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QUATERNIONS FOR GUTs

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Abstract

We derive an appropriate definition of transpose for quaternionic matrices and give a new panoramic review of the quaternionic groups. We aim to analyse possible quaternionic groups for GUTs.

1 Introduction

This paper aims to give a clear and succint classification of possible quaternionic groups for Grand Unification Theories. We know that the standard group theory which applies to elementary particle physics is understood to be complex, nevertheless we must observe that non-supersymmetric GUTs based on complex groups have run into difficulties. A stimulating possibility [1] is that a successful unification of the fundamental forces will require a generalization beyond the complex.

We will discuss in this paper unitary, special unitary, orthogonal (a new definition of transpose for quaternionic matrices overcomes previous difficulties) and symplectic groups on quaternions and complex linear quaternions. Appling a quaternionic group theory to elementary particle physics, our purpose is to obtain a set of groups for translating from complex to quaternionic quantum fields and to emphasize the potentialities of the quaternionic groups for focusing on a special class of GUTs. We conclude this brief introduction with an amusing storical note. Quaternions were discovered by Hamilton [2] on 16 October 1843. The Irish mathematician was so impressed by the new idea that he scratched the main formula of the new algebra on a stone bridge that he happened to be passing.

2 Quaternionic algebra and complex geometry

The quaternionic algebra over a field $\mathcal{F}$ is a set

$$\mathcal{H} = \{\alpha + i\beta + j\gamma + k\delta \mid \alpha, \beta, \gamma, \delta \in \mathcal{F}\} \tag{1}$$

with operation of multiplication defined according to the following rules for imaginary units:

$$i^2 = j^2 = k^2 = -1$$
$$ij = k \ , jk = i \ , \ ki = j \ ,$$
$$ji = -k \ , kj = -i \ , \ ik = -j \ .$$

Complex numbers can be constructed from the real numbers by introducing a quantity i whose square is -1:

$$c = r_1 + ir_2 \quad (r_m \in \mathcal{R} \quad m = 1, 2) \ ;$$

likewise, we can construct the quaternions[1] from the complex numbers in exactly the same way by introducing another quantity j whose square is -1

$$q = c_1 + jc_2 \quad (c_m \in \mathcal{C} \quad m = 1, 2)$$

[1]Quaternions, as used in this paper, will always mean *real quaternionic numbers* and never *complexified quaternions* ($\mathcal{F} = \mathcal{C}(1, \mathcal{I})$ in eq. (1) with $\mathcal{I}$ which commutes with i, j, k). A number of papers in the literature which discuss quantum mechanics equations based on complexified quaternions can be found in ref. [3].

and anticommutes with i ($\{i,\ j\} = 0 \Rightarrow k^2 = -1$).
We need three imaginary units $i,\ j,\ k$ because

$$ij = \alpha + i\beta + j\gamma \quad , \quad \alpha,\ \beta,\ \gamma \in \mathcal{R} \ ,$$

implies

$$i^2 j = i\alpha - \beta + ij\gamma = i\alpha - \beta + (\alpha + i\beta + j\gamma)\gamma = ... + j\gamma^2$$

and so gives the inconsistent relation

$$\gamma^2 = -1 \ .$$

In going from the complex numbers to the quaternions we lose the property of commutativity. In going from the quaternions to the next more complicated case (called octonionic numbers) we lose the property of associativity. The situation can be graphically represented by the following chart[4]

Name of field	Method of construction	Real dimension	Division algebra	Associa-tivity	Commu-tativity
Real	r	1	Yes	Yes	Yes
Complex	$c = r_1 + ir_2$	2	Yes	Yes	Yes
Quaternionic	$q = c_1 + jc_2$	4	Yes	Yes	No
Octonionic	$o = q_1 + \tilde{\imath}q_2$	8	Yes	No	No

We can immediatly shown the nonassociativity of the octonionic numbers in the previous "split" representation. We have seven imaginary units (the new imaginary unit $\tilde{\imath}$ anticommutes with the quaternionic imaginary units $i,\ j,\ k$)

$$i\ ,\ j\ ,\ k\ ,\ \tilde{\imath}\ ,\ I = i\tilde{\imath}\ ,\ J = j\tilde{\imath}\ ,\ K = k\tilde{\imath}\ \ .$$

It is straightforward to verify that

$$IJk = JKi = KIj = -1$$

$$(IJk = i\tilde{\imath}j\tilde{\imath}k = -i\tilde{\imath}^{\,2}jk = ijk = k^2 = -1)\ .$$

Associativity is dropped by following relations

$$i(\tilde{\imath}k) = -iK = J \ \ , \ \ (i\tilde{\imath})k = Ik = -J \ \ .$$

To complete this introduction to the quaternionic algebra, we introduce the quaternion conjugation operation denoted by $^+$ and defined by

$$1^+ = 1 \qquad (i,\ j,\ k)^+ = -\ (i,\ j,\ k) \ \ . \tag{2}$$

The previous definition implies

$$(\psi\varphi)^+ = \varphi^+\psi^+ \ \ ,$$

for ψ, φ quaternioninc functions. The definition of a conjugation operation which does not reverse the order of ψ, φ factors is given by

$$1^* = 1 \qquad (i,\ j,\ k)^* = j^+(i,\ j,\ k)j \ . \tag{3}$$

Remembering the noncommutativity of the quaternionic multiplication we must specify whether the quaternionic Hilbert space $V_{\mathcal{H}}$ is to be formed by right or by left multiplication of vectors by scalars. Besides we must specify if our scalars are quaternionic, complex or real numbers. We will follow the usual choice (see Adler [5], Horwitz and Biedenharn [6]) to work with a linear vector space under right multiplication by scalars.

Operators which act on states *only* from the left as in

$$\mathcal{O} \mid \psi >$$

are named quaternion linear operators, they obey

$$\mathcal{O}(\mid \psi > q) = (\mathcal{O} \mid \psi >)q \ , \tag{4}$$

for an arbitrary quaternion q. More general classes of operators such as complex or real linear operators can be introduced. We will use the notation $\mathcal{O}_c$ ($\mathcal{O}_r$) to indicate complex (real) linear operators. They act on quaternionic states as follows

$$\mathcal{O}_c(\mid \psi > c) = (\mathcal{O}_1 + \mathcal{O}_2 \mid i)(\mid \psi > c) = (\mathcal{O}_c \mid \psi >)c \ , \tag{5}$$

$$\mathcal{O}_r(\mid \psi > r) = (\mathcal{O}_1 + \mathcal{O}_2 \mid i + \mathcal{O}_3 \mid j + \mathcal{O}_4 \mid k)(\mid \psi > r) = (\mathcal{O}_r \mid \psi >)r \ , \tag{6}$$

for an arbitray complex c, real r ($\mathcal{O}_{1,\ 2,\ 3,\ 4}$ represent quaternion linear operators). The *barred* operators $\mathcal{O} \mid b$ act on quaternionic objects ψ as in

$$(\mathcal{O} \mid b)\phi \ \equiv \ \mathcal{O}\phi b \ .$$

There are three scalar products which can be used to define a real-valued norm $\parallel \psi \parallel$. We will call the binary mapping $< \psi \mid \varphi >$ of $V_{\mathcal{H}} \times V_{\mathcal{H}}$ into $\mathcal{H}$, defined by

$$< \psi \mid \varphi > = \int d^3x \ \psi^+\varphi \ ,$$

the quaternion scalar product (see Adler [1]), and the binary mapping $< \psi \mid \varphi >_c$ of $V_{\mathcal{H}} \times V_{\mathcal{H}}$ into $\mathcal{C}$, defined by

$$< \psi \mid \varphi >_c = \ \frac{1 - i \mid i}{2} \ < \psi \mid \varphi > \ , \tag{7}$$

the complex scalar product or complex geometry (as named by Rembieliński in ref. [7]). The complex scalar product, used by Horwitz and Biedenharn [6], in order to define consistently multiparticle quaternionic states, has then been applied in

papers on the Dirac equation [8], representations of $U(1,q)$ [9] and translations between Quaternion and Complex Quantum Mechanics [10].
The last trivial possibility is represented by a real scalar product, the binary mapping $< \psi \mid \varphi >_r$ of $V_{\mathcal{H}} \times V_{\mathcal{H}}$ into $\mathcal{R}$, defined by

$$< \psi \mid \varphi >_r = \frac{1 - i \mid i - j \mid j - k \mid k}{4} < \psi \mid \varphi > \quad .$$

In this paper we will use a linear quaternionic Hilbert space under right multiplication by complex scalars and will work with complex scalar products.

To conclude this section we recall that since $< \psi \mid \varphi >_c$ is the complex $\mathcal{C}(1,\ i)$ projection of $< \psi \mid \varphi >$, any transformation which is an invariance of $< \psi \mid \varphi >$ is automatically an invariance of $< \psi \mid \varphi >_c$ as well. Obviously a transformation which is an invariance of $< \psi \mid \varphi >_c$ is *not* automatically an invariance of $< \psi \mid \varphi >$. An example of that is given (see ref. [11] or for a brief review, section IV of this paper) by the quaternionic version of the electroweak group $U(1,\ q) \mid U(1,\ c)$. This group represents an invariance of $< \psi \mid \varphi >_c$ but not of $< \psi \mid \varphi >$.

3 A new possibility

In this section we give a *new* panoramic review of quaternionic groups. Why *new*? As elements of our matrices (given any two vector spaces V_n, V_m, every linear operator $\mathcal{O}$ from V_n to V_m can be represented by an $m \times n$ matrix) we will not use simple quaternions but complex linear quaternions or generalized quaternions as called in our previous works [12]

$$q_c = q_1 + q_2 \mid i \qquad (q_{1,\,2} \in \mathcal{H}) \quad . \tag{8}$$

Corresponding to our convention that $V_{\mathcal{H}}$ is a linear vector space under right multiplication by complex scalars, the most general linear one-dimensional operator which acts on quaternionic functions is in fact represented by (8).
The product of two complex linear quaternions q_c and p_c, in terms of quaternions $q_1,\ q_2,\ p_1$ and p_2, is given by

$$q_c p_c = q_1 p_1 - q_2 p_2 + (q_1 p_2 + q_2 p_1) \mid i \quad .$$

Before discussing the groups $Gl(n,\ q_c)$, we introduce a new definition of transpose for quaternionic matrices which will allow us to overcome previous difficulties (our definition which applies to standard quaternions will be extended to complex linear quaternions).
The customary convention of defining the transpose M^t of the matrix M is

$$(M^t)_{rs} = M_{sr} \quad .$$

In general, however, for quaternionic matrices MN one has

$$(MN)^t \neq N^t M^t \quad ,$$

whereas this statement holds as an equality for complex matrices. Defining an appropriate transpose for quaternionic numbers (which go back to usual definition for complex number $c^t = c$) we can overcome the just cited difficulty. The new transpose q^t of the quaternionic number

$$q = \alpha + i\beta + j\gamma + k\delta \qquad (\alpha, \beta, \gamma, \delta \in \mathcal{R})$$

is

$$q^t = \alpha + i\beta - j\gamma + k\delta \quad . \tag{9}$$

The transpose of a product of two quaternions q and p is the product of the transpose quaternions in reverse order (note that $q^t = -jq^+j$)

$$(qp)^t = p^t q^t \quad .$$

Our convention of defining the transpose M^t of the matrix M is

$$(M^t)_{rs} = {M_{sr}}^t$$

and so we have

$$(MN)^t = N^t M^t \quad .$$

Remembering the * conjugation defined in (3) we can write

$$M^+ = M^{*\,t} \quad .$$

Noting that under the transpose and quaternion conjugation operation we have $i^{\,t} = i$ and $i^{\,+} = -i$, we can immediatly generalize the definition of transpose and quaternion conjugation to complex linear quaternions as follows

$$q_c^t = q_1^t + q_2^t \mid i$$

$$(q_c^+ = q_1^+ - q_2^+ \mid i) \quad .$$

Introducing complex linear quaternions we create new possibilities in quaternionic quantum mechanics with complex geometry. For example we can always trivially relate an anti-Hermitian operator A to an Hermitian operator H by removing a factor $1 \mid i$

$$H = A \mid i \tag{10}$$

$$(< A\psi \mid \varphi >_c = - < \psi \mid A\varphi >_c \Rightarrow$$

$$-i < A\psi \mid \varphi >_c = i < \psi \mid A\varphi >_c = < \psi \mid A\varphi >_c i \Rightarrow$$

$$< H\psi \mid \varphi >_c = < \psi \mid H\varphi >_c) \quad .$$

This statement is not trivial in quaternionic quantum mechanics with quaternionic geometry (see Adler ref. [1] pag. 33).
In the literature we know an operator like that in (10), the momentum operator

$$-\vec{\partial} \mid i \quad ,$$

given by Rotelli in his paper on the quaternionic Dirac equation [8].

The classical groups which occupy a central place in group representation theory and have many applications in various branches of mathematics and physics are the unitary, special unitary, orthogonal and symplectic groups. So we will discuss in this paper the $U(n,\ q_c)$, $SU(n,\ q_c)$, $O(n,\ q_c)$, $Sp(n,\ q_c)$ subgroups of $Gl(n,\ q_c)$.

With complex linear quaternions we have the possibility to give a new definition of trace by

$$tr\ q_c = re(q_1) + i\ re(q_2)$$

which implies that for any two complex linear quaternions q_c and p_c

$$tr\ (q_c p_c) = tr\ (p_c q_c) \quad .$$

We know that the generators of the unitary, special unitary, orthogonal and symplectic groups must satisfy the following conditions[2]

$$\begin{array}{rcc} U(n) & : & A + A^{+} = 0 \quad , \\ SU(n) & : & A + A^{+} = 0 \ \ , \ \ tr\ A = 0 \quad , \\ O(n) & : & A + A^{t} = 0 \quad , \\ Sp(2n) & : & \mathcal{J}A + A^{t}\mathcal{J} = 0 \quad , \end{array}$$

where

$$\mathcal{J}_{2n\times 2n} = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ \text{-}1_{n\times n} & 0_{n\times n} \end{pmatrix} \quad .$$

So for the generators of the one-dimensional groups with complex linear quaternions we have

$$\begin{array}{rccccl} U(1, q_c) & : & q_c + q_c^{+} = 0 & \Rightarrow & A = i,\ j,\ k,\ 1 \mid i & ; \\ SU(1,\ q_c) & : & tr\ q_c = 0 & \Rightarrow & A = i,\ j,\ k & ; \\ O(1,\ q_c) & : & q_c + q_c^{t} = 0 & \Rightarrow & A = j,\ j \mid i & ; \\ Sp(1,\ q_c) & : & jq_c + q_c^{t}\, j = 0 & \Rightarrow & A = i,\ j,\ k,\ i \mid i,\ j \mid i,\ k \mid i & . \end{array}$$

Any complex linear quaternion group of dimension n is isomorphic to a complex representation of dimension $2n$. We give the transformation rule (for further detail see ref. [10])

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Longleftrightarrow \left[\frac{c_1 + c_4^*}{2} + j\frac{c_3 - c_2^*}{2} + (\frac{c_1 - c_4^*}{2i} + j\frac{c_3 + c_2^*}{2i}) \mid i \right] (z_1 + j z_2) \quad .$$

[2] A detailed classification of the real Lie algebras of linear Lie groups is given by Cornwell in the book of ref. [13] (vol. 2, pag. 392).

Remembering that a complex linear quaternions, in terms of real quantities, is expressed by

$$q_c = \alpha_1 + i\beta_1 + j\gamma_1 + k\delta_1 + (\alpha_2 + i\beta_2 + j\gamma_2 + k\delta_2) \mid i$$
$$\alpha_{1,\,2},\ \beta_{1,\,2},\ \gamma_{1,\,2},\ \delta_{1,\,2} \in \mathcal{R} \ ,$$

we have

$$\textit{complex linear quaternions} \supset \textit{quaternions} \supset \textit{complex} \ ,$$

and more

$$\textit{complex linear quaternions} \supset \textit{elements like } \alpha_1 + \alpha_2 \mid i \equiv c_{right} \ .$$

So

$$Gl(n\ q_c) \supset Gl(n\ q) \supset Gl(n\ c) \ ,$$
$$Gl(n\ q_c) \supset Gl(n\ c_{right}) \ .$$

We can now give the general formulas for counting the generators of generical n-dimensional groups as a function of n.

$\diamond$ Dimensionalities of groups $\diamond$

$$\boxed{\begin{array}{lccc} U(n,\ q_c) & : & 4n + 8\ \frac{n(n-1)}{2} & = & 4n^2 \\ U(n,\ q) & : & 3n + 4\ \frac{n(n-1)}{2} & = & n(2n+1) \\ U(n,\ c_{right}) & : & n + 2\ \frac{n(n-1)}{2} & = & n^2 \end{array}}$$

$$\boxed{\begin{array}{lcc} SU(n,\ q_c) & : & 4n^2 - 1 \\ SU(n,\ q) & \equiv & U(n,\ q) \\ SU(n,\ c_{right}) & : & n^2 - 1 \end{array}}$$

$$\boxed{\begin{array}{lcccc} O(n,\ q_c) & : & 2n + 8\ \frac{n(n-1)}{2} & = & 2n(2n-1) \\ O(n,\ q) & : & n + 4\ \frac{n(n-1)}{2} & = & n(2n-1) \\ O(n,\ c_{right}) & : & 2\ \frac{n(n-1)}{2} & = & n(n-1) \end{array}}$$

For the quaternionic symplectic groups we have

$$\mathcal{J}_{2n\times 2n} = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ \text{-}1_{n\times n} & 0_{n\times n} \end{pmatrix} \quad , \quad \mathcal{J}_{(2n+1)\times(2n+1)} = \begin{pmatrix} 0_{n\times n} & 0_{n\times 1} & 1_{n\times n} \\ 0_{1\times n} & j & 0_{1\times n} \\ \text{-}1_{n\times n} & 0_{n\times 1} & 0_{n\times n} \end{pmatrix} \ ,$$

so

$$\begin{array}{lcccc} Sp(2n,\ q_c) & : & 8n^2 + 2\ [\ 6n + 8\ \frac{n(n-1)}{2}\ ] & = & 4n(4n+1) \\ Sp(2n,\ q) & : & 4n^2 + 2\ [\ 3n + 4\ \frac{n(n-1)}{2}\ ] & = & 2n(4n+1) \\ Sp(2n,\ c_{right}) & : & 2n^2 + 2\ [\ 2n + 2\ \frac{n(n-1)}{2}\ ] & = & 2n(2n+1) \\ Sp(2n+1,\ q_c) & : & 4n(4n+1) + 2(8n) + 6 & = & 2(2n+1)\ [\ 2(2n+1) + 1\ ] \\ Sp(2n+1,\ q) & : & 2n(4n+1) + 2(4n) + 3 & = & (2n+1)\ [\ 2(2n+1) + 1\ ] \end{array}$$

The situation for the symplectic groups can be summarize as follows

$Sp(n,\ q_c)$	:	$2n(2n+1)$
$Sp(n,\ q)$	$\equiv$	$U(n,\ q)$
$Sp(2n,\ c_{right})$	:	$2n(2n+1)$

4 Quaternionic groups for GUTs

Finally we can apply quaternionic group theory to elementary particle physics. We must remark on an important point. A symmetry operation $\mathcal{S}$ of a system, described by $\mid \psi >$, is a mapping of $\mid \psi >$ into $\mid \psi' >$, which preserves all transition probabilities

$$\mathcal{S} \mid \psi >=\mid \psi' >$$

$$|< \varphi' \mid \psi' >|^2=|< \varphi \mid \psi >|^2 \quad .$$

In quaternionic quantum mechanics with complex geometry a '*quaternionic* | *complex*' phase

$$e^{i\alpha+j\beta+k\gamma} \mid e^{i\delta} \tag{11}$$

appears. We can immediatly prove that the previous transformation represents an invariance of $< \psi \mid \varphi >_c$

$$< \psi' \mid \varphi' >_c= e^{-i\delta} < \psi \mid \varphi >_c e^{i\delta} =< \psi \mid \varphi >_c$$

(the transformation (11) obviously does not represent an invariance of the quaternionic scalar product $< \psi \mid \varphi >$). So a quaternionic invariance group like that of the electroweak gauge group (for further details on the group $U(1,\ q)$ in quaternionic quantum mechanics with complex geometry see ref. [9]) naturally appears. In a recent work [11] we have studied the Higgs sector of the electroweak model from the point of view of quaternionic quantum mechanics with complex geometry. The Higgs fields are assumed to be four (two complex) and this coincides with the number of solutions of the standard Klein-Gordon equation within quaternionic quantum mechanics with complex geometry. The global invariance group of the one-component Klein-Gordon equation is $U(1,\ q) \mid U(1,\ c)$ isomorphic at the Lie algebra level with the Glashow-Salam-Weinberg group.

The aim of this paper is to extend our previous consideration about quaternionic electroweak models and to propose quaternionic groups for GUTs.

Within our formalism the peculiarity is the doubling of solutions (note that with complex scalar products $\mid \psi >$ and $\mid \psi > j$ are orthogonal states), so we have some problems in discussing the color group (three states: R, G, B).
There are three possibilities. The first one represents a *conservative* hypothesis, the second and the third ones represent interesting ideas with potential *predictive* powers.

- $SU(3,\ c_{right})$ for color group.
  We have the following doubling of states
  $$\begin{pmatrix} R \\ G \\ B \end{pmatrix} \quad , \quad j \begin{pmatrix} R \\ G \\ B \end{pmatrix} \quad .$$
  We need a 'new' quantum number to differentiate the previous solutions. The appropriate quantum number is represented by the weak isospin. So we can rewrite the previous solutions as follows
  $$\begin{pmatrix} u_R \\ u_G \\ u_B \end{pmatrix} \quad , \quad j \begin{pmatrix} d_R \\ d_G \\ d_B \end{pmatrix} \quad .$$
  Note that the *complex* group $SU(3,\ c_{right})$ does not mix u with jd, besides the *one-dimensional* quaternionic group $U(1,\ q)$ does not mix R, G, B. We are particularly pleased with that. So the color group $SU(3,\ c_{right})$ suggests the weak-isospin group $U(1,\ q)$. The gauge group for the standard model is[3]
  $$SU(3,\ c_{right}) \times U(1,\ q)_L \times U(1, c_{right})_Y$$
  (in this way using the color group $SU(3,\ c_{right})$ we have a translation between complex and quaternionic theories).

- $SU(3,\ c)$ for color group.
  We have always a doubling of states, but in this case the complex solutions transform like 3 whereas the j-complex solutions like 3^* (to see that it is sufficient to note that $ij = ji^*$). So working with the standard group $SU(3,\ c)$ we remark the possibilty of additional multiplets.
  The minimal grand unification group $SU(5,\ c)$ [14] will have (in our formalism) the following additional multiplets
  $$5 \ + \ j5^* \ ,$$
  $$10 \ + \ j10^* \ .$$
  This is an interesting result, in fact we know that a single unification point cannot be obtained within minimal (non-supersymmetric) $SU(5,\ c)$. The α_{strong} coupling misses the crossing point of the other two by more than eight standard deviations. In the 'quaternionic' version of $SU(5,\ c)$ additional multiplets of quark and leptons naturally appear and so we could find right unification

[3] In our Lagrangian we need a complex projection [11] for a Dirac Langrangian density in order to obtain the Dirac field equation and so the complex group $U(1,\ c_{right})$ will be always an invariance of our Lagrangian.

proprietis. In a work of Amaldi et al. [15] a non-supersymmetric $SU(5,\ c)$ model, based on additional split multiplets (split multiplet models also appear in ref. [16]), is proposed. Their model shows unification properties similar to the minimal supersymmetric extension of the standard model

$$\begin{array}{lcll} M_{thresold} & = & 10^{3.2\pm 0.9} & Gev\ , \\ M_{GUT} & = & 10^{16.0\pm 0.3} & Gev\ . \end{array}$$

- Quaternions for color group.
  Looking at charts of the previous section we can immediatly observe that the minimal quaternionic group candidate for color group is

  $$SU(2,\ q_c)\ .$$

  In fact its 15 generators contain the 8 generators of the standard color group $SU(3,\ c)$. In this case we have not a doubling of solutions, nevertheless we must note the appearance of an additional solution

  $$\left(\begin{array}{c} R + jG \\ B + jW \end{array} \right)\ .$$

  In this case we start with the gauge group $SU(2,\ q_c)$ and break down to the usual color group (in the quaternionic version). We need a *fourth* color.

What about the fourth color? The idea of a fourth color (as lepton number) was proposed in 1973 by Pati and Salam [17]. With quarks and leptons in one multiplet of a local gauge symmetry group G, baryon and lepton number conservations cannot be absolute. This line of reasoning had led Pati and Salam to predict in their paper [17] that the lightest baryon - the proton - must ultimately decay into leptons. The PS model was proposed before any grand unification scheme and so it constitutes really the forerunner of the GUT idea that quarks and leptons should belong to common representations of the gauge group.
Following the PS idea we can put the fermions of the first generation in the following multiplets

$$\left(\begin{array}{c} u_R + ju_G \\ u_B + j\nu_W \end{array} \right)\ ,\ \left(\begin{array}{c} d_R + jd_G \\ d_B + je_W \end{array} \right)$$

and propose an 'electrostrong' model based on the gauge group

$$SU(2,\ q_c) \times U(1,\ c_{right})\ .$$

If we wish to consider unification in the context of a bigger gauge group than $SU(5\ c)$ we must consider the group $SO(10,\ r)$. This group can break down to the standard model gauge group in many different chains of symmetry breaking. Chains preferred from the CERN LEP data include the PS model (for further details see

Deshpande et al. [18], Galli [19]). We can now immediatly translate [10] the PS model based on the complex group

$$SU(4) \times SU(2)_L \times SU(2)_R$$

by the quaternionic group

$$SU(2,\ q_c) \times U(1,\ q)_L \times U(1,\ q)_R \tag{12}$$

and propose a GUT model based on the group $O(5,\ q)$ which represents the minimal quaternionic group which contains the gauge group (12).

We conclude this paper with a completly new idea *inspired* by quaternions. We have discussed the problem concerning the odd number of colors. Before 1974, discovery of J/ψ (bound state of a charmed quark and a charmed antiquark) at Brookhven National Laboratory and at Stanford Linear Accelerator Center, we would have had the same problem with the flavor group. In that case the correct predictive hypothesis[4] should have been the choice of $SU(2,\ q_c)$ for flavor group and the choice of a new quark as fourth flavor

$$\begin{pmatrix} u + jd \\ s + jc \end{pmatrix} \quad .$$

So why do we not propose a white quark as fourth color? This possibility is currently under investigation [20]. An interesting quaternionic group for GUTs (only proposed here) appears natural if we believe in the existence of white quarks. The just cited quaternionic groups is

$$SU(3,\ q_c) \quad .$$

This group represents the natural quaternionic extension of the group $SU(5,\ c)$, is algebrically isomorphic to $SU(6,\ c)$ and contains the (new) color group and the electroweak group. In our forthcoming paper we will focus our attention on this group. Remembering that the anomaly for the representation R may be characterized by

$$tr\ (\{T^a,\ T^b\}T^c) = A(R)\ d^{abc} \quad ,$$

$$(T^a\ normalized\ generators\ of\ the\ representations\ R\ \ ,$$

$$d^{abc}\ symmetric\ structure\ constants\ of\ the\ Lie\ algebra)\ ,$$

[4]The conservative hypothesis is represented by using $SU(3,\ c_{right})$ for the flavor group, with the spin as new quantum number to differentiate the doubling of solutions

$$\begin{pmatrix} u_\uparrow \\ d_\uparrow \\ s_\uparrow \end{pmatrix} \quad , \quad j \begin{pmatrix} u_\downarrow \\ d_\downarrow \\ s_\downarrow \end{pmatrix} \quad .$$

and that for a representation given by the completely antisymmetric product of p fundamental representations of $SU(n,\ c)$, the coefficient $A(R)$ is

$$A(R) = \frac{(N-3)!(N-2p)}{(N-p-1)!(p-1)!} \quad ,$$

we have for $SU(6,\ c)$ (the complex counterpart of $SU(3,\ q_c)$) an anomaly cancellation when

$$p = \frac{N}{2} = 3 \quad .$$

That implies three vertical boxes in the Young tableaux and so a 20 dimensional representation. We now have 20 particles to accomodate in this representation, in fact we can add to the standard 16 particles of the first generation the following *new* four particles

$$u_{W\ L},\ u^c_{W\ L},\ d_{W\ L},\ d^c_{W\ L} \quad .$$

A last possibility concerning quaternion groups for GUTs is given by the choice of the quaternionic group $SU(3,\ q_c)$, but without requiring a fourth color. The unification of the standard coupling constants could appear through the split-multiplet mechanism for the complementary heavy fermions. The complex counterpart of $SU(3,\ q_c)$, namely $SU(6,\ c)$, is considered in detail in an interesting work of Chkareuli et al. [21]. We briefly summarize their results:

$SU(6,\ c)$ model with

- one family of complementary fermions

$$M_{intermediate\ breaking} = 5.4 \times 10^2\ Gev \quad , \quad M_{GU} = 1.3 \times 10^{16}\ Gev \quad ,$$

- two families of complementary fermions

$$M_{intermediate\ breaking} = 2.4 \times 10^9\ Gev \quad , \quad M_{GU} = 1.2 \times 10^{16}\ Gev \quad ,$$

- three families of complementary fermions

$$M_{intermediate\ breaking} = 3.9 \times 10^{11}\ Gev \quad , \quad M_{GU} = 1.3 \times 10^{16}\ Gev \quad .$$

5 Conclusions

In this paper we have given an informal panoramic review of the quaternionic groups. Our aim was to analyse possible quaternionic groups for GUTs. We have obtained a set of groups for translating from standard complex quantum fields to a particular version of quaternionic quantum fields and have proposed some new groups with potential predictive powers. In the following charts we list our results.

$\diamond$ Groups for translating from *cqm* to *qqm* with complex geometry $\diamond$

$SU(3,\ c_{right}) \times U(1,\ q)_L \times U(1,\ c_{right})$	standard model
$SU(2,\ q_{right}) \times U(1,\ q)_L \times U(1,\ q)_R$	PS standard model
$O(5,\ q)$	$SO(10,\ r)$ GUT model
$SU(3,\ q_c)$	split-multiplets provide for unification

$\diamond$ Groups with potential predictive power $\diamond$

$SU(5,\ c)$	split-multiplets, proposed by Amaldi et al. , naturally appear
$SU(3,\ q_c)$	flavor *inspires* fourth color, hypothetical existence of white quarks

We have proposed some quaternionic groups for GUTs and remark on their potentialities for focusing on a special class of standard complex GUTs (a detailed review of the complex groups for unified model building is given in ref. [22]). A further analysis of the quaternionic groups introduced here will be given in a more detailed work, where we will particularly focus our attention on the quaternionic group $SU(3,\ q_c)$.

Finally we wish to remember that we have another possibility to look at fundamental physics as proposed by Harari-Shupe [23]. We can think of quarks and leptons as composites of other more fundamental fermions, *preons*. A stimulating idea (within quaternionic quantum mechanics with quaternionic geometry) about this possibility is proposed by Adler (see [24] or [1] pag. 501). He suggests that the color degree of freedom postulated in the Harari-Shupe scheme could be sought in a noncommutative extension of standard quantum mechanics.

We hope that this paper emphasizes the nontriviality in the choice to adopt quaternions as the underlying number field and remarks on the possibile predictive power in using new mathematical formalism to describe theoretical physics[5]. Why i, j , k ? Why not ?

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[5] "The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attemps to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each succes in this direction, to try to interpret the new mathematical features in terms of physical entities" - Dirac [25].

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