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Quaternionic Lorentz group and Dirac equation

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Abstract. We formulate Lorentz group representations in which ordinary complex numbers are replaced by linear functions of real quaternions and introduce dotted and undotted quaternionic one-dimensional spinors. To extend to parity the space-time transformations, we combine these one-dimensional spinors into bi-dimensional column vectors. From the transformation properties of the two-component spinors, we derive a quaternionic chiral representation for the space-time algebra. Finally, we obtain a quaternionic bi-dimensional version of the Dirac equation.

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1 Introduction

The isomorphism between unitary quaternions and space rotations is extended to Lorentz boosts by means of linear functions of real quaternions. The use of linear functions of real quaternions in formulating Lorentz group representations lead to the introduction of two *inequivalent* one-dimensional spinors. In order to include parity, we construct bi-dimensional column vectors which satisfy a quaternionic version of the Dirac equation. For massless particles, this equation decouples into one-dimensional equations, which represent the quaternionic counterparts of the Weyl equations. This formulation of the Dirac equation allows to find a quaternionic chiral representation for the gamma matrices. This space-time matrix representation can be used in formulating quaternionic gauge models for the electroweak interaction. The use of linear functions of real quaternions and the assumption of a complex projection for the inner product [1,2], play a fundamental role to recover the mathematical properties of the Lorentz group [3], to define an appropriate momentum operator [4] and to obtain the right orthogonality between the solutions of the Dirac equation [5].

The paper is structured as follows. In section 2, we introduce the quaternionic Lorentz group [3] by using left/right operators [6,7]. Section 3 is intended to study quaternionic dotted and undotted spinors. In such a section, we give the quaternionic chiral representation for the gamma matrices. In section 4, we discuss the quaternionic Dirac equation [4,8,9,10,11] and justify the adoption of a complex projection for the inner product. In section 5, we give a brief introduction to quaternionic gauge models. Conclusions and out-looks are drawn in the final section.

2 Linear functions of real quaternions and Lorentz group

A generic quaternion q can be defined in terms of real [12] or complex [13] numbers as follows

$$q = a_0 + ia_1 + ja_2 + ka_3 = \xi + j\eta \ , \qquad a_{0,1,2,3} \in \mathbb{R} \ , \quad \xi, \eta \in \mathbb{C}(1, i) \ , \tag{1}$$

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where the imaginary units i, j and k satisfy

$$i^2 = j^2 = k^2 = ijk = -1 \ . \tag{2}$$

The conjugate of q, denoted by $\bar{q}$, is given by $\bar{q} = a_0 - ia_1 - ja_2 - ka_3 = \xi^* - j\eta$.

Due to the non-commutative nature of the quaternionic field, we have to distinguish between the *left* and *right* action of the imaginary units i, j, k. To do it, we introduce linear functions of real quaternions. For example, we can use the linear function $i\,(\quad)\,j$ to represent the simultaneous action on a quaternionic state q of the imaginary unit i from the left and of the imaginary unit j from the right, i.e. $i\,q\,j$. In order to shorten our notation, we shall rewrite linear functions of real quaternions in terms of the left/right operators [6,7]

$$\boldsymbol{L} \equiv (L_i, L_j, L_k) \quad \text{and} \quad \boldsymbol{R} \equiv (R_i, R_j, R_k) \ , \tag{3}$$

$$\boldsymbol{L} \ : \ \mathbb{H} \to \mathbb{H} \ , \quad \boldsymbol{L}\,q \equiv \boldsymbol{h}\,q \quad \text{and} \quad \boldsymbol{R} \ : \ \mathbb{H} \to \mathbb{H} \ , \quad \boldsymbol{R}\,q \equiv q\,\boldsymbol{h} \ , \qquad \boldsymbol{h} \equiv (i,j,k) \ .$$

In this formalism, the linear function of real quaternions $i\,(\quad)\,j$ is concisely expressed by L_iR_j. The algebra of left and right operators is given by

$$L_i^2 = L_j^2 = L_k^2 = L_iL_jL_k = R_i^2 = R_j^2 = R_k^2 = R_kR_jR_i = -\mathbf{1} \ ,$$

and by the commutation relations

$$[\, L_{i,j,k} \ , \ R_{i,j,k}\,] = 0 \ .$$

The Lorentz group $\mathsf{SO}(3,1)$ is characterized by six generators. The anti-hermitian generators associated to the space rotations, $\boldsymbol{\mathcal{A}}$, and the hermitian boost generators, $\boldsymbol{\mathcal{B}}$, satisfy the following commutation relations

$$\begin{array}{lll} \mathcal{A}_x = [\mathcal{A}_y, \mathcal{A}_z] \ , & \mathcal{A}_x = [\mathcal{B}_z, \mathcal{B}_y] \ , & \mathcal{B}_x = [\mathcal{A}_y, \mathcal{B}_z] = [\mathcal{B}_y, \mathcal{A}_z] \ , \\ \mathcal{A}_y = [\mathcal{A}_z, \mathcal{A}_x] \ , & \mathcal{A}_y = [\mathcal{B}_x, \mathcal{B}_z] \ , & \mathcal{B}_y = [\mathcal{A}_z, \mathcal{B}_x] = [\mathcal{B}_z, \mathcal{A}_x] \ , \\ \mathcal{A}_z = [\mathcal{A}_x, \mathcal{A}_y] \ , & \mathcal{A}_z = [\mathcal{B}_y, \mathcal{B}_x] \ , & \mathcal{B}_z = [\mathcal{A}_x, \mathcal{B}_y] = [\mathcal{B}_x, \mathcal{A}_y] \ . \end{array} \tag{4}$$

By using the left quaternionic imaginary units $\boldsymbol{L}$ and the right complex imaginary unit R_i, we can obtain a one-dimensional representation for the Lorentz generators [3]. The rotation generators are given in terms of the left-acting quaternionic imaginary units,

$$\mathcal{A}_x = L_i/2 \ , \quad \mathcal{A}_y = L_j/2 \ , \quad \mathcal{A}_z = L_k/2 \qquad \in \mathbb{H}^L \ . \tag{5}$$

The boost generators are represented by the joint action of the left-quaternionic imaginary units, i, j and k, and of the right-complex imaginary unit i

$$\mathcal{B}_x = \pm L_iR_i/2 \ , \quad \mathcal{B}_y = \pm L_jR_i/2 \ , \quad \mathcal{B}_z = \pm L_kR_i/2 \qquad \in \mathbb{H}^L \otimes \mathbb{C}^R \ . \tag{6}$$

These two *inequivalent* quaternionic representations for the boost generators imply two different transformation laws for quaternionic spinors,

$$s_+ = \exp\left[\boldsymbol{L}\cdot(\boldsymbol{\theta} + R_i\ \boldsymbol{\varphi})/2\right] \quad \text{and} \quad s_- = \exp\left[\boldsymbol{L}\cdot(\boldsymbol{\theta} - R_i\ \boldsymbol{\varphi})/2\right] \ , \tag{7}$$

where $\boldsymbol{\theta} \equiv (\theta_x, \theta_y, \theta_z)$ and $\boldsymbol{\varphi} \equiv (\varphi_x, \varphi_y, \varphi_z)$. For consistence, we introduce two quaternionic spinors, $q_\pm = \xi_\pm + j\eta_\pm$, which, under the action of the quaternionic unitary group, transform into $s_\pm\, q_\pm$.

For a detailed review of quaternionic group theory, we refer the reader to the Gilmore book [14] and refs. [7,15,16,17,18,19].

3 Dotted and undotted spinors

As remarked in the previous section, the two possible signs of $\boldsymbol{\mathcal{B}}$ in equation(6) imply

$$q_+ \ \to \ s_+\,q_+ = \exp\left[\boldsymbol{L}\cdot(\boldsymbol{\theta} + R_i\ \boldsymbol{\varphi})/2\right]\,q_+ \quad \text{and} \quad q_- \ \to \ s_-\,q_- = \exp\left[\boldsymbol{L}\cdot(\boldsymbol{\theta} - R_i\ \boldsymbol{\varphi})/2\right]\,q_- \ .$$

These *inequivalent* representations of the Lorentz group are characterized by two different types of one-component quaternionic spinors, $q_\pm$, which represent the quaternionic counterpart of the standard *dotted* and *undotted* bi-dimensional complex spinors. These spinors correspond to the representations $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ of the Lorentz group. The generators

$$\boldsymbol{\mathcal{A}}_1 = \tfrac{1}{2}\left(\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{B}}\, R_i\right) \quad \text{and} \quad \boldsymbol{\mathcal{A}}_2 = \tfrac{1}{2}\left(\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{B}}\, R_i\right) \ , \tag{8}$$

satisfy the following commutation relations

$$\mathcal{A}_{m,x} = [\,\mathcal{A}_{m,y}\,,\,\mathcal{A}_{m,z}\,]\ , \quad \mathcal{A}_{m,y} = [\,\mathcal{A}_{m,z}\,,\,\mathcal{A}_{m,x}\,]\ , \quad \mathcal{A}_{m,z} = [\,\mathcal{A}_{m,x}\,,\,\mathcal{A}_{m,y}\,]\ , \qquad {\scriptstyle m\,=\,1,\,2}\ ,$$

and

$$[\,\boldsymbol{\mathcal{A}}_1\,,\,\boldsymbol{\mathcal{A}}_2\,] = 0\ .$$

Thus, the quaternionic representations of the Lorentz group can be classified in terms of the representations of the left-acting group group $\mathsf{U}_1(1,\mathbb{H}^L)\otimes\mathsf{U}_2(1,\mathbb{H}^L)$. Quaternionic states will be identified by two angular momenta (j_1,j_2) corresponding to the generators $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_2$. In our case,

$$\begin{array}{lllll}\boldsymbol{\mathcal{A}} = \boldsymbol{L}/2\ , & \boldsymbol{\mathcal{B}} = +\boldsymbol{L}R_i/2 & \Rightarrow & \boldsymbol{\mathcal{A}}_1 = \boldsymbol{L}/2\ , \quad \boldsymbol{\mathcal{A}}_2 = 0\ , & \left(\frac{1}{2},0\right)\ ,\\ \boldsymbol{\mathcal{A}} = \boldsymbol{L}/2\ , & \boldsymbol{\mathcal{B}} = -\boldsymbol{L}R_i/2 & \Rightarrow & \boldsymbol{\mathcal{A}}_2 = \boldsymbol{L}/2\ , \quad \boldsymbol{\mathcal{A}}_1 = 0\ , & \left(0,\frac{1}{2}\right)\ .\end{array}$$

Under parity, velocity changes sign, hence the boost generators change sign, $\boldsymbol{\mathcal{B}}\to-\boldsymbol{\mathcal{B}}$. The rotation generators behave like axial vectors, $\boldsymbol{\mathcal{A}}\to\boldsymbol{\mathcal{A}}$. So, space inversion imply $(j_1,0)\leftrightarrow(0,j_2)$ and consequently $q_+\leftrightarrow q_-$. It is no longer sufficient to consider one-dimensional quaternionic spinors. We need to combine q_+ and q_- into bi-dimensional quaternionic column vectors

$$\psi = \begin{pmatrix} q_+ \\ q_- \end{pmatrix}\ . \tag{9}$$

The bi-dimensional spinor ψ is an *irreducible* representation of the Lorentz group *extended* by parity,

$$\psi \to \begin{pmatrix} s_+ & 0 \\ 0 & s_- \end{pmatrix}_{Lor} \begin{pmatrix} q_+ \\ q_- \end{pmatrix} \quad \text{and} \quad \psi \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{Par} \begin{pmatrix} q_+ \\ q_- \end{pmatrix}\ .$$

Under Lorentz boosts,

$$q_+ \to \exp\left(\boldsymbol{L}R_i\cdot\boldsymbol{\varphi}/2\right)\, q_+ = \exp\left(L_{\boldsymbol{n}}R_i\varphi/2\right)\, q_+ = \left(\cosh\tfrac{\varphi}{2} + L_{\boldsymbol{n}}R_i\sinh\tfrac{\varphi}{2}\right) q_+\ ,$$

where

$$L_{\boldsymbol{n}} \equiv \boldsymbol{L}\cdot\boldsymbol{n} = n_1L_i + n_2L_j + n_3L_k\ , \quad L^2_{\boldsymbol{n}} = -\boldsymbol{1}\ , \quad \varphi = \sqrt{\varphi_x^2+\varphi_y^2+\varphi_z^2}\ .$$

The q_- transformation is soon obtained from the previous one by changing $\varphi\to-\varphi$,

$$q_- \to \left(\cosh\tfrac{\varphi}{2} - L_{\boldsymbol{n}}R_i\sinh\tfrac{\varphi}{2}\right) q_-\ .$$

We can identify the original spinors with the particle at rest, $q_\pm(0)$, and the transformed spinors, $q_\pm(\boldsymbol{p})$, with the particle moving with momentum $\boldsymbol{p}$. By observing that

$$\cosh\varphi = \gamma = E/m\ , \quad \sinh\varphi = \beta\gamma = p/m\ , \qquad (c=1)\ ,$$

and

$$\cosh\tfrac{\varphi}{2} = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}} = \left(\frac{E+m}{2m}\right)^{\frac{1}{2}}\ , \quad \sinh\tfrac{\varphi}{2} = \left(\frac{\gamma-1}{2}\right)^{\frac{1}{2}} = \left(\frac{E-m}{2m}\right)^{\frac{1}{2}}\ ,$$

we find

$$q_+(\boldsymbol{p}) = \left[\left(\frac{E+m}{2m}\right)^{\frac{1}{2}} + L_{\boldsymbol{n}}R_i\left(\frac{E-m}{2m}\right)^{\frac{1}{2}}\right] q_+(0) = \left(E+m+\boldsymbol{L}R_i\cdot\boldsymbol{p}\right)\left[2m\left(E+m\right)\right]^{-\frac{1}{2}}\, q_+(0)$$

and

$$q_-(\boldsymbol{p}) = \left[\left(\frac{E+m}{2m} \right)^{\frac{1}{2}} - L_{\boldsymbol{n}} R_i \left(\frac{E-m}{2m} \right)^{\frac{1}{2}} \right] q_-(0) = (E + m - \boldsymbol{L}R_i \cdot \boldsymbol{p}) \left[2m\,(E+m) \right]^{-\frac{1}{2}} q_-(0) \; .$$

By recalling that (0) corresponds to the rest frame frame of the particle, we set $q_+(0) = q_-(0)$. So, we obtain

$$m\, q_+(\boldsymbol{p}) = (E + \boldsymbol{L}R_i \cdot \boldsymbol{p})\; q_-(\boldsymbol{p}) \quad \text{and} \quad m\, q_-(\boldsymbol{p}) = (E - \boldsymbol{L}R_i \cdot \boldsymbol{p})\; q_+(\boldsymbol{p}) \; .$$

These equations can be rewritten in matrix form as

$$\begin{pmatrix} -m & E + \boldsymbol{L}R_i \cdot \boldsymbol{p} \\ E - \boldsymbol{L}R_i \cdot \boldsymbol{p} & -m \end{pmatrix} \begin{bmatrix} q_+(\boldsymbol{p}) \\ q_-(\boldsymbol{p}) \end{bmatrix} = 0 \; . \tag{10}$$

By introducing the following complex linear quaternionic representation for the gamma matrices

$$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \quad \boldsymbol{\gamma} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{L}R_i \; , \tag{11}$$

we find

$$(\gamma^\mu p_\mu - m)\psi(\boldsymbol{p}) = 0 \; . \tag{12}$$

For massless particles, this equation decouples into one-dimensional equations,

$$(E + \boldsymbol{L}R_i \cdot \boldsymbol{p})\; q_-(\boldsymbol{p}) = 0 \quad \text{and} \quad (E - \boldsymbol{L}R_i \cdot \boldsymbol{p})\; q_+(\boldsymbol{p}) = 0 \; .$$

Since, for a massless particle $E = p$, these equations can also be rewritten as

$$\boldsymbol{L}R_i \cdot \hat{\boldsymbol{p}}\, q_-(\boldsymbol{p}) = -q_-(\boldsymbol{p}) \quad \text{and} \quad \boldsymbol{L}R_i \cdot \hat{\boldsymbol{p}}\, q_+(\boldsymbol{p}) = q_+(\boldsymbol{p}) \; . \tag{13}$$

The operator $\boldsymbol{L}R_i \cdot \hat{\boldsymbol{p}}$ measures the component of the spin in the direction of momentum and defines the *quaternionic helicity operator*.

4 Chiral and Dirac representations

The derivation of equation (12), which after quantization leads to a quaternionic version of the Dirac equation, differs from the original one formulated by Rotelli [4]. The equation presented in this paper is directly obtained from the transformation properties of the quaternionic Lorentz spinors, whereas the Rotelli derivation follows the original Dirac approach. In the Rotelli formulation, we have the following representation for the gamma matrices

$$\gamma^0_D \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; , \quad \boldsymbol{\gamma}_D \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{L} \; .$$

In this case, the space-time algebra $\mathsf{Cl}_{1,3}$ is given in terms of left-acting operators, $\gamma^\mu \in \mathcal{M}_2(\mathbb{H}^L)$. In our formulation, we find a complex linear set of gamma-matrices

$$\gamma^0_C \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \quad \boldsymbol{\gamma}_C \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{L}R_i \; .$$

Thus, we get a $\mathbb{C}$-linear quaternionic space-time algebra, $\gamma^\mu \in \mathcal{M}_2(\mathbb{H}^L \otimes \mathbb{C}^R)$. We observe that the $\mathbb{C}$-linearity is only hidden in the Rotelli approach. In the complex world, the Dirac equation reads indifferently as

$$i\partial_t \psi = H\psi \quad \text{or} \quad \partial_t \psi i = H\psi \; .$$

In the quaternionic world there is a clear difference in choosing a left or right position for the complex imaginary unit i. In fact, by requiring norm conservation

$$\partial_t \int d^3x\, \psi^\dagger \psi = 0 \; ,$$

we find that a left position of the imaginary unit i in the quaternionic Dirac equation, that is

$$L_i \partial_t \psi \equiv i \partial_t \psi = H \psi \; ,$$

implies

$$\partial_t \int d^3x\, \psi^\dagger \psi = \int d^3x\, \psi^\dagger [H, i] \psi$$

in general $\neq 0$ for a quaternionic Hamiltonian. A right position of the imaginary unit i,

$$R_i \partial_t \psi \equiv \partial_t \psi i = H \psi \; ,$$

ensures the norm conservation. By treating time and space in the same way, we gain the following definition for the energy/momentum operator

$$p^\mu \leftrightarrow R_i \partial^\mu \quad \Rightarrow \quad p^\mu \psi \leftrightarrow R_i \partial^\mu \psi \equiv \partial^\mu \psi i \; . \tag{14}$$

Finally, the complex linear quaternionic Dirac equation reads

$$R_i \gamma^\mu \partial_\mu \psi \equiv \gamma^\mu \partial_\mu \psi \, i = m \psi \; , \quad [\gamma^\mu, \gamma^\nu] = 2 g^{\mu\nu} \; . \tag{15}$$

Under Lorentz transformations, we get

$$\gamma^\mu a_{\mu\nu} \partial^\nu \mathcal{S}\, \psi \, i = m\, \mathcal{S}\, \psi \; , \quad a_{\mu\nu} \in \mathbb{R} \; .$$

In order to guarantee the covariance of the Dirac equation, we must have

$$a_{\mu\nu}\, \mathcal{S}^{-1}\, \gamma^\mu\, \mathcal{S} = \gamma_\nu \; .$$

For infinitesimal transformations, $a_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}$, we write

$$\mathcal{S} = \mathbf{1} - \tfrac{1}{4}\, \sigma_{\mu\nu}\, \omega^{\mu\nu}\, R_i \; .$$

A set of matrices $\sigma_{\mu\nu}$ satisfying this relation is given by $\frac{1}{2}\, [\gamma_\mu, \gamma_\nu]\, R_i$. A finite transformation is then given by

$$\mathcal{S} = \exp\left(\tfrac{1}{8}\, [\gamma^\mu, \gamma^\nu]\, \omega_{\mu\nu} \right) \; .$$

A fundamental ingredient in the formulation of quaternionic relativistic quantum mechanics is represented by the adoption of a *complex projection* of the inner product [1,4], necessary in order to guarantee that $R_i \boldsymbol{\partial}$ be an hermitian operator. In fact,

$$\int d^3x\, \varphi^\dagger R_i \boldsymbol{\partial} \psi = \int d^3x\, (R_i \boldsymbol{\partial} \varphi)^\dagger \psi \quad \Rightarrow \quad \int d^3x\, \varphi^\dagger \boldsymbol{\partial} \psi \, i = -i \int d^3x\, \boldsymbol{\partial} \varphi^\dagger \psi \; .$$

After integration by parts in the last integral, we find

$$\int d^3x\, \varphi^\dagger \boldsymbol{\partial} \psi \, i = i \int d^3x\, \varphi^\dagger \boldsymbol{\partial} \psi \; .$$

Due to the different position of the imaginary unit i and to the quaternionic nature of the wave functions, the previous relation is only satisfied by adopting a complex projection for the inner product,

$$\int d^3x \;\to\; \int_{\mathbb{C}} d^3x \; . \tag{16}$$

The use of a complex projection of the inner product also gives the right number of orthogonal solutions for the quaternionic Dirac equation [4].

5 Quaternionic electroweak gauge models

Let us rewrite equation (12) as

$$\Upsilon^\mu \partial_\mu \psi = m\psi \ , \quad \Upsilon^\mu \equiv \gamma^\mu R_i \ , \quad [\Upsilon^\mu, \Upsilon^\nu] = -2g^{\mu\nu} \ . \tag{17}$$

By using the quaternionic Dirac and chiral representations for the gamma matrices, we find

$$\Upsilon^0_D \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_i \ , \quad \boldsymbol{\Upsilon}_D \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{L} R_i \quad \text{and} \quad \Upsilon^0_C \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_i \ , \quad \boldsymbol{\Upsilon}_C \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{L} \ .$$

From the quaternionic space-time algebra, we can extract the generators of a right-acting complex $\mathsf{SU}(2)$ group. In fact, the matrices Υ^0, $\Upsilon^{123} = \Upsilon^1\Upsilon^2\Upsilon^3$ and $\Upsilon^5 = \Upsilon^0\Upsilon^{123}$ satisfy the following algebra

$$(\Upsilon^0)^2 = (\Upsilon^{123})^2 = (\Upsilon^5)^2 = \Upsilon^0\Upsilon^{123}\Upsilon^5 = -\mathbf{1}$$

and represent right-acting complex operators

$$(\, \Upsilon^0_D \ , \ \Upsilon^{123}_D \ , \ \Upsilon^5_D \,) \equiv \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R_i \ , \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_i \ , \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$$

and

$$(\, \Upsilon^0_C \ , \ \Upsilon^{123}_C \ , \ \Upsilon^5_C \,) \equiv \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_i \ , \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} R_i \right] \ .$$

The Dirac and chiral representations diagonalize respectively Υ^0 and Υ^5. By diagonalizing Υ^{123}, we gain a *new* set of gamma matrices

$$\Upsilon^0_M \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R_i \ , \quad \boldsymbol{\Upsilon}_M \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{L} R_i \ ,$$

quaternionic translation of the Majorana representation. Finally, the Dirac, Majorana and chiral representations can be associated to the three possible diagonalization choices for the generators of the right-acting complex group $\mathsf{SU}(2, \mathbb{C}^R)$, i.e. Υ^0, Υ^{123} and Υ^5.

We can write the first fermion family of the Salam-Weinberg model by 2×2 quaternionic matrices, Ψ_l (ν-e leptons) and Ψ_q (u-d quarks). The complex projection of the massless fermion electroweak Lagrangian [20]

$$\mathcal{L}_F = \left[\bar{\Psi}_l \Upsilon^\mu \partial_\mu \Psi_l + \bar{\Psi}_q \Upsilon^\mu \partial_\mu \Psi_q \right]_{\mathbb{C}} \tag{18}$$

is global invariant under the gauge group $\mathsf{SU}(2, \mathbb{C}^R) \otimes \mathsf{U}(1, \mathbb{C}^R)$. In the chiral representation, we have

$$\Psi_l = \begin{pmatrix} \nu_R & e_R \\ \nu_L & e_L \end{pmatrix} \quad \text{and} \quad \Psi_q = \begin{pmatrix} u_R & d_R \\ u_L & d_L \end{pmatrix} \ . \tag{19}$$

The Lagrangian (18), rewritten in terms of these spinors and of the quaternionic differential operators $\mathcal{D}_\pm = \partial_t R_i \pm \boldsymbol{L} \cdot \boldsymbol{\partial}$, reads

$$\begin{aligned} \mathcal{L}_F &= \left[\begin{pmatrix} \nu_R & e_R \\ \nu_L & e_L \end{pmatrix}^\dagger \begin{pmatrix} \mathcal{D}_- & 0 \\ 0 & \mathcal{D}_+ \end{pmatrix} \begin{pmatrix} \nu_R & e_R \\ \nu_L & e_L \end{pmatrix} + \begin{pmatrix} u_R & d_R \\ u_L & d_L \end{pmatrix}^\dagger \begin{pmatrix} \mathcal{D}_- & 0 \\ 0 & \mathcal{D}_+ \end{pmatrix} \begin{pmatrix} u_R & d_R \\ u_L & d_L \end{pmatrix} \right]_{\mathbb{C}} \\ &= \left[\begin{pmatrix} \nu_R & e_R \\ u_R & d_R \end{pmatrix}^\dagger \mathcal{D}_- \begin{pmatrix} \nu_R & e_R \\ u_R & d_R \end{pmatrix} + \begin{pmatrix} \nu_L & e_L \\ u_L & d_L \end{pmatrix}^\dagger \mathcal{D}_+ \begin{pmatrix} \nu_L & e_L \\ u_L & d_L \end{pmatrix} \right]_{\mathbb{C}} \\ &= [\Psi_R^\dagger \mathcal{D}_- \Psi_R + \Psi_L^\dagger \mathcal{D}_+ \Psi_L]_{\mathbb{C}} \ . \end{aligned} \tag{20}$$

In a forthcoming paper will be discussed in detail a quaternionic electroweak theory based on the Lagrangian (20) and the right-acting complex gauge group

$$g_2 \, \mathsf{SU}(2, \mathbb{C}^R)_L \otimes g_Y \, \mathsf{U}(1, \mathbb{C}^R)_Y \ .$$

Formulations of left/right symmetric models [21] require the generalization to the gauge group

$$g_{2,L} \, \mathsf{SU}(2, \mathbb{C}^R)_L \otimes g_{2,R} \, \mathsf{SU}(2, \mathbb{C}^R)_R \otimes g_1 \mathsf{U}(1, \mathbb{C}^R) \ .$$

Grand unification models and super-symmetric theories could require the choice of *effective* quaternionic gauge groups.

6 Conclusions

This work was intended as an attempt to motivate the use of quaternions in physics. In the last years, it was investigated the possibility to formulate quantum mechanics and field theory by using quaternionic wave functions and complex inner products. The use of a complex projection of the inner product [1,2] is a fundamental ingredient to obtain a consistent definition of energy/momentum operator [4]. Complex inner products also appear in the Dirac-Hestenes equation [22,23,24] and in many other mathematical and physical applications of Clifford algebras [25]. In the Hestenes work, the ordinary commutative i of standard quantum mechanics acquires a geometrical interpretation as the generator of rotations in a space-like plane. The electron spin and complex numbers are combined in a single geometric entity [22,23,24]. The Hestenes theory based on the Pauli algebra (even projection of the space-time algebra, STA) is algebraically isomorphic to Dirac theory. The Hestenes and Dirac theories are algebraically isomorphic but *not* equivalent. In fact, in STA formulation of the Dirac equation, we can give a geometric interpretation of the ordinary commutative imaginary unit i. Such an interpretation is *hidden* in the standard approach.

In terms of *complexified* quaternions, i.e. by introducing in $\mathbb{H}$ an imaginary unity η which commutes with the quaternionic imaginary units i, j and k, the Dirac spinor is represented by

$$\Psi = \psi_1 + j\,\psi_2 + \eta\;(\psi_3 + j\,\psi_4)\;\;,\quad \psi_{1,\ldots,4} \in \mathbb{C}(1,i)\;.$$

In this formalism [5], the Dirac-Hestenes equation reads

$$(\partial_t + \eta\,\boldsymbol{h}\cdot\boldsymbol{\partial})\,\Psi\,i = m\,\tilde{\Psi}\;, \tag{21}$$

where $\tilde{\Psi} = \psi_1 + j\,\psi_2 - \eta\;(\psi_3 + j\,\psi_4)$. In the rest frame, we find the following solutions

$$\Psi_1 = \exp[im\tau]\;,\;\;\Psi_2 = j\,\exp[im\tau]\;,\;\;\Psi_3 = \eta\,\exp[-im\tau]\;,\;\;\Psi_4 = \eta\,j\,\exp[-im\tau]\;.$$

The i-complex projection of inner products guarantees the orthogonality of the solutions of the complexified quaternionic Dirac equation. For example, we get

$$\left\{\overline{\Psi}_1\,\Psi_2\right\}_{\mathbb{C}} = \left\{\overline{\exp[im\tau]}\;j\,\exp[im\tau]\right\}_{\mathbb{C}} = \{j\,\exp[2im\tau]\}_{\mathbb{C}} = 0\;.$$

Many articles in the quaternionic literature use complexified quaternions to formulate Lorentz transformations and Dirac equation, see for example [5,8,26,27,28,29] and references therein. In the paper of ref. [8], the Lorentz group, realized by a subset of $\mathsf{SL}(2,\mathbb{H})$, is then *diagonalized* by introducing an imaginary unity η which commutes with the set $\{i,j,k\}$. In this paper, we have shown the possibility to formulate a quaternionic *one-dimensional* Lorentz group representation by using linear functions of real quaternions. The absence of η is replaced by the use of the right-acting imaginary unit R_i. Our quaternionic Lorentz group is isomorphic to $\mathsf{SL}(1,\mathbb{H}^L\otimes\mathbb{C}^R)$. By using the transformation properties of quaternionic spinors under space-time transformations we have obtained a *new* quaternionic representation from the gamma matrices. This representation, quaternionic counterpart of the standard chiral representation, allows to rewrite the fermion sector of electroweak Lagrangian in terms of the space-time differential operators $\mathcal{D}_\pm$. We have also showed that the Dirac, Majorana and chiral representations for the gamma matrices diagonalize respectively Υ^0, Υ^{123} and Υ^5, anti-hermitian generators of the right-acting electroweak gauge group.

A quaternionic formulation of the Dirac equation (12) by linear function of real quaternions is found in the papers of Morita [8] and Rotelli [4]. We recall that without the introduction of the complex projection of the inner product we cannot identify $\partial^\mu R_i$ as momentum operator and cannot recover four orthogonal solutions. The use of i-complex projection of quaternionic inner products is explicitly assumed in the Rotelli paper [4]. So, the Rotelli work plays a *fundamental* role in the formulation of a quaternionic quantum mechanics based on *complex inner products*. In such a formulation, the imaginary complex unit of the standard quantum mechanics is "translated" by the quaternionic imaginary unit i in the quaternionic wave functions and by the right-acting imaginary unit R_i in quaternionic operators [5]. Obviously, such an interpretation also extends to quaternionic quantum fields, where the commutation relations are now given in terms of the right-acting opertor R_i [30]. Finally, the introduction of complex inner products is also important in discussing quaternionic tensor products [31], group representations [7] and gauge models [32].

The use of complex projection of inner products, within formulations of physical theories based on non-commutative algebras (quaternions, complexified quaternions, STA, etc.), could play an important role in suggesting unification group (see the discussion on the electroweak model given in the previous section and ref. [25]) and searching for *hidden* geometric structures in relativistic wave equations. The last point represents the main motivation of the Hestenes work [22,23,24]. The correct geometric interpretation of the complex imaginary unit of the standard quantum mechanics still represents an open question in Clifford algebra. In fact, the complexified quaternionic Dirac equation [27,28,29] can also be written as

$$\eta \; (\partial_t \Psi \, j + \boldsymbol{h} \cdot \boldsymbol{\partial} \Psi \, k) = m \, \Psi \ . \tag{22}$$

In this formulation, the complex imaginary unit of standard quantum mechanics is identified with the imaginary unit η of the complexified quaternionic field, $L_\eta \Psi \equiv R_\eta \Psi = \eta \Psi$. In terms of STA, η represents a pseudo-scalar. We have "two" possible geometric interpretations of the complex imaginary unit of standard quantum mechanics, namely *bivector* if we select i-complex projections and *pseudo-scalar* if we choose η-complex projections of inner products.

In this paper, we have investigated a quaternionic formulation of Lorentz transformations and Dirac equation by means of linear functions of real quaternions. The main point is not the use of *real* against *complexified* quaternions but the use of a complex projection of quaternionic inner products. Complex projections represent the common link for many formulations of physical theories based on non-commutative algebras.

A different approach to quaternionic quantum mechanics, based on *quaternionic* inner products is extensively developed in the literature, see for example the discussion on foundations of quantum mechanics [33,34,35], CP-violation [36,37,38], quantum field dynamics [39,40] and preonic models [41].

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