, \ldots, m_{\rho} = f_{\rho}.
\]

Note that (7) is only meaningful for \( g \geq 2\rho \). For \( g \leq 2\rho - 1 \) we have:

**Lemma 1.2.** Let \( H \) be a semigroup of genus \( g \), and \( \rho \) be the number of even gaps of \( H \). If \( g \leq 2\rho - 1 \), then

\[
u_{\rho} \geq 4\rho - 2g + 1.
\]

**Proof.** From the proof of Lemma 1.1 we have that \( H \) has \( 2g - 3\rho \) even non-gaps in \([2, 4g - 4\rho]\). Consider the following sequence of even non-gaps:

\[
2u_{\rho} < \ldots < u_{\rho} + u_{4\rho-2g}.
\]

Since in this sequence we have \( 2g - 3\rho + 1 \) even non-gaps, then

\[
u_{\rho} + u_{4\rho-2g} \geq 4g - 4\rho + 2.
\]

Now, since \( u_{4\rho-2g} \leq 6g - 8\rho + 1 \) the proof follows.

2. \( \gamma \)-hyperelliptic Numerical Semigroups

In this section we deal with properties \((P_1), (P_2)\) and \((P_3)\) stated in §0. For \( i \in \mathbb{Z}^+ \) set

\[
d_i(H) := \gcd(m_1(H), \ldots, m_i(H)).
\]

**Theorem 2.1.** Let \( \gamma \in \mathbb{N}, H \) be a semigroup of genus \( g \geq 6\gamma + 4 \) if \( \gamma \geq 1 \). Then the following statements are equivalent:

(i) \( H \) is \( \gamma \)-hyperelliptic.

(ii) \( m_{2\gamma + 1}(H) = 6\gamma + 2 \).
Theorem 2.2. Let $\gamma \in \mathbb{N}$, $H$ be a semigroup of genus $g = 6\gamma + 5$ or $g \geq 6\gamma + 7$. Set $r := [(g + 1)/2] - \gamma - 1$. Then the following statements are equivalent:

(i) $H$ is $\gamma$-hyperelliptic.

(ii) $m_r(H) = g - 2$ if $g$ is even; $m_r(H) = g - 1$ if $g$ is odd.

(iii) $m_r(H) \leq g - 1 < m_{r+1}(H)$.

To prove these results we need a particular case of the result below. For $K$ a subset of a group we set $2K := \{a + b : a, b \in K\}$.

Lemma 2.3. [Fre, Thm. 1.10] Let $K = \{0 < m_1 < \ldots < m_i\} \subseteq \mathbb{Z}$ be such that $\gcd(m_1, \ldots, m_i) = 1$. If $m_i \geq i + 1 + b$, where $b$ is an integer satisfying $0 \leq b < i - 1$, then

$$\#2K \geq 2i + 2 + b.$$ 

Corollary 2.4. Let $H$ be a semigroup of genus $g$, and $i \in \mathbb{Z}^+$. If

$$d_i(H) = 1 \quad \text{and} \quad i \leq g + 1,$$

then we have

$$2m_i(H) \geq m_{3i-1}(H).$$

Proof. Let $K := \{0, m_1(H), \ldots, m_i(H)\}$. Then by (3), we can apply Lemma 2.3 to $K$ with $b = i - 2$.

Remark 2.5. Both the hypothesis $d_i(H) = 1$ and $i \leq g + 1$ of the corollary above are necessary. Moreover the conclusion of that result is sharp:

(i) Let $i = g + h$, $h \geq 2$. Then $2m_{g+h} = m_{3i-h}$.

(ii) Let $m_1 = 4$, $m_2 = 6$ and $m_3 = 8$. Then $d_3 = 2$ and $2m_3 = m_7$.

(iii) Let $m_1 = 5$, $m_2 = 10$, $m_3 = 15$, $m_4 = 18$, $m_5 = 20$. Then $2m_5 = m_{14}$.

Remark 2.6. (i) The Corollary above can also be proved by using results on the addition of residue classes: let $H$ and $i$ be as in 2.4; assume further that $2 \leq i \leq g - 2$ (the remaining cases are easy to prove), and consider $\tilde{K} := \{m_1, \ldots, m_i\} \subseteq \mathbb{Z}_{m_i}$ (i.e. a subset of the integers modulo $m_i$). Let $N := \#2K$. Then it is easy to see that

$$2m_i \geq m_{i+N}.$$ 

Consequently we have a proof of the above Corollary provided $N \geq 2i - 1$ (*). Since $m_i \geq 2i + 1$ (see (3)), we get (*) provided $m_i$ prime (Cauchy [6], Davenport [5], [17, Corollary 1.2.3]), or provided $\gcd(m_j, m_i) = 1$ for $j = 1, \ldots, i - 1$ (Chowla [16, Satz 114], [17, Corollary 1.2.4]). In general by using the hypothesis $d_i(H) = 1$ we can reduce the proof of the Corollary to the case $\gcd(m_{i-1}, m_i) = 1$. Then we apply Pillai's [20, Thm 1] generalization of Davenport and Chowla results (or Mann's result [17, Corollary 1.2.2]).
(ii) Let $H$ and $i$ be as above and assume that $2m_i = m_{3i-1}$. Then from (i) we have $N = \#2\tilde{K} = 2i - 1$. Thus by Kemperman [14, Thm 2.1] (or by [8, Thm. 1.11]) $2\tilde{K}$ satisfies one of the following conditions: (1) there exist $m, d \in \mathbb{Z}_{m_i}$, such that $2\tilde{K} = \{ m + dj : j = 0, 1, \ldots, N - 1 \}$, or (2) there exists a subgroup $F$ of $\mathbb{Z}_{m_i}$ of order $\geq 2$, such that $2\tilde{K}$ is the disjoint union of a non-empty set $C$ satisfying $C + F = C$, and a set $C'$ satisfying $C' \subseteq c + F$ for some $c \in C'$. For instance example (iii) of 2.5 satisfies property (2).

Set $m_j := m_j(H)$ for each $j$.

**Proof of Theorem 2.1.** By definition $H$ is hyperelliptic if and only if $m_1 = 2$. So let us assume that $\gamma \geq 1$.

(i) $\Rightarrow$ (ii): From Lemma 0.2 and (7) we find that $u_\gamma \geq 6\gamma + 3$ if $g \geq 5\gamma + 1$. Then (ii) follows from (8) and (4).

(ii) $\Rightarrow$ (i): We claim that $d_{2\gamma+1}(H) = 2$. For suppose that $d_{2\gamma+1}(H) \geq 3$. Then $6\gamma + 3 = m_{2\gamma+1} \geq 6\gamma + 3$ and so $m_1 \leq 2$, a contradiction. Hence $d_{2\gamma+1}(H) \leq 2$.

Now suppose that $d_{2\gamma+1}(H) = 1$. Then Corollary 2.4 implies

$$2(6\gamma + 2) = 2m_{2\gamma+1} \geq m_{6\gamma+2}.$$ 

But, since $g - 2 \geq 6\gamma + 2$, by (3) we would have

$$m_{6\gamma+2} \geq 2(6\gamma + 2) + 1$$

which leads to a contradiction. This shows that $d_{2\gamma+1}(H) = 2$. Now since $m_{2\gamma+1} = 6\gamma + 2$ we have that $m_\gamma \leq 4\gamma$. Moreover, there exist $\gamma$ even gaps of $H$ in $[2, 6\gamma + 2]$. Let $\ell$ be an even gap of $H$.

**Claim.** $\ell < m_\gamma$.

**Proof of the Claim.** Suppose that there exists an even gap $\ell$ such that $\ell > m_\gamma$. Take the smallest $\ell$ with such a property; then the following $\gamma$ even gaps: $\ell - m_\gamma, \ldots, \ell - m_1$ belong to $[2, m_\gamma]$. Thus, since $m_1 > 2$, we must have $\ell - m_\gamma = 2$. This implies that $H$ has $\gamma + 1$ even non-gaps in $[2, 6\gamma + 2]$, namely $\ell - m_\gamma, \ldots, \ell - m_1, \ell$, a contradiction.

This finishes the proof of Theorem 2.1.

**Remark 2.7.** From the proof of the above theorem and (8) we obtain the following result. Let $\gamma \in \mathbb{N}$ and $H$ be a semigroup of genus $g \geq 6\gamma + 4$ if $\gamma \geq 1$ and $g \geq 1$ if $\gamma = 0$. Then the following statements are equivalent:

(i) $H$ is $t$-hyperelliptic for some $t \in \{0, \ldots, \gamma\}$.

(ii) $m_{2\gamma+1}(H) \leq 6\gamma + 2$.

(iii) $\rho(H) \leq \gamma$. 

369
Proof of Theorem 2.2. (i) ⇒ (ii): Similar to the proof of (i) ⇒ (ii) of Theorem 2.1 (here we need \( g \geq 4\gamma + 3 \) (resp. \( g \geq 4\gamma + 4 \)) if \( g \) is odd (resp. even)).

Before proving the other implications we remark that \( g \leq 3r - 1 \): in fact, if \( g \geq 3r \) we would have \( g \leq 6\gamma + 6 \) (resp. \( g \geq 6\gamma + 3 \)) provided \( g \) even (resp. odd) - a contradiction.

(ii) ⇒ (iii): Let \( g \) be even and suppose that \( m_{r+1} = g - 1 \). Then by Corollary 2.4 we would have \( 2g - 2 = 2m_{r+1} \geq m_{3r+2} \) and hence \( g - 1 \geq 3r + 2 \). This contradicts the previous remark.

(iii) ⇒ (i): We claim that \( d_r(H) = 2 \). Suppose that \( d_r(H) \geq 3 \). Then we would have \( g - 1 \geq m_r \geq m_1 + 3(r - 1) \geq 3r - 1 \), which contradicts the previous remark. Now suppose that \( d_r(H) = 1 \). Then by Corollary 2.4 we would have

\[
2g - 2 \geq m_r \geq m_{3r-1},
\]

which again contradicts the previous remark.

Thus the number of even gaps of \( H \) in \([2, g - 1]\) is \( \gamma \), and \( m_\gamma \leq 4\gamma \). Let \( \ell \) be an even gap of \( H \). As in the proof of the Claim in Theorem 2.1 here we also have that \( \ell < m_\gamma \). Now the proof follows.

\[\square\]

Remark 2.8. From the proof of the above theorem and Remark 2.7 we have the following result. Let \( \gamma \in \mathbb{N} \), \( H \) be a semigroup of genus \( g = 6\gamma + 5 \) or \( g \geq 6\gamma + 7 \) and \( r = [(g+1)/2] - \gamma - 1 \). Then the following statements are equivalent:

(i) \( H \) is \( t \)-hyperelliptic for some \( t \in \{0, \ldots, \gamma\} \).

(ii) \( m_r(H) \leq g - 2 \) if \( g \) is even; \( m_r(H) \leq g - 1 \) if \( g \) is odd.

(iii) \( m_r(H) \leq g - 1 \).

(iv) \( \rho(H) \leq \gamma \).

Remark 2.9. The hypothesis on the genus in the above theorems and remarks is sharp. To see this let \( \gamma \geq 0 \) an integer, and let \( X \) be the curve defined by the equation

\[
y^4 = \prod_{j=1}^I (x - a_j),
\]

where the \( a_j \)'s are pairwise distinct elements of a field \( k \), \( I = 4\gamma + 3 \) if \( \gamma \) is odd; \( I = 4\gamma + 5 \) otherwise. Let \( P \) be the unique point over \( x = \infty \). Then \( H(P) = \{4, I\} \) and so \( g(H(P)) = 6\gamma + 3 \) (resp. \( 6\gamma + 6 \)), \( m_{2\gamma+1}(H(P)) = 6\gamma + 2 \) (resp. \( m_{2\gamma+2}(H(P)) = 6\gamma + 5 \)), and \( \rho(H(P)) = 2\gamma + 1 \) (resp. \( \rho(H(P)) = 2\gamma + 2 \)) provided \( \gamma \) odd (resp. \( \gamma \) even).

3. Weight of Semigroups

3.1. Bounding the weight. Let \( H \) be a semigroup of genus \( g \). Set \( m_j = m_j(H) \) for each \( j \) and \( \rho = \rho(H) \) (see (1)). Due to \( m_g = 2g \) (see (3)), the weight \( w(H) \) of \( H \) can be computed by

\[
w(H) = \frac{3g^2 + g}{2} - \sum_{j=1}^g m_j.
\]
Consequently the problem of bounding $w(H)$ is equivalent to the problem of bounding

$$S(H) := \sum_{j=1}^{g} m_j.$$ 

If $\rho = 0$, then we have $m_i = 2i$ for each $i = 1, \ldots, g$. In particular we have $w(H) = \binom{g}{2}$. Let $\rho \geq 1$ (or equivalently $f_1 \geq 4$). Then by (5) we have

$$S(H) = \sum_{f \in \mathcal{H}, f \leq g} 2f + \sum_{i=1}^{\rho} u_i,$$  

where

$$\mathcal{H} := \{f/2 : f \in H, f \text{ even}\}.$$  

Lemma 3.1. With the notation of §1 we have:

(i) If $f_1 = 4$, then $f_i = 4i$ for $i = 1, \ldots, \rho$.

(ii) If $f_1 \geq 6$, then $f_i \geq 4i + 2$ for $i = 1, \ldots, \rho - 2$, $f_{\rho-1} \geq 4\rho - 4$, $f_{\rho} = 4\rho$.

(iii) $f_i \leq 2\rho + 2i$ for each $i$.

(iv) $2g - 4i + 1 \leq u_i \leq 2g - 2i + 1$, for $i = 1, \ldots, \rho$.

Proof. By (4), we have

$$\mathcal{H} = \{\frac{f_1}{2}, \ldots, \frac{f_{\rho}}{2}\} \cup \{4\rho + i : i \in \mathbb{N}\}.$$  

Thus $\mathcal{H}$ is a semigroup of genus $\rho$. Then (i) is due to the fact that $f_1/2 = 2$ and (ii) follows from (3). Statement (iii) follows from (6).

(iv) The upper bound follows from $u_1 \leq 2g - 1$. To prove the lower bound we proceed by induction on $i$. The case $i = \rho$ follows from (7). Suppose that $u_i \geq 2g - 4i + 1$ but $u_{i-1} < 2g - 4(i - 1) + 1$, for $1 < i \leq \rho$. Then $u_i = 2g - 4i + 1$, $u_{i-1} = 2g - 4i + 3$, and there exists an odd gap $\ell$ of $H$ such that $\ell > u_{i-1}$. Without loss of generality we can assume $\ell - u_{i-1} = 2$. Set $I := [\ell - u_{i-1}, \ell - u_{\rho}] \subseteq [2, 4\rho - 2]$ and let $t$ be the number of even non-gaps of $H$ belonging to $I$. By the choice of $\ell$ we have that $\ell - u_{i-1} < f_1$. Now, since $\ell - u_{j} \in I$ for $j = i - 1, \ldots, \rho$ we also have that

$$\frac{u_{i-1} - u_{\rho} + 1}{2} \geq t + \rho - (i - 1) + 1.$$  

Thus $u_{\rho} \leq 2g - 2\rho - 2i - 2t + 1$. Now, since $u_{\rho} + f_{t+1} > u_{i-1}$ and since by statement (iii) $f_{t+i-1} \leq 2\rho + 2t + 2i - 2$, we have that the odd non-gaps $u_{\rho} + f_{t+1}, \ldots, u_{\rho} + f_{t+i-1}$ belong to $[\ell + 2, 2g - 1]$. This is a contradiction because $H$ would have $(\rho - i + 2) + (i - 1) = \rho + 1$ odd non-gaps. ■
Lemma 3.2. Let $H$ be a semigroup of genus $g$. With notation as in §1, we have

(i) $w(H) \geq \binom{g-2\rho}{2}$. Equality holds if and only if $f_1 = 2\rho + 2$ and $u_\rho = 2g - 2\rho + 1$.

(ii) If $g \geq 2\rho$, then $w(H) \leq \binom{g-2\rho}{2} + 2\rho^2$. Equality holds if and only if $H = \langle 4, 4\rho, 2g - 4\rho + 1 \rangle$.

(iii) If $g \leq 2\rho - 1$, then $w(H) \leq \binom{g+2\rho}{2} - 4g - 6\rho^2 + 8\rho$.

Proof. (i) By (9) we have to show that

$$S(H) \leq g^2 + (2\rho + 1)g - 2\rho^2 - \rho,$$

and that the equality holds if and only if $f_1 = 2\rho + 2$ and $u_\rho = 2g - 2\rho + 1$. Both the above statements follow from Lemma 3.1 (i), (iv).

(ii) Here we have to show that

$$S(H) \geq g^2 + (2\rho + 1)g - 4\rho^2 - \rho,$$  \(\dagger\)

and that equality holds if and only if $H = \langle 4, 4\rho + 2, 2g - 4\rho + 1 \rangle$.

Since $g \geq 2\rho$ by (4) we obtain

$$S(H) = \sum_{i=1}^{\rho} (f_i(H) + u_i(H)) + g^2 + g - 4\rho^2 - 2\rho. \quad (11)$$

Thus we obtain \(\dagger\) by means of Lemma 3.1 (ii), (iii) and (iv). Moreover the equality in \(\dagger\) holds if and only if $\sum_{i=1}^{\rho} (f_i + u_i) = 2\rho g + \rho$. Then the second part of (ii) also follows from the above mentioned results.

(iii) In this case, due to the proof of Lemma 1.1, instead of equation (11) we have

$$S(H) = \sum_{i=1}^{2g-3\rho} f_i + \sum_{i=2g-3\rho+1}^{g-\rho} (2i + 2\rho) + \sum_{i=1}^{\rho} u_i. \quad (12)$$

We will see in the next remark that in this case we have $f_1 \geq 6$. Thus by using Lemmas 1.2 and 3.1 (ii), (iv) we obtain

$$S(H) \geq 4\rho^2 - (2g + 7)\rho + g^2 + 5g,$$

where from it follows the proof.

Remark 3.3. (i) If $f_1 = 4$, then $g \geq 2\rho$. This follows because the biggest even gap of $H$ is $4(\rho - 1) + 2$. Moreover, one can determine $u_\rho, \ldots, u_1$ as follows: let $J \in \mathbb{N}$ satisfying the inequalities below

$$\max\{1, \frac{3\rho + 2 - g}{2}\} \leq J \leq \min\{\rho + 1, \frac{g - \rho + 3}{2}\},$$

provided $g$ even, otherwise replace $J$ by $\rho - J + 2$; then

$$\{u_\rho, \ldots, u_1\} = \{2g - 4\rho + 4J - 7 + 4i : i = 1, \ldots, \rho - J + 1\} \cup \{2g - 4J + 3 + 4i : i = 1, \ldots, J - 1\}.$$
(see [15, §3], [23, Remarks 2.5]). Consequently from (11) and (9) we obtain

\[ w(H) = \left( \frac{g-2\rho}{2} \right) + 2\rho^2 + 4\rho + 6 + 4J^2 - (4\rho + 10)J. \]

In particular we have

\[ \left( \frac{g-2\rho}{2} \right) + \rho^2 - \rho \leq w(H) \leq \left( \frac{g-2\rho}{2} \right) + 2\rho^2. \]

Let \( C \) be an integer such that \( 0 \leq 2C \leq \rho^2 + \rho \). Then \( w(H) = \left( \frac{g-2\rho}{2} \right) + \rho^2 - \rho + 2C \) if and only if \( 4 + 32C \) is a square. The lower bound is attained if and only if \( H = \langle 4, 4\rho + 2, 2g - 2\rho + 1, 2g - 2\rho + 3 \rangle \).

(ii) Let \( u_1 = 3 \). Then from (7) and Lemma 1.2 we find that \( g \in \{2\rho - 1, 2\rho, 2\rho + 1\} \). Moreover, in this case one can also obtain an explicit formula for \( w(H) \) ([12, Lemma 6]). Let \( g \equiv r \pmod{3} \), \( r = 0, 1, 2 \) and let \( s \) be an integer such that \( 0 \leq s \leq (g - r)/3 \). If \( r = 0, 1 \) (resp. \( r = 2 \)), then

\[ w(H) = \frac{g(g-1)}{3} + 3s^2 - gs - s \leq \frac{g(g-1)}{3} \]

resp.

\[ w(H) = \frac{g(g-2)}{3} + 3s^2 - gs + s \leq \frac{g(g-2)}{3}. \]

If \( r = 0, 1 \) (resp. \( r = 2 \)), equality occurs if and only if \( H = \langle 3, g + 1 \rangle \) (resp. \( H = \langle 3, g + 2, 2g + 1 \rangle \)).

Let \( g \leq 2\rho - 1 \). The way how we bound from below equation (12) is far away from being sharp. We do not know an analogous to the lower bound of Lemma 3.1 (iv) in this case. However, for certain range of \( g \) the bounds in 3.3 (ii) are the best possible:

**Lemma 3.4.** Let \( H \) be a semigroup of genus \( g \geq 11 \), \( r \in \{1, 2, 3, 4, 5, 6\} \) such that \( g \equiv r \pmod{6} \). Let \( \rho \) be the number of even gaps of \( H \). If

\[ \rho > \begin{cases} \frac{g-5}{6} - 1 & \text{if } r \neq 5, \\ \frac{g-5}{6} & \text{if } r = 5, \end{cases} \]

then

\[ w(H) \leq \begin{cases} \frac{g(g-2)}{3} & \text{if } r = 2, 5 \\ \frac{g(g-1)}{3} & \text{if } r = 1, 3, 4, 6. \end{cases} \]

If \( r = 2, 5 \) (resp. \( r \notin \{2, 5\} \)) equality above holds if and only if \( H = \langle 3, g + 2, 2g + 1 \rangle \) (resp. \( H = \langle 3, g + 1 \rangle \)).

**Proof.** We assume \( g \equiv 5 \pmod{6} \); the other cases can be proven in a similar way. By Remark 3.3 (ii) we can assume \( u_1 > 3 \), and then by (9) we have to prove that

\[ S(H) > \frac{2g^2 + 7g}{6}. \]  

\[ (*) \]
Now, since $\rho > (g - 5)/6$, by Remark 2.8 and Lemma 0.2 we must have

$$m_{\frac{g+1}{3}} = m_{\frac{g-5}{6}} \geq g.$$

(A) Let $S' := \sum_i m_i$, $(g+1)/3 \leq i \leq g$: Define

$$F := \{i \in \mathbb{N} : \frac{g+1}{3} \leq i \leq g, \ m_i \leq 2i + \frac{g-5}{3}\},$$

and let $f := \min(F)$. Then $f \geq (g+4)/3$, $m_f = 2f + \frac{g-5}{3}$, $m_{f-1} = 2f + \frac{g-8}{3}$. Thus for $g \geq i \geq f$, $d_i = 1$ and hence by Corollary 2.4, $2m_i \geq m_{3i-1} = g + 3i - 1$. In particular, $f \geq (g+7)/3$. Now we bound $S'$ in three steps:

Step (i). $(g+1)/3 \leq i \leq f - 1$: By definition of $f$ we have that $m_i \geq 2i + \frac{g-2}{3}$ and hence

$$\sum_i m_i \geq f^2 + \frac{g-5}{3}f - \frac{2g^2 - 2g - 4}{9}. \quad (13)$$

Step (ii). $f \leq i \leq (6f - g - 7)/3$: Here we have $m_i \geq m_f + i - f = i + f + \frac{g-5}{3}$. Hence

$$\sum_i m_i \geq \frac{5}{2}f^2 - \frac{4g + 37}{6}f - \frac{g^2 - 13g - 68}{18}.$$ 

Step (iii). $(6f - g - 4)/3 \leq i \leq g$: Here we have $m_i + m_{i+1} \geq g + 3i + 1$ for $i$ odd, $6f - g - 4 \leq i \leq g - 2$. Since $m_2 = 2g$ then we have

$$\sum_i m_i \geq -3f^2 + 6f + \frac{4g^2 + 2g - 8}{3}.$$

(B) Let $S'' := \sum_i m_i$, $1 \leq i \leq (g - 2)/3$: By Remark 2.7 and Lemma 0.2 we have that $m_i \geq 3i$ for $i$ odd, $i = 3, \ldots, (g - 2)/3$. First we notice that for $i$ odd and $3 \leq i \leq (g - 8)/3$ we must have $m_{i+1} \geq 3i + 3$. Otherwise we would have $d_{i+1} = 1$ and hence by Corollary 2.4 and (3) we would have $2m_{i+1} \geq m_{3i+2} \geq 6i + 5$, a contradiction.

**Claim.** Let $i$ be odd and $3 \leq i \leq (g - 8)/3$. If $m_i = 3i$ or $m_{i+1} = 3i + 3$, then $m_1 = 3$.

**Proof of the Claim.** It is enough to show that $d_i = 3$ or $d_{i+1} = 3$. Suppose that $m_i = 3i$. Since $i$ is odd, $d_i$ is one or three. Suppose $d_i = 1$. Then by Corollary 2.4 we have $6i = 2m_i \geq m_{3i-1}$. Let $\ell \in G(H)$. Suppose that $\ell > m_{3i-1}$, then $\ell \leq m_{3i-1} + 3$. In fact if $\ell > m_{3i-1} + 3$, by choosing the smallest $\ell > m_{3i-1}$ we would have $3i + 2$ gaps in $[1, 6i]$, namely, $1, 2, 3, \ell - m_{3i-1}, \ldots, \ell - m_1$, a contradiction. Then it follows that $g \leq 3i + 1 + 3 = 3i + 4$.

Now suppose that $m_{i+1} = 3i + 3$; as in the previous proof here we also have that $d_{i+1} > 1$. Suppose that $d_{i+1} = 2$. Then $m_1 > 3$ and hence $m_i = 3i + 1$. Since we know that $m_{i+1} \geq 3i + 6$, then the even number $\ell = 3i + 5$ is a gap of $H$. Then we would find $2i + 2$ even numbers in $[2, 3i + 3]$, namely $m_1, \ldots, m_{i+1}$, and $\ell - m_{i+1}, \ldots, \ell - m_1$, a contradiction. Hence $d_{i+1} = 3$ and then $m_1 = 3$. \[ \]
Then, since we assume $u_1 > 3$, we have $m_i + m_{i+1} \geq 6i + 5$ for $i$ odd $3 \leq i \leq (g-8)/3$, $m_{x-3} \geq g - 2$, and so

\[
\sum_{i=1}^{(g-2)/3} m_i \geq \sum_{j=1}^{(g-1)/6} (12j + 10) + m_1 + m_2 + m_{x-3} \geq \frac{g^2 + g - 78}{6} + m_1 + m_2. \tag{14}
\]

Summing up (i), (ii), (iii) and (B) we get

\[
S(H) \geq 3f^2 - (2g + 11)f + \frac{22g^2 + 32g - 206}{18} + m_1 + m_2.
\]

The function $\Gamma(x) := 3x^2 - (2g + 11)x$ attains its minimum for $x = (2g + 11)/6 < (g + 7)/3 \leq f$. Suppose that $f \geq (g + 13)/3$. Then we find

\[
S(H) \geq \frac{7g^2 + 7g - 60}{6} + m_1 + m_2.
\]

We claim that $m_1 + m_2 > 11$. Otherwise we would have $m_3 = m_1 + m_2 = 10$ which is impossible. From the claim we get (\ast\ast\ast).

In all the computations below we use the fact that $2g \leq (m - 1)(n - 1)$ whenever $m, n \in H$ with $\gcd(m, n) = 1$ (see e.g. Jenkins [10]).

Now suppose that $f = (g + 10)/3$. Here we find

\[
S(H) \geq \frac{7g^2 + 7g - 72}{6} + m_1 + m_2.
\]

Suppose that $m_1 + m_2 \leq 12$ (otherwise the above computations imply (\ast\ast\ast). If $g > 11$, then $m_3 \geq 13$ and so $m_3 = m_1 + m_2 \in \{9, 11, 12\}$. If $m_1 + m_2 = 9$, then $g \leq 6$; if $m_1 + m_2 = 11$ then $g \leq 10$; if $m_1 + m_2 = 12$ then $g \leq 11$ or $m_1 = 4$, $m_2 = 8$. Let $s$ denote the first odd non-gap of $H$. Then $2g \leq 3(s - 1)$ and so $s > (2g + 2)/3$. In the interval $[4, (2g + 2)/3]$ does not exist $h \in H$ such that $h \equiv 2 \pmod{4}$: In fact if such a $h$ exists then we would have $4p + 2 \leq (2g - 4)/3$ or $p \leq (g - 5)/6$. Consequently $m_3 = 12, \ldots, m_{(g+1)/6} = (2g + 2)/3$. Thus we can improve the computation in (14) by summing it up $\sum_{i=1}^{j} (4i + 1)$, where $j = (g - 5)/12$ or $j = (g - 11)/12$. Then we get (\ast). If $g = 11$, the first seven non-gaps are $\{4, 8, 10, 12, 14, 15, 16\}$ or $\{5, 7, 10, 12, 14, 15, 16\}$. In both cases the computation in (13) increases at least by one, and so we obtain (\ast).

Finally let $f = (g + 7)/3$. Here we find

\[
S(H) \geq \frac{7g^2 + 7g - 78}{6} + m_1 + m_2,
\]

and we have to analyze the cases $m_1 + m_2 \in \{9, 11, 12, 13\}$. This can be done as in the previous case. This finishes the proof of Lemma 3.4.

\section*{3.2. The equivalence (P\textsubscript{1}) $\Leftrightarrow$ (P\textsubscript{4})} We are going to characterize $\gamma$-hyperelliptic semigroups by means of weights of semigroups. We begin with the following result, which has been proved by Garcia for $\gamma \in \{1, 2\}$ [9, Lemmas 8 and 10].
Theorem 3.5. Let $\gamma \in \mathbb{N}$ and $H$ a semigroup whose genus $g$ satisfies (2). Then the following statements are equivalent:

(i) $H$ is $t$-hyperelliptic for some $t \in \{0, \ldots, \gamma\}$.

(ii) $w(H) \geq \left(\frac{g^2 - 2\gamma}{2}\right)$.

Theorem 3.6. Let $\gamma$, $H$ and $g$ be as in Theorem 3.5. The following statements are equivalent:

(i) $H$ is $\gamma$-hyperelliptic.

(ii) $\left(\frac{g^2 - 2\gamma}{2}\right) \leq w(H) \leq \left(\frac{g^2 - 2\gamma}{2}\right) + 2\gamma^2$.

(iii) $\left(\frac{g^2 - 2\gamma}{2}\right) \leq w(H) < \left(\frac{g^2 - 2\gamma + 2}{2}\right)$.

Proof of Theorem 3.5. (i) $\Rightarrow$ (ii): By Lemma 0.2 and Lemma 3.2 (i) we have $w(H) \geq \left(\frac{g^2 - 2t}{2}\right)$. This implies (ii).

(ii) $\Rightarrow$ (i): Suppose that $H$ is not $t$-hyperelliptic for any $t \in \{0, \ldots, \gamma\}$. We are going to prove that $w(H) < \left(\frac{g^2 - 2\gamma}{2}\right)$ which, by (9), is equivalent to prove that:

$$\sum_{i=1}^{n} m_i > g^2 + (2\gamma + 1)g - 2\gamma^2 - \gamma.$$  (*)&

Case 1: $g$ satisfies the hypothesis of Lemma 3.4. From that lemma we have $S(H) \geq (7g^2 + 5g)/3$ and then we get (*) provided

$$g > \gamma := 12\gamma + 1 + \sqrt{96\gamma^2 + 1}.$$

We notice that $\gamma^2 + 4\gamma + 3 \geq \gamma$ if $\gamma \geq 7$. For $\gamma = 1, 4, 6$ we need respectively $g > 11$, $g > 44$ and $g > 65$. By noticing that $11, 44$ and $65$ are congruent to $2$ modulo $3$, we can use $g = 11, g = 44$ and $g = 65$ because in these cases $S(H) \geq (7g^2 + 7g)/3$. For the other values of $\gamma$ we obtain the bounds of (2).

Case 2: $g$ does not satisfy the hypothesis of Lemma 3.4. Here we have $g \geq 6\rho + 5$. Hence by Lemma 0.2 and (8) we must have $\rho \geq \gamma + 1$. From (11) and Lemma 3.1 we also have $S(H) \geq g^2 + (2\rho + 1)g - 4\rho^2 - \rho$. The function $\Gamma(\rho) := (2g - 1)\rho - 4\rho^2$ satisfies

$$\Gamma(\rho) \geq \Gamma(\gamma + 1) = (2\gamma + 2)g - 4\gamma^2 - 9\gamma - 5,$$

for $\gamma + 1 \leq \rho \leq [2(g - 1)/4] - \gamma - 1$. Thus we obtain condition (*) provided

$$g \geq \gamma^2 + 4\gamma + 3.$$  &

Remark 3.7. Let $H$ be a semigroup of genus $g$, $r$ the number defined in Lemma 3.4. Put $c := (g - 5)/6$ if $r = 5$, and $c := (g - r)/6 - 1$ otherwise.

From the proof of Case 2 of the above result we see that $S(H) \geq g^2 + 3g - 5$ whenever $1 \leq \rho(H) \leq (g - 3)/2$. Hence this result and Lemma 3.4 imply

$$w(H) \leq \begin{cases} 
(g^2 - 5g + 10)/2 & \text{if } \rho(H) \leq c \\
\min\{(g^2 - 5g + 10)/2, (g - 1)g/3\} & \text{if } c < \rho(H) \leq (g - 3)/2 \\
(g - 1)g/3 & \text{if } \rho(H) > (g - 3)/2.
\end{cases}$$

376
**Proof of Theorem 3.6.** (i) ⇒ (ii) follows from Lemma 3.2. (ii) ⇒ (iii) follows from the hypothesis on \( g \).

(iii) ⇒ (i): By Theorem 3.5 we have that \( H(P) \) is \( t \)-hyperelliptic for some \( t \in \{0, \ldots, \gamma\} \). Then by Lemma 3.2 and hypothesis we have

\[
\frac{(g - 2\gamma + 2)}{2} > w(H) \geq \frac{(g - 2t)}{2},
\]

from where it follows that \( t = \gamma \).

**Remark 3.8.** The hypothesis on \( g \) in the above two theorems is sharp:

(i) Let \( \gamma \geq 7 \) and consider \( H := \langle 4, 4(\gamma + 1), 2g - 4(\gamma + 1) + 1 \rangle \) where \( g \) is an integer satisfying \( \max\{4\gamma + 4, \frac{12 + 6\gamma - 3}{2}\} < g \leq \gamma^2 + 4\gamma + 2 \). Then \( H \) has genus \( g \) and \( \rho(H) = \gamma + 1 \). In particular \( H \) is not \( \gamma \)-hyperelliptic. By Lemma 3.2 (ii) we have \( w(H) = \left(g - 2(\gamma + 1)\right) + 2(\gamma + 1)^2 \). Now it is easy to check that \( w(H) \) satisfies Theorem 3.5 (ii) and Theorem 3.6 (iii).

(ii) Let \( \gamma \leq 6 \) and consider \( H := \langle 3, g + 1 \rangle \), where \( g = 10, 22, 33, 43, 55, 64 \) if \( \gamma = 1, 2, 3, 4, 5, 6 \) respectively. \( H \) has genus \( g \) and it can be easily checked that \( H \) is not \( \gamma \)-hyperelliptic by means of inequality (7) and Lemma 1.1. Moreover \( w(H) = g(g - 1)/3 \) (see Remark 3.3 (ii)). Now it is easy to check that \( w(H) \) satisfies Theorem 3.5 (ii) and Theorem 3.5 (iii).

(iii) The semigroups considered in (i) and (ii) are also Weierstrass semigroups (see Komeda [15], Maclachlan [18, Thm. 4]).

### 3.3. Weierstrass weights

In this section we apply Theorem 3.6 in order to characterize double coverings of curves by means of weights of Weierstrass semigroups. Specifically we strengthen [23, Theorem B] and hence all its corollaries. The basic references for the discussion below are Parkas-Kra [7, III.5] and Stöhr-Voloch [22, §1].

Let \( X \) be a non-singular, irreducible, projective and non-hyperelliptic algebraic curve of genus \( g \) over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). Let \( \pi : X \to \mathbb{P}^{g-1} \) be the morphism induced by the canonical linear system on \( X \). To any \( P \in X \) we can associate the sequence \( j_i(P) \) \( (i = 0, \ldots, g - 1) \) of intersection multiplicities at \( \pi(P) \) of \( \pi(X) \) with hyperplanes of \( \mathbb{P}^{g-1} \). This sequence is the same for all but finitely many points (the so called Weierstrass points of \( X \)). These points are supported by a divisor \( W \) in such a way that the Weierstrass weight at \( P \), \( v_P(W) \), satisfies

\[
v_P(W) \geq w(P) := \sum_{i=1}^{g-1} (j_i(P) - \epsilon_i),
\]

where \( 0 = \epsilon_0 < \ldots < \epsilon_{g-1} \) is the sequence at a generic point. One has \( j_i(P) \geq \epsilon_i \) for each \( i \), and from the Riemann-Roch theorem it follows that \( G(P) := \{j_i(P) + 1 : i = 0, \ldots, g - 1\} \) is the set of gaps of a semigroup \( H(P) \) of genus \( g \) (the so called Weierstrass semigroup at \( P \)). Moreover, we have

\[
v_P(W) = \square(P) \quad (\ast)
\]

if the following condition holds:

\[
\det\left(\begin{pmatrix} j_i(P) \\ \epsilon_j \end{pmatrix}\right) \neq 0 \pmod{p}.
\]
$X$ is called \textit{classical} if $\varepsilon_i = i$ for each $i$. This is the case if $p = 0$ or $p > 2g - 2$, and here we have $(\ast)$ for each $P$. For a classical curve, the number $w(P)$ is just the weight $w(H(P))$ of the semigroup $H(P)$ defined in §0. The following result strengthen [23, Thm.B]. The proof follows from [23, Thm.A], [24, Thm.A], and Theorem 3.6.

\textbf{Theorem 3.9.} Let $X$ be a curve whose genus $g$ satisfies (2). Then the following statements are equivalent:

(i) $X$ is a double covering of a curve of genus $\gamma$.

(ii) There exists $P \in X$ such that

\[ \binom{g - 2\gamma}{2} \leq w(H(P)) \leq \binom{g - 2\gamma}{2} + 2\gamma^2. \]

(iii) There exists $P \in X$ such that

\[ \binom{g - 2\gamma}{2} \leq w(H(P)) < \binom{g - 2\gamma + 2}{2}. \]

Remark 3.8 says that the bound for $g$ above is the best possible. Further applications of §3.1 and §3.2 will be published elsewhere [25].

\textbf{References}


TORRES


Fachbereich 6
Mathematik und Informatik
Universität GH Essen
D - 45117 Essen, Germany
Email: fernando.torres@uni-essen.de

Received January 9, 1996
and in final form July 5, 1996