ON THE WEIGHT OF NUMERICAL SEMIGROUPS

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Abstract. We investigate the weights of a family of numerical semigroups by means of even gaps and the Weierstrass property for such a family.

1. Introduction

Let \( \mathcal{X} \) be a (non-singular, projective, irreducible) curve of genus \( g \geq 2 \) defined over an algebraically closed field \( K \) of characteristic zero. The Weierstrass semigroup (or the non-gaps) at a point \( P \) is the set \( H(P) = \{ \ell_1 < \cdots < \ell_g \} \) \( (\ell_i = \ell_i(P)) \) are the gaps at \( P \). The Weierstrass gap theorem asserts that \( \ell_g \leq 2g - 1 \), see e.g. [4]. Let \( \mathcal{X} \) be a double covering of a curve \( \tilde{\mathcal{X}} \) of genus \( \gamma \); i.e., there exists a morphism \( \pi : \mathcal{X} \to \tilde{\mathcal{X}} \) of degree two. Assume that \( P \in \mathcal{X} \) is ramified (thus \( g \geq \gamma \)) and set \( \tilde{P} = \pi(P) \). If \( g \) is large enough with respect to \( \gamma \), properties of \( H(P) \) characterize the morphism \( \pi \) (see Theorem 1.1). To be more precise let us first recall that \( 2h \in H(P) \) iff \( h \in H(\tilde{P}) \) ([9]). Then \( H(P) \) has exactly \( \gamma \) even gaps which are contained in \( [2, 4\gamma] \) and thus \( \gamma \) odd non-gaps in \( [3, 2g - 1] \), say \( u_\gamma < \cdots < u_1 \) \((u_i = u_i(P))\). Set \( 2\tilde{H}(\tilde{P}) := \{ 2h : h \in H(\tilde{P}) \} \). Therefore the semigroup \( H(P) \) is of the form

\[
H(P) = 2\tilde{H}(\tilde{P}) \cup \{ u_\gamma, \ldots, u_1 \} \cup \{ 2g + i : i \in \mathbb{N}_0 \}.
\]

The Weierstrass weight at \( P \) is \( w(P) := \sum_{i=1}^{\gamma} (\ell_i - 1) \). The following result is our starting point (see Problems 1.2, 2.5, 2.8). Throughout this paper let \( F(\gamma)(*) \) be the function defined by \( F(0) = 2, F(1) = 11, F(2) = 23, F(3) = 34, F(4) = 44, F(5) = 56, F(6) = 65 \) and \( F(\gamma) = \gamma^2 + 4\gamma + 3 \) for \( \gamma \geq 7 \).

Theorem 1.1. ([10], [9], [8], [5], [24]) Let \( \gamma \geq 0 \) be an integer and \( \mathcal{X} \) a curve of genus \( g \geq F(\gamma) \). The following statements are equivalents:

(I) There exists a point \( P \) in \( \mathcal{X} \) such that the Weierstrass semigroup \( H(P) \) is of the form (1.1);

(II) There exists a point \( P \) in \( \mathcal{X} \) such that the Weierstrass weight satisfies

\[
\left( \frac{g - 2\gamma}{2} \right) \leq w(P) \leq \left( \frac{g - 2\gamma}{2} \right) + 2\gamma^2;
\]

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The curve $\mathcal{X}$ is a double covering of a curve of genus $\gamma$ (for short, we say that $\mathcal{X}$ is $\gamma$-hyperelliptic).

The following problem arises.

**Problem 1.2.** Find the true values of $w(P)$ in (1.2).

A point $P \in \mathcal{X}$ is called *Weierstrass* if $w(P) > 0$. This concept has an important role in the study of the geometry of curves (see [2] for a beautiful exposition on several topics on Weierstrass points). Let $W$ denote the set of Weierstrass points. By means of Hürwitz’s Wronskian method [6] it can be shown that $W$ is the support of a divisor $\mathcal{W}$ on $\mathcal{X}$ and that $w(P)$ is the multiplicity of $P$ in $\mathcal{W}$; moreover $\sum_P w(P) = (g - 1)g(g + 1)$. Hürwitz, among other things, was concerned about the following matters:

1. On the number of Weierstrass points of $\mathcal{X}$ (which have to do with bounds on weights);
2. Suppose that $\mathcal{X} \subseteq \mathbb{P}^{g-1}(K)$ is non-hyperelliptic and let $P \in \mathcal{X}$; since $(\ell_1(P) - 1, \ldots, \ell_g(P) - 1)$ is the sequence of all multiplicity of intersections of hyperplanes and $\mathcal{X}$ at $P$, it is natural to look for the sequence that can be appear in this way. This is equivalent to ask on arbitrary semigroups $H$ that occur as a Weierstrass semigroups; i.e. whether $H = H(P)$ for some $P \in \mathcal{X}$ (for short, we say that $H$ satisfies the *Weierstrass property*).

Counting Weierstrass points is important e.g. in getting bounds on the number of automorphisms of curves [6] or on the study of constellations of curves [22]. Question two was raised approximately in 1893; further historical accounts can be read in [3]. Long after that, in 1980 Büchweitz [1] showed that not every semigroup satisfies the Weierstrass property (see also [13] and [24]).

In Section 2 we investigate weights of arbitrary numerical semigroup by means of even gaps. In this context the analogue of Theorem 1.1 is Theorem 2.6; Problem 1.2 is related then to Problems 2.5, 2.8. Theorem 2.16 subsume several values of weights. In Section 3 we investigate the Weierstrass property of the semigroups arising in Section 2; here we are mainly concern with the cases $\gamma = 0, 1, 2, 3, 4$. A typical application of our results is for example the non-existence of a 2-hyperelliptic curve of genus $g \geq 23$ with a ramification point of weight $(g-4)+6$ (Example 3.3).

2. **On Problem 1.2**

In this section we consider Problem 1.2 above in the context of Numerical Semigroup Theory only (as was already mentioned in the introduction there are numerical semigroups that cannot be realized as Weierstrass semigroups).

Let $H = \{0 = m_0 < m_1 < m_2 < \cdots \}$ ($m_i = m_i(H)$) be an arbitrary (numerical) semigroup; i.e., $H$ is a subsemigroup of the nonnegative integers $\mathbb{N}_0$ such that its complement
is finite. Let $G(H) := \mathbb{N}_0 \setminus H = \{\ell_1 < \cdots < \ell_g\}$ ($\ell_i = \ell_i(H)$). Following geometrical settings we say that $m_1$, the elements of $H$, the elements of $G(H)$ and $g = g(H)$ are respectively the multiplicity, the non-gaps, the gaps and the genus of $H$. The ‘Weierstrass gap theorem’ is also true here; in particular, $m_{g+i} = 2g + i$ for all $i \in \mathbb{N}_0$ ([1], [18]). The number

$$w(H) := \sum_{i=1}^{g} (\ell_i - i) = \frac{1}{2} (3g^2 + g) - \sum_{i=1}^{g} m_i$$

is the weight of $H$. We say that $H$ is $\gamma$-hyperelliptic if it has exactly $\gamma$ even gaps. In this case, for $g \geq 2\gamma$, $H$ satisfies property (1.1), [7] (cf. [2]), [23]; i.e., there exists a unique semigroup $\tilde{H}$ of genus $\gamma$ and $\gamma$ odd numbers $3 \leq u_\gamma < \cdots < u_1 \leq 2g - 1$ ($u_i = u_i(H)$) such that

$$H = 2\tilde{H} \cup \{u_\gamma, \ldots, u_1\} \cup \{2g + i : i \in \mathbb{N}_0\}.$$ 

As a matter of fact, $\tilde{H} = \{h/2 : h \in H, h \equiv 0 \pmod{2}\}$. We say that $H$ is a double covering of $\tilde{H}$. Let $\tilde{H} = \{0 = \tilde{m}_0 < \tilde{m}_1 < \tilde{m}_2 < \cdots\}$. After some computations from (2.1) we have the following.

**Lemma 2.1.** Notation as above. For $g \geq 2\gamma$, the weight of a $\gamma$-hyperelliptic semigroup $H$ can be computed by any of the formulas below:

$$w(H) = \left(\frac{g - 2\gamma}{2}\right) + (2g + 2\gamma + 1)\gamma - \sum_{i=1}^{\gamma} (2\tilde{m}_i + u_i)$$

(2.2)

$$= \left(\frac{g - 2\gamma}{2}\right) + (2g - \gamma)\gamma - \sum_{i=1}^{\gamma} u_i + 2w(\tilde{H}).$$

**Corollary 2.2.** Let $\gamma \geq 0$ be an integer and $H_1$ and $H_2$ semigroups of genus $g \geq 2\gamma$ which are double coverings of a semigroup $\tilde{H}$ of genus $\gamma$. Then

$$w(H_1) = w(H_2) \quad \text{iff} \quad \sum_{i=1}^{\gamma} u_i(H_1) = \sum_{i=1}^{\gamma} u_i(H_2).$$

From (2.2) we have:

**Lemma 2.3.** ([25]) For a $\gamma$-hyperelliptic semigroup $H$ of genus $g \geq 2\gamma$, $w(H) \equiv (g - 2\gamma)^2$ (mod 2).

**Lemma 2.4.** ([23], [26]) Notation as above.

1. For each $i = 1, \ldots, \gamma$, $2\tilde{m}_i + u_i \geq 2g + 1$, especially we get $u_\gamma \geq 2g - 4\gamma + 1$. Thus if $g \geq 4\gamma$, then $m_i = 2\tilde{m}_i$ for $i = 1, \ldots, \gamma$.
2. Let $2\tilde{m}_1 \geq 4$. Then $u_1 + 2\tilde{m}_1 = 2g + 1$ iff $u_1 = 2g - 3$ and $2\tilde{m}_1 = 4$; in this case, for each $i = 1, \ldots, \gamma$, $u_i + 2\tilde{m}_i = 2g + 1$.
3. Let $u_\gamma = 2g - 2\gamma - 1$. If $u_1 = 2g - 3$, then for $i = 1, \ldots, \gamma$, $u_i = 2g - 2i - 1$. If $u_1 = 2g - 1$, let $k := \max\{i \in \{1, \ldots, \gamma - 1\} : u_i = 2g - 2i + 1\}$. Then $u_i = 2g - 2i + 1$ for $i = 1, \ldots, k$ and $u_i = 2g - 2i - 1$ for $i = k + 1, \ldots, \gamma$. 
(4) Let \( u_\gamma = 2g - 2\gamma - 3 \) and \( u_1 = 2g - 1 \). Let \( k := \max \{i \in \{1, \ldots, \gamma - 1\} : u_i = 2g - 2i + 1\} \) and \( s := \min \{i \in \{k + 1, \ldots, \gamma\} : u_i = 2g - 2i - 3\} \) (thus \( s \geq k + 1 \)). Then \( u_i = 2g - 2i + 1 \) for \( i = 1, \ldots, k \); \( u_i = 2g - 2i - 1 \) for \( i = k + 1, \ldots, s - 1 \) and \( u_i = 2g - 2i - 3 \) for \( i = s, \ldots, \gamma \).

Proof. (1) It follows from the fact that all the numbers in the sequence \( u_i < u_i + 2\tilde{m}_1 < \cdots < u_i + 2\tilde{m}_i \) are odd non-gaps of \( H \).

(2) Here the odd numbers \( u_\gamma < u_\gamma + 2\tilde{m}_1 < \cdots < u_2 + 2\tilde{m}_1 \) are the odd non-gaps of \( H \). Therefore \( u_{\gamma - 1} = u_\gamma + 2\tilde{m}_1, \ldots, u_1 = u_2 + 2\tilde{m}_1 \), so that \( u_\gamma + (\gamma - 1)2\tilde{m}_1 = u_1 \). We have \( u_1 \leq 2g - 1 \). If \( \tilde{m}_1 \geq 3 \), \( u_\gamma \leq 2g - 6\gamma - 5 \) which is a contradiction according to Item (1). Then the result follows.

(3) If \( u_1 = 2g - 3 \), clearly \( u_i = 2g - 2i - 1 \) for \( i = 1, \ldots, \gamma \). Let \( u_1 = 2g - 1 \) and \( k \) the number defined as above. Then \( u_{k+1} \leq 2g - 2k - 1 \). We claim that \( u_{k+1} \leq 2g - 2k - 3 \), otherwise \( u_{k+1} = 2g - 2k - 1 = 2g - 2(k + 1) + 1 \) a contradiction with the definition of \( k \). Now in the interval \( [2g - 2\gamma - 1, 2g - 2k - 3] \) there are \( \gamma - k \) odd numbers as well as \( \gamma - k \) odd non-gaps. The result now follows.

(4) Let \( k \) and \( s \) be the numbers defined above. Arguing as in Item (3), \( u_{k+1} \leq 2g - 2k - 3 \). In the interval \( [u_\gamma, u_s] \) there are \( \gamma + 1 - s \) odd numbers and the same number of odd non-gaps. Thus \( u_i = 2g - 2i - 3 \) for \( i = s, \ldots, \gamma \). If \( s = k + 1 \) the result is clear; otherwise, \( u_{s-1} \geq 2g - 2s - 1 \) and thus \( u_{s-1} \geq 2g - 2s + 1 \) by the definition of \( s \). In the interval \( [2g - 2s + 1, 2g - 2k - 3] \) we have both \( s - (k + 1) \) odd numbers and odd non-gaps and the proof is complete. \( \square \)

The analogous of Theorem 1.1 is formulated as follows. Recall that \( F(\gamma) \) is the function defined by (\( * \)) in the introduction. Let \( \gamma \geq 0 \) and \( g \) be integers. Let \( H \) be a semigroup of genus \( g \geq 2\gamma \). Consider the following statements:

(I) \( H \) is a \( \gamma \)-hyperelliptic;

(II) The weight \( w(H) \) satisfies

\[
(\frac{g - 2\gamma}{2}) \leq w(H) \leq (\frac{g - 2\gamma}{2}) + 2\gamma^2.
\]

Then (I) implies (II) which follows from Lemmas 2.1 and 2.4(1).

Problem 2.5. Find the true values of weights of \( \gamma \)-hyperelliptic semigroups \( H \) (of genus \( g \geq 2\gamma \)) in (2.3).

Concerning the converse we have the following.

Theorem 2.6. ([23]) (II) implies (I) provided that \( g \geq F(\gamma) \).

Thus if we impose the condition \( g \geq F(\gamma) \) in Problem 2.5, then all the possible value attained in (2.3) will be from \( \gamma \)-hyperelliptic semigroups only.
Remark 2.7. The bound $F(\gamma)$ on $g$ is sharp [5], [23].

Problem 2.8. Which semigroups satisfying (2.3) are Weierstrass?

The border cases in (2.3) follow from Lemma 2.4 (1) (2).

Proposition 2.9. ([23]) Let $\gamma \geq 0$ be an integer and $H$ a semigroup of genus $g \geq 2\gamma$. Then:

1. $w(H) = \binom{g}{2}$ iff $2 \in H$;
2. Let $\gamma \geq 1$. Then $w(H) = \binom{g-2\gamma}{2}$ iff $m_i = 2\gamma + 2i$ and $u_i = 2g - 2i + 1$ for $i = 1, \ldots, \gamma$;
3. Let $\gamma \geq 1$. Then $w(H) = \binom{g-2\gamma}{2} + 2\gamma^2$ iff $m_1 = 4$ and $u_1 = 2g - 3$. In this case, $H = \langle 4, 4\gamma + 2, 2g - 4\gamma + 1 \rangle$.

Next we compute the weights of semigroups of multiplicity 4 and 6.

Proposition 2.10. ([26]) Let $\gamma \geq 1$ be an integer and $H$ a $\gamma$-hyperelliptic semigroup of genus $g \geq 3\gamma$ of multiplicity $m_1 = 4$. Then

1. $w(H) \in \{ \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k : k = 1, \ldots, \gamma + 1 \}$;
2. $w(H) = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k$ iff $H = \langle 4, 4\gamma + 2, 2g - 2\gamma - 2k + 3, 2g - 2\gamma + 2k + 1 \rangle$.

Proof. (1) By Proposition 2.9 (3) we shall assume $k \leq \gamma$. We have to show that $w(H) \notin \{ \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k : k = 1, \ldots, \gamma \}$ gives a contradiction. We have that $u_1 \geq 2g - 3$ (as $m_1 = 4$) and thus the aforementioned result allows to define the integer $j := \max\{ i \in \{ 1, \ldots, \gamma - 1 \} : u_i = 2g - 2i + 1 \}$. Now, the weight of a hyperelliptic semigroup of genus $\gamma$ is $\gamma^2 - \gamma$; then $w(H) \geq \binom{g-2\gamma}{2} + \gamma^2 - \gamma$ by (2.2). It follows that

$$w(H) = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k + 2n$$

for some integer $n \in \{ 1, \ldots, k - 1 \}$; finally Lemma 2.1 implies

$$j^2 - (2\gamma + 1)j + (\gamma^2 + \gamma - k^2 + k - 2n) = 0,$$

which contradicts the fact that $j$ is an integer.

(2) The case $k = \gamma + 1$ is just Proposition 2.9 (3). Let now $k \in \{ 1, \ldots, \gamma \}$ and $w(H) = \binom{g-2\gamma}{2} + \gamma^2 - \gamma + k^2 - k$. From Proposition 2.9 (3) there is $j = \max\{ i \in \{ 1, \ldots, \gamma \} : u_i = 2g - 2i + 1 \}$, therefore if $j < \gamma$ we have $u_{j+l} = 2g - 2j - 4l + 1$ for each $l = 1, \ldots, \gamma - j$. And so from Lemma 2.1 we have

$$j^2 - (2\gamma + 1)j + (\gamma^2 + \gamma - k^2 + k) = 0.$$ 

Therefore, $j = \gamma - k + 1$. In particular, $u_{\gamma} = 2g - 2\gamma - 2k + 3$ and $u_{\gamma-k} = 2g - 2\gamma + 2k + 1$. So $H_k := \langle 4, 4\gamma + 2, 2g - 2\gamma - 2k + 3, 2g - 2\gamma + 2k + 1 \rangle \subseteq H$. Since $g \geq 3\gamma$ it follows that $H_k$ is a $\gamma$-hyperelliptic semigroup. Let $4e_2 + 2, 4e_1 + 1, 4e_3 + 3$ be the smallest integers in
A semigroup \( \tilde{H} \) of genus \( \gamma \) of multiplicity \( \tilde{m}_1 = 3 \) has the following form and weight [8].

For each \( k = 0, 1, \ldots, [\gamma/3] \),

1. If \( \gamma \equiv 0 \) (mod 3), \( \tilde{H} = \tilde{H}_k = \{3i : i = 1, \ldots, 2\gamma/3\} \cup \{\gamma - 2 + 3k + 3s : s = 1, \ldots, \gamma/3 - k\} \cup \{2\gamma - 1 + 3s - 3k : s = 1, \ldots, k\} \cup \{2\gamma + i : i \geq 0\} \) and \( w(\tilde{H}) = \gamma(\gamma - 1)/3 + 3k^2 - k\gamma - k \);

2. If \( \gamma \equiv 1 \) (mod 3), \( \tilde{H} = \tilde{H}_k = \{3i : i = 1, \ldots, (2\gamma - 1)/3\} \cup \{\gamma - 2 + 3k + 3s : s = 1, \ldots, (\gamma + 2)/3 - k\} \cup \{2\gamma - 1 + 3s - 3k : s = 1, \ldots, k\} \cup \{2\gamma + i : i \geq 0\} \) and \( w(\tilde{H}) = \gamma(\gamma - 1)/3 + 3k^2 - k\gamma - k \);

3. If \( \gamma \equiv 2 \) (mod 3), \( \tilde{H} = \tilde{H}_k = \{3i : i = 1, \ldots, (2\gamma - 1)/3\} \cup \{\gamma - 1 + 3k + 3s : s = 1, \ldots, (\gamma + 1)/3 - k\} \cup \{2\gamma - 2 + 3s - 3k : s = 1, \ldots, k\} \cup \{2\gamma + i : i \geq 0\} \) and \( w(\tilde{H}) = \gamma(\gamma - 2)/3 + 3k^2 - k\gamma + k \).

Therefore a direct computation via (2.2) shows the following.

**Proposition 2.11.** Let \( \gamma \geq 1 \) be an integer and \( H \) a \( \gamma \)-hyperelliptic semigroup of genus \( g \geq 2\gamma \) of multiplicity \( m_1 = 6 \). Let \( u_\gamma < \cdots < u_1 \) be the odd non-gaps of \( H \). Then

\[
w(H) = \left( \frac{g - 2\gamma}{2} \right) + 2\gamma g - \sum_{i=1}^{\gamma} u_i + I_k(\gamma),
\]

where \( k = 0, 1, \ldots, [\gamma/3] \), \( I_k(\gamma) = 2(3k^2 - k\gamma - k) - \gamma(\gamma + 2)/3 \) if \( \gamma \equiv 0, 1 \) (mod 3) and \( I_k(\gamma) = 2(3k^2 - k\gamma + k) - \gamma(\gamma + 4)/3 \) otherwise.

We obtain an improvement on (2.3), namely:

**Proposition 2.12.** ([26]) Let \( \gamma \geq 1 \) be an integer and \( H \) a \( \gamma \)-hyperelliptic semigroup of genus \( g \).

1. If \( g \geq 3\gamma \) and \( m_1 = 4 \), then either \( w(H) = \left( \frac{g - 2\gamma}{2} \right) + 2\gamma^2 \), \( w(H) = \left( \frac{g - 2\gamma}{2} \right) \) or

\[
\left( \frac{g - 2\gamma}{2} \right) + \gamma^2 - \gamma \leq w(H) \leq \left( \frac{g - 2\gamma}{2} \right) + 2(\gamma^2 - \gamma);
\]

2. If \( g \geq 2\gamma \) and \( m_1 \geq 6 \), then \( w(H) \leq \left( \frac{g - 2\gamma}{2} \right) + 2\gamma^2 - (2\gamma - 4) \).

**Proof.** (1) It follows from Proposition 2.10.

(2) We have \( \tilde{m}_i \geq 2i + 1 \) for \( i = 1, \ldots, \gamma - 2 \), \( \tilde{m}_{\gamma - 1} \geq 2(\gamma - 1) \) ([18]) and \( u_i \geq 2\gamma - 4i + 1 \) for \( i = 1, \ldots, \gamma \) ([23]). Thus \( \sum_{i=1}^{\gamma}(2\tilde{m}_i + u_i) \geq 2g\gamma + 3\gamma - 4 \). Now the inequality follows from (2.2). \( \square \)

**Proposition 2.13.** Let \( H \) and \( \tilde{H} \) be semigroups of genus \( g \) and \( \gamma \) respectively with \( g \geq 2\gamma \). Suppose that \( H \) is a double covering of \( \tilde{H} \).
(1) If \(u_\gamma = u_\gamma(H) = 2g - 2\gamma + 1\), then \(w(H) = \binom{g-2\gamma}{2} + 2w(\tilde{H})\).

(2) If \(u_\gamma = u_\gamma(H) = 2g - 4\gamma + 1\), then \(w(H) = \binom{g-2\gamma}{2} + 2\gamma + 4w(\tilde{H})\).

Proof. In both cases the odd non-gaps are determined by \(u_\gamma\). We have respectively \(u_i = u_\gamma + 2\tilde{m}_i\) (\(i = 1, \ldots, \gamma - 1\)) and \(u_i = 2g - 2i + 1\) (\(i = 1, \ldots, \gamma\)); now the result follows from (2.2).

Proposition 2.14. Let \(H\) be a \(\gamma\)-hyperelliptic semigroup of genus \(g \geq 2\gamma\) and multiplicity \(m_1 = 2\gamma + 2\). Suppose in addition that \(u_\gamma = 2g - 2\gamma - 1\). Let \(k\) be the integer defined in Lemma 2.4 (3). (For the case \(u_1 = 2g - 3\), we let \(k = 0\)). Then

\[w(H) = \binom{g-2\gamma}{2} + 2(\gamma - k).\]

Proof. \(H\) is a double covering of a semigroup of multiplicity \(\gamma + 1\) and the result follows from (2.2) and Lemma 2.4 (3).

In a similar way we obtain:

Proposition 2.15. Let \(H\) be a \(\gamma\)-hyperelliptic semigroup of genus \(g \geq 2\gamma\) and multiplicity \(m_1 = 2\gamma + 2\). Suppose in addition that \(u_\gamma = 2g - 2\gamma - 3\). Let \(k\) and \(s\) be the numbers defined in Lemma 2.4 (4). Then

\[w(H) = \binom{g-2\gamma}{2} + 2(2\gamma - k - s + 1).\]

Proof. \(H\) is a double covering of a semigroup of multiplicity \(\gamma + 1\) and \(m_1 + u_\gamma = 2g - 1\). In particular \(w(\tilde{H}) = 0\) and \(u_1 = 2g - 1\). Therefore from (2.2) and Lemma 2.4 (4) we have

\[w(H) = \binom{g-2\gamma}{2} + (2g - \gamma)\gamma - \sum_{i=1}^{\gamma} u_i\]
\[= \binom{g-2\gamma}{2} + (2g - \gamma)\gamma - \sum_{i=1}^{\gamma} (2g - 2i) - k + (\gamma - s - k) + 3s\]
\[= \binom{g-2\gamma}{2} + 2(2\gamma - k - s + 1).\]

We subsume next some values in (2.3) obtained so far. For \(H\) a \(\gamma\)-hyperelliptic semigroup of genus \(g \geq 2\gamma\) set \(D(H) := w(H) - \binom{g-2\gamma}{2}\).

Theorem 2.16. Notation as above. The function \(D(H)\) attains the following values:

1. \(D(H) = 2k\) for \(k = 0, 1, \ldots, 2\gamma\);
2. \(D(H) = 2\gamma + 4k\) for \(k = 1, \ldots, \gamma - 1\);
3. If further \(g \geq 3\gamma\), \(D(H) = 2\gamma^2 - (\gamma + k)(\gamma + 1 - k)\) for \(k = 1, \ldots, \gamma + 1\).
Proof. Let $k$ be an integer with $0 \leq k \leq \gamma - 1$. Let $\tilde{H}_k$ be the semigroup of genus $\gamma$ with gaps $1, \ldots, \gamma - 1$ and $\gamma + k$ so that $w(\tilde{H}_k) = k$. Thus Proposition 2.13 (or Proposition 2.14) implies Item (1) for $k = 0, \ldots, \gamma$. From these $\tilde{H}_k$’s, Item (2) follows. Item (1) for $k = \gamma + 1, \ldots, 2\gamma$ follows from Proposition 2.15. Item (3) follows from Proposition 2.10.

\[ \square \]

3. ON THE WEIERSTRASS PROPERTY

In this section we will find out semigroups $H$ satisfying both (2.3) and the Weierstrass property. Let $\gamma \geq 0$ and $g \geq 2\gamma$ be integers. For such semigroups recall that $D(H) = w(H) - \left(\frac{g - 2\gamma}{2}\right)$ and that $F(\gamma)$ is the function defined in the introduction via ($\ast$). Let $S(H) := \{2g + i : i \in \mathbb{N}_0\}$.

Example 3.1. Let $\gamma = 0$. Then $D(H) = 0$. This case is only possible if $2 \in H$ and the semigroup is Weierstrass for $g \geq F(0)$; see e.g. [4].

Example 3.2. Let $\gamma = 1$. From Lemma 2.3 $D(H) = 0$ or $D(H) = 2$. By Proposition 2.9 the former case occurs iff $m_1 = 4$ and $u_1 = 2g - 1$ and the second iff $m_1 = 4$ and $u_1 = 2g - 3$. In both cases $H$ is Weierstrass [17].

Example 3.3. Let $\gamma = 2$ and assume $g \geq F(2)$. From Lemma 2.3, $D(H) \in \{2i : i = 0, 1, 2, 3, 4\}$. We shall show that $D(H) \neq 6$. Here we have $m_1 \in \{4, 6\}$. If $m_1 = 4$, $D(H) = 8$ or $D(H) \leq 4$ by Proposition 2.12(1). Let $m_1 = 6$. Then $H$ is a double covering of the semigroup $\tilde{H} = \{0, 3, 4, 5, \ldots\}$ and hence $D(H) = 4g - 4 - u_2 - u_1$ by (2.2). We know that $u_2 \geq 2g - 7$ (Lemma 2.4 (1)). Thus $u_2 + u_1 \geq 4g - 8$ and $D(H) \leq 4$.

All the values $D(H) = 0, 2, 4, 8$ occur only if $H$ is 2-hyperelliptic (Theorem 2.6). As a matter of fact all these semigroups are Weierstrass [17], [5], [19]. We observe that if $\tilde{H} = \{0, 3, 4, 5, \ldots\}$, $H_1 := 2\tilde{H} \cup \{2g - 7, 2g - 1\} \cup S(H)$ and $H_2 := 2\tilde{H} \cup \{2g - 5, 2g - 3\} \cup S(H)$ then $w(H_1) = w(H_2)$ by Corollary 2.2.

Example 3.4. Any semigroup of multiplicity 4 is Weierstrass [17]. Thus all the values in Theorem 2.16(3) arise from Weierstrass semigroups.

Example 3.5. If $m_1 = 2\gamma + 2$ and $u_\gamma = 2g - 2\gamma + 1$, then $w(H) = \left(\frac{g - 2\gamma}{2}\right)$ (Proposition 2.13 (1)). By using the theory of Fuchsian groups and Lewittes’ theorem on fixed points of automorphisms, it can be shown that $H$ is Weierstrass [21]. Next we give a direct proof for $\gamma = 3$. We use ideas from [19]. Thus at least for $g \geq 11$ we have to show that $H = 2(4, 5, 6, 7) \cup \{2g - 5, 2g - 3, 2g - 1\} \cup S(H)$ is Weierstrass.

Let $g \geq 11$ and $0 \leq r \leq 7$ be integers such that $g + r$ is odd. Let $a(x)$ be a degree four polynomial in $K[x]$ and let $b$ be a constant such that the roots of $a(x) := ax^2 - b^2$, say $a_1, \ldots, a_8$, are pairwise different. Let $b_1, \ldots, b_{g-10} \in K \setminus \{a_1, \ldots, a_8\}$ pairwise different.

We consider the curves $X$ and $X_r$ defined respectively by

\[ y^2 = a(x), \quad z^4 = a(x)(x - a_1)^2 \cdots (x - a_r)^2(x - b_1)^2 \cdots (x - b_{g-10})^2. \]
Then \( \mathcal{X}_r \) is a double covering of \( \mathcal{X} \). There are two points \( R_\infty, R'_\infty \in \mathcal{X} \) over \( x = \infty \); since \( \gcd(4, 2r + 2g - 12) = 2 \), there exist just two points \( S_\infty \) and \( S'_\infty \) in \( \mathcal{X}_r \) over \( R_\infty \) and \( R'_\infty \) respectively. We shall show that \( H(S_\infty) = H \). Observe that \( H(R_\infty) = \langle 4, 5, 6, 7 \rangle \) and thus the even non-gaps of \( H(S_\infty) \) are \( \{0, 8, 10, 12, 14\ldots\} \).

**Claim 1.** By applying the Riemann-Hurwitz formula it follows that the genus of \( \mathcal{X}_r \) is \( g \).

**Claim 2.** The numbers \( 2g - 5, 2g - 3, 2g - 1 \) belong to \( H(S_\infty) \).

**Proof.** (Claim 2) We compute some divisors (cf. [19]). Let \( P_i \in \mathcal{X}_r \) be the unique point over \( x = a_j \) and \( Q_i, Q'_i \) the two points in \( \mathcal{X}_r \) over \( x = b_i \). We permute the roots of \( a(x) \) such way that \( \alpha(a_j) = b \) for \( j = 1, 2, 3, 4 \) and we write \( f := y - \alpha(x) + b \). Then

- \( \text{div}(x - a_j) = 4P_i - 2S_\infty - 2S'_\infty \);
- \( \text{div}(x - b_i) = 2Q_i + 2Q'_i - 2S_\infty - 2S'_\infty \);
- \( \text{div}(y) = 2P_i + \cdots + 2P_8 - 8S_\infty - 8S'_\infty \);
- \( \text{div}(z) = 3P_1 + \cdots + 3P_r + P_{r+1} + \cdots + P_8 + (Q_1 + Q'_1) + \cdots + (Q_{g-10} + Q'_{g-10}) - (g + r - 6)(S_\infty + S'_\infty) \);
- \( \text{div}(y + \alpha(x)) = 8S'_\infty - 8S_\infty \);
- \( \text{div}(f) = 2P_1 + \cdots + 2P_4 - 8S'_\infty \).

We choose \( r \) such that \( g + r \equiv 7 \pmod{8} \). Let \( h := z(y + \alpha(x))^n \) where \( n = (g + r + 1)/8 \) if \( r \leq 4 \) and \( n = (g + r - 7)/8 \) if \( r \geq 5 \). Then \( \text{div}_\infty(h) = (g + r - 6 + 8n)S_\infty \) iff \( 8n - (g + r - 6) \geq 0 \). For each such integer \( r \) we shall give in the table bellow functions \( h_1, h_2 \) and \( h_3 \) such that their pole divisors are respectively \( (2g - 5)S_\infty, (2g - 3)S_\infty \) and \( (2g - 1)S_\infty \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( h )</td>
<td>( (x - a_1)h )</td>
<td>( (x - a_1)^2h )</td>
</tr>
<tr>
<td>1</td>
<td>( fh/(x - a_1) )</td>
<td>( h )</td>
<td>( (x - a_1)h )</td>
</tr>
<tr>
<td>2</td>
<td>( fh/(x - a_1)(x - a_2) )</td>
<td>( (x - a_2)h_1 )</td>
<td>( h )</td>
</tr>
<tr>
<td>3</td>
<td>( fh/(x - a_1)(x - a_2)(x - a_3) )</td>
<td>( (x - a_3)h_1 )</td>
<td>( (x - a_2)h_2 )</td>
</tr>
<tr>
<td>4</td>
<td>( fh/(x - a_1)(x - a_2)(x - a_3)(x - a_4) )</td>
<td>( (x - a_4)h_1 )</td>
<td>( (x - a_3)h_2 )</td>
</tr>
<tr>
<td>5</td>
<td>( (x - a_6)(x - a_7)(x - a_8)h/y )</td>
<td>( h/f )</td>
<td>( (x - a_5)h_2 )</td>
</tr>
<tr>
<td>6</td>
<td>( (x - a_7)(x - a_8)h/y )</td>
<td>( h/f )</td>
<td>( (x - a_6)h_2 )</td>
</tr>
<tr>
<td>7</td>
<td>( (x - a_7)(x - a_8)h/y )</td>
<td>( (x - a_7)h_1 )</td>
<td>( (x - a_6)h_2 )</td>
</tr>
</tbody>
</table>
where $A(x) := (x-a_1) \cdot \cdots \cdot (x-a_r)(x-b_1) \cdot \cdots \cdot (x-b_{(g-9)/2})$. Then $\mathcal{X}_r$ is a double covering of $\mathcal{X}$ and let $D_\infty$ be the pole divisor of the function $x$ on $\mathcal{X}_r$. Let $P_i \in \mathcal{X}_r$ be the unique point over $x = a_i$ and $Q_1^i, Q_1'^i, Q_1''^i, Q_1'''^i$ the four different points over $x = b_i$. We shall show that $H(P_1) = H$. As a matter of fact the even non-gaps of $H(P_1)$ are precisely the set $2(3, 4)$, because the unique point $p_1$ of $\mathcal{X}$ over $x = a_1$ is a hyperflex point on the non-singular curve $\mathcal{X}$. It is not difficult to see that:

- $\text{div}(x-a_1) = 8P_1 - D_\infty$;
- $\text{div}(y) = 2(P_1 + P_2 + P_3 + P_4) - D_\infty$;
- $\text{div}(z) = 5P_1 + \cdots + 5P_r + P_{r+1} + \cdots + P_4 + (Q_1 + Q_1' + Q_1'' + Q_1''') + \cdots + (Q_{(g-9)/2} + Q_{(g-9)/2}' + Q_{(g-9)/2}'') - \frac{g+2r-7}{4}D_\infty$.

Let $F_1 := z/(x-a_1)^a$. Then $\text{div}_\infty(F_1) = (8n-5)P_1$ if $2n - \frac{g+2r-7}{2} \geq 0$. First, we consider the case $g \equiv 1 \pmod{4}$. We set $n = (g-1)/4$. Here $r = 2$ implies $2g - 7 \in H(P_1)$. With $F_2 = (x-a_3)(x-a_4)F_1/y^2$ and $F_3 = yF_1(x-a_1)$ we find that $2g - 3$ and $2g - 1$ belong to $H(P_1)$ respectively. In the case $g \equiv 3 \pmod{4}$ we set $n = (g-3)/4$. Then the proof with $r = 1$ similar to that of the case $g \equiv 1 \pmod{4}$ with $r = 2$ works well. Notice that (2.2) implies $D(H) = 6$.

**Example 3.7.** Let us consider a $\gamma$-hyperelliptic semigroup of genus $g$ with $n := u_\gamma = 2g - 4\gamma + 1$. Suppose that $H$ is the double covering of $\tilde{H} = \{0, m_1, \ldots, m_\gamma = 2\gamma, 2\gamma + 1, \ldots\}$. Thus $u_{\gamma-1}(H) = n + 2m_1, \ldots, u_1(H) = n + 2m_{\gamma-1}$ so that $H = 2\tilde{H} + n\mathbb{N}_0$. Let $c(\tilde{H})$ denote the conductor of $\tilde{H}$; i.e., the least integer c such that $c + h \in \tilde{H}$ for all $h \in \mathbb{N}_0$. We look for the hypotheses in [11, Thm. 2.2], namely $n \geq 2c(\tilde{H}) - 1$ and $n \neq 2m_1 - 1$. Since $c(\tilde{H}) \leq 2\gamma$, both conditions are satisfied for $g \geq 4\gamma$. Therefore if $\tilde{H}$ is Weierstrass so $H$ does, loc. cit. There are several sufficient conditions in order that $\tilde{H}$ be Weierstrass; e.g.,

- $m_1(\tilde{H}) \leq 5 ([20], [17], [15])$;
- $g(\tilde{H}) \leq 8 ([14], [12])$;
- Either $w(\tilde{H}) \leq g(\tilde{H})/2$ or $g(\tilde{H})/2 < w(\tilde{H}) \leq g(\tilde{H}) - 1$ and $2m_1(\tilde{H}) > \ell_g(\tilde{H}) ([3], [16])$.

**Example 3.8.** Let us consider further values of $D(H)$ in Theorem 2.6. We let $\tilde{H}$ be a $\rho$-hyperelliptic of genus $\gamma$. For example for $\rho = 1$, $\gamma \geq 3$, let $w(\tilde{H}) = (g-2)^2, (g-2)^2 + 2$; thus the example above and Proposition 2.13 (2) show that there are Weierstrass semigroups $H$ with $D(H) = 2\gamma^2 - (8\gamma - 12), 2\gamma^2 - (8\gamma - 20)$.

**Example 3.9.** Let us consider the case $\gamma = 3, g \geq F(3)$. Arguing as in Example 3.3 ($\gamma = 2$) we can rule out the possibility $D(H) = 16$. Thus by Lemma 2.3 $D(H) \in \{2i : i = 0, \ldots, 9\} \setminus \{16\}$. We have the following table:
Example 3.10. Let $\gamma = 4$ and $g \geq F(4)$. From Lemma 2.3 and Proposition 2.12 $D(H) \in \{2i : i = 0, 1, \ldots, 16\} \setminus \{30\}$.

1. For $k = 1, 2, 3, 4, 5$, $D(H) = 32 - (4+k)(5-k) = 32, 24, 18, 14, 12$ by using a Weierstrass semigroup of multiplicity 4.

2. Let $u_4 = 2g - 15$ and $\tilde{H}$ a semigroup of genus 4 (which is always Weierstrass). Thus $H = 2\tilde{H} + u_4\mathbb{N}$ is Weierstrass (cf. Example 3.7) and $D(H) = 8 + 4w(\tilde{H})$ by Proposition 2.13 (2). Now $w(\tilde{H}) \leq 4(4 - 1)/3$ ([8]) and in fact there exists $\tilde{H}$ such that $w(\tilde{H}) = 0, 1, 2, 3, 4$. Thus $D(H) = 24, 20, 16, 12, 8$ occur for $H$ a Weierstrass semigroup.

3. If $u_4 = 2g - 7$, via the $\tilde{H}$’s in (2), Proposition 2.13 (1) gives $D(H) = 8, 6, 4, 2, 0$.

4. If $m_1 = 10$ and $u_4 = 2g - 11$, Proposition 2.15 gives $D(H) = 12, 10, 8, 6, 4, 2, 0$.

In contrast to the case $\gamma = 3$, the following questions remain open: (a) Is there exists $H$ such that $D(H) = 28, 26, 22$? (b) Is any semigroup in Item 3 above Weierstrass?

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