# Real linear quaternionic operators

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**Abstract**

In a recent paper [J.Math.Phys. **42**, 2236–2265 (2001)], we discussed differential operators within a quaternionic formulation of quantum mechanics. In particular, we proposed a practical method to solve quaternionic and complex linear second order differential equations with constant coefficients. In this paper, we extend our discussion to real linear quaternionic differential equations. The method of resolution is based on the Jordan canonical form of quaternionic matrices associoated to real linear differential operators.

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## I. Introduction

Before going into the details of the discussion of $\mathbb{R}$ linear quaternionic differential operators, we briefly recall the technique used to solve $\mathbb{H}$ and $\mathbb{C}$ linear differential equations with constant quaternionic coefficients and show the difficulties in extending the method of resolution to the $\mathbb{R}$-linear case. We do not attempt a formal discussion of quaternionic theory of differential equations. Instead, we take an operational and intuitive approach. For the convenience of the reader and to make our exposition as self-contained as possible, we follow the mathematical notation and terminology used in our previous paper [1]. In particular, the operators

$$L_\mu = (1, L_i, L_j, L_k) \quad \text{and} \quad R_\mu = (1, R_i, R_j, R_k) \ , \quad \mu = 0, 1, 2, 3 \ ,$$

will denote the left and right action on quaternionic functions of real variable, $\varphi(x)$, of the imaginary units $i, j$ and $k$. To shorten notation, we shall use the upper-script $\mathbb{X} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ to indicate the $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ linearity (from the right) of $n$-order quaternionic differential operators, $\mathcal{D}_n^{\mathbb{X}}$, and $n$-dimensional quaternionic matrices, $M_n^{\mathbb{X}}$.

The general solution of $n$-order $\mathbb{C}$ linear homogeneous ordinary differential equations

$$\mathcal{D}_n^{\mathbb{C}} \varphi(x) = 0 \ , \tag{1}$$

where

$$\mathcal{D}_n^{\mathbb{C}} = \frac{d^n}{dx^n} - \sum_{p=0}^{n-1} a_{\mathbb{C}}^{(p)} \frac{d^p}{dx^p} = \frac{d^n}{dx^n} - \sum_{p=0}^{n-1} \left[ \sum_{\mu=0}^{3} \sum_{\nu=0}^{1} a_{\mu\nu}^{(p)} L_\mu R_\nu \right] \frac{d^p}{dx^p} , \quad a_{\mu\nu}^{(p)} \in \mathbb{R} ,$$

has the form

$$\varphi(x) = \sum_{m=1}^{2n} \varphi_m(x)\, c_m , \tag{2}$$

where $\{\varphi_1(x), \ldots, \varphi_{2n}(x)\}$ represent $2n$ quaternionic particular solutions linearly independent over $\mathbb{C}(1,i)$ and $c_m$ are complex constants determined by the initial values of the function $\varphi(x)$ and its derivatives

$$\varphi(x_0) = \varphi_0 , \quad \frac{d\varphi}{dx}(x_0) = \varphi_1 , \quad \ldots , \quad \frac{d^{n-1}\varphi}{dx^{n-1}}(x_0) = \varphi_{n-1} \quad \in \mathbb{H} . \tag{3}$$

The solution of Eq. (1) is given by

$$\varphi(x) = \sum_{p=1}^{n} \left\{ \exp\left[ M_n^{\mathbb{C}} \, (x - x_0) \right] \right\}_{1p} \varphi_{p-1} = \sum_{p=1}^{n} \left\{ S_n^{\mathbb{C}} \exp\left[ J_n^{\mathbb{C}} \, (x - x_0) \right] (S_n^{\mathbb{C}})^{-1} \right\}_{1p} \varphi_{p-1} , \tag{4}$$

where

$$M_n^{\mathbb{C}} = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ a_{\mathbb{C}}^{(0)} & a_{\mathbb{C}}^{(1)} & a_{\mathbb{C}}^{(2)} & \cdot & \cdot & \cdot & a_{\mathbb{C}}^{(n-1)} \end{pmatrix} \tag{5}$$

and $J_n^{\mathbb{C}}$ represents the Jordan canonical form of $M_n^{\mathbb{C}}$. Note that $J_n^{\mathbb{C}} = D_n^{\mathbb{C}} + N_n^{\mathbb{C}}$, where $D_n^{\mathbb{C}}$ is diagonal and $N_n^{\mathbb{C}}$ is nilpotent. The Jordan matrix $J_n^{\mathbb{C}}$ can be determined by solving the right complex eigenvalue problem [2–4] for $M_n^{\mathbb{C}}$.

It is worth pointing out that from Eq. (4) we can recover the form of the particular solutions $\varphi_m(x)$. For example, in the case of a null nilpotent matrix, the general solution (2) can be rewritten in terms of left acting quaternionic coefficients ($u_p$ and $v_p$), complex exponentials ($\exp[\, z_p\, x\,]$ and $\exp[\, w_p\, x\,]$), and right acting complex constants ($c_p$ and $\tilde{c}_p$) determined by the initial conditions. Explicitly, we find

$$\varphi(x) = \sum_{p=1}^{n} \left\{ u_p \, \exp[\, z_p\, x\,]\, c_p + v_p \, \exp[\, w_p\, x\,]\, \tilde{c}_p \right\} , \tag{6}$$

where the complex coefficients $\{z_1, w_1, \ldots, z_n, w_n\}$ represent the right eigenvalues of the quaternionic matrix $M_n^{\mathbb{C}}$. In the case of equal complex eigenvalues and equal quaternionic coefficients the remaining particular solutions are determined by using the nilpotent matrix $N_n^{\mathbb{C}}$.

For $\mathbb{H}$ linear quaternionic matrices, $M_n^{\mathbb{H}}$, it is possible to show that $v_p = u_p\, j$ and $w_p = \bar{z}_p$. Consequently, for $\mathbb{H}$ linear quaternionic differential operators, the general solution (6) reduces to

$$\varphi(x) = \sum_{p=1}^{n} u_p \, \exp[\, z_p\, x\,]\, (c_p + j\tilde{c}_p) = \sum_{p=1}^{n} \exp[\, q_p\, x\,]\, h_p , \tag{7}$$

where $q_p = u_p z_p u_p^{-1}$. The initial conditions (3) shall fix the $n$ quaternionic constants $h_p$.

Due to the $\mathbb{R}$ linearity, the general solution of $n$-order homogeneous ordinary differential equations with quaternionic constant coefficients which appear on the left and on the right has the form

$$\varphi(x) = \sum_{s=1}^{4n} \varphi_s(x)\, r_s \ , \tag{8}$$

where $\{\varphi_1(x)\, , \, \dots \, , \, \varphi_{4n}(x)\}$ represent $4n$ quaternionic particular solutions, linearly independent over $\mathbb{R}$, and $r_s$ are real constants fixed by Eqs. (3). The question still unanswered is how to determine the particular solutions $\varphi_s(x)$. The natural choice of left acting quaternionic coefficients and real exponentials

$$\varphi_s(x) = u_s \exp\left[\, \lambda_s\, x \,\right] \ ,$$

does not represent a satisfactory answer. In fact, such particular solutions are at most valid for the real part of the eigenvalue spectrum of the matrix $M_n^{\mathbb{R}}$.

# II. Real linear quaternionic differential equations

Let us consider the second order $\mathbb{R}$ linear quaternionic differential equation

$$\mathcal{D}_2^{\mathbb{R}}\, \varphi(x) = 0 \ , \tag{9}$$

where

$$\mathcal{D}_2^{\mathbb{R}} = \frac{d^2}{dx^2} - a_{\mathbb{R}}^{(1)} \frac{d}{dx} - a_{\mathbb{R}}^{(0)} = \frac{d^2}{dx^2} - \sum_{\mu,\nu=0}^{3} a_{\mu\nu}^{(1)} L_\mu R_\nu \frac{d}{dx} - \sum_{\mu,\nu=0}^{3} a_{\mu\nu}^{(0)} L_\mu R_\nu \ .$$

By introducing the $\mathbb{R}$ linear quaternionic matrix

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ a_{\mathbb{R}}^{(0)} & a_{\mathbb{R}}^{(1)} \end{pmatrix}$$

and the quaternionic column vector

$$\Phi = \begin{bmatrix} \varphi \\ \frac{d}{dx}\varphi \end{bmatrix} \ ,$$

we can rewrite Eq. (9) in matrix form

$$\tfrac{d}{dx}\, \Phi(x) = M_2^{\mathbb{R}}\, \Phi(x) \ . \tag{10}$$

The real matrix counterpart of the quaternionic operator $M_2^{\mathbb{R}}$, from now on denoted by $M_8[\mathbb{R}]$, has an 8-dimensional eigenvalue spectrum characterized by real numbers and/or complex conjugate pairs (the translation tables are given in the appendix). Let $J_8[\mathbb{C}]$ be the complex Jordan form of $M_8[\mathbb{R}]$,

$$J_8[\mathbb{C}] = R_{2m} \oplus Z_n \oplus \bar{Z}_n \ , \quad m + n = 4 \ , \tag{11}$$

where $R_{2m}$ and $Z_n$ represent the matrix Jordan blocks containing, respectively, the real and complex eigenvalues of $M_8[\mathbb{R}]$. By using an appropriate similarity matrix

$$S_8[\mathbb{C}] = \begin{pmatrix} S_{2m\times 2m}^{(1)}[\mathbb{R}] & S_{2m\times n}^{(4)}[\mathbb{C}] & \bar{S}_{2m\times n}^{(4)}[\mathbb{C}] \\ S_{n\times 2m}^{(2)}\ [\mathbb{R}] & S_{n\times n}^{(5)}\ [\mathbb{C}] & \bar{S}_{n\times n}^{(5)}\ [\mathbb{C}] \\ S_{n\times 2m}^{(3)}\ [\mathbb{R}] & S_{n\times n}^{(6)}\ [\mathbb{C}] & \bar{S}_{n\times n}^{(6)}\ [\mathbb{C}] \end{pmatrix} \ , \tag{12}$$

we can rewrite $M_8[\mathbb{R}]$ as product of three complex matrices, that is $S_8[\mathbb{C}]J_8[\mathbb{C}]S_8^{-1}[\mathbb{C}]$. The problem is now represented by the impossibility to translate the single terms of the previous matrix product by $\mathbb{R}$ linear quaternionic matrices. To overcome this difficulty we introduce the complex matrix

$$W_8[\mathbb{C}] = \mathbf{1}_{2m} \oplus \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes \mathbf{1}_n \right] . \tag{13}$$

An easy algebraic calculation shows that

$$M_8[\mathbb{R}] = T_8[\mathbb{R}]\, J_8[\mathbb{R}]\, T_8^{-1}[\mathbb{R}]$$

where

$$T_8[\mathbb{R}] = S_8[\mathbb{C}]\, W_8[\mathbb{C}] = \begin{pmatrix} S^{(1)}_{2m\times 2m}[\mathbb{R}] & \sqrt{2}\,\mathrm{Re}\left\{S^{(4)}_{2m\times n}[\mathbb{C}]\right\} & -\sqrt{2}\,\mathrm{Im}\left\{S^{(4)}_{2m\times n}[\mathbb{C}]\right\} \\ S^{(2)}_{n\times 2m}[\mathbb{R}] & \sqrt{2}\,\mathrm{Re}\left\{S^{(5)}_{n\times n}[\mathbb{C}]\right\} & -\sqrt{2}\,\mathrm{Im}\left\{S^{(5)}_{n\times n}[\mathbb{C}]\right\} \\ S^{(3)}_{n\times 2m}[\mathbb{R}] & \sqrt{2}\,\mathrm{Re}\left\{S^{(6)}_{n\times n}[\mathbb{C}]\right\} & -\sqrt{2}\,\mathrm{Im}\left\{S^{(6)}_{n\times n}[\mathbb{C}]\right\} \end{pmatrix} , \tag{14}$$

and

$$J_8[\mathbb{R}] = W_8^{-1}[\mathbb{C}]\, J_8[\mathbb{C}]\, W_8[\mathbb{C}] = R_{2m} \oplus \begin{pmatrix} \mathrm{Re}[Z_n] & -\mathrm{Im}[Z_n] \\ \mathrm{Im}[Z_n] & \mathrm{Re}[Z_n] \end{pmatrix} . \tag{15}$$

The real Jordan canonical form $J_8[\mathbb{R}]$ can be decomposed into the sum of three commuting real matrices, that is the diagonal matrix

$$D_8[\mathbb{R}] \;=\; \mathrm{Diag}\left\{\lambda_1\,,\,...\,,\,\lambda_m\,,\,\tilde{\lambda}_1\,,\,...\,,\,\tilde{\lambda}_m\,,\,\mathrm{Re}\,[z_1]\,,\,...,\,\mathrm{Re}\,[z_n]\,,\,\mathrm{Re}\,[z_1]\,,\,...\,,\,\mathrm{Re}\,[z_n]\right\} ,$$

the anti-symmetric matrix

$$A_8[\mathbb{R}] \;=\; \mathbf{0}_{2m} \oplus \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Diag}\,\{\mathrm{Im}\,[z_1]\,,\,...\,,\,\mathrm{Im}\,[z_n]\} \right] ,$$

and the nilpotent matrix $N_8[\mathbb{R}]$ (a lower triangular matrix whose only nonzero elements are ones which appear in some of the sub-diagonal positions). The real Jordan form $J_8[\mathbb{R}] = D_8[\mathbb{R}] + A_8[\mathbb{R}] + N_8[\mathbb{R}]$ and the similarity transformation $T_8[\mathbb{R}]$ can be now translated in their real linear quaternionic counterparts $J_2^{\mathbb{R}} = D_2^{\mathbb{R}} + A_2^{\mathbb{R}} + N_2^{\mathbb{R}}$ and $T_2^{\mathbb{R}}$. The matrix solution of Eq.(10) then reads

$$\begin{aligned} \Phi(x) &= \exp[M_2^{\mathbb{R}}\,(x-x_0)]\Phi(x_0) \\ &= T_2^{\mathbb{R}}\,\exp[J_2^{\mathbb{R}}\,(x-x_0)]\,(T_2^{\mathbb{R}})^{-1}\,\Phi(x_0) \\ &= T_{\mathbb{R}}\,\exp[D_{\mathbb{R}}\,(x-x_0)]\,\exp[A_{\mathbb{R}}\,(x-x_0)]\,\exp[N_{\mathbb{R}}\,(x-x_0)]\,(T_2^{\mathbb{R}})^{-1}\,\Phi(x_0)\;. \end{aligned} \tag{16}$$

In the case of a null nilpotent matrix (explicit examples are given in subsection A), it can be verified by direct calculations that the general solution of real linear quaternionic differential equations with constant coefficients is

$$\begin{aligned} \varphi(x) \;=\; & \sum_{r=1}^{m} \left\{ u_r\,\exp[\lambda_r\,x]\,\alpha_r + \tilde{u}_r\,\exp[\tilde{\lambda}_r\,x]\,\beta_r \right\} + \\ & \sum_{p=1}^{n} \{v_p\,\cos[b_p\,x] - \tilde{v}_p\,\sin[b_p\,x]\}\,\exp[a_p\,x]\,\gamma_p + \\ & \sum_{p=1}^{n} \{\tilde{v}_p\,\cos[b_p\,x] + v_p\,\sin[b_p\,x]\}\,\exp[a_p\,x]\,\delta_p\;, \quad m+n=4\;, \end{aligned} \tag{17}$$

where $a_p = \mathrm{Re}\,[z_p]$, $b_p = \mathrm{Im}\,[z_p]$ and $u_r, \tilde{u}_r, v_p, \tilde{v}_p \in \mathbb{H}$. The 8 real constants $\alpha_r$, $\beta_r$, $\gamma_p$, and $\delta_p$ are fixed by the quaternionic initial conditions $\varphi(x_0) = \varphi_0$ and $\frac{d\varphi}{dx}(x_0) = \varphi_1$. The important point to note here is that the particular solutions corresponding to the complex eigenvalue $z = a + ib$ are given by a quaternionic combination of $\cos[b\,x]\,\exp[a\,x]$ and $\sin[b\,x]\,\exp[a\,x]$, namely

$$\{v\,\cos[b\,x] - \tilde{v}\,\sin[b\,x]\}\exp[a\,x] \quad \text{and} \quad \{\tilde{v}\,\cos[b\,x] + v\,\sin[b\,x]\}\exp[a\,x]\ .$$

For $\mathbb{C}$ linear quaternionic differential operators (an explicit example is given in subsection B), the general solution (17) reduces to

$$\varphi(x) = \sum_{r=1}^{m} u_r\,\exp[\lambda_r\,x]\,c_r + \sum_{p=1}^{n} v_p\,\exp[z_p\,x]\,d_p\ ,\quad c_r\,,\,d_p \in \mathbb{C}\ . \tag{18}$$

## A. Null nilpotent matrix

Let us solve the second order $\mathbb{R}$ linear quaternionic differential equation

$$\left[\frac{d^2}{dx^2} - L_iR_j\,\frac{d}{dx} - L_jR_i\right]\varphi(x) = 0\ , \tag{19}$$

with initial conditions $\varphi(0) = j$ and $\frac{d\varphi}{dx}(0) = k$. The matrix operator corresponding to this equation is

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ L_jR_i & L_iR_j \end{pmatrix}\ . \tag{20}$$

By using the matrix $T_2^{\mathbb{R}}$

$$\begin{array}{rcl}
\{T_2^{\mathbb{R}}\}_{11} & = & \frac{1+\sqrt{5}}{4}\,(L_i + L_kR_j - L_iR_k - L_kR_i) - \frac{1-\sqrt{5}}{4}\,(1 - L_jR_j + L_jR_i - R_k)\ ,\\
\{T_2^{\mathbb{R}}\}_{12} & = & \frac{\sqrt{3}}{2\sqrt{2}}\,(1 + L_iR_i - L_jR_i - L_k) - \frac{1}{2\sqrt{2}}\,(L_i + R_i + L_j + L_kR_i)\ ,\\
\{T_2^{\mathbb{R}}\}_{21} & = & \frac{1}{2}\,(L_iR_i + L_kR_k - L_jR_k + R_i - L_j + R_j + L_k + L_iR_j)\ ,\\
\{T_2^{\mathbb{R}}\}_{22} & = & \frac{1}{2\sqrt{2}}\,(L_jR_k - L_kR_j - L_iR_k + R_j)\ ,
\end{array}$$

and its inverse

$$\begin{array}{rcl}
\{(T_2^{\mathbb{R}})^{-1}\}_{11} & = & \frac{1}{4\sqrt{5}}\,(1 - L_jR_j + L_kR_j - L_i - L_kR_i - L_iR_k + L_jR_i + R_k)\ ,\\
\{(T_2^{\mathbb{R}})^{-1}\}_{12} & = & \frac{5+\sqrt{5}}{40}\,(L_iR_i + L_kR_k + L_iR_j - L_k) + \frac{5-\sqrt{5}}{40}\,(L_j - R_j - L_jR_k - R_i)\ ,\\
\{(T_2^{\mathbb{R}})^{-1}\}_{21} & = & \frac{1}{2\sqrt{6}}\,(1 + L_iR_i - L_jR_i + L_k)\ ,\\
\{(T_2^{\mathbb{R}})^{-1}\}_{22} & = & \frac{1}{4\sqrt{6}}\,(L_jR_j - L_kR_k + L_iR_j + R_k) + \frac{1}{4\sqrt{2}}\,(L_kR_j - L_jR_k + L_iR_k + R_j)\ ,
\end{array}$$

we can rewrite $M_2^{\mathbb{R}}$ in terms of

$$D_2^{\mathbb{R}} = \frac{1}{2}\begin{pmatrix} L_iR_i + \sqrt{5}\,L_kR_k & 0 \\ 0 & L_jR_j \end{pmatrix} \quad \text{and} \quad A_2^{\mathbb{R}} = \frac{\sqrt{3}}{2}\begin{pmatrix} 0 & 0 \\ 0 & R_j \end{pmatrix}\ .$$

From Eq. (16) we obtain the following solution

$$\begin{array}{rcl}
\varphi(x) & = & \{T_2^{\mathbb{R}}\}_{11}\,\exp\left[\frac{L_iR_i+\sqrt{5}\,L_kR_k}{2}\,x\right]\left[\{(T_2^{\mathbb{R}})^{-1}\}_{11}\,j + \{(T_2^{\mathbb{R}})^{-1}\}_{12}\,k\right] +\\
& & \{T_2^{\mathbb{R}}\}_{12}\,\exp\left[\frac{L_jR_j+\sqrt{3}\,R_j}{2}\,x\right]\left[\{(T_2^{\mathbb{R}})^{-1}\}_{21}\,j + \{(T_2^{\mathbb{R}})^{-1}\}_{22}\,k\right]\\
& = & \left\{\frac{i-j}{2\sqrt{5}}\left(\sinh\left[\frac{\sqrt{5}}{2}\,x\right] - \sqrt{5}\,\cosh\left[\frac{\sqrt{5}}{2}\,x\right]\right) + \frac{1+k}{\sqrt{3}}\,\sin\left[\frac{\sqrt{3}}{2}\,x\right]\right\}\exp[x/2] +\\
& & \left\{\frac{k-1}{\sqrt{5}}\,\sinh\left[\frac{\sqrt{5}}{2}\,x\right] + \frac{i+j}{2}\left(\cos\left[\frac{\sqrt{3}}{2}\,x\right] + \frac{1}{\sqrt{3}}\,\sin\left[\frac{\sqrt{3}}{2}\right]\right)\right\}\exp[-x/2]\ .
\end{array} \tag{21}$$

In the case of a null nilpotent matrix, the general solution of real linear quaternionic differential equations can be written in terms of the eigenvalue spectrum of $M_8[\mathbb{R}]$, real matrix counterpart of $M_2^{\mathbb{R}}$. In this particular case, the eigenvalue spectrum is

$$\left\{\frac{1+\sqrt{5}}{2}\,,\ \frac{1-\sqrt{5}}{2}\,,\ -\frac{1+\sqrt{5}}{2}\,,\ -\frac{1-\sqrt{5}}{2}\,,\ \frac{1+i\sqrt{3}}{2}\,,\ -\frac{1-i\sqrt{3}}{2}\,,\ \frac{1-i\sqrt{3}}{2}\,,\ -\frac{1+i\sqrt{3}}{2}\right\}\ .$$

Consequently, Eq. (17) becomes

$$\varphi(x) = u_1 \exp\left[\tfrac{1+\sqrt{5}}{2}\,x\right]\,\alpha_1 + u_2 \exp\left[\tfrac{1-\sqrt{5}}{2}\,x\right]\,\alpha_2 + \tilde{u}_1 \exp\left[-\tfrac{1+\sqrt{5}}{2}\,x\right]\,\beta_1 + \tilde{u}_2 \exp\left[-\tfrac{1-\sqrt{5}}{2}\,x\right]\,\beta_2 +$$
$$\left\{v_1 \cos\left[\tfrac{\sqrt{3}}{2}\,x\right] - \tilde{v}_1 \sin\left[\tfrac{\sqrt{3}}{2}\,x\right]\right\}\exp\left[x/2\right]\,\gamma_1 + \left\{\tilde{v}_1 \cos\left[\tfrac{\sqrt{3}}{2}\,x\right] + v_1 \sin\left[\tfrac{\sqrt{3}}{2}\,x\right]\right\}\exp\left[x/2\right]\,\delta_1 +$$
$$\left\{v_2 \cos\left[\tfrac{\sqrt{3}}{2}\,x\right] - \tilde{v}_2 \sin\left[\tfrac{\sqrt{3}}{2}\,x\right]\right\}\exp\left[-x/2\right]\,\gamma_2 + \left\{\tilde{v}_2 \cos\left[\tfrac{\sqrt{3}}{2}\,x\right] + v_2 \sin\left[\tfrac{\sqrt{3}}{2}\,x\right]\right\}\exp\left[-x/2\right]\,\delta_2\ .$$

By direct calculations, we can determine the quaternionic coefficients $u_{1,2}$, $\tilde{u}_{1,2}$, $v_{1,2}$ and $\tilde{v}_{1,2}$. We find

$$\begin{array}{lcl} u_1 = i - j & , & \tilde{u}_1 = 1 - k\ ,\\ u_2 = i - j & , & \tilde{u}_2 = 1 - k\ ,\\ v_1 = 1 + k & , & \tilde{v}_1 = 0\ ,\\ v_2 = i + j & , & \tilde{v}_2 = 0\ . \end{array}$$

The particular solutions corresponding to the complex eigenvalues of $M_8[\mathbb{R}]$ are thus given by

$$z_1 = \tfrac{1+i\sqrt{3}}{2}\,,\ \bar{z}_1\ :\quad (1+k)\,\exp\left[x/2\right]\ \begin{array}{c}\cos\left[\tfrac{\sqrt{3}}{2}\,x\right]\\ \\ \sin\left[\tfrac{\sqrt{3}}{2}\,x\right]\end{array}\ ,$$

and

$$z_2 = -\,\tfrac{1-i\sqrt{3}}{2}\,,\ \bar{z}_2\ :\quad (i+j)\,\exp\left[-\,x/2\right]\ \begin{array}{c}\cos\left[\tfrac{\sqrt{3}}{2}\right]\\ \\ \sin\left[\tfrac{\sqrt{3}}{2}\right]\end{array}\ ,$$

The initial conditions $\varphi(0) = j$ and $\frac{d\varphi}{dx}(0) = k$ make the solution completely determined,

$$\begin{array}{lclclclclclclcl} \alpha_1 & = & \frac{1-\sqrt{5}}{4\sqrt{5}} & , & \beta_1 & = & \frac{1}{2\sqrt{5}} & , & \gamma_1 & = & 0 & , & \delta_1 & = & \frac{1}{\sqrt{3}} & ,\\ \alpha_2 & = & -\frac{1+\sqrt{5}}{4\sqrt{5}} & , & \beta_2 & = & -\frac{1}{2\sqrt{5}} & , & \gamma_2 & = & \frac{1}{2} & , & \delta_2 & = & \frac{1}{2\sqrt{3}} & . \end{array}$$

Let us now consider the following second order $\mathbb{R}$ linear quaternionic differential equation

$$\left[\frac{d^2}{dx^2} - (L_iR_i + L_jR_j)\,\frac{d}{dx} + R_k - L_k\right]\varphi(x) = 0\ . \tag{22}$$

The eigenvalue spectrum of the real matrix counterpart of

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1\\ L_k - R_k & L_iR_i + L_jR_j \end{pmatrix} \tag{23}$$

is

$$\{2\,,\,-2\,,\,0\,,\,0\,,\,1+i\,,\,-(1-i)\,,\,1-i\,,\,-(1+i)\,\}\ .$$

In this case $v_1 \cos x\, \exp[x]$ and $v_2 \cos x\, \exp[-x]$ does not represent (non trivial) particular solutions. In fact, in order to satisfy Eq. (22) we have to impose the following constraints on the quaternionic coefficients $v_{1,2}$

$$\begin{array}{rcl} [L_iR_i + L_jR_j - 2]\ v_1 = 0 & \Rightarrow & v_1 = i\,\beta + j\,\gamma\ ,\\ [L_iR_i + L_jR_j + L_k - R_k]\ v_1 = 0 & \Rightarrow & v_1 = 0\ ,\\ [L_iR_i + L_jR_j + 2]\ v_2 = 0 & \Rightarrow & v_2 = \alpha + j\,\delta\ ,\\ [L_iR_i + L_jR_j - L_k + R_k]\ v_2 = 0 & \Rightarrow & v_2 = 0\ . \end{array}$$

The particular solutions corresponding to the complex eigenvalues $z_1 = 1 + i$ and $z_2 = -(1+i)$ are

$$z_1\,,\ \bar{z}_1\ :\quad \exp\left[x\right]\ \begin{array}{c} j\,\cos\left[x\right] - i\,\sin\left[x\right]\\ \\ i\,\cos\left[x\right] + j\,\sin\left[x\right]\end{array}\ ,$$

and

$$z_2\,,\,\bar{z}_2\;:\qquad \exp[-x]\quad \begin{array}{c} j\,\cos[x]+i\,\sin[x] \\ \\ i\,\cos[x]-j\,\sin[x] \end{array}\qquad .$$

The general solution is given by

$$\begin{aligned}\varphi(x) &= \exp[-2\,x]\;\alpha_1+k\,\exp[2\,x]\;\alpha_2+\beta_1+k\,\beta_2+\\ &\quad \{j\,\cos[x]-i\,\sin[x]\}\,\exp[x]\;\gamma_1+\{i\,\cos[x]+j\,\sin[x]\}\,\exp[x]\;\delta_1+\\ &\quad \{j\,\cos[x]+i\,\sin[x]\}\,\exp[-x]\;\gamma_2+\{i\,\cos[x]-j\,\sin[x]\}\,\exp[-x]\;\delta_2\;.\end{aligned}$$

## B. Complex linear case

We now determine the solution of the second order $\mathbb{C}$ linear quaternionic differential equation

$$\left[\frac{d^2}{dx^2}-L_jR_i\right]\varphi(x)=0\;. \tag{24}$$

The eigenvalue spectrum of the real matrix counterpart of

$$M_2^{\mathbb{R}}=\begin{pmatrix} 0 & 1\\ L_jR_i & 0\end{pmatrix} \tag{25}$$

is

$$\{1\,,\,-1\,,\,1\,,\,-1\,,\,i\,,\,i\,,\,-i\,,\,-i\,\}\;\;.$$

The general solution is then given by

$$\begin{aligned}\varphi(x) &= u_1\,\exp[x]\;\alpha_1+u_2\,\exp[-x]\;\alpha_2+\tilde{u}_1\,\exp[x]\;\beta_1+\tilde{u}_2\,\exp[-x]\;\beta_2+\\ &\quad \{v_1\,\cos[x]-\tilde{v}_1\,\sin[x]\}\;\gamma_1+\{\tilde{v}_1\,\cos[x]+v_1\,\sin[x]\}\;\delta_1+\\ &\quad \{v_2\,\cos[x]-\tilde{v}_2\,\sin[x]\}\;\gamma_2+\{\tilde{v}_2\,\cos[x]+v_2\,\sin[x]\}\;\delta_2\;.\end{aligned} \tag{26}$$

By direct calculations, we can determine the quaternionic coefficients $u_{1,2}$, $\tilde{u}_{1,2}$, $v_{1,2}$ and $\tilde{v}_{1,2}$. We find

$$\begin{array}{lcl} u_1=k-1 & , & \tilde{u}_1=j-i\;,\\ u_2=k-1 & , & \tilde{u}_2=j-i\;,\\ v_1=1+k & , & \tilde{v}_1=0\;,\\ v_2=i+j & , & \tilde{v}_2=0\;.\end{array}$$

Consequently, Eq. (26) becomes

$$\varphi(x)=(k-1)\;[\exp[x]\;c_1+\exp[-x]\;c_2]+(1+k)\;\{\exp[ix]\;d_1+\exp[-ix]\;d_2\}\;\;. \tag{27}$$

The initial conditions $\varphi(0)=j$ and $\frac{d\varphi}{dx}(0)=k$ fix the following solution

$$\varphi(x)=\{(j-i)\,\cosh x+(k-1)\,\sinh x+(i+j)\,\exp[-ix]\}\,/2\;. \tag{28}$$

## C. Non null nilpotent matrix

As a last example we consider the differential equation

$$\left[\frac{d^2}{dx^2}-(L_iR_j+L_jR_i)\right]\varphi(x)=0\;. \tag{29}$$

The eigenvalue spectrum of the real matrix counterpart of

$$M_2^{\mathbb{R}}=\begin{pmatrix} 0 & 1\\ L_iR_j+L_jR_i & 0\end{pmatrix} \tag{30}$$

is

$$\left\{0\,,\,0\,,\,0\,,\,0\,,\,\sqrt{2}\,,\,-\sqrt{2}\,,\,i\sqrt{2}\,,\,-i\sqrt{2}\right\}\;.$$

By using an appropriate matrix $T_2^{\mathbb{R}}$, we can rewrite $M_2^{\mathbb{R}}$ in terms of

$$D_2^{\mathbb{R}}=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}0 & 0\\ 0 & L_jR_j+L_kR_k\end{array}\right)\,,\;A_2^{\mathbb{R}}=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}0 & 0\\ 0 & L_i-R_i\end{array}\right)\;\text{and}\;N_2^{\mathbb{R}}=\frac{1}{2}\left(\begin{array}{cc}L_i-L_kR_j & 0\\ 0 & 0\end{array}\right)\,,$$

$M_2^{\mathbb{R}}=T_2^{\mathbb{R}}\,(D_2^{\mathbb{R}}+A_2^{\mathbb{R}}+N_2^{\mathbb{R}})\,(T_2^{\mathbb{R}})^{-1}$. Notwithstanding its diagonal form, $N_2^{\mathbb{R}}$ represents a nilpotent matrix. In fact, an easy algebraic calculation shows that $(N_2^{\mathbb{R}})^2=0$. This implies in the general solution the explicit presence of the real variable $x$. Explicitly, we have

$$\begin{aligned}\varphi(x)\;&=\;\alpha_1+k\,\alpha_2+(\tilde{\alpha}_1+k\,\tilde{\alpha}_2)\;x\,+\\&\quad(i-j)\,\left\{\exp\left[\sqrt{2}\,x\right]\;\beta_1+\exp\left[-\sqrt{2}\,x\right]\;\beta_2\right\}+\\&\quad(i+j)\,\left\{\cos\left[\sqrt{2}\,x\right]\gamma_1+\sin\left[\sqrt{2}\,x\right]\;\delta_1\right\}\;.\end{aligned}$$

To make this solution determined, let us specify certain supplementary constraints, $\varphi(0)=j$ and $\frac{d\varphi}{dx}(0)=k$,

$$\varphi(x)=kx+\tfrac{j-i}{2}\;\cosh\left[\sqrt{2}\,x\right]+\tfrac{i+j}{2}\;\cos\left[\sqrt{2}\,x\right]\;. \tag{31}$$

By changing the initial conditions, $\varphi(0)=k$ and $\frac{d\varphi}{dx}(0)=j$, we find

$$\varphi(x)=k+\tfrac{j-i}{2\sqrt{2}}\;\sinh\left[\sqrt{2}\,x\right]+\tfrac{i+j}{2\sqrt{2}}\;\sin\left[\sqrt{2}\,x\right]\;. \tag{32}$$

# III. Conclusions

In this paper, we have extended the resolution method of $\mathbb{H}$ and $\mathbb{C}$ linear ordinary differential equations with quaternionic constant coefficients [1] to $\mathbb{R}$ linear differential operators. By a matrix approach, we have shown that particular solutions of differential equations with constant quaternionic coefficients which appear both on the left and on the right can be given in terms of the eigenvalues of the matrix $M_{4n}[\mathbb{R}]$, representing the real counterpart of the quaternionic operators, $M_n^{\mathbb{R}}$, associated to the $\mathbb{R}$ linear quaternionic differential equation. In correspondence to the eigenvalue $z=a+ib$, we have found the following particular solutions

$$\{q\,\cos\,[b\,x]-p\,\sin\,[b\,x]\}\exp\,[a\,x]\quad\text{and}\quad\{p\,\cos\,[b\,x]+q\,\sin\,[b\,x]\}\exp\,[a\,x]\;\;,\quad q\,,\,p\in\mathbb{H}\;. \tag{33}$$

When $q=p$ or $p=0$ these solutions reduce to

$$q\,\cos\,[b\,x]\,\exp\,[a\,x]\quad\text{and}\quad q\,\sin\,[b\,x]\,\exp\,[a\,x]\;\;.$$

For $\mathbb{C}$ linear differential operators, the particular solutions (33) couple to give

$$q\,\exp\,[(a+ib)\,x]\;\;.$$

Our discussion can be viewed as a preliminary step towards a full understanding of the role that quaternions could play in analysis, linear algebra and, consequently, in physical applications. A complete theory of quaternionic differential operators (as well of the quaternionic eigenvalue problem) is at present far from being conclusive and deserves further investigations.

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# Appendix. Real translation tables

**Table 1**.

| | | | |
|---|---|---|---|
| $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $1$ | $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ $R_i$ | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $R_j$ | $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $R_k$ |
| $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $L_i$ | $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $L_i\,R_i$ | $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $L_i\,R_j$ | $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ $L_i\,R_k$ |
| $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ $L_j$ | $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ $L_j\,R_i$ | $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $L_j\,R_j$ | $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ $L_j\,R_k$ |
| $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $L_k$ | $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ $L_k\,R_i$ | $\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ $L_k\,R_j$ | $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $L_k\,R_k$ |

**Example**.

$$L_i - 2\,R_j - 3\,L_iR_k \quad \rightarrow \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} - 2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - 3 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \quad \begin{pmatrix} 0 & -1 & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

**Table 2**.

| $1/4$ | $1$ | $L_iR_i$ | $L_jR_j$ | $L_kR_k$ |
|---|---|---|---|---|
| $a_{00}$ | $+$ | $-$ | $-$ | $-$ |
| $a_{11}$ | $+$ | $-$ | $+$ | $+$ |
| $a_{22}$ | $+$ | $+$ | $-$ | $+$ |
| $a_{33}$ | $+$ | $+$ | $+$ | $-$ |
| $1/4$ | $L_i$ | $R_i$ | $L_jR_k$ | $L_kR_j$ |
| $a_{01}$ | $-$ | $-$ | $+$ | $-$ |
| $a_{10}$ | $+$ | $+$ | $+$ | $-$ |
| $a_{23}$ | $-$ | $+$ | $-$ | $-$ |
| $a_{32}$ | $+$ | $-$ | $-$ | $-$ |

| $1/4$ | $L_j$ | $R_j$ | $L_iR_k$ | $L_kR_i$ |
|---|---|---|---|---|
| $a_{02}$ | $-$ | $-$ | $-$ | $+$ |
| $a_{20}$ | $+$ | $+$ | $-$ | $+$ |
| $a_{13}$ | $+$ | $-$ | $-$ | $-$ |
| $a_{31}$ | $-$ | $+$ | $-$ | $-$ |
| $1/4$ | $L_k$ | $R_k$ | $L_iR_j$ | $L_jR_i$ |
| $a_{03}$ | $-$ | $-$ | $+$ | $-$ |
| $a_{30}$ | $+$ | $+$ | $+$ | $-$ |
| $a_{12}$ | $-$ | $+$ | $-$ | $-$ |
| $a_{21}$ | $+$ | $-$ | $-$ | $-$ |

**Example**.

$$A := \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & \text{-1} & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & \text{-1} \\ 0 & 1 & 1 & 0 \end{pmatrix} .$$

$$\begin{aligned} A \quad \rightarrow \quad & (a_{01}/4)\ [-L_i - R_i + L_jR_k - L_kR_j] + \\ & (a_{02}/4)\ [-L_j - R_j - L_iR_k + L_kR_i] + \\ & (a_{10}/4)\ [+L_i + R_i + L_jR_k - L_kR_j] + \\ & (a_{13}/4)\ [+L_j - R_j - L_iR_k - L_kR_i] + \\ & (a_{20}/4)\ [+L_j + R_j - L_iR_k + L_kR_i] + \\ & (a_{23}/4)\ [-L_i + R_i - L_jR_k - L_kR_j] + \\ & (a_{31}/4)\ [-L_j + R_j - L_iR_k - L_kR_i] + \\ & (a_{32}/4)\ [+L_i - R_i + L_jR_k - L_kR_j] \\ \rightarrow \quad & L_i - 2\,R_j - 3\,L_iR_k\ . \end{aligned}$$