Local Hypercomplex Analyticity

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The notion of analyticity is studied in the context of hypercomplex numbers. A critical review of the problems arising from the conventional approach is given. We describe a local analyticity condition which yields the desired type of hypercomplex solutions. The result is the definition of a generalized complex analyticity to hypercomplex space.

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I. INTRODUCTION

When in 1843, searching for a new numerical field that in three dimensions operates in analogy to complex numbers in two dimensions, the Irish mathematician William Rowan Hamilton discovered quaternions [1], he probably was not aware of what difficulties hide behind quaternions and what hostilities would arise in the Science community against such non commutative numbers.

With the notable exception of Hamilton himself, quaternions have always met difficulties in finding a physical incarnation [2]. Notwithstanding their (implicit) technical use in the *Treatise of Electricity and Magnetism* by Maxwell [3], where the ∇-operator is given in terms of the three quaternionic imaginary units i, j, k

$$\nabla = i\partial_x + j\partial_y + k\partial_z \ ,$$

quaternions were quickly moved aside by the subsequent and practical vectorial algebra formulated of Gibbs and Heaviside [4] in the 1880's. Gibbs and Heaviside, starting from quaternionic products, introduced the usual three dimensional scalar and vector product. It was these two products which were then preferentially applied in Physics, in alternative to the quaternionic product. Thus, quaternions fell in disuse in Physics not many decades after they were introduced.

While the intrinsic non commutativity of Quantum Mechanics should have encouraged the use of the quaternionic field as the natural underlying numerical field [5], the difficulties encountered in quaternionic calculations, together with the natural conservativism of the Science community have discriminated against the application of Hamilton's number field. In particular, problems arise in the appropriate definitions of tensor product spaces, in the corrected definition of determinant and transpose for quaternionic matrices, in the reproduction of standard trace theorems, and in the passage from hermitian to antihermitian operators, etc.

Nevertheless, after the fundamental paper of Horwitz and Biedenharn on Gauge Theories & Quantization [6] and Adler's work on Quaternionic Quantum Mechanics & Quantum Fields [7,8], quaternions have gained new impulse in Quantum Physics. Much recent progress in quaternionic applications to Mathematics and Physics have been obtained and thus quaternions are considered with more interest nowadays. Among the recent applications of quaternions we recall quaternionic formulations for Tensor Products [9], Group Representations [10], Special Relativity [11], Dark Matter [12], Relativistic Quantum Mechanics [13,14], Scattering & Decay Problems [15], Field Theory [16], Hidden Symmetry [17], Electroweak Model [18], GUTs [19], Preonic Model [20], etc. Experimental tests, which might highlight the possible quaternionic nature of wave functions, have also been proposed [21].

Within Mathematics itself an important question remains pertinent: How is the well known complex analyticity condition,

$$\partial_{x_0} f(z, \bar{z}) = -i\partial_{x_1} f(z, \bar{z}) \qquad [\ z = x_0 + ix_1 \quad (x_0, x_1 \in \mathcal{R})\ ] \ ,$$

best extended to hypercomplex numbers including quaternions?

In this paper, we start with a critical review of the conventional approach to quaternionic analysis, showing the disadvantages of defining a quaternionic derivative in the standard form

$$\partial_q \equiv \alpha\partial_{x_0} + \beta\partial_{x_1} + \gamma\partial_{x_2} + \delta\partial_{x_3} \ ,$$

with $\alpha,\ \beta, \gamma,\ \delta$ "constant" quaternionic numbers.

The main goal of our work is the demonstration that a new generalization of complex analyticity to quaternionic numbers can be easily obtained. This occurs thanks to a local representation of the complex imaginary unit, that yields the possibility of defining a *local quaternionic derivative* which overcomes some previous problems. Such a generalization is easily extended to octonionic numbers.

This paper is structured as follows: In Section II, we introduce the quaternionic algebra and the *barred operators*. In Section III, after a brief introduction to complex differentiability we discuss the standard quaternionic approaches. In Section IV, we recall some of the essential aspects of Fueter's calculus. Section V is directed to the study of "global" analyticity conditions. In this Section we show the impossibility of finding a suitable quaternionic derivative by adopting the standard viewpoint. Section VI deals with the "local" hypercomplex version of the Cauchy-Riemann equations. Our conclusions are drawn in the final Section.

II. QUATERNIONIC ALGEBRA AND BARRED OPERATORS

Let us introduce the quaternionic algebra by trying to follow the conceptual approach of Hamilton. We begin by looking for numbers of the form $x_1 + ix_2 + jx_3$, with $i^2 = j^2 = -1$, which will do for three-dimensional space what complex numbers have done for the plane. Influenced by the existence of a complex number norm

$$\bar{z}z = (\mathrm{Re}z)^2 + (\mathrm{Im}z)^2 \ ,$$

when we look at its generalization

$$(x_1 - ix_2 - jx_3)(x_1 + ix_2 + jx_3) = x_1^2 + x_2^2 + x_3^2 - (ij + ji)x_2x_3 \ ,$$

to obtain a real number, it is necessary to adopt the anticommutative law of multiplication for the imaginary units. We now note that with only two imaginary units we have no chance of constructing a new numerical field, because assuming

$$ij = \alpha_1 + i\alpha_2 + j\alpha_3 \ ,$$
$$ji = \beta_1 + i\beta_2 + j\beta_3 \ ,$$

and

$$ij = -ji \ ,$$

we find

$$\alpha_{1,2,3} = \beta_{1,2,3} = 0 \ .$$

Thus we must introduce a third imaginary unit $k \neq i \neq j$, with

$$k = ij \ .$$

This noncommutative field is therefore characterized by three imaginary units $i,\ j,\ k$ which satisfy the following multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1 \ . \tag{1}$$

Numbers of the form

$$q = x_0 + ix_1 + jx_2 + kx_3 \qquad (\ x_{0,1,2,3} \in \mathcal{R}\ ) \ , \tag{2}$$

are called (real) *quaternions*. They are added, subtracted and multiplied according to the usual laws of arithmetic, except for the commutative law of multiplication.

An important difference between quaternionic and complex numbers is related to the definition of the conjugation operation. Whereas with complex numbers we can define only one type of conjugation

$$i \to -i \ ,$$

working with quaternionic numbers we can introduce different conjugation operations. Indeed, with three imaginary units we have the possibility to define besides the standard conjugation

$$\bar{q} = x_0 - ix_1 - jx_2 - kx_3 \ , \tag{3}$$

the six new operations

$$\begin{array}{lcl} (i,j,k) & \to & (-i,+j,+k) \ , \ (+i,-j,+k) \ , \ (+i,+j,-k) \ ; \\ (i,j,k) & \to & (+i,-j,-k) \ , \ (-i,+j,-k) \ , \ (-i,-j,+k) \ . \end{array}$$

These last six conjugations can be concisely expressed in terms of q and $\bar{q}$ by

$$\begin{array}{lcl} q & \to & -i\bar{q}i \ , \ -j\bar{q}j \ , \ -k\bar{q}k \ , \\ q & \to & -iqi \ , \ -jqj \ , \ -kqk \ . \end{array}$$

It might seem that the only independent conjugation be represented by $\bar{q}$. Nevertheless, $\bar{q}$ can also be expressed in terms of q

$$\bar{q} = -\frac{1}{2} \left(q + iqi + jqj + kqk \right) \ . \tag{4}$$

So we conclude that while within a complex theory z and $\bar{z}$ represent independent variables, in a quaternionic theory all the seven conjugations can be defined as involutions of the quaternionic variable q. We will return to this point later.

Let us now introduce formally the so-called *barred operators*, already implicit in the previous formulas. In recent years the left/right-action of the quaternionic imaginary units, expressed by barred operators [22], has been very useful in overcoming difficulties owing to the noncommutativity of quaternionic numbers. Among the successful applications of barred operators we mention the one-dimensional quaternionic formulation of Lorentz boosts [11] and new possibilities for Quaternionic Group Theories [19]. Partially-barred quaternions also appear in Quantum Mechanics, e.g they allow an appropriate definition of the momentum operator, in a quaternionic version of Dirac's equation [13] and Electroweak Model [18].

The most general barred quaternionic operator is represented by

$$q_0 + q_1 \mid i + q_2 \mid j + q_3 \mid k \qquad (\ q_{0,1,2,3} \in \mathcal{H} \ ) \ . \tag{5}$$

where $1 \mid i$, $1 \mid j$ and $1 \mid k$ indicate the right-action of the quaternionic imaginary units. The application of $q_1 \mid q_2$ to a third quaternion q_3 is by definition $q_1 q_3 q_2$. The multiplication rules are simple once seen. For example, in the multiplication of two particular barred quaternions, $j \mid k$ and $i \mid j$, we find

$$(j \mid k) \times (i \mid j) = ji \mid jk = -k \mid i \ .$$

We also observe that in this example $j \mid k$ and $i \mid j$ represent commuting quaternionic barred operators.

Let us now analyze the link between complex/quaternion numbers and real matrices. We know that the action of a complex number on z can be given in terms of real matrices. In fact, identifying z with the vector column

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \ , \tag{6}$$

the action on z by the complex number $a + ib$ is expressed by the following real matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \ . \tag{7}$$

Obviously this involves only particular real matrices. This corresponds to the fact that complex numbers are characterized by *two* real parameters whereas the most general 2×2 real matrix by four. The restriction in Eq. (7) precludes one from obtaining from (6) the complex conjugate

$$\bar{z} \equiv \begin{pmatrix} x_0 \\ \text{-}x_1 \end{pmatrix} \ ,$$

and whence we see again, in another way, that z and $\bar{z}$ represent independent variables. The situation, drastically changes for quaternions. If we represent a quaternionic number, q, by the column vector

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} ,$$

the most general action we can perform on this column vector is obviously given by a 4×4 real matrix (16 real parameters). But, due to the noncommutative nature of quaternions, the most general transformation on quaternionic numbers is represented by the barred quaternions (5), characterized by just 16 real parameters. So we have a *full* connection between 4×4 real matrices and quaternions. The situation can be concisely summarized as follows

Field	State's Real Parameters	Operator's Real Parameters	Real Matrices
$\mathcal{C}$	two	two	$O(2) \in Gl(2)$
$\mathcal{H}$	four	sixteen	$Gl(4)$

There is another important distinction between complex and quaternionic numbers. The requirement that a function of a complex variable $z = x_0 + ix_1$ should be a complex polynomial function $f(x_0, x_1) + ig(x_0, x_1)$ picks out a proper subset of the polynomial functions. However, the corresponding requirement of a function of a quaternionic variable, q, namely that it should be a sum of monomials of the type

$$a_o q a_1 ... a_{z-1} q a_r \qquad (a_n \in \mathcal{H}) \; .$$

places *no* restriction on the function. In contrast to the complex case the coordinates $x_0, \vec{x}$ can themselves be written as quaternionic polynomials

$$\begin{aligned} x_0 &= P_r \, q \equiv \frac{1 - i \mid i - j \mid j - k \mid k}{4} \, q \; , \\ (\; ix_1, \, jx_2, \, kx_3 \;) &= (\; -iP_r i \, q, \, -jP_r j \, q, \, -kP_r k \, q \;) \equiv (\; P_i \, q \; , \, P_j \, q \; , \, P_k \, q \;) \; , \end{aligned}$$

or explicitly,

$$\begin{aligned} x_0 &= \frac{1}{4} \, (1 - i \mid i - j \mid j - k \mid k) \, q \; , \\ x_1 &= -\frac{i}{4} \, (1 - i \mid i + j \mid j + k \mid k) \, q \; , \\ x_2 &= -\frac{j}{4} \, (1 + i \mid i - j \mid j + k \mid k) \, q \; , \\ x_3 &= -\frac{k}{4} \, (1 + i \mid i + j \mid j - k \mid k) \, q \; , \end{aligned}$$

and so every real polynomial in $(x_0, \vec{x})$ can be written as quaternionic polynomial in q by combinations of

$$q \quad , \quad -iqi \quad , \quad -jqj \quad , \quad -kqk \quad .$$

Thus, a theory of quaternionic power series will be the same as a theory of real power series functions in $\mathcal{R}^4$.

Consequently, while working within complex numbers we have no possibility to express $\bar{z}$ by z, so that it is necessary to consider a general function f as $f(z, \bar{z})$,

$$f_0(x_0, x_1) + i f_1(x_0, x_1) \rightarrow f(z, \bar{z}) \qquad (\; f_{0,1} \in \mathcal{R} \; , \; f \in \mathcal{C} \;) \; .$$

In quaternionic theories, due to the noncommutativity, $\bar{q}$ (and all other conjugates) can be written in terms of q, by quaternionic barred operators, e.g.

$$\bar{q} = -\frac{1 + i \mid i + j \mid j + k \mid k}{2} \, q \; ,$$

thus, we can in all generality refer to $f(q)$ instead of $f(q, \bar{q}, ...)$,

$$f_0(x_0, \vec{x}) + i f_1(x_0, \vec{x}) + j f_2(x_0, \vec{x}) + k f_3(x_0, \vec{x}) \rightarrow f(q) \qquad (\; f_{0,1,2,3} \in \mathcal{R} \; , \; f \in \mathcal{H} \;) \; .$$

III. QUATERNIONIC DIFFERENTIABILITY

It is well known that a point in the two-dimensional real Euclidean space (x_0, x_1) can be naturally expressed by a complex number

$$z = x_0 + ix_1$$

and that the most general conformal coordinate transformation in two dimension is analytic in this complex coordinate. In four dimensions, coordinates can be represented by a quaternion as naturally as, in two dimensions, they are unified into a complex number. Thus, it is natural to ask whether the notion of analyticity can be extended to quaternions.

Before discussing hypercomplex analyticity, let us briefly recall the traditional approach to the notion of complex analyticity. A function whose arguments are complex numbers $x_0 \pm ix_1$ is defined as

$$f(z, \bar{z}) = u(x_0, x_1) + iv(x_0, x_1) \ , \tag{8}$$

where $u(x_0, x_1)$ and $v(x_0, x_1)$ are real functions of the real arguments x_0 and x_1. The standard theory of analytic functions deals only with a restricted but important class of functions of two real variables, namely, those functions that satisfy the requirement of being "differentiable". Though differentiability of a function of a complex variable places, as we said, a severe limitation on the functions one is allowed to consider, it leads to a theory of these functions that is both elegant and extremely powerful. We aim to extend the standard discussion on the differentiability of functions of a complex variable to quaternionic functions of a *quaternionic* (and even octonionic) variable.

The derivative of a function of a complex variable with respect to the *argument* z is formally defined in the same way as it is for functions of a real variable (complex numbers commute and so the position of the factor $1/\Delta z$ below is not relevant)

$$\partial_z f(z, \bar{z}) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z, \bar{z} + \Delta \bar{z}) - f(z)}{\Delta z} \ , \tag{9}$$

with $\Delta z = \Delta x_0 + i\Delta x_1$. The limit in the previous equation will, in general, depend upon the order on which Δx_0, Δx_1 tend to zero. We say that a function of a complex argument is *differentiable* if the limit (9) exists, is finite, and does not depend on the manner in which one takes the limit. Whereas in the case of a real variable, one can approach a given point only in two ways (either from the left or from the right), a point in the complex plane can be reached from an infinite number of directions. Thus, instead of the single requirement for a function of a real variable,

$$Left \ \text{ limit } \quad \equiv \quad Right \ \text{ limit} \ ,$$

an infinite number of such requirements has to be satisfied in order to ensure the differentiability of a function of a complex variable. One can understand therefore, why the property of being differentiable is, in the case of functions of a complex variable, very much more restrictive than it is for functions of a real variable.

The *necessary* and *sufficient* condition for a function of a complex variable to be differentiable at a given point is that, at that point, it obeys the Cauchy-Riemann conditions. Such conditions are obtained by imposing the requirement that the right-hand side of (9) yields the same result, whatever the order in which the limit $\Delta x_0 \, \Delta x_1 \to 0$ is taken. By first setting $\Delta x_1 = 0$ and then taking the limit $\Delta x_0 \to 0$ we find

$$\partial_z f(z, \bar{z}) = \partial_{x_0} u(x_0, x_1) + i\partial_{x_0} v(x_0, x_1) \ . \tag{10}$$

In the other case, in which we first set $\Delta x_0 = 0$ and then take the limit $\Delta x_1 \to 0$ we find

$$\partial_z f(z, \bar{z}) = -i\partial_{x_1} u(x_0, x_1) + \partial_{x_1} v(x_0, x_1) \ . \tag{11}$$

By equating the real and imaginary parts of Eqs. (10, 11), we obtain the Cauchy-Riemann conditions:

$$\partial_{x_0} u = \partial_{x_1} v \qquad \text{and} \qquad \partial_{x_1} u = -\partial_{x_0} v \ ,$$

or equivalently

$$\partial_{x_0} f(z, \bar{z}) = -i\partial_{x_1} f(z, \bar{z}) \ . \tag{12}$$

The real and imaginary parts of a differentiable function separately satisfy the Laplace equation

$$(\partial^2_{x_0} + \partial^2_{x_1})u = (\partial^2_{x_0} + \partial^2_{x_1})v = 0 \ ,$$

and are therefore *harmonic* functions of two variables (the converse is *not* necessarily true).

The net result of (12) is that an analytic f does not depend upon $\bar{z}$, is expandable in a power series in z, and this in turn justifies the *a priori* dubious convention of use of the terminology *functions of a complex variable*. Indeed, defining the complex derivative by

$$\partial_z \equiv \alpha \partial_{x_0} + \beta \partial_{x_1} \qquad (\alpha, \ \beta \in \mathcal{C}) \ , \tag{13}$$

and requiring that

$$\partial_z z = 1 \qquad \text{and} \qquad \partial_z \bar{z} = 0 \ , \tag{14}$$

we obtain

$$\partial_z \equiv \frac{1}{2} \left(\partial_{x_0} - i \partial_{x_1} \right) \ . \tag{15}$$

The corresponding conjugate complex derivative is

$$\partial_{\bar{z}} \equiv \frac{1}{2} \left(\partial_{x_0} + i \partial_{x_1} \right) \ .$$

The Cauchy-Riemann condition can then be written in compact form as,

$$\partial_{\bar{z}} f(z, \bar{z}) = 0 \qquad \left[\, \frac{1}{2} \left(\partial_{x_0} + i \partial_{x_1} \right) \, \right] \ . \tag{16}$$

Thus, these conditions have as a consequence that a mathematical expression defining a z-differentiable function can depend explicitly only on $z = x_0 + ix_1$ but not on $\bar{z} = x_0 - ix_1$. We also obtain the complex Laplace equation

$$\Box f(z) = 0 \qquad (\, \Box \equiv \partial_z \partial_{\bar{z}} \,) \ . \tag{17}$$

What happens in higher dimensions? Which is the quaternionic counterpart of ∂_z? Are there problems due to the noncommutativity of quaternions?

There is a substantial body of literature on attempted constructions of theories of analytic functions of both real (and even complexified) quaternions. In seeking to construct a differential and integral calculus of quaternionic functions the first step should be the definition of a derivative. A straightforward extension of the complex derivative could be

$$\partial_{\bar{q}} \equiv \frac{1}{2} \left(\partial_{x_0} + \frac{i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}}{3} \right) \ ,$$

where $\partial_{\bar{q}}$ has been defined in such a way that

$$\partial_{\bar{q}} q = 0 \qquad \text{and} \qquad \partial_{\bar{q}} \bar{q} = 1 \ ,$$

in analogy with the above complex case. Consequently the Cauchy-Riemann condition would be generalized to

$$\partial_{x_0} f(q) = - \frac{i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}}{3} \, f(q) \ . \tag{18}$$

However, due to the noncommutativity of quaternionic numbers, the only solution to Eq. (18) in the form of a power series of q is

$$f(q) = c_1 + q c_2 \qquad (\, c_1, \ c_2 \in \mathcal{H} \,) \ .$$

So we find only linear functions in q as solution. Even if these are not the only solutions. This is for too restrictive to yield any practical use. We also note that (18) does *not* reduce to (12) for complex functions, and indeed does not include the standard complex solutions $f(z)$.

A somewhat different approach extends the concept of holomorphism, which for hypercomplex numbers need not coincide with analyticity. Working with a noncommutative field we need to define a left or right derivative. In fact in the quaternionic version of Eq. (9) we must specify the position of the factor $1/\Delta q$. A right quaternionic derivative of the function $f(q)$ might be formed by requiring (in analogy to the complex case) that the limit

$$\partial_q f(q) = \lim_{\Delta q \to 0} \; [f(q + \Delta q) - f(q)]/\Delta q \tag{19}$$

exists and be independent of path for all increments Δq. By considering four linearly independent increments Δx_0, $i\Delta x_1$, $j\Delta x_2$, $k\Delta x_3$ we can derive a set of partial differential equations to be satisfied relating the components of $f(q)$

$$\partial_{x_0} f(q) = -i\partial_{x_1} f(q) = -j\partial_{x_2} f(q) = -k\partial_{x_3} f(q) \; . \tag{20}$$

This approach also leads to nothing productive since, even for the simple function q^2, the

$$\lim_{\Delta q \to 0} \; \Delta f/\Delta q$$

is not independent of the variation Δq. Even here we do not encompass complex analyticity. Indeed for functions independent of both x_2 and x_3,the above equations allow only constant functions as solutions.

IV. FUETER ANALYTICITY

Among the different approaches to hypercomplex analyticity, the most important appears to be that of Fueter [23] and his school in Zürich in the 1930's. He showed that a third order analytic equation yields generalizations of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion [24,25]. Whatismore, the Fueter quaternionic analyticity [26] has the appreciable virtue of selecting something less restrictive than Eqs. (18,20).

To construct series with quaternionic coefficients endowed with the ring property we have to consider the general quaternionic polynomial

$$\begin{aligned} p(q) = {} & a_0 + a_1 q + q a_2 + a_3 q a_4 + \\ & b_1 q^2 + q^2 b_2 + a_3 q^2 a_4 + b_5 q b_6 q b_7 + \\ & \ldots \end{aligned} \tag{21}$$

which is just a finite sum of quaternionic monomials $m(q)$, e. g.

$$m(q) = \alpha_0 q \alpha_2 q \ldots \alpha_{r-1} q \alpha_r$$

of degree r and with constant quaternions $\alpha_{0,\ldots,r}$. The polynomial $p(q)$ is called general quaternionic since it has no holomorphic property. Fueter considered various subgroups of the general mapping (21), the so-called Weiestrass-like series

$$L(q) = \sum_n q^n a_n \quad \text{or} \quad R(q) = \sum_n a_n q^n \qquad [\; a_n \in \mathcal{H} \;] \; ,$$

corresponding, respectively, to left and right holomorphic mappings.

In analogy to the complex case, four derivative operators are introduced by

$$\begin{aligned} \mathcal{D}^L_{\bar{q}} f(q) &\equiv \partial_{x_0} f(q) + i\partial_{x_1} f(q) + j\partial_{x_2} f(q) + k\partial_{x_3} f(q) \; , \\ \mathcal{D}^L_{q} f(q) &\equiv \partial_{x_0} f(q) - i\partial_{x_1} f(q) - j\partial_{x_2} f(q) - k\partial_{x_3} f(q) \; , \end{aligned} \tag{22}$$

$$\begin{aligned} \mathcal{D}^R_{\bar{q}} f(q) &\equiv \partial_{x_0} f(q) + \partial_{x_1} f(q) i + \partial_{x_2} f(q) j + \partial_{x_3} f(q) k \; , \\ \mathcal{D}^R_{q} f(q) &\equiv \partial_{x_0} f(q) - \partial_{x_1} f(q) i - \partial_{x_2} f(q) j - \partial_{x_3} f(q) k \; . \end{aligned} \tag{23}$$

The functions

$$l(q) = \mathcal{D}^L_{\bar{q}} \mathcal{D}^L_{q} L(q) = \Box L(q) \quad \text{and} \quad r(q) = \mathcal{D}^R_{\bar{q}} \mathcal{D}^R_{q} R(q) = \Box R(q) \; ,$$

represent, for Fueter, left and right analytic functions since they are annihilated, respectively by the operators $\mathcal{D}^L_{\bar{q}}$ and $\mathcal{D}^R_{\bar{q}}$

$$\mathcal{D}^L_{\bar{q}} l(q) = \mathcal{D}^R_{\bar{q}} r(q) = 0 \; .$$

So the functions $L(q)$ and $R(q)$ satisfy some Cauchy-Riemann-like condition, but of the *third order* in derivatives instead of the first. The "first order" Laplace equations $\Box L(q) = 0$ and $\Box R(q) = 0$ do not hold in general; however the "second order" Laplace equations $\Box^2 L(q) = 0$ and $\Box^2 R(q) = 0$ are valid.

Note that given the essential third order nature of the Fueter equations there is no simple limit to the complex case. We shall return to this in the conclusions.

V. GLOBAL QUATERNIONIC DERIVATIVE

Returning to first order analytic equations, let us analyze a more general case than that of Section III. Consider arbitrary coefficients $\in \mathcal{H}$ of the "real" derivatives

$$\partial_q \equiv \alpha \partial_{x_0} + \beta \partial_{x_1} + \gamma \partial_{x_2} + \delta \partial_{x_3} \qquad (\ \alpha,\ \beta, \gamma,\ \delta \in \mathcal{H}\ )\ . \tag{24}$$

We find an immediate problem if we require that our quaternionic derivative satisfies

$$\partial_q q = 1 \qquad \text{and} \qquad \partial_q q^2 = 2q\ . \tag{25}$$

In fact we obtain the following constraints for the quaternionic coefficients $\alpha,\ \beta,\ \gamma,\ \delta$

$$\alpha + \beta i + \gamma j + \delta k = 1\ ,$$
$$\alpha i - \beta = i\ ,\quad \alpha j - \gamma = j\ ,\quad \alpha k - \delta = k\ .$$

Combining the last three equations we have

$$3\alpha + \beta i + \gamma j + \delta k = 3\ ,$$

which, when coupled with the first one gives the trivial solution

$$\alpha = 1$$

and consequently

$$\beta = \gamma = \delta = 0\ .$$

Allowing barred quaternionic coefficients for our quaternionic derivatives, a possible solution to the constraints coming from Eq. (25) is given by

$$\alpha = 1 - i \mid i - j \mid j - k \mid k\ ,$$
$$2\beta = k \mid j - j \mid k\ ,\quad 2\gamma = i \mid k - k \mid i\ ,\quad 2\delta = j \mid i - i \mid j\ .$$

An immediate check verifies that

$$(\alpha,\ \beta,\ \gamma,\ \delta) \times 1 = (4,\ -i,\ -j,\ -k)\ ,$$
$$(\alpha,\ \beta,\ \gamma,\ \delta) \times (i,\ j,\ k) = (0,\ -1,\ 0,\ 0\ ;\ 0,\ 0,\ -1,\ 0\ ;\ 0,\ 0,\ 0,\ -1)\ ,$$

and so the constraints on $\alpha,\ \beta,\ \gamma,\ \delta$ are respected. Nevertheless, the constraints coming from the requirement that the quaternionic derivative of q^3 be $3q^2$ are

$$\alpha 1 + \beta i + \gamma j + \delta k = 1\ ,$$
$$\alpha i - \beta 1 = i\ ,\quad \alpha j - \gamma 1 = j\ ,\quad \alpha k - \delta 1 = k\ ,$$
$$\alpha 1 + \beta i + \tfrac{1}{3}\left(\gamma j + \delta k\right) = \alpha 1 + \gamma j + \tfrac{1}{3}\left(\beta i + \delta k\right) = \alpha 1 + \delta k + \tfrac{1}{3}\left(\beta i + \gamma k\right) = 1\ ,$$
$$\beta j + \gamma i = \beta k + \delta i = \gamma k + \delta j = 0\ .$$

Immediately we obtain

$$\beta i + \gamma j = \beta i + \delta k = \gamma j + \delta k = 0\ ,$$

and so again only the trivial solution exists

$$\alpha = 1 \qquad \text{and} \qquad \beta = \gamma = \delta = 0\ .$$

We conclude this Section with the result that it is not possible to obtain a satisfactory generalization (e.g. Weiestrass-like series in q) of the complex derivative to the quaternionic field *if the quaternionic derivative is defined in terms of constant quaternionic (even barred) coefficients.* In the next Section we will show how the solution arises by defining "local" quaternionic coefficients.

VI. LOCAL ANALYTICITY

We rewrite the quaternion q by introducing a "local" imaginary unit *iota*, in the following form

$$q = x_0 + ix_1 + jx_2 + kx_3 \rightarrow x_0 + \iota x \ . \tag{26}$$

with

$$x \equiv |\vec{x}| \qquad [\ \vec{x} \equiv (x_1,\ x_2,\ x_3)\ ]$$

and

$$\iota \equiv \frac{ix_1 + jx_2 + kx_3}{x} \ .$$

The quaternion q assumes a more familiar form (a complex number) and it becomes natural to define the quaternionic derivative in terms of the local imaginary unit ι as

$$\partial_q = \frac{1}{2}\left(\partial_{x_0} - \iota \partial_x\right) \ . \tag{27}$$

With ι held fixed (derivative within the ι-complex plane) we have,

$$x\partial_x \equiv x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} \ .$$

In terms of the global quaternionic imaginary units i, j and k Eq. (27) reads

$$\partial_q = \frac{1}{2}\left(\partial_{x_0} - \vec{Q}\cdot\hat{x}\ \hat{x}\cdot\vec{\partial}\right) \ , \tag{28}$$

where

$$\begin{aligned}\vec{Q} &\equiv (i,\ j,\ k)\ ,\\ \vec{\partial} &\equiv (\partial_{x_1},\ \partial_{x_2},\ \partial_{x_3})\ ,\\ \hat{x} &\equiv \vec{x}/x\ .\end{aligned}$$

This is a "local" derivative because the coefficients of $\partial/\partial_{1,2,3}$ depend upon the point q. To check the consistence of our quaternionic derivative it is sufficient to observe that

$$\begin{aligned}\partial_x(\iota x) &= \partial_x(ix_1 + jx_2 + kx_3)\\ &= \left(\frac{x_1}{x}\,\partial_{x_1} + \frac{x_2}{x}\,\partial_{x_2} + \frac{x_3}{x}\,\partial_{x_3}\right)(ix_1 + jx_2 + kx_3)\\ &= \frac{ix_1 + jx_2 + kx_3}{x}\\ &= \iota\ .\end{aligned}$$

We are now able to extend to quaternionic fields the Cauchy-Riemann conditions. Explicitly, we find the following quaternionic analyticity requirements:

$$\partial_{x_0} f(q) = -\iota\partial_x f(q) \ , \tag{29}$$

con $f(q) = f_0 + if_1 + jf_2 + kf_3$ ed $f_{0,1,2,3}$ real functions. Eq. (29) can be also rewritten in terms of $(i,\ j,\ k)$ as follows

$$\partial_{x_0} f_0 + \vec{Q}\cdot\partial_{x_0}\vec{f} = \hat{x}\cdot\mathcal{D}\vec{f} - \vec{Q}\cdot\left(\hat{x}\mathcal{D}f_0 + \hat{x}\wedge\mathcal{D}\vec{f}\right) \ , \tag{30}$$

where

$$\begin{aligned}\vec{f} &\equiv (f_1,\ f_2,\ f_3)\ ,\\ \mathcal{D} &\equiv \hat{x}\cdot\vec{\partial}\ .\end{aligned}$$

The solutions to Eq. (29) are immediate from complex analysis. They consist of all polinomials in q (the "complex" q) with arbitrary right acting quaternionic coefficients. Thus we re-obtain the Fueter solutions from a local differential equation. These solutions also contain the "classical" complex solutions and Eq. (29) reduces to the standard complex equation when applied to a function independent of x_2 and x_3.

VII. CONCLUSIONS

It is useful to recall that one of the difficulties of defining quaternionic analyticity is that the simplest extensions to four dimensions of complex analyticity tend to restrict the solutions too drastically. Another is that some generalizations (18,20) do not yield the standard complex limit.

One solution found by Fueter involves a third order analyticity condition and yields as solutions a class of polinomials in q with right (or left according to the derivative operator used) quaternionic coefficients. These solutions $L(q)$ and $R(q)$ are appealing generalizations of the standard complex Taylor series, while the associated functions $l(q)$ and $r(q)$ satisfy integral properties which are a generalization of those of complex functions. This occurs because these latter functions satisfy a global first order differential equation, but as a consequence they are not simple polinomial series in q. Furthermore, while any quaternionic function has a complex limit by, say, setting x_2 and x_3 to zero this does not imply that the partial derivatives with respect to these missing variables is identically null. Similarly while the first order analyticity condition reduce to a complex form, the integral equations involving hypersurfaces in q-space do not automatically become become line integrals in a complex plane. Thus $l(q)$ and $r(q)$ do not include complex analytic functions.

We have shown in this work that the polinomial Fueter solutions $L(q)$ and $R(q)$ can be derived from a *local* complex analyticity equation, i.e. by requiring merely ι-complex analyticity in the natural complex plane defined by the point (variable) q and the real axis. We observe, however, that while q only involves (by definition) ι, $f(q)$ is more general because of the unrestricted nature of the quaternionic coefficients in its Taylor expansion.

The extensions of our approach to octonions is straightforward given the first order nature of our analyticity equation. The corresponding definition of ι is

$$\iota \quad \equiv \quad \frac{e_1 x_1 + e_2 x_2 + ... + e_7 x_7}{x} \; .$$

where $e_1 ... e_7$ are the imaginary unit vectors and x the norm of the "vector" part of the octonionic point. The solutions are natural extensions of the quaternionic case.

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