Duffin-Kemmer-Petiau equation
on the quaternion field

Stefano DE LEO [a)]

Università di Lecce, Dipartimento di Fisica
Istituto di Fisica Nucleare, Sezione di Lecce
Lecce, 73100, ITALY

Abstract

We show that the Klein-Gordon equation on the quaternion field is equivalent to a pair of DKP equations. We shall also prove that this pair of DKP equations can be taken back to a pair of *new* KG equations. We shall emphasize the important difference between the standard and the *new* KG equations. We also present some qualitative arguments, concerning the possibility of interpreting *anomalous* solution, within a quaternion quantum field theory.

a) e-mail: *Deleos@Le.Infn.It*

1 Introduction

Working in first quantization, the free-particle Duffin[1]-Kemmer[2]-Petiau[3] equation can be written as:

$$\beta^\mu \partial_\mu \psi = m\psi \tag{1}$$

where the β^μ are 16×16 complex matrices satisfying the condition

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = -g^{\mu\nu}\beta^\lambda - g^{\lambda\nu}\beta^\mu. \tag{2}$$

This algebra splits into five (spin 0), ten (spin 1) and one-dimensional (trivial) representations[4].

Historically, the loss of interest in the DKP stems from the equivalence of the DKP equation to the KG and the Proca equations[5, 6], in addition to the greater algebraic complexity of the DKP formulation. We show how this equation leads to interesting results on the quaternion field.
Before doing it we must remember that a basic theorem[7] in the foundation of quantum mechanics states that a general quantum mechanical system can be represented as a vector space with scalar coefficients drawn from the real, the complex, the quaternion[8, 9, 10] or the octonion[11] fields (the scalar coefficients form a division algebra). We assume a non commutative (but associative) multiplication in this paper, and so our analysis applies to the quaternion field. We show (in the simplest case - spin 0) the non equivalence between the *minimal* DKP (spin 0) and the KG equation. In fact, we find that the KG equation is equivalent to a pair of DKP equations.

Quaternions are a generalization of the complex numbers

$$q = z_1 + jz_2 \tag{3}$$

$$(z_m \in \mathcal{C}(1,i) \quad m = 1,2)$$

with

$$i^2 = j^2 = k^2 = -1 \; ; \qquad ijk = -1 \,. \tag{4}$$

In our formalism the momentum operator is defined as

$$p^\mu = \partial^\mu | i \tag{5}$$

where a *bared* operator $A|b$ acts as follows upon the quaternion column matrix ϕ

$$(A|b)\phi \equiv A\phi b \ . \tag{6}$$

We anticipate that the only b term appearing in this formalism is i (together, of course, with the trivial identity). We emphasize that a characteristic of this formalism is the absolute need of a complex scalar product (CSP), defined in terms of the quaternion counterpart (QSP) by

$$CSP = \frac{1-i|i}{2}\, QSP \tag{7}$$

The CSP first introduced in 1984 by Horwitz and Biedenharn[12], in order to define consistently multiparticle quaternion states, has been then justified by papers on Dirac equation[13], representations of $U(1,q)$[14] and translations between Quaternion and Complex Quantum Mechanics[15].
However, we remember that a different approach is formulated by Adler who uses the QSP (his fundamental results are quoted in the recent book of ref.[9]) and by Morita[16] who uses complex quaternions (or *biquaternions*) which contain an additional commuting imaginary $\mathcal{I} = \sqrt{-1}$.
Let us now examine the simplest possible case (spin 0) for the DKP equation.
We start with the second order KG equation:

$$(\partial_\mu \partial^\mu + m^2)\varphi = 0 \tag{8}$$

which can be rewritten into the form of a first-order matrix differential equation

$$\beta^\mu \partial_\mu \psi = m\psi$$

with

$$\psi \;=\; \begin{pmatrix} \frac{\partial_t \varphi}{m} \\ \frac{\partial_x \varphi}{m} \\ \frac{\partial_y \varphi}{m} \\ \frac{\partial_z \varphi}{m} \\ \varphi \end{pmatrix} \tag{9}$$

and

$$\beta^0 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad \beta^1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

(10)

$$\beta^2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} \quad , \quad \beta^3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

By returning to the CSP we must note that 1 and j are two orthogonal solutions, therefore we have the appearance of all four standard Dirac free-particle solutions[13] notwithstanding the two-component structure of the wave function and two orthogonal solutions, corresponding to spin up and spin down, in the Schrödinger equation (*belated theoretical discovery of spin*). However we also find two solutions (for a given four-moment) for the KG equation with the result that in addition to the normal scalar, we discover an *anomalous* scalar[17]. Such doubling of solutions is also in the DKP equation but contrary to the KG equation, it is a matrix equation and we can therefore hope to reduce it on the quaternion field.

In the next Section we shall find the quaternion matrix which reduces the 5×5 complex matrix into two 3×3 quaternion matrix blocks. In the subsequent Section we study the pair of quaternion DKP equations and prove their equivalence. In Section IV we explicitly find the *new* KG equations corresponding to the pair of quaternion DKP equations. The physical significance of these results and the possible applications of the 'quaternion' relativistic equation with *anomalous* solution in quantum field theory are discussed in our conclusions in the final Section.

2 A *particular* quaternion reduction of the 5×5 complex β^μ matrices

In our precedent paper[15] we have defined a set of rules for translating from standard Quantum Mechanics (QM) to a particular version of Quaternion Quantum Mechanics (QQM). We must emphasize that this translation is limited to *even* complex matrices. How can we translate the 5×5 complex matrices of the DKP equation now?

We prove in this section with a *particular* reduction on the quaternion field that the 5×5 complex β^μ matrices split into two "overlapping" 3×3 quaternion matrix blocks. We have in this *particular* reduction the following relation between complex $[cmd]$ and quaternion $[qmd]$ matrix dimensions:

$$(2n + 1)\,[cmd] \quad \rightarrow \quad 2\,[overlapping\ blocks] \times (n+1)\,[qmd]\,. \qquad (11)$$

To demonstrate the relation (11) we start with the standard DKP equation and analyse the simplest case (spin 0). We thus work with 5×5 complex matrices. Since we use a complex geometry (CSP) we have normal ($\sim a + ib$) and anomalous ($\sim j(a + ib)$) solutions.

We shall find the explicit form of the matrix necessary to reduce the complex matrices on the quaternion field. Not to disturb our next considerations by a complicate mathematical language let us indicate with

$$
\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \qquad (12)
$$

the normal solutions and with

$$
\begin{pmatrix} j \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ j \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ \cdot \\ j \\ \cdot \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ j \\ \cdot \end{pmatrix}, \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ j \end{pmatrix} \qquad (13)
$$

the anomalous solutions.

In the next Section we shall explicitly find the solutions of the DKP equation (spin 0). We shall have only two normal and two anomalous solutions but for the considerations of this Section we can neglect it. By returning to normal and anomalous solutions (12), (13) we transform the first ones in the following column matrices

$$
\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \quad
\begin{pmatrix} j \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ j \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}
\tag{14}
$$

and the second ones in

$$
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ j \\ \cdot \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ j \end{pmatrix} , \quad
\begin{pmatrix} \cdot \\ \cdot \\ j \\ \cdot \\ \cdot \end{pmatrix}
\tag{15}
$$

by the matrix

$$
S \;=\; \begin{pmatrix}
a & ja & \cdot & \cdot & \cdot \\
\cdot & \cdot & a & ja & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 \\
-jd & d & \cdot & \cdot & \cdot \\
\cdot & \cdot & -jd & d & \cdot
\end{pmatrix}
\tag{16}
$$

where

$$
a = \frac{1 - i|i}{2} \quad , \quad d = \frac{1 + i|i}{2}
$$

are *generalized*[15] quaternions.

We have effected this transformation with the hope to find two DKP equation, one corresponding to the normal solutions (now representing by (14) and the other corresponding to the anomalous solutions (now representing by (15)).

By using the matrix (16) we can rewrite the DKP equation (1) in the following way

$$
(S\beta^\mu S^+ \partial_\mu - m)S\psi = 0 \;,
\tag{17}
$$

7

where

$$
S^+ = \begin{pmatrix} a & \cdot & \cdot & ja & \cdot \\ -jd & \cdot & \cdot & d & \cdot \\ \cdot & a & \cdot & \cdot & ja \\ \cdot & -jd & \cdot & \cdot & d \\ \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} \quad ,
$$

$$(ja)^+ = (j - ji \mid i)^+ = (j + k \mid i)^+ = -j + k \mid i = -jd \ , \ a^+ = a \ , \ d^+ = d \ ,$$

and

$$S^+ S = 1 \ .$$

The similarity transformation is therefore

$$S\beta^\mu_{ocm}S^+ = \beta^\mu_{nqm}$$

where

$$ocm = \text{old complex matrices}$$
$$nqm = \text{new quaternion matrices}$$

We are ready to give explicitly the new quaternion matrices obtained by the S matrix (this reduction is for us very important if we want kill the anomalous solution).

The new quaternion matrices are

$$
\beta^0_{nqm} = \begin{pmatrix} \cdot & \cdot & a & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ -a & \cdot & \cdot & -ja & \cdot \\ 0 & 0 & -jd & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad
\beta^1_{nqm} = \begin{pmatrix} \cdot & \cdot & ja & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ -jd & \cdot & \cdot & d & \cdot \\ 0 & 0 & d & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \end{pmatrix}
$$

$$
\beta^2_{nqm} = \begin{pmatrix} \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & a & 0 & 0 \\ \cdot & a & \cdot & \cdot & ja \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & -jd & \cdot & \cdot \end{pmatrix} \quad , \quad
\beta^3_{nqm} = \begin{pmatrix} \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & ja & 0 & 0 \\ \cdot & -jd & \cdot & \cdot & d \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & d & \cdot & \cdot \end{pmatrix}
$$

(18)

which we can split into

$$\beta^0 = \begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -a & \cdot & \cdot \end{pmatrix} \quad , \quad \beta^1 = j \begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -d & \cdot & \cdot \end{pmatrix}$$

(19)

$$\beta^2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & a & \cdot \end{pmatrix} \quad , \quad \beta^3 = j \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & -d & \cdot \end{pmatrix}$$

and

$$\tilde{\beta}^0 = -j \begin{pmatrix} \cdot & a & \cdot \\ d & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad , \quad \tilde{\beta}^1 = \begin{pmatrix} \cdot & d & \cdot \\ d & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

(20)

$$\tilde{\beta}^2 = j \begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -d & \cdot & \cdot \end{pmatrix} \quad , \quad \tilde{\beta}^3 = \begin{pmatrix} \cdot & \cdot & d \\ \cdot & \cdot & \cdot \\ d & \cdot & \cdot \end{pmatrix}$$

Now the β^μ matrices act on the normal solutions and the $\tilde{\beta}^\mu$ matrices on the anomalous solutions. Note that $a = \frac{1-i|i}{2}$, therefore the β^μ matrices (19) act on the third element of the column matrix by cancelling the j, k part. The form of the solution of the DKP equation with β^μ matrices is like

$$\begin{pmatrix} z_1 + j z_2 \\ z_3 + j z_4 \\ z_5 \end{pmatrix}$$

(21)

with

$$z_m \in \mathcal{C} \quad m = 1, 2, ..., 5 \ .$$

In the same way (now we must note that $d = \frac{1+i|i}{2}$) we can prove that the form of the solution of the DKP equation with $\tilde{\beta}^\mu$ matrices (20) is like

$$\begin{pmatrix} j \tilde{z}_5 \\ \tilde{z}_1 + j \tilde{z}_2 \\ \tilde{z}_3 + j \tilde{z}_4 \end{pmatrix}$$

(22)

with

$$\tilde{z}_m \in \mathcal{C} \quad m = 1, 2, ..., 5 \ .$$

9

We shall prove in the next Section that the solutions to the DKP equations can be written as

normal DKP equation

$$\psi_{normal} \quad = \quad \begin{pmatrix} \frac{(\partial_t + j\partial_x)\varphi}{m} \\ \frac{(\partial_y + j\partial_z)\varphi}{m} \\ \varphi \end{pmatrix} \tag{23}$$

anomalous DKP equation

$$\psi_{anomalous} \quad = \quad \begin{pmatrix} j\varphi \\ \frac{(\partial_t + j\partial_x)\varphi}{m} \\ \frac{(\partial_y + j\partial_z)\varphi}{m} \end{pmatrix} \tag{24}$$

with φ complex wave functions.

3   The quaternion DKP equation

We analyse the DKP equation (1) with the β^μ matrices given by (19). We can now proceed in the standard manner. The plane wave solutions are of the form (n.b. the ordering)

$$\psi(\vec{x}, t) = u(\vec{p}\,)e^{-ipx}$$

with the 'spinors' u satisfies the relation

$$(\beta^\mu p_\mu | i + m)u = 0 \,. \tag{25}$$

To simplify our consideration we start with $\vec{p} = 0$ and find the solutions to the eq.(25). We have

$$-\beta^0 p_0 u i = mu \,.$$

Explicitly

$$ap_0 \begin{pmatrix} -u_3 \\ 0 \\ u_1 \end{pmatrix} i \quad = \quad m \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \,. \tag{26}$$

Therefore if $p_0 = +m$ the plane wave solution is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} e^{-imt} \tag{27}$$

whereas if $p_0 = -m$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} +i \\ 0 \\ 1 \end{pmatrix} e^{+imt} \ . \tag{28}$$

Note that the normalization condition is now:

$$-\psi^+ \eta \beta^0 \psi i$$

with

$$\eta = -(2(\beta^0)^2 + 1)$$

(in fact we must remember that, for the DKP equation, the continuity equation is

$$-\partial_\mu (\psi^+ \eta \beta^\mu \psi i) = 0 \ ) \ .$$

Now we are ready to find the DKP solutions for $\vec{p} \neq 0$. By remembering the eq.(25) and the explicit representation of β^μ matrices (19) we have

$$\begin{pmatrix} -p_0 a u_3 i + p_x j a u_3 i \\ p_y a u_3 i + p_z j a u_3 i \\ p_0 a u_1 i - p_x j d u_1 i + p_y a u_2 i - p_z j d u_1 i \end{pmatrix} = m \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{29}$$

and therefore

$$u_1 = -\frac{ip_0 + kp_x}{m} (u_3)_\mathcal{C}$$

$$u_2 = \frac{ip_y - kp_z}{m} (u_3)_\mathcal{C} \tag{30}$$

$$u_3 = (u_3)_\mathcal{C}$$

with $\mathcal{C}$ we indicate the complex projection

$$(q)_\mathcal{C} = \frac{1 - i|i}{2} q \ .$$

11

To find, in the limit $\vec{p} \to 0$, the solutions (27, 28) we pose $(u_3)_\mathcal{C} = \frac{1}{\sqrt{2}}$

$$\psi_{(p_0 = +|p_0|)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{i|p_0|+kp_x}{m} \\ \frac{ip_y - kp_z}{m} \\ 1 \end{pmatrix} e^{-ipx} \tag{31}$$

$$\psi_{(p_0 = -|p_0|)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{i|p_0|-kp_x}{m} \\ \frac{ip_y - kp_z}{m} \\ 1 \end{pmatrix} e^{-ipx} . \tag{32}$$

The normal solution are now represented by (31, 32). In the same way we can prove that the anomalous solutions are represented by

$$\psi_{(p_0 = +|p_0|)} = \frac{1}{\sqrt{2}} \begin{pmatrix} j \\ -\frac{i|p_0|+kp_x}{m} \\ \frac{ip_y - kp_z}{m} \end{pmatrix} e^{-ipx} \tag{33}$$

$$\psi_{(p_0 = -|p_0|)} = \frac{1}{\sqrt{2}} \begin{pmatrix} j \\ \frac{i|p_0|-kp_x}{m} \\ \frac{ip_y - kp_z}{m} \end{pmatrix} e^{-ipx} . \tag{34}$$

We conclude this Section by noting that the representations of the matrices (19, 20) are equivalent. In fact one can prove that the following matrix

$$S = \begin{pmatrix} \cdot & \cdot & j \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix} \tag{35}$$

transforms the β^μ in the $\tilde{\beta}^\mu$ matrices and consequently we can connect the normal to the anomalous solutions by

$$\psi_{anomalous} = S\psi_{normal} \tag{36}$$

With help of the DKP equation we can kill the anomalous solution. Therefore we find, with quaternions, an important result:

"*the KG equation is equivalent to a pair of DKP equations*".

12

4 The *new* KG equation on the quaternion field

Our purpose, in this Section, is to show that with quaternions we have the possibility to choose either the standard or the *new (modified) KG* equation. Before doing it, we must find the *new KG* equation on the quaternion field. We start with the equation (1) (with matrices (20) and anomalous solution (24)). If we explicitly write this equation we have

$$(-ja\partial_t + d\partial_x)\frac{(\partial_t + j\partial_x)\varphi}{m} + (ja\partial_y + d\partial_z)\frac{(\partial_y + j\partial_z)\varphi}{m} = m(j\varphi) \qquad (37)$$

By noting that d kills the complex wave function φ and that $ja = dj$ we can rewrite the previous equation in the following way

$$ja(\partial_\mu\partial^\mu + m^2)\varphi = d(\partial_\mu\partial^\mu + m^2)\tilde{\varphi} = 0 \qquad (38)$$

with

$$\tilde{\varphi} = j\varphi \; pure \text{ quaternion function.}$$

It is simple to adapt it to DKP equation with matrices (19) and normal solution (23) finding

$$a(\partial_\mu\partial^\mu + m^2)\varphi = 0 \qquad (39)$$

These equation can be related by a similarity transformation, in fact:

$$-j\,\frac{1 - i|i}{2}\,j \;\; = \;\; \frac{1 + i|i}{2} \qquad (40)$$

and, consequently, we can go from standard to anomalous solution by j multiplication.

The situation in QQM (Quaternion Quantum Mechanics) can be summarize as follows:

Standard Klein-Gordon equation (four plane wave solutions)

$$\alpha(\partial_\mu\partial^\mu + m^2)\phi = 0 \;\;, \qquad (41)$$

$$(\alpha = 1) \;,$$

e^{-ipx} positive and negative energy ,
je^{-ipx} complex-orthogonal solutions ;

New Klein-Gordon equation (two plane wave solutions)

$$\alpha(\partial_\mu \partial^\mu + m^2)\phi = 0 \tag{42}$$

$(\alpha = \frac{1-i|i}{2}$ complex solutions e^{-ipx} ,
$\alpha = \frac{1+i|i}{2}$ *pure* quaternion solutions $je^{-ipx})$,

obviously ϕ is a quaternion function.

From now on we use the *new* Klein-Gordon equation when we want kill the anomalous solutions and the standard K-G equation when we want consider normal and anomalous solution. It is important to note that in our formalism the DKP equation is replaced by the *new* K-G equation.

5   Conclusions

In this work we have studied the Duffin-Kemmer-Petiau equation from the point of view of Quaternion Quantum Mechanics with a complex geometry. In Complex Quantum Mechanics the Klein-Gordon equation is equivalent to the *spin* 0 part of the Duffin-Kemmer-Petiau equation (which presents a greater algebraic complexity). In our formalism the DKP equation (which kills the anomalous solutions) is *not* equivalent to the "standard" KG equation $(\partial_\mu \partial^\mu \phi + m^2 \phi = 0)$.
Therefore, for the first time in Quaternion Quantum Mechanics, the *new* (or *modified*) Klein-Gordon equation appears.

Our consideration have been limited to the *spin 0 part* of the Duffin-Kemmer-Petiau equation but it is very simple to extend them to the *spin 1 part* of the Duffin-Kemmer-Petiau equation (there the standard and *new* Proca equations should appear).

Our goal has been to demonstrate the equivalence between the *quaternion* Duffin-Kemmer-Petiau equation and the *new* Klein- Gordon equation.

The situation in the Quaternion Quantum Mechanics can be summarize as follows:

standard Dirac equation - 4×4 complex γ^μ matrices -	contains the anomalous solutions
quaternion Dirac equation - see ref. [13] -	kills the anomalous solutions
standard Klein-Gordon equation - see eq. (41) -	contains the anomalous solutions
new Klein-Gordon equation - see eq. (42) -	kills the anomalous solutions.

Nevertheless we are *not* obliged to kill the anomalous solutions. The correct equations and the corresponding Lagrangian is in practice determined only when the number of particles in the theory is fixed. For example in the Higgs sector of the Electroweak theory[18, 19, 20] we used in our recent paper[21] the standard Klein-Gordon equation which automatically contains for us four particles corresponding to the four Higgs $\mathcal{H}^0$, $\mathcal{H}^+$, $\overline{\mathcal{H}}^0$, $\mathcal{H}^-$ before spontaneous symmetry breaking. We shall also *resuscitate* the standard Dirac equation in the fermion sector of the Electroweak theory where we interpret the normal solution as the *neutrino*-field and the anomalous solution as the *electron*-field. The operator which distinguish between normal and anomalous solutions in the *Quaternion Electroweak Theory* (*QET*) is represented by the third component $(i|i)$ of weak-isospin[21]. A complete and detailed *QET* will be presented in our forthcoming paper.

Before this work we had the possibility of choice between equations with and without anomalous solutions only for the fermion equation whereas for the bosonic equation we were always obliged to consider equation *with* anomalous solutions. Now we have finally the possibility to work in Quaternion Quantum Mechanics using bosonic equation *without* anomalous solutions. Nevertheless we remember (also once) that the equations with anomalous solutions (very important in the *Quaternion Electroweak Theory*) don't disappear.

We hope that this paper points out the non triviality in the choice to adopt quaternions as the underlying number field.

References

[1] *Duffin RJ* - Phys. Rev. **54**, 1114 (1938).

[2] *Kemmer N* - Proc. R. Soc. **166**, 127 (1938).

[3] *Petiau G* - Ac. R. Belg. Cl. Mém. Collect 8 **16**, No.2 (1936).

[4] *Géhéniau J* - Ac. R. Belg. Cl. Mém. Collect 8 **18**, No.1 (1938).

[5] *Fischbach E, Nieto MM and Scott CK* - Prog. Theor. Phys. **48**, 574 (1972).

[6] *Krajcik RA and Nieto MM* - Phys. Rev. **D 10**, 4049 (1974).

[7] *Birkhoff G and von Neumann J* - Ann. Math. **37**, 823 (1936).

[8] *Filkestein D, Jauch M et al.* - J. Math. Phys. **3**, 207 (1962).

[9] *Adler SL* - Quaternion quantum mechanics and quantum fields Oxford UP, Oxford, 1995.

[10] *Adler SL* - Phys. Rev. Letts **55**, 783 (1985),
     Adler SL - Comm. Math. Phys. **104**, 611 (1986),
     Adler SL - Nuc. Phys. **B 415**, 195 (1994).

[11] *Günaydin M, Gürsey F* - Lett. Nuovo Cim. **6**, 401 (1973),
     Günaydin M, Gürsey F - J. Math. Phys. **14**, 1651 (1973),
     Günaydin M, Gürsey F - Phys. Rev. **D 9**, 3387 (1974).

[12] *Horwitz LP and Biedenharn* - Ann. of Phys. **157**, 432 (1984).

[13] *Rotelli P* - Mod. Phys. Lett. **A 4**, 933 (1989).

[14] *De Leo S and Rotelli P* - Nuovo Cim. 110**B**, 33 (1995).

[15] *De Leo S and Rotelli P* - Prog. Theor. Phys. **92**, 917 (1994).

[16] *Morita K* - Prog. Theor. Phys. **67**, 1860 (1981); **68**, 2159 (1982);
     Morita K - Prog. Theor. Phys. **70**, 1648 (1983); **72**, 1056 (1984);
     Morita K - Prog. Theor. Phys. **73**, 999 (1984); **75**, 220 (1985);
     Morita K - Prog. Theor. Phys. **90**, 219 (1993).

[17] *De Leo S and Rotelli P* - Phys. Rev. **D 45**, 575 (1992).

[18] *Salam A* - Proc. 8th Nobel Symposium
Weak and Electromagnetic Interactions, ed. Svartholm 367 (1968).

[19] *Weinberg S* - Phys. Rev. Lett. **19**, 1264 (1967).

[20] *Glashow SL* - Nucl. Phys. **22**, 579 (1961).

[21] *De Leo S and Rotelli P*
Quaternion Higgs and the Electroweak Gauge Group
hepth/9506179 - to appear in Int. Journal of Mordern Physics A.