# Relative helicity phases in planar Dirac scattering

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We study planar Dirac scattering for an electrostatic stratified barrier potential. The general expressions for transmitted and reflected waves are derived. Of particular interest is the information upon relative helicity phases. We also briefly discuss some possible applications.

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## I. INTRODUCTION

One of the objectives of this work is the derivation of the general formulas for spinor plane wave scattering from a stratified barrier potential. The potential is assumed to be the zero component of a four-vector potential, also known in the literature as an "electrostatic" potential [1]. Our approach is three-dimensional but the assumed stratified direction ($x$ axis) of the potential together with the incoming three-momentum determines the scattering plane within which lie both the reflected and transmitted momenta. We shall call this the $x$-$y$ plane and hence, without lose of generality, we consider planar scattering, both diffusion and tunneling [2–13]. Our spinors are Dirac spinors. We shall also operate with helicity amplitudes since helicity is a good quantum number for Dirac spinors [14].

Amongst the features that emerges for the reflected waves is the presence of both helicity *flip* and *nonflip* terms. Furthermore, the corresponding probabilities depend explicitly upon the relative phase of the incoming helicity terms, when both are present. This allows one to *measure* this relative phase. As far as we know this has not been noted previously.

Overall phase factors are of no physical significance and indeed cannot be measured in any way. However, relative phases can be important and influence experiments. We therefore start by recalling the significance of some relative phases in diverse areas of particle physics [14–16]. The relative phases that are most common in recent particle literature are those between different but related decay amplitudes [17–20]. They are measurable when a multiple decay chain involves alternative paths resulting in the same final states. They are thus coherent and must be added before squaring. However, the example in this paper involves the relative phases between states, so let us limit our discussion to phases of this type. Optical interference phenomena [21–24] are the obvious antecedents. A classic example is the two-slit experiment with possibly the inclusion of a transparent plate before one of the slits to modify the relative phase between contributions [25]. Another is that encountered in oscillation phenomena such as in neutrino physics [26,27]. In oscillations, two or more mass eigenstates contained in a "flavor" eigenstate [28,29]—for kaons this flavor would be the strong hypercharge [30]—develop differently in space and time resulting in relative phases and consequent flavor superpositions. A third example stems from an isospin calculation [31]. Let us determine, for example, the ratio of cross sections,

$$R = \frac{\sigma(pp \to \pi^+ d)}{\sigma(np \to \pi^0 d)}, \tag{1}$$

where $d$ is the singlet deuteron. The incoming states, seen in an arbitrary Lorentz frame, formally represent particle beams, even if a fixed target characterizes the laboratory frame. There is no problem for the cross section in the numerator; it involves pure $I = 1$ states. But how do we represent the "$np$" beams in the denominator? The natural choice is the tensor product of the neutron and proton isospin kets, $|n\rangle \otimes |p\rangle$, and this choice yields the result of $R = 2$ [32]. However, there is no *operational distinction* between "$np$" and "$pn$," although they are mathematically distinct, indeed they represent orthogonal states. Now, if one uses "$pn$," $|p\rangle \otimes |n\rangle$, instead of "$np$," one again obtains $R = 2$. Nevertheless, the most general beam-beam representation is an admixture of both, including an arbitrary relative phase, that is,

$$\cos\theta |n\rangle \otimes |p\rangle + e^{i\alpha} \sin\theta |p\rangle \otimes |n\rangle. \tag{2}$$

Using

$$|n\rangle \otimes |p\rangle = (|1\rangle + |0\rangle)/\sqrt{2} \quad \text{and}$$
$$|p\rangle \otimes |n\rangle = (|1\rangle - |0\rangle)/\sqrt{2},$$

where $|0\rangle$ and $|1\rangle$ represent isospin $I = 0$ and $I = 1$ states, the superposition in Eq. (2) can be written as follows:

$$[(\cos\theta + e^{i\alpha}\sin\theta)\,|1\rangle + (\cos\theta - e^{i\alpha}\sin\theta)\,|0\rangle]/\sqrt{2}. \tag{3}$$

In terms of the mixing angle $\theta$ and of the relative phase $\alpha$, the ratio of cross sections $R$ is then

$$R = 2/[1 + \cos\alpha \sin 2\theta]. \tag{4}$$

This shows that $R = 2$ is obtained not only for a pure $|n\rangle \otimes |p\rangle$ state or a pure $|p\rangle \otimes |n\rangle$ state, but for a general linear combination of these state if the relative phase is $\alpha = \pi/2$.

Finally, another type of relative phase is that between quark eigenstates used in the reduction of a $2 \times 2$ unitary matrix to the Cabibbo rotation form [33]. A similar use is made in the reduction of the $3 \times 3$ unitary Kobayashi-Maskawa (KM) matrix to its standard form with three real angles and one complex phase [34]. The CP violating phase in the KM matrix is a very important phase but it is *not* a relative phase between two states, it appears between decay amplitudes

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[17–20]. The quark relative phases discussed here, although theoretically essential, are not experimentally measurable. This distinguishes them from the other examples given above.

This paper is structured as follows. In the next section, we introduce our formalism, discuss the Dirac energy zones for a stratified electrostatic potential, and give the general formulas of the plane wave functions in the free and potential regions. In Sec. III, by using a matrix approach, we explicitly calculate the reflection and transmission coefficients for planar scattering. In Sec. IV, we study the relative phase between the two helicity eigenstates of an incoming plane wave and analyze how it modifies the probabilities of the reflected helicities. Our conclusions are drawn in the final section.

## II. FORMALISM

We treat a potential $V_0$ transforming as the zero component of a Lorentz four-vector and thus leading to the appearance of factors such as $E - V_0$. The Dirac Hamiltonian in the presence of a constant electrostatic potential reads [15]

$$H_D^{(\mathrm{el})} = -i\,\boldsymbol{\alpha}\cdot\nabla + \beta\,m + V_0. \tag{5}$$

This can be contrasted, for example, with a scalar potential analysis in which the potential $V_s$ adds to the mass leading to factors such as $m + V_s$,

$$H_D^{(\mathrm{sc})} = -i\,\boldsymbol{\alpha}\cdot\nabla + \beta\,(m + V_s). \tag{6}$$

There are multiple possible potential forms, including Yukawa potentials [35]. None of these others will be discussed in this work.

In a previous study [12], for this potential, we showed that spin flip, which is rigorously absent in one-dimensional scattering, is *present* in two-dimensional or planar scattering with the exception of "head-on" collisions. Head-on collisions reduce the problem to the one-dimensional case. We warn here of a possible confusion. For a head-on collision the reflected wave has an inverted momentum, so spin flip corresponds to helicity conservation. Thus absence of spin flip implies *total* helicity flip for head-on collisions.

In the following, we use the Pauli-Dirac representation of the Dirac matrices. Since the scattering plane is chosen as the $x$-$y$ plane, the incoming particle four-momentum reads

$$p^{\mu}_{\mathrm{in}} = (E,\,p_1,\,p_2,\,0),$$

with $E^2 = p_1^2 + p_2^2 + m^2$. The reflected momentum is

$$p^{\mu}_{\mathrm{ref}} = (E,-p_1,\,p_2,\,0).$$

In the barrier region, $0 < x < L$, the four-momentum $q^{\mu}$ is obtained from the incident momentum by the substitution $E \to E - V_0$ and observing that the discontinuity along the $x$ axis implies $q_2 = p_2$,

$$q^{\mu} = (E - V_0,\,q_1,\,p_2,\,0).$$

The $q_1$ is determined by the relation $(E - V_0)^2 = q_1^2 + p_2^2 + m^2$. In the potential region the solutions can be oscillatory in $x$ ($q_1^2 > 0$), this happens for the following energy zones:

$$E > V_0 + \sqrt{p_2^2 + m^2} \quad \text{(diffusion)},$$

$$m < E < V_0 - \sqrt{p_2^2 + m^2} \quad \text{(Klein zone)},$$

or evanescent ($q_1^2 < 0$), when

$$V_0 - \sqrt{p_2^2 + m^2} < E < V_0 + \sqrt{p_2^2 + m^2} \quad \text{(tunneling)}.$$

The Klein energy zone for a step is consistent with Klein's suggestion of the creation of particle-antiparticle pairs, with antiparticles propagating in the potential region, which they see (because of their opposite charge) as a potential well. In a previous work [7], we have strongly argued that this phenomena (Klein pair production), if it occurs for a step potential, must *necessarily* also occur for a barrier. However, for simplicity we avoid this Klein zone in this paper. As an aside, we note on the contrary, that Klein pair production is *not* compatible with the kinematics of a scalar potential.

Returning to our analysis, the outgoing (transmitted) momentum is identical to the incoming (incident) momentum $p^{\mu}_{\mathrm{out}} = p^{\mu}_{\mathrm{in}}$. The helicity operator is $\boldsymbol{\Sigma}\cdot\mathcal{P}/|\boldsymbol{p}|$, with $\boldsymbol{\mathcal{P}}$ the three-momentum operator and $\boldsymbol{\Sigma} = \mathrm{diag}[\boldsymbol{\sigma},\boldsymbol{\sigma}]$. The two planar eigenstates of this operator, with eigenvalues $\pm 1$, are explicitly

$$\psi_{\pm}[p_1,E,x]\,\exp[i\,(p_2\,y - E\,t)], \tag{7}$$

with

$$\psi_{\pm}[p_1,E,x] = \frac{1}{2}\sqrt{\frac{E+m}{E}}\left(\pm 1,\frac{p_1 + i\,p_2}{\sqrt{E^2 - m^2}},\sqrt{\frac{E-m}{E+m}},\pm\frac{p_1 + i\,p_2}{E+m}\right)^t \exp[i\,p_1\,x].$$

The plane-wave solution is thus divided into three spatial regions, with $\exp[i\,(p_2\,y - E\,t)]$ common to all three. For the free potential region before the barrier, region I with $x < 0$, we have

$$\begin{aligned}\Psi_{\mathrm{I}}[p_1,E,x] &= I_-\psi_-[p_1,E,x] + I_+\psi_+[p_1,E,x]\\ &\quad + R_-\psi_-[-p_1,E,x] + R_+\psi_+[-p_1,E,x],\end{aligned} \tag{8}$$

where the $\pm 1$ helicity amplitudes are indicated by $I_{\pm}$ for the incoming plane waves, moving from left to right in the positive $x$ direction, and by $R_{\pm}$ for the reflected waves, moving from right to left. For the barrier potential, region II with $0 < x < L$, we have

$$\begin{aligned}&\Psi_{\mathrm{II}}[q_1,E - V_0,x]\\ &\quad = A_-\psi_-[q_1,E - V_0,x] + A_+\psi_+[q_1,E - V_0,x]\\ &\qquad + B_-\psi_-[-q_1,E - V_0,x] + B_+\psi_+[-q_1,E - V_0,x],\end{aligned} \tag{9}$$

with $A_{\pm}$ and $B_{\pm}$ the helicity amplitudes for plane waves traveling to and from the $x = L$ potential discontinuity. Finally, for the free potential region after the barrier, region III with $x > L$, we have

$$\Psi_{\mathrm{III}}[p_1,E,x] = T_-\psi_-[p_1,E,x] + T_+\psi_+[p_1,E,x], \tag{10}$$

with $T_{\pm}$ the helicity amplitudes for the outgoing plane waves moving from left to right.

### III. REFLECTION AND TRANSMISSION COEFFICIENTS

The continuity equations are

$$\Psi_{\rm I}[p_1,E,0] = \Psi_{\rm II}[q_1,E-V_0,0] \quad \text{and} \quad \Psi_{\rm II}[q_1,E-V_0,L] = \Psi_{\rm III}[p_1,E,L]. \tag{11}$$

The above continuity equations imply, after eliminating the $A$ and $B$ terms,

$$\begin{bmatrix} I_+ - I_- + R_+ - R_- \\ \frac{p_1+i\,p_2}{\sqrt{E^2-m^2}}\,(I_+ + I_-) + \frac{-p_1+i\,p_2}{\sqrt{E^2-m^2}}\,(R_+ + R_-) \\ \sqrt{\frac{E-m}{E+m}}\,(I_+ + I_- + R_+ + R_-) \\ \frac{p_1+i\,p_2}{E+m}\,(I_+ - I_-) + \frac{-p_1+i\,p_2}{E+m}\,(R_+ - R_-) \end{bmatrix} = M \begin{bmatrix} T_+ - T_- \\ \frac{p_1+i\,p_2}{\sqrt{E^2-m^2}}\,(T_+ + T_-) \\ \sqrt{\frac{E-m}{E+m}}\,(T_+ + T_-) \\ \frac{p_1+i\,p_2}{E+m}\,(T_+ - T_-) \end{bmatrix} \exp[i\,p_1 L],$$

where

$$M = S\,D^*\,S^{-1},$$

with

$$S = \begin{pmatrix} 1 & -1 & 1 & -1 \\ \frac{q_1+ip_2}{\sqrt{(E-V_0)^2-m^2}} & \frac{q_1+ip_2}{\sqrt{(E-V_0)^2-m^2}} & \frac{-q_1+ip_2}{\sqrt{(E-V_0)^2-m^2}} & \frac{-q_1+ip_2}{\sqrt{(E-V_0)^2-m^2}} \\ \sqrt{\frac{E-V_0-m}{E-V_0+m}} & \sqrt{\frac{E-V_0-m}{E-V_0+m}} & \sqrt{\frac{E-V_0-m}{E-V_0+m}} & \sqrt{\frac{E-V_0-m}{E-V_0+m}} \\ \frac{q_1+ip_2}{E-V_0+m} & -\frac{q_1+ip_2}{E-V_0+m} & \frac{-q_1+ip_2}{E-V_0+m} & -\frac{-q_1+ip_2}{E-V_0+m} \end{pmatrix},$$

and

$$D = \mathrm{diag}[\exp(iq_1L), \exp(iq_1L), \exp(-iq_1L), \exp(-iq_1L)].$$

After some algebra we find

$$M = \begin{bmatrix} a_- & 0 & 0 & b_+ \\ 0 & a_+ & b_+ & 0 \\ 0 & b_- & a_- & 0 \\ b_- & 0 & 0 & a_+ \end{bmatrix},$$

where

$$a_\pm = \cos(q_1L) \pm \frac{p_2}{q_1}\sin(q_1L) \quad \text{and}$$
$$b_\pm = -i\,\frac{E-V_0\pm m}{q_1}\sin(q_1L).$$

Defining $R$ and $\widetilde{R}$ as the reflected helicity conserving and changing amplitudes and recalling that there is no spin flip for the transmitted wave, which implies a unique transmitted amplitude $T$, it can be proven that the incident, reflected, and transmitted amplitudes are

$(I_+,I_-)$ (incident),

$(I_+R + I_-\widetilde{R}, I_-R + I_+\widetilde{R}) = (R_+,R_-)$ (reflected),

$(I_+T, I_-T) = (T_+,T_-)$ (transmitted).

The explicit expression for $R$, $\widetilde{R}$, and $T$ confirm our previous published results and are explicitly

$$\widetilde{R} = i\,\frac{m\,V_0(p_1+ip_2)}{q_1(E^2-m^2)}\sin(q_1L)T\exp(ip_1L),$$
$$R = -\frac{p_2V_0E(p_1+ip_2)}{q_1p_1(E^2-m^2)}\sin(q_1L)T\exp(ip_1L), \tag{12}$$
$$T = \exp(-ip_1L)\Big/\left[\cos(q_1L) + i\,\frac{EV_0-p_1^2}{p_1q_1}\sin(q_1L)\right].$$

They satisfy the necessary probability conserving feature,

$$|R|^2 + |\widetilde{R}|^2 + |T|^2 = 1. \tag{13}$$

Again as in previous articles we emphasize that our $R$, $\widetilde{R}$, and $T$ are amplitudes and not probabilities as sometimes defined in the literature. If the incoming plane wave is normalized conventionally then,

$$|I_+|^2 + |I_-|^2 = 1,$$

and consequently probability conservation yields

$$|R_+|^2 + |R_-|^2 + |T_+|^2 + |T_-|^2 = 1. \tag{14}$$

Obviously each reflected helicity amplitude $R_\pm$ is the sum of the conserved helicity term $I_\pm R$ plus and helicity flipped contribution $I_\mp\widetilde{R}$. From the explicit expressions for $R$, $\widetilde{R}$, and $T$ we recognize the well-known feature of resonance which occurs when

$$\sin(q_1L) = 0 \Leftrightarrow q_1L = n\,\pi \quad (n = 1,2,\ldots).$$

For these particular cases $R_\pm = 0$ and $T_\pm = I_\pm$ (total transmission). It is worth recalling here that resonance phenomena is only valid for diffusion when coherence dominates. For finite spatial wave packets, coherence will break down if the barrier length $L$ grows sufficiently larger than the wave-packet dimensions or for glancing impacts (high incident angles). In either case multiple reflected and transmitted wave packets appear at regular intervals. Resonance is then lost. It is also absent for tunneling phenomena but because of a different cause. Tunneling is consistent with the above formulas but with the significant difference of an *imaginary* value for $q_1$. This follows from

$$q_1^2 = (E-V_0)^2 - p_2^2 - m^2 < 0,$$

since, for tunneling, evanescent $x$ waves exist in region II. In this case,

$$\sin(q_1 L) \to i \sinh(|q_1 L|),$$

and can never be zero. It is to be noted that since, by definition, for head-on collisions $p_2 = 0$, the helicity conserving factor $R = 0$. Consequently, total helicity flip occurs for the reflection of head-on collisions.

## IV. RELATIVE PHASES ANALYSIS

Consider, for the moment an incoming polarized beam say $I_+$, with $I_- = 0$. This could be achieved via a Stern-Gerlach apparatus. Then from our results the transmitted wave will also be polarized:

$$T_+ = I_+ T \quad \text{and} \quad T_- = 0,$$

but this will not be the case for the reflected wave, indeed,

$$R_+ = I_+ R \quad \text{and} \quad R_- = I_+ \widetilde{R}.$$

However, these two helicity states will have a definite relative phase since $\widetilde{R}/R$ is imaginary. The exact value of this ratio depends upon momentum, but it does not depend upon the values of $V_0$ nor $L$, the barrier parameters. We thus have a means of producing reflected waves with known relative phase and calculable strengths.

Now, consider incoming waves, with a fixed relative phase. The above reflected waves provides an example. Let

$$I_+ = |I_+|\, e^{i\,\alpha} \quad \text{and} \quad I_- = |I_-|\, e^{i\,\beta},$$

so that the relative phase of interest is $\alpha - \beta$. We subject this source to scattering by our barrier. Again the transmitted waves are of limited interest since they carry over the same relative phase between $T_+$ and $T_-$. On the other hand the situation is more complex for the reflected waves,

$$\begin{aligned} |R_+|^2 &= |I_+ R|^2 + |I_- \widetilde{R}|^2 - 2|I_+ I_- R \widetilde{R}| \sin(\alpha - \beta), \\ |R_-|^2 &= |I_- R|^2 + |I_+ \widetilde{R}|^2 + 2|I_+ I_- R \widetilde{R}| \sin(\alpha - \beta), \end{aligned} \tag{15}$$

which explicitly depend upon the relative phase. Thus, a second Stern-Gerlach apparatus followed by intensity (flux) measurement will yield the relative incoming phase.

Our knowledge of relative phases is almost nonexistent. For example, in a broad particle beam of constant density is the relative phase (when both helicities are present) the same for all particles or is it completely arbitrary? In the latter case the measurement of relative phase as described above would yield a null mean value. Nevertheless, even in this latter case, the reflection of a *polarized* beam will provide a fixed relative phase, for given incident angle and beam energy, for each individual particle in the beam. We believe these predictions are worthy of experimental verification.

## V. CONCLUSIONS

In this work, we have studied the general planar diffusion (inclusive of tunneling) of plane waves by an "electrostatic" potential barrier. We have provided the explicit helicity amplitudes for both reflection and transmission. Since we consider only plane waves we are in the limit of total coherence and this is evidenced by the prediction of resonances in diffusion, when no reflection occurs. We have confirmed some previous results, such as the absence of spin flip for head-on collisions corresponding to total helicity flip. There is a generalization of this result $(\widetilde{R} = 0)$, for any incident angle (fixed $p_2/p_1$) the ratio $R/\widetilde{R}$ tends to zero in the zero momentum limit. We also confirm that the transmitted amplitudes are always proportional to the incoming helicity amplitudes; no helicity flip occurs.

The most interesting part of our results is that the probabilities of the reflected helicity waves depend in a simple manner upon the relative phase of the incoming helicity amplitudes. They, or at least their average (if not constant), may hence be determined experimentally. Furthermore, if the incoming wave is polarized, the reflected waves will have a fixed relative phase of $\pi/2$ and modulus determined by the scattering angle and the beam energy. Unfortunately, the apparatus described in this paper does not furnish a method for creating helicity amplitudes with any chosen relative phase (excluding $\pi/2$) but only for the determination of any such phase.

As emphasized above these results are valid only in the case of dominant coherence. However, we have for diffusion a simple manner to confirm this situation by identifying resonance phenomena. At certain incident angles or for certain momenta or, for certain barrier lengths (with other variables fixed) total transmission must occur or at least dominate.

Thus any experimental test of our predictions must involve diffusion of a particle beam of say electrons (initially polarized), by a stratified electrostatic potential. After testing for coherence by means of resonance, the first prediction to verify should be the absence of spin flip for head-on collisions. Subsequently, the explicit predictions for the transmitted and reflected amplitudes can be tested. If our predictions are confirmed, we suggest that a systematic study of the means of altering the incoming relative phase, such as with the help of electromagnetic fields could be attempted. It is very hard for us to suggest any practical use of the above results. However, neither can we exclude any. For example, we recall the significant multiple practical applications of the related tunnel effect, such as in the electron microscope, beyond its initial importance as an example of the failure of classical physics.

We know very little experimentally about relative phases. Here, we have considered helicity amplitudes but they are of theoretical interest also in isospin or other analysis. For example, they may have a significant effect upon oscillation formulas in particle physics. Finally, we recall that the formulas derived are quite general and are valid both for diffusion (and Klein) and, when $q_1$ is imaginary, for tunneling. Integration over a convolution function of the incoming momentum allows one to study wave-packet sources but this generally alters the physics involved, for example, as we have already pointed out resonance phenomena no longer survive.

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