ODD DIMENSIONAL TRANSLATION BETWEEN COMPLEX AND QUATERNIONIC QUANTUM MECHANICS

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We complete the rules of translation between standard complex quantum mechanics (CQM) and quaternionic quantum mechanics (QQM) with a complex geometry. In particular we describe how to reduce ($2n$+1)-dimensional complex matrices to *overlapping* (n+1)-dimensional quaternionic matrices with generalized quaternionic elements. This step resolves an outstanding difficulty with reduction of purely complex matrix groups within quaternionic QM and avoids *anomalous* eigenstates. As a result we present a more complete translation from CQM to QQM and viceversa.

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I. INTRODUCTION

The possibility of using quaternions to describe standard quantum mechanics received a major thrust with the adoption by Horwitz and Biedenharn [1] of a complex scalar product (complex geometry [2]). These authors showed that this assumptions permits the definition (in analogy with standard theory) of tensor products between single particle wave functions without encountering intractable problems of interpretation and definition due to the non commuting multiplications of quaternionic wave functions[♯1].

A second important step in this objective of translation was achieved with the introduction of the so-called *barred* operators [3], and specifically of $q \mid i$ (q-quaternion, i-one of the three imaginary units) which acts upon a (quaternionic) wave function ψ by

$$(q \mid i)\psi = q\psi i \quad . \tag{1}$$

Originally these generalized quaternions were introduced to define an acceptable (i-complex hermitian) momentum operator $(-\boldsymbol{\partial} \mid i)$ within a two-component quaternionic Dirac equation [4].

The distinction between left acting or right acting quaternions is of course justified by the non commutativity of quaternions. Instead of proceeding to consider the (possibly future) generalization to $1 \mid j$ and $1 \mid k$ terms, it was noted by the authors in a previous work [5] that generalized quaternions i. e. $q_1 + q_2 \mid i$ (indicated generically by $\mathcal{H} \mid \mathcal{C}$) depend upon eight real numbers, the same degree of freedom of the most general two by two complex matrix. In fact the already well known identifications of i, j and k with $-\frac{i}{2}\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$ the Pauli matrices) and of course 1 (in $\mathcal{H}$) with the 2-dimensional unit matrix can thus be extended to the most general 2-dimensional complex matrix.

A particular set of rules for translation is given by

$$M = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \quad \Longrightarrow \quad M = q_1 + q_2 \mid i \qquad [c_{1,\,\dots,\,4} \in \mathcal{C}(1,\ i) \ \text{ and } \ q_{1,\,2} \in \mathcal{H}] \quad , \tag{2a}$$

with

$$2q_1 = c_1 + c_4^* + j(c_3 - c_2^*) \quad ,$$
$$2iq_2 = c_1 - c_4^* - j(c_3 + c_2^*) \quad .$$

Equivalent to:

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♯1 A minor criticism of the formalism of Horwitz and Biedenharn is that quaternions are used only for single particle wave functions in their paper. In practise they use the complex translation for multiparticle states, an unnecessary choice.

$$M = z_1 + j\tilde{z}_1 + (z_2 + j\tilde{z}_2) \mid i \quad \Longrightarrow \quad M = \begin{pmatrix} z_1 + iz_2 & -(\tilde{z}_1^* + i\tilde{z}_2^*) \\ \tilde{z}_1 + i\tilde{z}_2 & z_1^* + iz_2^* \end{pmatrix} \quad , \tag{2b}$$

This augments the so called symplectic rule for state vectors (column matrices)

$$\psi_a = \begin{pmatrix} z_a \\ \tilde{z}_a \end{pmatrix} \quad \Longleftrightarrow \quad \psi_a = z_a + j\tilde{z}_a \quad , \tag{3a}$$

$$\psi_a^\dagger = (z_a^* \;\; \tilde{z}_a^*) \quad \Longleftrightarrow \quad \psi_a^\dagger = z_a^* - j\tilde{z}_a \quad , \tag{3b}$$

which in turn is consistent only within a complex geometry

$$\psi_1^\dagger \psi_2 = z_1^* z_1 + z_2^* z_2 \quad \Longleftrightarrow \quad (\psi_1^\dagger \psi_2)_c = \frac{1 - i \mid i}{2} \, \psi_1^\dagger \psi_2 \quad . \tag{4}$$

We recall also that the rule for the tensor product of two state vectors ψ_1 and ψ_2 is thus automatic

$$\psi_1 \otimes \psi_2 = \begin{pmatrix} z_1 z_2 \\ z_1 \tilde{z}_2 \\ \tilde{z}_1 z_2 \\ \tilde{z}_1 \tilde{z}_2 \end{pmatrix} \quad \Longleftrightarrow \quad \psi_1 \otimes \psi_2 = \begin{pmatrix} z_2 z_1 + j\tilde{z}_2 z_1 \\ z_2 \tilde{z}_1 + j\tilde{z}_2 \tilde{z}_1 \end{pmatrix} = \begin{pmatrix} \psi_2 z_1 \\ \psi_2 \tilde{z}_1 \end{pmatrix} \quad . \tag{5}$$

In the same way we may derive the rules for taking the tensor product of group elements, for example,

$$g_1 \otimes g_2 = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \otimes \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} c_1 z_1 & c_1 z_2 & c_2 z_1 & c_2 z_2 \\ c_1 z_3 & c_1 z_4 & c_2 z_3 & c_2 z_4 \\ c_3 z_1 & c_3 z_2 & c_4 z_1 & c_4 z_2 \\ c_3 z_3 & c_3 z_4 & c_4 z_3 & c_4 z_3 \end{pmatrix} \tag{6a}$$

can be translated as follows

$$g_1 \otimes g_2 = (q_1 + q_2 \mid i) \otimes (p_1 + p_2 \mid i) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} (p_1 + p_2 \mid i) \quad , \tag{6b}$$

where

$$\begin{aligned}
\alpha_1 &= (q_1 + iq_2)_c + (q_1 + iq_2)_c^* - i[(q_1 + iq_2)_c - (q_1 + iq_2)_c^*] \mid i \quad , \\
\alpha_2 &= (jq_1 - kq_2)_c^* + (jq_1 - kq_2)_c - i[(jq_1 - kq_2)_c^* - (jq_1 - kq_2)_c] \mid i \quad , \\
\alpha_3 &= -(jq_1 + kq_2)_c - (jq_1 + kq_2)_c^* + i[(jq_1 + kq_2)_c - (jq_1 + kq_2)_c^*] \mid i \quad , \\
\alpha_4 &= (q_1 - iq_2)_c^* + (q_1 - iq_2)_c - i[(q_1 - iq_2)_c^* - (q_1 - iq_2)_c] \mid i \quad .
\end{aligned}$$

This translation is only partial because it connects e. g. n-dimensional quaternionic representations of quaternionic groups with *even* $2n$-dimensional complex representations of corresponding (isomorphic) complex groups and viceversa. Indeed we have presented elsewhere [3] a comparison between the groups[♯2] $U(1,\ q)$ and $SU(2,\ c)$. Odd complex representations of the complex groups are excluded. There is also a problem with the irreducibility of odd dimensional complex representations of quaternionic groups (see below).

We note, in passing, that while matrix operators may contain generalized quaternions, the state vectors (wave functions in general) contain only quaternionic elements. This asymmetry correctly reproduces the *real* degrees of freedom between n-component complex column matrices and n-dimensional complex square matrices. This is because a quaternion has 4 real degrees of freedom but a generalized quaternion 8.

[♯2] With generalized quaternions we can also confront $U(2,\ c)$ with $U(1,\ q_c)$, where $q_c \in \mathcal{H} \mid \mathcal{C}$.

Degrees of freedom on (n even)	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{H}$	$\mathcal{H}\mid\mathcal{C}$
ψ	$2n$	n	$n/2$	
		$\uparrow\downarrow$	$\nwarrow\searrow$	
M	$2n^2$	n^2		$(n/2)^2$

As described in previous articles [3,5] the above translation is inadequate for odd dimensional complex representations be they for groups (operators in general) or for state vectors. In particular this problem first arose for the representation of odd dimensional spin states i. e. bosons (s=0,1, ...). It is perhaps instructive to describe briefly the situation previous to this work for these odd dimensional states.

The first discovery was the existence of *anomalous* solutions [6] for bosonic quaternionic dynamical equations (Klein-Gordon, Maxwell, ...). This simply followed from the complex geometry which imposes the orthogonality of ψ and ψj. If ψ is chosen as the purely complex solution of a given bosonic equation then the purely quaternionic (j-k) solutions are given by ψj. Or more precisely, by $j\psi$ if we wish to confront solutions with identical 4-momentum. This feature is already present when considering the eigenvalue solutions of an odd dimensional matrix equation. For example the (quaternionic) eigenstates of the standard spin-1 eigenvalue equation are

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}j\\0\\0\end{pmatrix}, \begin{pmatrix}0\\j\\0\end{pmatrix}, \begin{pmatrix}0\\0\\j\end{pmatrix} \quad , \tag{7}$$

the last three (anomalous) cannot mix under rotations with the former because of the complex nature of the 3-dimensional rotation generators. Thus the vector space is *reducible* but not apparently the operator form[♯3]. This at least was the belief up to know. Without the simultaneous reduction of states and operators it was not clear if these group representations - complex odd dimensional acting upon a quaternionic space - could be neglected. They appeared to exclude a complete translation between CQM and QQM which was good or bad news according to ones point of view.

There existed at least one way of bypassing this problem by eliminating the anomalous solutions [7,8]. Assuming of course that the anomalous solutions are an unwanted feature. This was to use "spinor" (fermionic-like) equations for bosons such as the Kemmer equation which contains spin 1 plus spin 0 solutions. The corresponding Duffin-Kemmer-Petiau β-matrices (being even dimensional 16×16) have a reduced form of 8×8 quaternionic representations. In this procedure anomalous solutions do not appear for the same reason that they do not appear for the Dirac equation. Actually the β-complex matrices are formed of a 10-dimensional (spin 1) and 5-dimensional (spin 0) and 1-dimensional (unphysical $\beta=0$) block structure. Thus while the spin 1 case is automatic because even-dimensional, the spin 0 case is handled by using the trick of adding the unphysical solution and working formally with 6×6 matrices. We have shown elsewhere that the Kemmer quaternionic equation is thus not equal to the Klein-Gordon quaternionic equation etc., because it does not have the anomalous solutions. In fact it corresponds to the modified equations obtained by complex-projecting the original quaternionic equations.

The above trick uses explicitly the dynamical particle equations. It has limited the odd dimensional states to purely complex ones and thus apparently made the corresponding representations irreducible. *This is not the correct interpretation.* First because we must be able to discuss the group representations of e. g. $SU(2,\ c)$ without any reference to dynamical equations. Secondly because, in fact, the odd dimensional complex representations *are* reducible with generalized quaternions thanks to the overlapping feature described below. Hence the problem of having a reducible vector space for a non-reducible matrix representation will be partially eliminated.

[♯3]Fermionic complex operators (but acting within a quaternionic space) being even-dimensional matrices are reduced to half the dimension with generalized quaternions, with the maintenance of the same number of total solutions.

II. OVERLAPPING REPRESENTATIONS

We shall describe this technique first by deriving the situation for spin 1 and then extracting the general rules for any odd dimensional matrix. The three complex antihermitian generators of spin 1 A_m $(m = 1,\ 2,\ 3)$ are in standard form

$$A_1 = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad A_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \text{-}1 & 0 \\ 1 & 0 & \text{-}1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad A_3 = -i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \text{-}1 \end{pmatrix} \quad . \tag{8}$$

These have normal/anomalous solutions shown in eq. (7).

Thus each eigenvalue is degenerate and the vector space represented by the column matrices is reducible to two three dimensional subspaces. The conventional form of reductions of the 3×3 generators is not possible because it would require the division of the 3×3 matrices A_m into two distinct blocks one of 2×2 (quaternionic in general and this is possible) and one of 1×1 (an $\mathcal{H} \mid \mathcal{C}$ number) and this can be explicitly excluded. Hermitian generators of $SU(2)$ (spin) within the numbers $\mathcal{H} \mid \mathcal{C}$ exist, they are $i \mid i,\ j \mid i,\ k \mid i$ [♯4]. However, these correspond to spin 1/2 eigenvalues and not spin 1. It is easy to convince oneself that no one-dimensional spin 1 representations exist in $\mathcal{H} \mid \mathcal{C}$.

This is an example of the reduction problem mentioned previously. Now we shall show explicitly that there exists a generalized quaternionic similarity matrix S $(S^\dagger = S^{-1})$ such that the A_m are reduced to two overlapping 2×2 block forms so that one element, the (2,2)-element, is common to both blocks. However if this element if not identically zero it is always a composite of two terms one of which annihilates one of the corresponding eigenvectors. Thus the two blocks may be unlocked and studied separately.

Explicitly an S matrix such as that described above is given by:

$$S = \begin{pmatrix} a & ja & 0 \\ 0 & 0 & 1 \\ d & -jd & 0 \end{pmatrix} \quad , \tag{9a}$$

$$S^\dagger = \begin{pmatrix} a & 0 & d \\ -jd & 0 & ja \\ 0 & 1 & 0 \end{pmatrix} \quad , \tag{9b}$$

with

$$\begin{array}{lcl} 2a = 1 - i \mid i & \quad \text{extinguishes quaternionic elements} & , \\ 2d = 1 + i \mid i & \quad \text{extinguishes complex elements} & , \end{array}$$

whence,

$$a^2 + d^2 = 1\ ,\ a + d = 1\ ,\ da = ad = 0\ ,\ (ja)^\dagger = -jd\ ,\ ja = dj \quad .$$

The transformed generators $\tilde{A}_m = S A_m S^\dagger$ are then given by

$$\tilde{A}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} k & ka & 0 \\ kd & 0 & -ka \\ 0 & -kd & -k \end{pmatrix} \quad , \tag{10a}$$

$$\tilde{A}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} j & -ja & 0 \\ -jd & 0 & ja \\ 0 & jd & -j \end{pmatrix} \quad , \tag{10b}$$

[♯4] Note that $i,\ j,\ k$ are *antihermitian* generators and only within $\mathcal{H} \mid \mathcal{C}$ do they have straightforward hermitian equivalents. With a quaternionic geometry one is restricted to only antihermitian generators [9].

$$
\tilde{A}_3 = -i \begin{pmatrix} a & 0 & \vline & 0 \\ \cline{4-4} 0 & \multicolumn{1}{|c}{-1} & \vline & 0 \\ \cline{1-2} 0 & \multicolumn{1}{|c}{0} & & d \end{pmatrix} \quad . \tag{10c}
$$

In $\tilde{A}_3$ the (2,2)-element can be written conveniently as $i(a+d)$, i. e. containing a sum of projection operators. The corresponding transformed state vectors with eigenvalues (with respect to $\tilde{B}_3 \mid i$ or $\tilde{C}_3 \mid i$) $+1,\ 0,\ -1$ are respectively

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} j \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \begin{pmatrix} 0 \\ j \\ 0 \end{pmatrix} \quad .
$$

We observe that the first triplet consists of vectors of the form

$$
\begin{pmatrix} q \\ z \\ 0 \end{pmatrix} \quad ,
$$

while the second triplet of vectors of the form

$$
\begin{pmatrix} 0 \\ jz \\ q \end{pmatrix} \quad .
$$

Naturally the two triplets remain orthogonal with a complex geometry and furthermore the separate sets of reduced 2×2 quaternionic blocks do not perform any rotation upon one or other set of triplets. Actually the two sets of reduced 2×2 generators are connected by a similarity transformation and thus are in turn equivalent. Explicitly the sets of 2×2 reduced quaternionic representations are:

$$
\tilde{B}_1 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} k & ka \\ kd & 0 \end{pmatrix} \quad , \quad \tilde{B}_2 = \tfrac{-1}{\sqrt{2}} \begin{pmatrix} -j & ja \\ jd & 0 \end{pmatrix} \quad , \quad \tilde{B}_3 = -ia \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ;
$$

$$
\tilde{C}_1 = \tfrac{-1}{\sqrt{2}} \begin{pmatrix} 0 & ka \\ kd & k \end{pmatrix} \quad , \quad \tilde{C}_2 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & ja \\ jd & -j \end{pmatrix} \quad , \quad \tilde{C}_3 = -id \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad .
$$

The corresponding state vectors with eigenvalues $+1,\ 0,\ -1$ respectively are

$$
\begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} j \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ j \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \begin{pmatrix} j \\ 0 \end{pmatrix} \quad .
$$

It is of course natural to ask what the translation of these reduced 2×2 generators to complex form yields. The answer, which seems obvious a posteriori is the 4×4 complex generators of $SU(2)$ *reducible* to spin-1 $\oplus$ spin-0. Not, of course, the irreducible 4×4 representations which would correspond to spin 3/2. Because of our derivation we would be tempted to identify the spin-0 element as a member of an independent triplet, but this has no physical foundation. What is significant is that the reduction is not perfect in the sense that it brings along a *scalar* partner.

These results lead to the following consequences: One is a mechanical (automatic) way of reducing *any* odd dimensional (otherwise irreducible) complex matrix with quaternions into *overlapping block* structure. The second is the physical significance of this procedure. For the first, we propose to add an extra row and column of zero's to our matrix thus making it become an even matrix and then applying the *translation* of this complex matrix to quaternions. This is a formal *trick* since we began with a complex odd dimensional matrix operating upon a quaternionic space i. e. considered as a quaternionic matrix without need of translation and with only a question about its reducibility. Nevertheless, this trick always yields one or other of the overlapping block forms. For the spin-1 case studied above this procedure gives e. g.

$$
-i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \longrightarrow \quad -i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad ia \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{11a}
$$

or/and,

$$
-\,i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \text{-1} \end{pmatrix} \quad\longrightarrow\quad -\,i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-1} \end{pmatrix} \quad\Longrightarrow\quad id\begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix} \quad . \tag{11b}
$$

This procedure avoids the need of determining S explicitly each time and is very simple. We also note that the resulting 2×2 generators exactly reproduce the tensor product of the generators of spin $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$. This brings us to the physical interpretation which will be given in the conclusions.

We complete this section by giving explicitly the quick-rules for translating (and not now reducing) a generic odd dimensional complex matrix:

$$
M \;=\; \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & c_1 & c_2 & c_3 \\ \cdots & c_4 & c_5 & c_6 \\ \cdots & c_7 & c_8 & c_9 \end{pmatrix} \quad .
$$

As suggested above, we add an extra row and column of zero's to our odd dimensional complex matrix

$$
M \;\to\; \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & c_1 & c_2 & c_3 & 0 \\ \cdots & c_4 & c_5 & c_6 & 0 \\ \cdots & c_7 & c_8 & c_9 & 0 \\ \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \quad ,
$$

after that we can immediately translate the blocks

$$
\begin{pmatrix} c_1 & c_2 \\ c_4 & c_5 \end{pmatrix} \quad , \quad \begin{pmatrix} c_3 & 0 \\ c_6 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} c_7 & c_8 \\ 0 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} c_9 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad ...
$$

by the standard rules for even dimensional translation (2a). The final result brings to

$$
M = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & c_1 & c_2 & c_3 \\ \cdots & c_4 & c_5 & c_6 \\ \cdots & c_7 & c_8 & c_9 \end{pmatrix} \quad\Longrightarrow\quad M = \begin{pmatrix} \ddots & \vdots & \vdots \\ \cdots & Q_1 & Q_2 \\ \cdots & Q_3 & Q_4 \end{pmatrix} \quad , \tag{12}
$$

$$
\begin{aligned}
2Q_1 &= c_1 + c_5^* + j(c_4 - c_2^*) - i[c_1 - c_5^* - j(c_4 + c_2^*)] \mid i \quad ,\\
2Q_2 &= (c_3 + jc_6)(1 - i \mid i) \quad ,\\
2Q_3 &= (1 - i \mid i)(c_7 - jc_8^*) \quad ,\\
2Q_4 &= c_9(1 - i \mid i) \quad .
\end{aligned}
$$

III. CONCLUSIONS

We began by admitting embarrassment with purely complex odd dimensional matrix representations of a group (our example was $SU(2) \sim U(1,\ q)$) acting upon a quaternionic space. The space representation was reducible, the generators (and hence group) representations were not. Furthermore, we could not translate odd dimensional complex vector spaces in CQM as was possible for even dimensional spaces. Now we suggest a cure for both these problems. First the odd dimensional complex matrices (within quaternionic space) are reducible if we allow for overlapping block structures. A *remnant* of say the anomalous solutions survives in the form of a singlet (scalar) state. We have suggest the procedure of adding an extra null line and column and then formally translating as a rapid way of deriving the reduced overlapping blocks. Even from this viewpoint the extra lines correspond to a one-dimensional zero generator (the added corner element) which implies the existence of an additional scalar. In the case of translations of odd dimensional operators from CQM to QQM we repeat this procedure of adding an extra null line and column and then translating. The complex state vectors will, of course, also acquire an extra element. This again corresponds to adding by hand an extra scalar particle. Whether in a given theory this particle is physical or not is beyond the limits of this paper. It depends upon the dynamics of the situation e. g. recall the case of the Kemmer equation.

However, one cannot elude the impression that quaternions invite *even numbers of particle states*. Thus for spin 1 we are obliged to add a scalar state even if we avoid the full number of anomalous copies. Perhaps it is only an accident that at the fundamental level the number of Higgs (before spontaneous breaking) is four as are the number of gauge particles as are the number of particle (antiparticle) fermions per family (at least of left handed nature). We thus have no difficulty in dealing and translating the Salam-Weinberg model [10]. We apparently have difficulty with color (for $SU(3)$ triplet states) but here we have two possibilities. If we include other multiplicative groups such as $SU(2)_{weak}$ then the overall dimensions may remain even and be translated as above. In this case the only price we pay is our inability to assign a *natural* quaternionic group to $SU(3)_{color}$. Alternatively, and far more ambitions, we would be tempted to use 4×4 (reducible) representations of $SU(3)_{color}$ before translating. This corresponds to passing to $SU(4)_{color} \sim SU(2,\ q_c)$ or in terms of its $SU(3)_{color}$ subgroup to consider 3 primary colors plus white (color singlet). In this case we must identify the *white* quark with a physical particle, for example the neutrino as in the Pati-Salam model [11], or else seek refuge in spontaneous symmetry breaking to take the white quark mass to values beyond present experimental limits.

Finally we wish to remember that within quaternionic quantum mechanics with *quaternionic geometry* there is a stimulating possibility to look at fundamental physics as proposed by Adler [9,12]. He suggest that the color degree of freedom postulated in the Harari-Shupe scheme [13] could be sought in a *noncommutative* extension of standard quantum mechanics.

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