Hypercomplex Group Theory

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Due to the noncommutative nature of quaternions and octonions we introduce *barred operators*. This objects give the opportunity to manipulate appropriately the hypercomplex fields. The standard problems arising in the definitions of transpose, determinant and trace for quaternionic and octonionic matrices are immediately overcome. We also investigate the possibility to formulate a *new approach* to Hypercomplex Group Theory (HGT). From a mathematical viewpoint, our aim is to highlight the possibility of looking at new hypercomplex groups by the use of barred operators as fundamental step toward a clear and complete discussion of HGT.

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I. INTRODUCTION

Complex numbers have played a dual role in Physics, first as a technical tool in resolving differential equation (e.g. in classical optics) or via the theory of analytic functions for performing real integrations, summing series, etc.; secondly in a more essential way in the development of Quantum Mechanics (and later Field Theory) characterized by complex wave functions and for fermions by complex wave equations. With quaternions, for the first type of application, i.e. as a means to simplify calculations, we can quote the original work of Hamilton [1], but this only because of the late development of vector algebra by Gibbs and Heaviside [2]. Even Maxwell used quaternions as a tool in his calculations, e.g. in the *Treatise of Electricity and Magnetism* [3] we find the ∇-operator expressed by the three quaternionic imaginary units.

Notwithstanding the Hamilton's conviction that quaternions would soon play a role comparable to, if not greater than, that of complex numbers the use of quaternions in Physics was very limited [4]. Nevertheless, in the last decades, we find a renewed interest in the application of noncommutative fields in Mathematics and Physics. In Physics, we quote quaternionic versions of Gauge Theories [5–8], Quantum Mechanics and Fields [9–14], Special Relativity [15]. In Mathematics, we find applications of quaternions for Tensor Products [16,17], Group Representations [18]. Nonassociative numbers are difficult to manipulate, nevertheless, the use of the octonionic field within Quantum Mechanics [19], in particular in a formulation the Dirac Equation [20], and in Group Theory [21] has recently appeared.

In this paper we aim to give a *new* panoramic review of hypercomplex groups. We use the adjective "new" since the elements of our matrices will not be simple quaternions or octonions but *barred hypercomplex operators*.

In Physics, particularly Quantum Mechanics, we are accustomed to distinguishing between "states" and "operators". Even when the operators are represented by numerical matrices, the squared form of operators distinguishes them from the column structure of the spinors states. Only for one-component fields and operators is there potential confusion. In extending Quantum Mechanics defined over the complex field to quaternions or even octonions, it has almost always been assumed that matrix operators contain elements which are "numbers" indistinguishable from those of the state vectors. *This is an unjustified limitation.* In fact, (noncommutative) hypercomplex theories require barred operators [22].

This paper is organized as follows: In section II, we introduce the quaternionic and octonionic algebras. In section III, we show that the noncommutative nature of the quaternionic and octonionic fields suggest the use of barred operators. We also give a brief review on the recent applications of barred operators in Mathematics and Physics. In section IV and V, we find the appropriate definitions of transpose, trace and determinant for quaternionic and octonionic matrices. Such sections also contain the *new* classification of hypercomplex groups. Our conclusions are drawn in the final section.

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II. HYPERCOMPLEX ALGEBRAS

Complex numbers can be constructed from the real numbers by introducing a quantity e_1 whose square is -1:

$$c = r_1 + e_1 r_2 \qquad (r_{1,2} \in \mathcal{R}) \; .$$

Likewise, we can construct the quaternions from the complex numbers in exactly the same way by introducing another quantity e_2 whose squared is -1,

$$q = c_1 + e_2 c_2 \qquad (c_{1,2} \in \mathcal{C}) \; ,$$

and which anticommutes with e_1 ($e_1 e_2 = -e_2 e_1 = e_3$). We wish to emphasize the need of *three anticommuting* imaginary units in constructing the quaternionic field (only two imaginary units are not sufficient to obtain Hamilton's field).

In introducing the quaternionic algebra, let us follow the conceptual approach of Hamilton. In 1843, the Irish mathematician attempted to generalize the complex field in order to describe the rotations in the three-dimensional space. He began by looking for numbers of the form

$$x + e_1 y + e_2 z \; ,$$

with $e_1^2 = e_2^2 = -1$. Hamilton's hope was to do for three-dimensional space what complex numbers do for the plane. Influenced by the existence of a complex number norm

$$c^* c = (\mathrm{Re}\, c)^2 + (\mathrm{Im}\, c)^2 \; ,$$

when he looked at its generalization

$$(x - e_1 y - e_2 z)(x + e_1 y + e_2 z) = x^2 + y^2 + z^2 - (e_1 e_2 + e_2 e_1) yz \; ,$$

to obtain a real number, he had to adopt the anticommutative law of multiplication for the imaginary units. Nevertheless, as remarked before, with only two imaginary units we have no chance of constructing a new numerical field, because assuming

$$e_1 e_2 = \alpha_0 + e_1 \alpha_1 + e_2 \alpha_2 \qquad (\alpha_{0,1,2} \in \mathcal{R}) \; ,$$
$$e_2 e_1 = \beta_0 + e_1 \beta_1 + e_2 \beta_2 \qquad (\beta_{0,1,2} \in \mathcal{R}) \; ,$$

and

$$e_1 e_2 = -e_2 e_1 \; ,$$

we find the relation

$$\alpha_{0,1,2} = \beta_{0,1,2} = 0 \; .$$

Thus, we must introduce a third imaginary unit $e_3 \neq e_{1,2}$, with

$$e_3 = e_1 e_2 = -e_2 e_1 \; .$$

This noncommutative field is therefore characterized by three imaginary units e_1, e_2, e_3 which satisfy the following multiplication rules

$$e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1 \; . \tag{1}$$

Numbers of the form

$$q = x_0 + e_1 x + e_2 y + e_3 z \qquad (\, x_0, x, y, z \in \mathcal{R} \,) \; , \tag{2}$$

are called (real) *quaternions*. They are added, subtracted and multiplied according to the usual laws of arithmetic, except for the commutative law of multiplication.

Similarly to rotations in a plane that can be concisely expressed by complex number, a rotation about an axis passing through the origin and parallel to a given unitary vector $\hat{u} \equiv (u_x, u_y, u_z)$ by an angle α can be obtained taking the following *quaternionic* transformation

$$\exp\left(\frac{\alpha}{2}\,\vec{e}\cdot\vec{u}\right)\,\vec{e}\cdot\vec{r}\,\exp\left(-\frac{\alpha}{2}\,\vec{e}\cdot\vec{u}\right) \;\; ,$$

where $\vec{e} \equiv (e_1, e_2, e_3)$ and $\vec{r} \equiv (x, y, z)$. In section III, we shall see how the quaternionic number q in Eq. (2), with the identification $x_0 \equiv ct$, can be used to formulate a one-dimensional version of the Lorentz group [15]. This gives the natural generalization of Hamilton's idea

complex/plane $\quad\rightarrow\quad$ pure imaginary quaternions/space $\quad\rightarrow\quad$ quaternions/space-time ,

completing the unification of algebra and geometry.

Let us now consider the conjugate of q

$$q^{\dagger} = x_0 - e_1 x - e_2 y - e_3 z \;. \tag{3}$$

We observe that $q^{\dagger}q$ and $qq^{\dagger}$ are both equal to the real number

$$N(q) = x_0^2 + x^2 + y^2 + z^2 \;\; ,$$

which is called the norm of q. When $q \neq 0$, we can define

$$q^{-1} = q^{\dagger}/N(q) \;\; ,$$

so the quaternions form a zero-division ring. Such a noncommutative number field is denoted, in Hamilton honour, by $\mathcal{H}$.

An important difference between quaternionic and complex numbers is related to the definition of the conjugation operation. Whereas with complex numbers we can define only one type of conjugation

$$e_1 \rightarrow -e_1 \;\; ,$$

working with quaternionic numbers we can introduce different conjugation operations. Indeed, with three imaginary units we have the possibility to define besides the standard conjugation (3), the six new operations

$$\begin{array}{lcl} (e_1, e_2, e_3) & \rightarrow & (-e_1, +e_2, +e_3) \;,\; (+e_1, -e_2, +e_3) \;,\; (+e_1, +e_2, -e_3) \;; \\ (e_1, e_2, e_3) & \rightarrow & (+e_1, -e_2, -e_3) \;,\; (-e_1, +e_2, -e_3) \;,\; (-e_1, -e_2, +e_3) \;. \end{array}$$

These last six conjugations can be concisely represented by q and $q^{\dagger}$ as follows

$$\begin{array}{lcl} q & \rightarrow & -e_1 q^{\dagger} e_1 \;,\; -e_2 q^{\dagger} e_2 \;,\; -e_3 q^{\dagger} e_3 \;, \\ q & \rightarrow & -e_1 q e_1 \;,\; -e_2 q e_2 \;,\; -e_3 q e_3 \;. \end{array}$$

It could seem that the only independent conjugation be represented by $q^{\dagger}$. Nevertheless, $q^{\dagger}$ can also be expressed in terms of q, in fact

$$q^{\dagger} = -\frac{1}{2}\,(q + e_1 q e_1 + e_2 q e_2 + e_3 q e_3) \;. \tag{4}$$

In going from the complex numbers to the quaternions we lose the property of commutativity (e.g. $e_1 e_2 = -e_2 e_1$). In going from the quaternions to the next more complicated division algebra, we also lose the property of associativity. Octonionic numbers can be constructed from the quaternions by introducing a new imaginary unit e_4 which anticommutes with the quaternionic imaginary units e_1, e_2 and e_3,

$$o = q_1 + e_4 q_2 \qquad (q_{1,2} \in \mathcal{H}) \;.$$

We can immediately show the nonassociativity of the octonionic numbers in the previous "split" representation. In fact, by starting from the seven imaginary units

$$e_1,\; e_2,\; e_3,\; e_4 \;, e_5 = e_1 e_4,\; e_6 = e_2 e_4,\; e_7 = e_3 e_4 \;,$$

it is straightforward to verify that

$$e_5e_6e_3 = e_6e_7e_1 = e_7e_5e_2 = -1 \ ,$$

in fact imposing associativity we have for example

$$e_5e_6e_3 = e_1e_4e_2e_4e_3 = -e_1e_2e_4^2e_3 = e_1e_2e_3 = -1 \ .$$

Associativity fails in the following relations

$$e_1(e_4e_3) = -e_1e_7 = e_6 \qquad \text{and} \qquad (e_1e_4)e_3 = e_5e_3 = -e_6 \ .$$

We now summarize our notation for the octonionic algebra and introduce useful elementary properties to manipulate the nonassociative numbers. An octonionic number will be represented by

$$o = r_0 + \sum_{m=1}^{7} r_m e_m \qquad (\ r_{0,\dots,7} \ \text{ reals} \ ) \ , \tag{5}$$

where e_m are elements obeying the noncommutative and nonassociative algebra

$$e_m e_n = -\delta_{mn} + \epsilon_{mnp} e_p \qquad (\ m, \ n, \ p = 1, \dots, 7 \ ) \ , \tag{6}$$

with ϵ_{mnp} totally antisymmetric and equal to unity for the seven combinations

$$123, \ 145, \ 176, \ 246, \ 257, \ 347 \text{ and } 365$$

(each cycle represents a quaternionic subalgebra). Amongst the different (equivalent) possibilities, we choose the previous combinations to uniform the notation of this paper with the notation which appears in recent works [19–21].

The norm, $N(o)$, for the octonions is defined by

$$N(o) = (o^\dagger o)^{\frac{1}{2}} = (o o^\dagger)^{\frac{1}{2}} = (r_0^2 + \dots + r_7^2)^{\frac{1}{2}} \ , \tag{7}$$

with the octonionic conjugate $o^\dagger$ given by

$$o^\dagger = r_0 - \sum_{m=1}^{7} r_m e_m \ . \tag{8}$$

The inverse is then

$$o^{-1} = o^\dagger / N(o) \qquad (\ o \neq 0 \ ) \ . \tag{9}$$

We can define an *associator* (analogous to the usual algebraic commutator) as follows

$$\{x, \ y, \ z\} \equiv (xy)z - x(yz) \ , \tag{10}$$

where, in each term on the right-hand, we must, first of all, perform the multiplication in brackets. Note that for real, complex and quaternionic numbers the associator is trivially null. For octonionic imaginary units we have

$$\{e_m, \ e_n, \ e_p\} \equiv (e_m e_n) e_p - e_m (e_n e_p) = 2\epsilon_{mnps} e_s \ , \tag{11}$$

with ϵ_{mnps} totally antisymmetric and equal to unity for the seven combinations

$$1247, \ 1265, \ 2345, \ 2376, \ 3146, \ 3157 \text{ and } 4576 \ .$$

Working with octonionic numbers the associator (10) is in general non-vanishing, however, an “alternative condition” is fulfilled

$$\{x, \ y, \ z\} + \{z, \ y, \ x\} = 0 \ . \tag{12}$$

III. BARRED OPERATORS

Due to the noncommutative nature of quaternions we must distinguish between q_1q_2 and q_2q_1. Thus, it is appropriate to consider left/right-actions for our imaginary units e_1, e_2 and e_3. We introduce *barred operators* [22] to represent, in a compact way, the right-action of the three quaternionic imaginary units. Explicitly, we write

$$1 \mid e_1 \ , \ 1 \mid e_2 \ , \ 1 \mid e_3 \tag{13}$$

to identify the right multiplication of e_1, e_2, e_3 and so

$$(1 \mid e_m)q \equiv qe_m \qquad (m = 1, 2, 3) \ .$$

In this formalism, the most general transformation on quaternions will be given by

$$q_0 + q_1 \mid e_1 + q_2 \mid e_2 + q_3 \mid e_3 \qquad (q_{0,1,2,3} \in \mathcal{H}) \ . \tag{14}$$

In the last few years the left/right-action of the quaternionic numbers, expressed by barred operators (14), has been very useful in overcoming difficulties owing to the noncommutativity of quaternions. Among the successful applications of barred operators we mention the one-dimensional quaternionic formulation of Lorentz boosts. Explicitly, the quaternionic generators of the Lorentz group are

$$\begin{array}{ll}
\text{boost } (ct, x) & \frac{e_3|e_2 - e_2|e_3}{2} \ , \\
\text{boost } (ct, y) & \frac{e_1|e_3 - e_3|e_1}{2} \ , \\
\text{boost } (ct, z) & \frac{e_2|e_1 - e_1|e_2}{2} \ , \\
\text{rotation around } x & \frac{e_1 - 1|e_1}{2} \ , \\
\text{rotation around } y & \frac{e_2 - 1|e_2}{2} \ , \\
\text{rotation around } z & \frac{e_3 - 1|e_3}{2} \ .
\end{array}$$

The four real quantities which identify the space-time point (ct, x, y, z) are represented by the quaternion

$$q = ct + e_1x + e_2y + e_3z \ .$$

We know that in analogy to the connection between the rotation group $O(3)$ and the special unitary group $SU(2)$, there is a natural correspondence between the Lorentz group $O(3, 1)$ and the special linear group $SL(2)$. The use of barred operators (14) gives us the possibility to extend the connection between the special unitary group $SU(2)$ and the unitary quaternions by allowing a *one-dimensional* quaternionic version of the special linear group $SL(2)$ (a detailed discussion is found in ref. [15]). We also note that barred operators (14) have 16 real parameters, the same number of parameters which appear in 4×4 real matrices. This suggests a correspondence between such barred quaternionic operators and generic 4-dimensional real matrices (appendix A).

New possibilities, coming out from the use of barred operators, also appear in Quantum Mechanics and Field Theory, e.g. they allow an appropriate definition of the momentum operator [12], quaternionic version of standard relativistic equations [12,13], Lagrangian formalism [23], electroweak model [7] and grand unification theories [8].

Let us now discuss the algebra of barred operators and introduce some elementary relations and definitions which will be useful in the following sections. Remembering the noncommutativity of the quaternionic multiplication, we must specify if our scalar factors are quaternionic, complex or real numbers. Operators which act on states *only* from the left (i.e. quaternionic numbers) will be named *quaternionic linear operator* and will be simply indicated by q. Obviously, from these more general classes of operators, such as complex or real linear quaternionic operators, can be constructed. For example, the barred operator (14) represents a *real linear quaternionic operator*. It will be denoted by $\mathcal{Q}_r$,

$$\mathcal{Q}_r = q_0 + q_1 \mid e_1 + q_2 \mid e_2 + q_3 \mid e_3 \ .$$

To complete the list of possible barred operators we give an explicit example of *complex linear quaternionic operator*

$$\mathcal{Q}_c \equiv q_0 + q_1 \mid e_1 \ .$$

In this section, we shall deal with the algebra of real linear quaternionic operators

$$\mathcal{Q}_r \supset \mathcal{Q}_c \supset q \ .$$

Our considerations and conclusions can be immediately translated to complex linear quaternionic operator.

The product of two barred operators $\mathcal{Q}_r$ and $\mathcal{P}_r$ in terms of quaternions $q_{0,1,2,3}$ and $p_{0,1,2,3}$ is given by

$$\begin{aligned}\mathcal{Q}_r\mathcal{P}_r = & \, q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 + \\ & (q_0p_1 + q_1p_0 - q_2p_3 + q_3p_2) \mid e_1 + \\ & (q_0p_2 + q_2p_0 - q_3p_1 + q_1p_3) \mid e_2 + \\ & (q_0p_3 + q_3p_0 - q_1p_2 + q_2p_1) \mid e_3 \ .\end{aligned}$$

The "full" conjugation operation is defined by changing the sign of our left/right quaternionic imaginary units, i.e.

$$(e_1, e_2, e_3)^\dagger = -(e_1, e_2, e_3) \qquad \text{and} \qquad (1 \mid e_1, 1 \mid e_2, 1 \mid e_3)^\dagger = -(1 \mid e_1, 1 \mid e_2, 1 \mid e_3) \ .$$

The previous definition implies

$$\begin{aligned}[(q_1 \mid q_2)(p_1 \mid p_2)]^\dagger & = (q_1p_1 \mid p_2q_2)^\dagger = p_1^\dagger q_1^\dagger \mid q_2^\dagger p_2^\dagger \\ & = (p_1 \mid p_2)^\dagger (q_1 \mid q_2)^\dagger \ ,\end{aligned}$$

and so

$$(\mathcal{Q}_r\mathcal{P}_r)^\dagger = \mathcal{P}_r^\dagger \mathcal{Q}_r^\dagger \ .$$

In section IV, dealing with quaternionic matrices we shall distinguish between real linear quaternionic groups, $GL(n, \mathcal{Q}_r)$, and complex linear quaternionic groups, $GL(n, \mathcal{Q}_c)$. For a clear and complete discussion of standard quaternionic groups, $GL(n, q)$, the reader is referred to Gilmore's book [24]. The use of barred operators give new opportunities in HGT. Let us observe as follows. The so-called "symplectic" complex representation of a quaternion (state) q

$$q = c_1 + e_2c_2 \qquad (c_{1,2} \in \mathcal{C}) \ ,$$

by a complex column matrix, is

$$q \ \leftrightarrow \ \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \ . \tag{15}$$

The operator representation of e_1, e_2 and e_3 consistent with the above identification

$$e_1 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \ , \qquad e_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \ , \qquad e_3 \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i\sigma_1 \ , \tag{16}$$

has been known since the discovery of quaternions. It permits any quaternionic number or matrix to be translated into a complex matrix, *but not necessarily viceversa*. Eight real numbers are required to define the most general 2×2 complex matrix but only four are needed to define the most general quaternion. In fact since every (non-zero) quaternion has an inverse, only a subclass of invertible 2×2 complex matrices are identifiable with quaternions. Complex linear quaternionic operators complete the translation [22]. The barred quaternionic imaginary unit

$$1 \mid e_1 \ \leftrightarrow \ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \ ,$$

adds four additional degrees of freedom, obtained by matrix multiplication of the corresponding matrices,

$$1 \mid e_1 \ , \ e_1 \mid e_1 \ , \ e_2 \mid e_1 \ , \ e_3 \mid e_1 \ ,$$

and so we have a set of rules for translating from any 2×2 complex matrices to $\mathcal{Q}_c$-barred operators. This opens new possibilities for quaternionic numbers, see for example the one-dimensional version of the Glashow group [7]. Obviously this translation does not apply to odd-dimensional complex matrices [25].

We conclude this section by extending our considerations on barred operators to the octonionic field. Here, the situation is more delicate. Due to the nonassociativity of octonions we must distinguish between *left* and *right* barred operator. The natural octonionic extension of (14) should be

$$o_0 + \sum_{m=1}^{7} o_m \mid e_m \qquad (o_{0,\dots,7} \text{ octonions}) \; , \tag{17}$$

but, due to the nonassociativity, this operator is not a well defined object. For example, the triple product $o_1 o e_1$ could be either $(o_1 o) e_1$ or $o_1(o e_1)$. So, in order to avoid ambiguities, we need to define *left/right-barred operators*. Left barred operators will be indicated by

$$o_1 \;) \; e_1 + o_2 \;) \; e_2 + \dots + o_7 \;) \; e_7 \; ,$$

in similar way we introduce right barred operators

$$o_1 \; (\; e_1 + o_2 \; (\; e_2 + \dots + o_7 \; (\; e_7 \; .$$

Their action on a generic quaternionic number o is respectively represented by

$$(o_1 o) e_1 + (o_2 o) e_2 + \dots + (o_7 o) e_7 \; ,$$

and

$$o_1 (o e_1) + o_2 (o e_2) + \dots + o_7 (o e_7) \; .$$

Nevertheless, there are barred operators in which the nonassociativity does not apply, like

$$1 \;) \; e_m = 1 \; (\; e_m \equiv 1 \mid e_m \; ,$$

or

$$e_m \;) \; e_m = e_m \; (\; e_m \equiv e_m \mid e_m \; .$$

The counting of independent barred operators should be 106, explicitly

$$\begin{array}{lr} 1, \; e_m, \; 1 \mid e_m & (15 \text{ elements}) \; , \\ e_m \mid e_m & (7) \; , \\ e_m \;) \; e_n \quad (m \neq n) & (42) \; , \\ e_m \; (\; e_n \quad (m \neq n) & (42) \; , \\ m = 1, 2, \dots, 7 & \end{array}$$

Yet, we can prove that each right-barred operator can be expressed by a suitable combination of left-barred operators. The proof is based on the correspondence between the left-right barred octonionic operators and generic 8×8 real matrices [21]. In appendix B, we report, for the sake of completeness, the translation rules between octonionic left-right barred operators and 8×8 real matrices. Thus, to represent the most general octonionic operator, we need only left-barred objects

$$o_0 + \sum_{m=1}^{7} o_m \;) \; e_m \; , \tag{18}$$

reducing to 64 the previous 106 elements. Barred operators (18), which will be denoted by $\mathcal{O}_r$, represent *real linear octonionic operators*. From these, we can immediately extract *complex linear octonionic operator*, $\mathcal{O}_c$, and obviously the standard *octonionic linear operators*, o,

$$\mathcal{O}_r \supset \mathcal{O}_c \supset o \; .$$

The classification of octonionic/quaternionic operator, given in this section, can be concisely summarized as follows

Number Field	Real Linear B.O.	Complex Linear B.O.	State
Octonionic	$\mathcal{O}_r$ (64)	$\mathcal{O}_c$ (16)	o (8)
Quaternionic	$\mathcal{Q}_r$ (16)	$\mathcal{Q}_c$ (8)	q (4)

B.O. $\leftrightarrow$ Barred Operator,

in parenthesis we recall the number of real parameters characterizing the respective barred operator.

IV. QUATERNIONIC GROUPS

Every set of basis vectors in V_n can be related to every other coordinate system by an $n \times n$ non singular matrix. The $n \times n$ matrix groups involved in changing bases in the vector spaces $\mathcal{R}_n$, $\mathcal{C}_n$ and $\mathcal{H}_n$ are called *general linear groups* of $n \times n$ matrices over the reals, complex and quaternions

$$\begin{array}{ccccc} GL(n,r) & \rightarrow & GL(n,c) & \rightarrow & GL(n,q) \\ & & & & \downarrow \\ & & & & GL(n,\mathcal{Q}_c) \\ & & & & \downarrow \\ & & & & GL(n,\mathcal{Q}_r) \ . \end{array}$$

Before discussing the groups $GL(n,\mathcal{Q}_r)$ and $GL(n,\mathcal{Q}_c)$, we introduce a new definition of transpose for quaternionic matrices which will allow us to overcome the difficulties due to the noncommutative nature of the quaternionic field (our definition, applying to standard quaternions, will be extended to complex and real linear quaternions).

The customary convention of defining the transpose M^t of the matrix M is

$$(M^t)_{rs} = M_{sr} \ .$$

In general, however, for two quaternionic matrices M and N one has

$$(MN)^t \neq N^t M^t \ ,$$

whereas this statement hold as an equality for complex matrices. For example, the matrices

$$\left[\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \right]^t = \begin{pmatrix} m_1 n_1 + m_2 n_3 & m_3 n_1 + m_4 n_3 \\ m_1 n_2 + m_2 n_4 & m_3 n_2 + m_4 n_4 \end{pmatrix} \tag{19}$$

and

$$\begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}^t \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}^t = \begin{pmatrix} n_1 m_1 + n_3 m_2 & n_1 m_3 + n_3 m_4 \\ n_2 m_1 + n_4 m_2 & n_2 m_3 + n_4 m_4 \end{pmatrix} \tag{20}$$

are equal only if we use a commutative number field. How can we define orthogonal quaternionic groups? By looking at the previous example, we see that the problem arises in the different position of factors $m_{1,2,3,4}$ and $n_{1,2,3,4}$ in the elements of our matrices. The solution is very simple once seen. It is possible to give a quaternionic transpose which reverses the order of factors and *goes back* to the usual definition for complex numbers, $c^t = c$. We define

$$q^t = x_0 + e_1 x_1 - e_2 x_2 + e_3 x_3 \ . \tag{21}$$

In this way, the transpose of a product of two quaternions q and p is the product of the transpose quaternions in reverse order

$$(qp)^t = p^t q^t \ .$$

The proof is straightforward if we recognize the following relation between transpose q^t and conjugate $q^\dagger$,

$$q^t = -e_2 q^\dagger e_2 \ .$$

Thus, in the quaternionic world the convention of defining the transpose M^t of the matrix M will be

$$(M^t)_{rs} = {M_{sr}}^t \ ,$$

or equivalently

$$M^t = -e_2 M^\dagger e_2 \ , \tag{22}$$

where $M^\dagger$ is defined in the standard way as

$$(M^\dagger)_{rs} = {M_{sr}}^\dagger \ .$$

With this new definition of quaternionic transpose, the relation

$$
\begin{aligned}
(MN)^t &= -e_2(MN)^\dagger e_2 = -e_2 N^\dagger M^\dagger e_2 \\
&= (-e_2 N^\dagger e_2)(-e_2 M^\dagger e_2) \\
&= N^t M^t
\end{aligned}
$$

also holds for noncommutative numbers.

Noting that under the transpose operation we have $e_{1,3}^t = e_{1,3}$ and $e_2^t = -e_2$, we can immediately generalize the definition of transpose conjugation to complex and real linear quaternionic operators

$$
\begin{aligned}
\mathcal{Q}_c^t &= q_0^t + q_1^t \mid e_1 \ , \\
\mathcal{Q}_r^t &= q_0^t + q_1^t \mid e_1 - q_2^t \mid e_2 + q_3^t \mid e_3 \ .
\end{aligned}
$$

The fundamental property of reversering the order of factors for the transpose of quaternionic products

$$
\begin{aligned}
(\mathcal{Q}_c\mathcal{P}_c)^t &= [q_0p_0 - q_1p_1 + (q_0p_1 + q_1p_0) \mid e_1]^t \\
&= p_0^tq_0^t - p_1^tq_1^t + (p_0^tq_1^t + p_1^tq_0^t) \mid e_1 \\
&= (p_0^t + p_1^t \mid e_1)(q_0^t + q_1^t \mid e_1) \\
&= \mathcal{P}_c^t\mathcal{Q}_c^t \ , \\
(\mathcal{Q}_r\mathcal{P}_r)^t &= p_0^tq_0^t - p_1^tq_1^t - p_2^tq_2^t - p_3^tq_3^t + \\
&\quad (p_0^tq_1^t + p_1^tq_0^t + p_2^tq_3^t - p_3^tq_2^t) \mid e_1 - \\
&\quad (p_0^tq_2^t + p_2^tq_0^t + p_3^tq_1^t - p_1^tq_3^t) \mid e_2 + \\
&\quad (p_0^tq_3^t + p_3^tq_0^t + p_1^tq_2^t - p_2^tq_1^t) \mid e_3 \\
&= (p_0^t + p_1^t \mid e_1 - p_2^t \mid e_2 + p_3^t \mid e_3) \times \\
&\quad (q_0^t + q_1^t \mid e_1 - q_2^t \mid e_2 + q_3^t \mid e_3) \\
&= \mathcal{P}_r^t\mathcal{Q}_r^t
\end{aligned}
$$

is again preserved.

In discussing the classification of the classical (matrix) groups, it is necessary to introduce one additional concept: the *metric*. A metric function on a vector space is a mapping of a pair of vectors into a number field $\mathcal{F}$ ($\mathcal{F} \equiv \mathcal{R}/\mathcal{C}$ for real/complex linear operators, see below). Let us now recall the following theorem: *The subset of transformations of basis in V_n which preserves the mathematical structure of a metric forms a subgroup of general linear groups.*

	bilinear symmetric		*orthogonal*
Groups preserving	bilinear antisymmetric	metrics are called	*symplectic*
	sesquilinear symmetric		*unitary* .

The previous theorem is valid for all real and complex metric-preserving matrix groups. It is also valid for quaternionic groups that preserve sesquilinear metrics, since two quaternions obey $(q_1q_2)^\dagger = q_2^\dagger q_1^\dagger$. It is not true for quaternionic matrices and bilinear metrics, since two quaternions do not generally commute. Nevertheless, it is still possible to associate subgroups of $GL(n,q)$ with groups that preserve bilinear metrics. In the literature this is done in the following way. "*Each quaternion in $GL(n,q)$ is replaced by the corresponding 2×2 complex matrix using the translation rules (16). The subset of matrices in this complex $2n \times 2n$ matrix representation of $GL(n,q)$ that leaves invariant a bilinear metric forms a group, since the theorem is valid for bilinear metrics on complex linear vector spaces. We can associate an $n \times n$ quaternion-valued matrix with each $2n \times 2n$ complex-valued matrix in the resulting groups that preserve bilinear metrics in the space $\mathcal{C}_{2n}$, which is a representation for the space $\mathcal{H}_n$*" - *Gilmore* [24].

Once we write our complex matrix, we can trivially obtain the generators of complex orthogonal groups in a standard manner and then we can translate back into quaternionic language. But this is *surely a laborious procedure*. Defining an appropriate transpose for quaternionic numbers (21), we can overcome the just-cited difficulty. Besides, using the symplectic representation (15), the most general transformation (on quaternionic states) will be necessarily represented by complex linear quaternionic operators, $\mathcal{Q}_c$, and for the invariant metric we have to require a "complex" projection

$$
(q^tq)_c = [(c_1 - e_2c_2^*)(c_1 + e_2c_2)]_c = c_1^2 + c_2^2 \ .
$$

We wish to emphasize that the introduction of the imaginary unit $1 \mid e_1$ in complex linear quaternionic operators

$$
(1 \mid e_1)^\dagger = -1 \mid e_1 \ ,
$$

necessarily implies a complex inner product. The "new" imaginary units $1 \mid e_1$ represents an antihermitian operator, and so it must verify

$$
\begin{array}{ccc}
\int\ (A\psi)^\dagger\varphi & = & -\int\ \psi^\dagger A\varphi \\
\downarrow & & \downarrow \\
\int\ (\psi e_1)^\dagger\varphi & = & -\int\ \psi^\dagger\varphi e_1\ .
\end{array}
$$

The previous relation is true only if we adopt a *complex projection*

$$\int_c \equiv \frac{1-e_1\mid e_1}{2}\int\ ,$$

for the inner products

$$\int_c (\psi e_1)^\dagger\varphi = -e_1\int_c \psi^\dagger\varphi = -\int_c \psi^\dagger\varphi e_1\ .$$

The generators of the unitary and orthogonal groups satisfy the following constraints

Groups:	Generators:
Unitary	$A+A^\dagger=0$,
Orthogonal	$A+A^t=0$.

For one-dimensional quaternionic groups, we find

Groups:	Generators:
$U(1,q)$	$e_1\ ,e_2\ ,e_3\ ,$
$U(1,\mathcal{Q}_c)$	$e_1\ ,e_2\ ,e_3\ ,1\mid e_1\ ,$
$U(1,\mathcal{Q}_r)$	$e_1\ ,e_2\ ,e_3\ ,1\mid e_1\ ,1\mid e_2\ ,1\mid e_3\ ,$
$O(1,q)$	$e_2\ ,$
$O(1,\mathcal{Q}_c)$	$e_2\ ,e_2\mid e_1\ ,$
$O(1,\mathcal{Q}_r)$	$e_2\ ,e_1\mid e_2\ ,e_3\mid e_2\ ,1\mid e_2\ ,e_2\mid e_1\ ,e_2\mid e_3\ .$

At this point, we make a number of observations:

1. - The difference between orthogonal and unitary groups is manifest for complex linear quaternionic groups because of the different numbers of generators.

2. - Orthogonal and unitary real linear quaternionic groups have the same number of generators.

3. - The groups $U(n_+,n_-,r)$ and $O(n_+,n_-,r)$ are identical (there is no difference between bilinear and sesquilinear metrics in a real vector space) and this suggest a possible link between $U(n,\mathcal{Q}_r)$ and $O(n,\mathcal{Q}_r)$.

4. - For real linear quaternionic groups, the invariant metric requires a "real" projection (note that $1\mid e_{1,2,3}$ represent antihermitian operators only for real inner products).

Let us show the "real" invariant metric for $U(1,\mathcal{Q}_r)$ and $O(1,\mathcal{Q}_r)$,

$$
\begin{aligned}
(q^\dagger q)_r &= [(x_0-e_1x_1-e_2x_2-e_3x_3)(x_0+e_1x_1+e_2x_2+e_3x_3)]_r = x_0^2+x_1^2+x_2^2+x_3^2\ ,\\
(q^t q)_r &= [(x_0+e_1x_1-e_2x_2+e_3x_3)(x_0+e_1x_1+e_2x_2+e_3x_3)]_r = x_0^2-x_1^2+x_2^2-x_3^2\ .
\end{aligned}
$$

We can immediately recognize the invariant metric of $O(4,r)$ and $O(2,2,r)$. To complete the analogy between one-dimensional real linear quaternionic operators and 4-dimensional real matrices, we observe that besides the subgroups related to the $\dagger$-conjugation (where the sign of all three imaginary units is changed) and the t-conjugation (where only $e_2\to -e_2$), we can define a new subgroup which leaves invariant the following real metric

$$(q^\dagger g q)_r\ ,$$

where

$$g=-\frac{1}{2}\left(1+e_1\mid e_1+e_2\mid e_2+e_3\mid e_3\right)\ .$$

Explicitly,

$$(q^\dagger g q)_r = [(x_0-e_1x_1-e_2x_2-e_3x_3)(x_0-e_1x_1-e_2x_2-e_3x_3)]_r = x_0^2-x_1^2-x_2^2-x_3^2\ .$$

So the new subgroup, $\tilde{O}(1,\mathcal{Q}_r)$, represents the one-dimensional quaternionic counterpart of the Lorentz group [15]. We observe that the $*$-conjugation, obtained by changing the sing of two imaginary units,

$$q^* = x_0 - e_1x_1 - e_2x_2 + e_3x_3 \ ,$$

also gives a real invariant metric related to $O(3,1,r)$

$$(q^*q)_r = [(x_0 - e_1x_1 - e_2x_2 + e_3x_3)(x_0 + e_1x_1 + e_2x_2 + e_3x_3)]_r = x_0^2 + x_1^2 + x_2^2 - x_3^2 \ .$$

Nevertheless, it is not suitable in defining a new subgroup because

$$(qp)^* = q^*p^* \ .$$

The classical groups which occupy a central place in group representation theory and have many applications in various branches of Mathematics and Physics are the unitary, special unitary, orthogonal, and symplectic groups. In order to define special groups, we must define an appropriate trace for our matrices. In fact, for noncommutative numbers the trace of the product of two numbers is not the trace of the product with reversed factors. With complex linear quaternions we have the possibility to give a new definition of "complex" trace (Tr) by

$$Tr\ \mathcal{Q}_c = \mathrm{Re}\, q_0 + e_1\, \mathrm{Re}\, q_1 \ . \tag{23}$$

Such a definition implies that for any two complex linear quaternionic operators $\mathcal{Q}_c$ and $\mathcal{P}_c$

$$Tr\ (\mathcal{Q}_c\mathcal{P}_c) = Tr\ (\mathcal{P}_c\mathcal{Q}_c) \ .$$

For real linear quaternions we need to use the standard definition of "real" trace (tr)

$$tr\ \mathcal{Q}_r = \mathrm{Re}\, q_0 \ , \tag{24}$$

since the previous "complex" definition (23) gives

$$Tr\ (\mathcal{Q}_r\mathcal{P}_r) \neq Tr\ (\mathcal{P}_r\mathcal{Q}_r) \ .$$

Explicitly, we find

$$\begin{aligned} Tr\ (\mathcal{Q}_r\mathcal{P}_r) &= \mathrm{Re}\ (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + \\ &\quad e_1\ \mathrm{Re}\ (q_0p_1 + q_1p_0 - q_2p_3 + q_3p_2) \ , \\ Tr\ (P_r\mathcal{Q}_r) &= \mathrm{Re}\ (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + \\ &\quad e_1\ \mathrm{Re}\ (p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2) \ . \end{aligned}$$

We recall that the generators of the unitary, special unitary, orthogonal groups must satisfy the following conditions [26]

$$\begin{array}{lll} U(n) & A + A^\dagger = 0 \ , & \\ SU(n) & A + A^\dagger = 0 \ , & Tr\ A = 0 \ , \\ O(n) & A + A^t = 0 \ . & \end{array}$$

These conditions also apply for quaternionic groups.

For complex symplectic groups we find

$$Sp(2n) \qquad \mathcal{J}A + A^t\mathcal{J} = 0 \ ,$$

where

$$\mathcal{J} = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix} \ .$$

Working with quaternionic numbers, we can construct a group preserving a non-singular antisymmetric metric, for n odd as well as n even. Thus for quaternionic symplectic groups we have

$$Sp(n) \qquad \mathcal{J}A + A^t\mathcal{J} = 0 \ ,$$

with

$$\mathcal{J}_{2n\times 2n} = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix} \ , \qquad \mathcal{J}_{(2n+1)\times(2n+1)} = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{1}_{n\times n} \\ \mathbf{0}_{1\times n} & e_2 & \mathbf{1}_{1\times n} \\ -\mathbf{1}_{n\times n} & \mathbf{0}_{n\times 1} & \mathbf{0}_{n\times n} \end{pmatrix} \ .$$

The generators of one-dimensional groups with complex and real linear quaternions are

$U(1,\mathcal{Q}_c)$	$\mathcal{Q}_c+\mathcal{Q}_c^\dagger=0$,		$\rightarrow$	$A=e_{1,2,3},\ 1 \mid e_1$,
$SU(1,\mathcal{Q}_c)$	$\mathcal{Q}_c+\mathcal{Q}^\dagger=0$,	$Tr\ \mathcal{Q}_c=0$,	$\rightarrow$	$A=e_{1,2,3}$,
$O(1,\mathcal{Q}_c)$	$\mathcal{Q}_c+\mathcal{Q}_c^t=0$,		$\rightarrow$	$A=e_2,\ e_2 \mid e_1$,
$Sp(1,\mathcal{Q}_c)$	$e_2\mathcal{Q}_c+\mathcal{Q}_c^t e_2=0$,		$\rightarrow$	$A=e_{1,2,3},\ e_{1,2,3} \mid e_1$,
$U(1,\mathcal{Q}_r)$	$\mathcal{Q}_r+\mathcal{Q}_r^\dagger=0$,		$\rightarrow$	$A=e_{1,2,3},\ 1 \mid e_{1,2,3}$,
$O(1,\mathcal{Q}_r)$	$\mathcal{Q}_r+\mathcal{Q}_r^t=0$,		$\rightarrow$	$A=e_2,\ e_{1,3} \mid e_2,\ 1 \mid e_2,\ e_2 \mid e_{1,3}$,
$\tilde{O}(1,\mathcal{Q}_r)$	$g\mathcal{Q}_r+\mathcal{Q}_r^\dagger g=0$,		$\rightarrow$	$A=e_k-1 \mid e_k,\ e_i \mid e_j - e_j \mid e_i,$ $i,j,k=1,2,3$ and $i\neq j$,
$Sp(1,\mathcal{Q}_r)$	$e_2\mathcal{Q}_r+\mathcal{Q}_r^t e_2=0$,		$\rightarrow$	$A=e_{1,2,3},\ 1 \mid e_2,\ e_{1,2,3} \mid e_{1,3}$.

We conclude our classification of quaternionic groups giving the general formulas for counting the generators of generic n-dimensional groups as function of n.

Dimensionalities of quaternionic groups

$U(n,q)$	$\leftrightarrow$	$USp(2n,c)$	$n(2n+1)$,
$U(n,\mathcal{Q}_c)$	$\leftrightarrow$	$U(2n,c)$	$4n^2$,
$U(n,\mathcal{Q}_r)$	$\leftrightarrow$	$O(4n,r)$	$2n(4n-1)$,
$SU(n,q)$	$\equiv$	$U(n,q)$	
$SU(n,\mathcal{Q}_c)$	$\leftrightarrow$	$SU(2n,c)$	$4n^2-1$,
$SU(n,\mathcal{Q}_r)$	$\equiv$	$U(n,\mathcal{Q}_r)$	
$O(n,q)$	$\leftrightarrow$	$SO^*(2n,c)$	$n(2n-1)$,
$O(n,\mathcal{Q}_c)$	$\leftrightarrow$	$O(2n,c)$	$2n(2n-1)$,
$O(n,\mathcal{Q}_r)$	$\leftrightarrow$	$O(2n_+,2n_-,r)$	$2n(4n-1)$,
$\tilde{O}(n,\mathcal{Q}_r)$	$\leftrightarrow$	$O(3n_+,n_-,r)$	$2n(4n-1)$,
$Sp(n,q)$	$\leftrightarrow$	$USp(2n,c)$	$n(2n+1)$,
$Sp(n,\mathcal{Q}_c)$	$\leftrightarrow$	$Sp(2n,c)$	$2n(2n+1)$,
$Sp(n,\mathcal{Q}_r)$	$\leftrightarrow$	$Sp(4n,r)$	$2n(4n+1)$.

V. OCTONIONIC GROUPS

The introduction of real linear quaternionic barred operators, with their 16 real parameters, naturally suggests a link between $\mathcal{Q}_r$ and a generic 4×4 real matrix. Besides, complex linear quaternionic operators, $\mathcal{Q}_c$, are characterized by four complex parameters, the same number of parameters which characterizes two-dimensional complex matrices.

If we try to repeat such a counting for real linear octonionic barred operators, due to the nonassociativity, we find 106 real parameters. Nevertheless, as remarked in section III, we can express right-barred operators by left-barred operators reducing to 64 the previous counting. So, we don't have apparently any problem in repeating the considerations of section IV. Yet, for complex linear octonionic operators, this will not be so easy (the nonassociativity of the octonionic field gives some drawbacks).

Let us try to generalize the symplectic complex representation (15) of a quaternionic state to an octonionic state

$$o=c_1+e_2c_2+e_4c_3+e_6c_4 \qquad (c_{1,2,3,4}\in\mathcal{C}) ,$$

by the complex matrix

$$o \leftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} . \tag{25}$$

We immediately see that the barred quaternionic imaginary unit $1 \mid e_1$, when applied to an octonionic state, gives

$$(1 \mid e_1)\, o = e_1c_1+e_2(e_1c_2)+e_4(e_1c_3)+e_6(e_1c_4) .$$

We don't have problems with the nonassociativity of octonions because $e_1e_2e_3,\ e_1e_4e_5,\ e_1e_7e_6$ form quaternionic subalgebras and so the associativity is restored. Thus, the barred imaginary unit $1 \mid e_1$ can be identified with the 4-dimensional complex matrix $i\mathbb{1}_{4\times 4}$. The only difference between quaternions and octonions is thus represented (as expected) by the dimension of the complex matrix. In order to complete the translation we need to find the octonionic counterpart of (16). It is here that we have problems. Let us observe the following embarrassing situation. The action of e_2 on an octonionic state is

$$\begin{aligned} e_2 o &= e_2 c_1 + e_2(e_2 c_2) + e_2(e_4 c_3) + e_2(e_6 c_4) \\ &= -c_2 + e_2 c_1 - e_4 c_4^* + e_6 c_3^* \ . \end{aligned}$$

So we should have

$$e_2 o \ \leftrightarrow \ M_{(e_2)} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} -c_2 \\ c_1 \\ -c_4^* \\ c_3^* \end{pmatrix} \ .$$

Obviously we cannot find a complex (left-acting) matrix $M_{(e_2)}$ which corresponds to the octonionic imaginary unit e_2. We also note that the octonionic imaginary unit e_2 does not represent an antihermitian operator, as desired to generalize the quaternionic translation rules (16). In fact, for nonassociative numbers, we have

$$\begin{array}{ccc} \int_c \ (A\psi)^\dagger \varphi & & -\int_c \ \psi^\dagger (A\varphi) \\ \downarrow & & \\ \int_c \ (e_2\psi)^\dagger \varphi & & \downarrow \\ \downarrow & & \\ -\int_c \ (\psi^\dagger e_2)\varphi & \stackrel{?}{=} & -\int_c \ \psi^\dagger (e_2\varphi) \ . \end{array}$$

For complex linear octonionic operators e_1 and $1 \mid e_1$ represent antihermitian imaginary units, whereas the remaining $e_{2,\dots,7}$ are not antihermitian operators. The proof is quoted in appendix C.

We wish to obtain the octonionic generalization of "e_2", which translates to

$$e_2 \ \leftrightarrow \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \tag{26}$$

when considered a quaternionic number and which obviously represents an antihermitian operator in the octonionic world. The solution is

$$\text{“}e_2\text{”} \ \equiv \ e_2 + \frac{e_3 \ ) \ e_1 - e_3 \ (\ e_1}{2} \ \leftrightarrow \ \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{27}$$

Eq. (27) represents the octonionic counterpart of Eq. (26), and goes back to the usual definition for quaternionic numbers, in fact

$$\frac{e_3 \ ) \ e_1 - e_3 \ (\ e_1}{2} \, q \equiv 0 \ .$$

In a similar manner we can construct the octonionic counterpart of the quaternionic hermitian operator

$$e_3 \mid e_1 \ \leftrightarrow \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

Explicitly,

$$\frac{e_3 \ ) \ e_1 + e_3 \ (\ e_1}{2} \ \leftrightarrow \ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ .$$

In a previous paper [21] we discussed the link between $GL(4,c)$ and complex linear octonionic operators. Nevertheless, in that paper we were not able to connect directly the 16 (complex) basis elements of $GL(4,c)$ with complex linear octonionic operators. "*The 32 (real) basis elements of $GL(4,c)$ can be extracted from the 64 generators of $GL(8,r)$*" [21]. We can now directly connect the antihermitian operators

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad ...$$

with the complex linear octonionic operators

$$e_2 + \frac{e_3\ )\ e_1 - e_3\ (\ e_1}{2} \quad , \quad e_4 + \frac{e_5\ )\ e_1 - e_5\ (\ e_1}{2} \quad , \quad e_6 - \frac{e_7\ )\ e_1 - e_7\ (\ e_1}{2} \quad , \quad ...$$

and the hermitian operators

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad ...$$

with

$$\frac{e_3\ )\ e_1 + e_3\ (\ e_1}{2} \quad , \quad \frac{e_5\ )\ e_1 + e_5\ (\ e_1}{2} \quad , \quad \frac{e_7\ )\ e_1 + e_7\ (\ e_1}{2} \quad , \quad ...$$

In this way we can immediately recognize the antihermitian operators as generators of $U(1, \mathcal{O}_c)$, and so we obtain the octonionic (one-dimensional) counterpart of $U(4, c)$.

We conclude this section with some considerations concerning the octonionic groups.

1. - The nonassociativity is not a problem for real/complex octonionic operators because we have a correspondence between such octonionic operators and (associative) general linear groups, $GL(8, r)/GL(4, c)$.

2. - By our translation rules, we can immediately obtain the octonionic counterpart of $GL(8n, r)$ and $GL(4n, c)$. Thus, we can translated a part of the standard real/complex groups.

3. - We must admit some technical difficulties in manipulating the octonionic field, see for example the nonantihermiticity (for complex linear operators) of the "standard" octonionic imaginary units $e_{2,...,7}$.

4. - The considerations of section IV (transpose, trace, ...) can be extended to octonionic fields. For complex linear octonionic operators, we must work with the new octonionic imaginary units "e_2", "e_4", "e_6", etc.

VI. CONCLUSIONS

The more exciting possibility that quaternionic or octonionic equations will eventually play a significant role in Mathematics and Physics is synonymous, for some physicist, with the advent of a revolution in Physics comparable to that of Quantum Mechanics.

For example, Adler suggested [28] that the color degree of freedom postulated in the Harari-Shupe model [29,30] (where we can think of quarks and leptons as composites of other more fundamental fermions, preons) could be sought in a noncommutative extension of the complex field. Surely a stimulating idea. Nevertheless, we think that it would be very strange if standard Quantum Mechanics did not permit a quaternionic or octonionic description other than in the trivial sense that complex numbers are contained within the quaternions or octonions.

In the last few years much progress has been achieved in manipulating such fields. We quote the quaternionic version of electroweak theory [7], where the Glashow group is expressed by the one-dimensional quaternionic group $U(1, q) \mid U(1, c)$, quaternionic GUTs [8] and Special Relativity, where the Lorentz group is represented by $\tilde{O}(1, \mathcal{Q}_r)$. We also recall new possibilities related to the use of octonions in Quantum Mechanics [19], in particular in writing a one-dimensional octonionic Dirac equation [20]. The link between octonionic and quaternionic versions of standard Quantum Physics is represented by the use of a complex geometry [31].

In this paper we observe that beyond the study of matrix groups with "simple" quaternionic elements, q, one can consider more general groups with matrix elements of the form $\mathcal{Q}_{r/c}$. To the best of our knowledge these more general matrix groups have not been studied in the literature. We overcome the problems arising in the definitions of transpose, determinant and trace for quaternionic matrices. For octonionic fields we must admit a more complicated situation, yet our discussion can be also proposed for nonassociative numbers.

Finally, we hope that this paper emphasizes the possibility of using hypercomplex numbers in Mathematics and Physics and it represents an important step towards a complete discussion of Hypercomplex Matrix Algebras.

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APPENDIX A
QUATERNIONIC BARRED OPERATORS AND 4 × 4 REAL MATRICES

We give the translation rules between quaternionic barred operators and 4×4 real matrices:

$$\boldsymbol{e_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad \boldsymbol{1} \mid \boldsymbol{e_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} ,$$

$$\boldsymbol{e_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \qquad \boldsymbol{1} \mid \boldsymbol{e_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ,$$

$$\boldsymbol{e_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \qquad \boldsymbol{1} \mid \boldsymbol{e_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

The remaining rules can be easily constructed noting that

$$\begin{aligned} \boldsymbol{e_m} &\leftrightarrow L_m , \\ \boldsymbol{1} \mid \boldsymbol{e_m} &\leftrightarrow R_m , \\ \boldsymbol{e_m} \mid \boldsymbol{e_n} &\leftrightarrow M^L_{mn} \equiv R_n L_m , \\ & \qquad M^R_{mn} \equiv L_m R_n . \end{aligned}$$

APPENDIX B
OCTONIONIC LEFT-RIGHT BARRED OPERATORS AND 8 × 8 REAL MATRICES

In this appendix we give the translation rules between octonionic left-right barred operators and 8×8 real matrices. In order to simplify our presentation we introduce the following notation:

$$\{ a, b, c, d \}_{(1)} \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} , \quad \{ a, b, c, d \}_{(2)} \equiv \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{pmatrix} , \tag{28a}$$

$$\{ a, b, c, d \}_{(3)} \equiv \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix} , \quad \{ a, b, c, d \}_{(4)} \equiv \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} , \tag{28b}$$

where $a,\ b,\ c,\ d$ and 0 represent 2×2 real matrices.

From now on, with σ_1, σ_2, σ_3 we represent the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{29}$$

The only necessary translation rules that we need to know explicitly are the following

$$
\begin{array}{llllllllllllll}
\boldsymbol{e_1} & \leftrightarrow & \{ -i\sigma_2, & -i\sigma_2, & -i\sigma_2, & i\sigma_2 \}_{(1)} & , & \mathbf{1} \mid \boldsymbol{e_1} & \leftrightarrow & \{ -i\sigma_2, & i\sigma_2, & i\sigma_2, & -i\sigma_2 \}_{(1)} & , \\
\boldsymbol{e_2} & \leftrightarrow & \{ -\sigma_3, & \sigma_3, & -\mathbb{1}, & \mathbb{1} \}_{(2)} & , & \mathbf{1} \mid \boldsymbol{e_2} & \leftrightarrow & \{ -\mathbb{1}, & \mathbb{1}, & \mathbb{1}, & -\mathbb{1} \}_{(2)} & , \\
\boldsymbol{e_3} & \leftrightarrow & \{ -\sigma_1, & \sigma_1, & -i\sigma_2, & -i\sigma_2 \}_{(2)} & , & \mathbf{1} \mid \boldsymbol{e_3} & \leftrightarrow & \{ -i\sigma_2, & -i\sigma_2, & i\sigma_2, & i\sigma_2 \}_{(2)} & , \\
\boldsymbol{e_4} & \leftrightarrow & \{ -\sigma_3, & \mathbb{1}, & \sigma_3, & -\mathbb{1} \}_{(3)} & , & \mathbf{1} \mid \boldsymbol{e_4} & \leftrightarrow & \{ -\mathbb{1}, & -\mathbb{1}, & \mathbb{1}, & \mathbb{1} \}_{(3)} & , \\
\boldsymbol{e_5} & \leftrightarrow & \{ -\sigma_1, & i\sigma_2, & \sigma_1, & i\sigma_2 \}_{(3)} & , & \mathbf{1} \mid \boldsymbol{e_5} & \leftrightarrow & \{ -i\sigma_2, & -i\sigma_2, & -i\sigma_2, & -i\sigma_2 \}_{(3)} & , \\
\boldsymbol{e_6} & \leftrightarrow & \{ -\mathbb{1}, & -\sigma_3, & \sigma_3, & \mathbb{1} \}_{(4)} & , & \mathbf{1} \mid \boldsymbol{e_6} & \leftrightarrow & \{ -\sigma_3, & \sigma_3, & -\sigma_3, & \sigma_3 \}_{(4)} & , \\
\boldsymbol{e_7} & \leftrightarrow & \{ -i\sigma_2, & -\sigma_1, & \sigma_1, & -i\sigma_2 \}_{(4)} & , & \mathbf{1} \mid \boldsymbol{e_7} & \leftrightarrow & \{ -\sigma_1, & \sigma_1, & -\sigma_1, & \sigma_1 \}_{(4)} & .
\end{array}
$$

The remaining rules can be easily constructed remembering that

$$
\begin{array}{rcl}
\boldsymbol{e_m} & \leftrightarrow & L_m \; , \\
\mathbf{1} \mid \boldsymbol{e_m} & \leftrightarrow & R_m \; , \\
\boldsymbol{e_m} \mid \boldsymbol{e_m} & \leftrightarrow & M^L_{mm} \; \equiv \; R_m L_m \; , \\
 & & M^R_{mm} \; \equiv \; L_m R_m \; , \\
\boldsymbol{e_m} \;)\; \boldsymbol{e_n} & \leftrightarrow & M^L_{mn} \; \equiv \; R_n L_m \; , \\
\boldsymbol{e_m} \;(\; \boldsymbol{e_n} & \leftrightarrow & M^R_{mn} \; \equiv \; L_m R_n \; .
\end{array}
$$

Following this procedure any matrix representation of right/left barred operators can be obtained. Using Mathematica [27], we proved the linear independence of the 64 elements which represent the most general octonionic operator

$$o_0 + \sum_{m=1}^{7} o_m \;)\; e_m \; .$$

Our left barred operators form a complete basis for any 8×8 real matrix and this establishes the isomorphism between $GL(8)$ and left barred octonions.

APPENDIX C
ANTIHERMITICITY PROPERTIES OF COMPLEX LINEAR OCTONIONIC OPERATORS

Let us consider the action of barred operators on octonionic functions

$$\psi = c_1 + e_2 c_2 + e_4 c_3 + e_6 c_4 \qquad (c_{1,\dots,4} \in \mathcal{C}) \; .$$

With the notation

$$e_2 \;\; \rightarrow \;\; \{-c_2, \; c_1, \; -c_4^*, \; c_3^*\} \; ,$$

we will indicate

$$e_2 \psi \;\; = \;\; -c_2 + e_2 c_1 - e_4 c_4^* + e_6 c_3^* \; .$$

The action of a generic octonionic barred operators can be expressed by suitable combinations of he action of barred operators $\boldsymbol{e_m}$ and $\mathbf{1} \mid \boldsymbol{e_m}$

$$
\begin{array}{llllllllllllll}
\boldsymbol{e_1} & \rightarrow & \{ e_1 c_1, & -e_1 c_2, & -e_1 c_3, & -e_1 c_4 \} & , & \mathbf{1} \mid \boldsymbol{e_1} & \rightarrow & \{ e_1 c_1, & e_1 c_2, & e_1 c_3, & e_1 c_4 \} & , \\
\boldsymbol{e_2} & \rightarrow & \{ -c_2, & c_1, & -c_4^*, & c_3^* \} & , & \mathbf{1} \mid \boldsymbol{e_2} & \rightarrow & \{ -c_2^*, & c_1^*, & c_4^*, & -c_3^* \} & , \\
\boldsymbol{e_3} & \rightarrow & \{ -e_1 c_2, & -e_1 c_1, & -e_1 c_4^*, & e_1 c_3^* \} & , & \mathbf{1} \mid \boldsymbol{e_3} & \rightarrow & \{ e_1 c_2^*, & -e_1 c_1^*, & e_1 c_4^*, & -e_1 c_3^* \} & , \\
\boldsymbol{e_4} & \rightarrow & \{ -c_3, & c_4^*, & c_1, & -c_2^* \} & , & \mathbf{1} \mid \boldsymbol{e_4} & \rightarrow & \{ -c_3^*, & -c_4^*, & c_1^*, & c_2^* \} & , \\
\boldsymbol{e_5} & \rightarrow & \{ -e_1 c_3, & e_1 c_4^*, & -e_1 c_1, & -e_1 c_2^* \} & , & \mathbf{1} \mid \boldsymbol{e_5} & \rightarrow & \{ e_1 c_3^*, & -e_1 c_4^*, & -e_1 c_1^*, & e_1 c_2^* \} & , \\
\boldsymbol{e_6} & \rightarrow & \{ -c_4, & -c_3^*, & c_2^*, & c_1 \} & , & \mathbf{1} \mid \boldsymbol{e_6} & \rightarrow & \{ -c_4^*, & c_3^*, & -c_2^*, & c_1^* \} & , \\
\boldsymbol{e_7} & \rightarrow & \{ e_1 c_4, & e_1 c_3^*, & -e_1 c_2^*, & e_1 c_1 \} & , & \mathbf{1} \mid \boldsymbol{e_7} & \rightarrow & \{ -e_1 c_4^*, & -e_1 c_3^*, & e_1 c_2^*, & e_1 c_1^* \} & .
\end{array}
$$

We can immediately verify that e_1 represents an antihermitian operator. The antihermiticity of e_1 is shown if

$$\int_c \psi^\dagger (e_1 \varphi) = - \int_c (e_1 \psi)^\dagger \varphi \; . \tag{30}$$

In the previous equation the only nonvanishing terms are represented by *diagonal* terms ($\sim c_1^\dagger z_1, \; c_2^\dagger z_2, \; c_3^\dagger z_3, \; c_4^\dagger z_4$). In fact, *off-diagonal* terms, like $c_2^\dagger z_3, \; c_3^\dagger z_4$, are killed by the complex projection,

$$\begin{aligned} (c_2^\dagger e_2)[e_1(e_4 z_3)] &\sim (\alpha_2 e_2 + \alpha_3 e_3)(\alpha_4 e_4 + \alpha_5 e_5) \sim \alpha_6 e_6 + \alpha_7 e_7 \; , \\ [(c_3^\dagger e_4) e_1](e_6 z_4) &\sim (\beta_4 e_4 + \beta_5 e_5)(\beta_6 e_6 + \beta_7 e_7) \sim \beta_2 e_2 + \beta_3 e_3 \; , \\ & \qquad\qquad [\; \alpha_{2,\dots,7} \text{ and } \beta_{2,\dots,7} \in \mathcal{R} \;] \; . \end{aligned}$$

The diagonal terms give

$$\begin{aligned} \int_c \psi^\dagger (e_1 \varphi) &= c_1^\dagger [e_1 z_1] - (c_2^\dagger e_2)[e_1(e_2 z_2)] - (c_3^\dagger e_4)[e_1(e_4 z_3)] - (c_4^\dagger e_6)[e_1(e_6 z_4)] \; , \\ - \int_c (e_1 \psi)^\dagger \varphi &= [c_1^\dagger e_1] z_1 - [(c_2^\dagger e_2) e_1](e_2 z_2) - [(c_3^\dagger e_4) e_1](e_4 z_3) - [(c_4^\dagger e_6) e_1](e_6 z_4) \; . \end{aligned}$$

The parenthesis in the previous equations are not of relevance since

$$\begin{array}{lll} c_1^\dagger e_1 z_1 & (1, \; e_1) & \text{is a complex number} \; , \\ c_2^\dagger e_2 e_1 e_2 z_2 & (\text{subalgebra } 123) \; , & \\ c_3^\dagger e_4 e_1 e_4 z_3 & (\text{subalgebra } 145) \; , & \\ c_4^\dagger e_6 e_1 e_6 z_4 & (\text{subalgebra } 176) & \text{are quaternionic numbers} \; . \end{array}$$

The antihermiticity proof of $1 \mid e_1$ is very similar to that of e_1. The above-mentioned demonstration does not work for the imaginary units e_2, ... , e_7 (breaking the symmetry between the seven octonionic imaginary units). In fact for e_2 we find

$$\begin{aligned} \int_c \psi^\dagger (e_2 \varphi) &= [(c_1^\dagger - c_2^\dagger e_2 - c_3^\dagger e_4 - c_4^\dagger e_6)(-z_2 + e_2 z_1 - e_4 z_4^* + e_6 z_3^*)]_c \\ &= -c_1^\dagger z_2 + c_2^\dagger z_1 + c_3^\dagger z_4^* + c_4^\dagger z_3^* \; , \\ - \int_c (e_2 \psi)^\dagger \varphi &= [(c_2^\dagger + c_1^\dagger e_2 - c_4^\dagger e_4 + c_3^\dagger e_6)(z_1 + e_2 z_2 + e_4 z_3 + e_6 z_4)]_c \\ &= c_2^\dagger z_1 - c_1^\dagger z_2 + c_4^\dagger z_3 - c_3^\dagger z_4 \; . \end{aligned}$$

A similar proof works for $e_{3,\dots,7}$.

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