Quaternionic Commutations

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Abstract

Given n quaternions we investigate the extent of non-commutativity of their multiple products, commutators and exponential products.

1 Introduction

The field of quaternions, $\mathbb{H}$, is not commutative. However, it has some weak commutation properties, which we wish to point out in this letter.

First, given n quaternions $q1$ through qN all the multicommutators of the form

$$C(q1, \cdots, qN; \sigma) := [q_{\sigma(1)}, [q_{\sigma(2)}, \cdots, [q_{\sigma(n-1)}, q_{\sigma(n)}] \cdots]] \ ,$$

parametrized by the various permutations $\sigma \in S_n$, are pairwise equal up to a $\pm$ sign.

Secondly, all the multiproducts of the form

$$P(q1, \cdots, qN; \sigma) := q_{\sigma(1)} q_{\sigma(2)} \cdots q_{\sigma(n)} \tag{1}$$

for which σ is cyclic (consists of a single cycle) are mutually similar. Thus, the $n!$ multiproducts defined by $q1$ through qN occupy at most $(n-1)!$ similarity classes.

Finally, the Campbell-Baker-Hausdorf formula for the product of two non-commuting exponentials, which in general contains an infinite sum of non-commuting words in p and q, reduces over the quaternions to a simple 4-term sum.

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Perhaps the most basic "commutation property" of quaternions is the commutability under the norm sign. Recall that the norm (or absolute value) $|q|$ of $q = q_0 + q_1 i + q_2 j + q_3 k$ is defined by $|q|^2 = q\bar{q} = q_0^2 + q_1^2 + q_2^2 + q_3^2$. It satisfies the two commutation identities

$$|pq| = |qp| = |q|\,|p| \qquad \text{and} \qquad |1 - pq| = |1 - qp| \ . \tag{2}$$

2 Basic product and commutation formulas

Our notation for quaternions is the standard one, see [1]. The quaternion $q \in \mathbb{H}$ is represented over the reals as

$$q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \ , \qquad\qquad q_m \in \mathbb{R} \ .$$

We shall use the more concise vector notation $q = q_0 + \mathbf{h} \cdot \mathbf{q}$, where $\mathbf{h} = (i, j, k)$ and $\mathbf{q} = (q_0, q_1, q_2) \in \mathbb{R}^3$. Using the inner and outer products in $\mathbb{R}^3$, denoted here respectively by $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, we may write a concrete formula for the product of two quaternions:

$$ab = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) + \mathbf{h} \cdot (a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}). \tag{3}$$

As for the commutator, it follows from (3) that

$$[a, b] = 2\mathbf{h} \cdot (\mathbf{a} \times \mathbf{b}). \tag{4}$$

It also follows from (4) that a quaternion q is a commutator if and only if $q_0 := Re[q] = 0$.

The commutator formula in (4) easily generalizes to the n-commutators $C(q1, \cdots, qN; \sigma)$ defined in the Introduction, obtaining the formula

$$C(q1, \cdots, qN; \sigma) = 2^m sgn(\sigma) \mathbf{h} \cdot (\mathbf{q1} \times \cdots \times \mathbf{qN}). \tag{5}$$

Thus, the various commutators defined by $q1$ through qN are all equal up to a $\pm$ sign.

3 Similarity

Two quaternions p and q are called similar if

$$q = s^{-1} p\, s \ , \qquad \text{for some } s \in \mathbb{H} \ .$$

It can be seen that similarity of p and q amounts to the two conditions $|p| = |q|$ and $Re[p] = Re[q]$. Now we ask whether the two quaternion products

$p := P(q1, \cdots, qN; \sigma_1)$ and $q := P(q1, \cdots, qN; \sigma_2)$ are similar. The first condition $|p| = |q|$ is guaranteed in view of (2). Thus, similarity of p and q in (1) is reduced to the second condition, $Re[q] = Re[p]$.

The condition for similarity of $2, 3$- and 4-products is given below. The general case is still open.

Lemma. *In the non-commutative field of quaternions:*

(i) For all p, q the products pq and qp are always similar;

(ii) If in (1) q is obtained from p by a primitive permutation ($\sigma(i) = i + k \ mod(n)$) then p and q are similar;

(iii) $p = abc$ and $q = acb$ are similar if and only if the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ are linearly dependent.

(iv) $p = abcd$ and $q = adcb$ are similar if and only if $a_0\alpha - b_0\beta + c_0\gamma - d_0\delta = 0$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Proof. (i) Assume that $p = ab$ and $q = ba$ for some $a, b \in \mathbb{H}$. If $b = 0$ then $p = q = 0$ and there is nothing to prove. Otherwise, the identity $bpb^{-1} = q$ holds, implying that p and q are similar.

(ii) Follows directly from (i).

(iii) By (ii), the six terms $P(a, b, c; \sigma)$ $(\sigma \in S_3)$ can occupy at most two similarity classes, represented by abc and acb. The only requirement for similarity of these two terms is that $Re(abc) = Re(acb)$. Direct calculation based on (3) gives

$$Re(abc) = (ab)_0 c_0 - (ab) \cdot \mathbf{c} =$$
$$= a_0 b_0 c_0 - [a_0(\mathbf{b} \cdot \mathbf{c}) + b_0(\mathbf{a} \cdot \mathbf{c}) + c_0(\mathbf{a}.\mathbf{b})] - b\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}. \tag{6}$$

It follows that

$$Re[abc] - Re[acb] = 2\,(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = 2\ \det[\mathbf{a}, \mathbf{b}, \mathbf{c}],$$

proving the claim. It is observed that for a generic triple (a, b, c) the terms abc and acb are not similar.

(iv) The case $n = 4$ has six different similarity classes, represented by $abcd$, $abdc$, $acbd$, $acdb$, $adbc$, $adcb$. All six expressions have the same norm, hence one only need compare their real parts.

Some cases can be reduced to $n = 3$. For example, considering ab a fixed vector we have

$$Re[abcd] - Re[abdc] = 2\,[a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a} \times \mathbf{b}] \times \mathbf{d} \cdot \mathbf{c}$$

The vectorial part of ab is $a_0\mathbf{b}+b_0\mathbf{a}+\mathbf{a}\times\mathbf{b}$ and we have the same result obtained for $n=3$ as expected. The same holds to $abcd$ and $adbc$. In this case bc is the fixed vector.

The pair $abcd$ and $acbd$ must be analysed. We find

$$Re[abcd]-Re[adcb]=-2\left[(a_0\mathbf{b}+b_0\mathbf{a})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(c_0\mathbf{d}+d_0\mathbf{c})\right] \quad (7)$$

Two cases should be analysed. If $span\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}=\mathbb{R}^2$, (7) is null, as is easy to check. If $span\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}=\mathbb{R}^3$ consider

$$\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}+\delta\mathbf{d}=0\ , \quad \alpha,\beta,\gamma,\delta\in\mathbb{R}\ . \quad (8)$$

Substituting (8) into (7) yields

$$a_0\alpha-b_0\beta+c_0\gamma-d_0\delta=0\ ,$$

which completes the proof. •

Remarks:

1. With item (ii) we may reduce the number of similarity classes involving permutations of a given n-product from $n!$ to $(n-1)!$ at most. Can this number be reduced further?

2. We did not investigate the extent to which our observations generalize to other rings of interest. Note that (i,ii) will hold in every integral domain.

3. Investigate the equality of norms for the various expressions of the form $P(q1,\cdots,qN;\sigma)-t$, $t\in\mathbb{R}$. It follows from (3) that cyclic permutation has no effect on the norm. What is the geometric interpretation? What about non-cyclic permutations?

4. We should search the conditions for true equality between two multi-products.

4 The Campbell-Baker-Hausdorff formula

As is well known, the formula $Exp[p]Exp[q]=Exp[p+q]$ does not hold in general, when p and q do not commute. The well-known CBH (Campbell-Baker-Hausdorff) formula contains an additive correction term within the right hand side exponential which allows the equality to be restored. In the general case, the correction term is a certain infinite sum involving non-commuting words of arbitrary length made of the letters p and q; however, in the case of quaternions this formula should be closed.

A detailed discussion of the quaternionic CBH formula and its closed form will appear in a forthcoming paper [Ref.: Quaternionic exponentials, in preparation].

5 Exponential derivative

Due the non commutativity of quaternions

$$\{\exp[\psi(x)]\}' = \psi'(x) \exp[\psi(x)] \ . \tag{9}$$

does not hold. Let us see how the expression (9) is changed. From the different ways to write a quaternion function we have chosen the more convenient one for this section. In order to find the derivative of the exponential function we shall consider

$$\psi(x) = f(x) + I(x)g(x)$$

where

$$f(x) = \psi_0(x) \ , \quad g(x) = \sqrt{\psi_1^2(x) + \psi_2^2(x) + \psi_3^2(x)}$$

are real functions of x and

$$I(x) = \frac{i\psi_1(x) + j\psi_2(x) + k\psi_3(x)}{\sqrt{\psi_1^2(x) + \psi_2^2(x) + \psi_3^2(x)}}$$

with $|I(x)|^2 = -1$. Note that

$$\{I(x), I'(x)\} := I(x)I'(x) + I'(x)I(x) = 0 \tag{10}$$

where $'$ denotes the first derivative. Consider $\mathcal{F}'(x) = f'(x) + I(x)g'(x)$. Omitting the x variable we have

$$\psi = f + Ig \quad \rightarrow \quad \psi' = f' + Ig' + I'g = \mathcal{F}' + I'g \ .$$

Using power series expansion of the exponential function [2] we find

$$e^{\psi} = e^f e^{Ig} = e^f \sum_{n=0}^{\infty} \frac{[Ig]^n}{n!} = e^f \{\cos g + I \sin g\} \tag{11}$$

Thus, the exponential function derivative, using (11) and the property (10), is given by

$$\begin{aligned} [e^{\psi}]' &= \mathcal{F}' e^{\psi} + I' e^f \sin g \\ &= [\mathcal{F}' + I'g - I'g] e^{\psi} + I' e^f \sin g \\ &= \psi' e^{\psi} - I' \left\{ g e^{\psi} - e^f \sin g \right\} \end{aligned}$$

References

[1] Cohen N and De Leo S, The quaternionic determinant, *Elec. J. Lin. Alg.* **7**, 100–111 (2000).

[2] Hamilton W R, *Elements of Quaternions*, vol I, 3^{rd} Ed, Chelsea Publishing Co., New York (1969).