Relativistic tunneling through opaque barriers

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(Received 22 October 2010; revised manuscript received 20 December 2010; published 28 February 2011)

We propose an analytical study of relativistic tunneling through opaque barriers. We obtain a closed formula for the phase time. This formula is in excellent agreement with the numerical simulations and corrects the standard formula obtained by the stationary phase method. An important result is found when the upper limit of the incoming energy distribution coincides with the upper limit of the tunneling zone. In this case, the phase time is *proportional* to the barrier width.

DOI: 10.1103/PhysRevA.83.022111 PACS number(s): 03.65.Xp, 03.65.Ta

I. INTRODUCTION

The question of how long it takes a particle to tunnel through a barrier potential has been the issue of intriguing discussions in the last several decades. Since the first studies regarding the tunneling process [1,2], several tunneling time definitions have been proposed [3–5]. However, there is no general agreement about a satisfactory definition. Considering a time-dependent description in terms of wave packets, Hartman [6] used the stationary phase method (SPM), previously applied to scattering problems [7,8], to estimate the instant in which the transmitted peak appears in the free region after the barrier. This time definition is known as *phase time*. One can also introduce a time integration over the probability of finding the particle inside the barrier. This average time spent in the potential region, regardless of whether transmission or reflection occurs, is known as *dwell time* [9,10]. Phase and dwell times have a well-established mutual relation [3,11]. The important point to be noted here is that, in the opaque limit, both predict the independence of tunneling times on the barrier width, the so-called Hartman effect [6]. It is also possible to investigate transit times by describing kinematical paths in the potential region [12–14] or introducing a new degree of freedom, such as the Larmor clock [9], and time-oscillating barriers [15]. Recently, a time operator has been considered [16–18], which is canonically conjugated to the energy operator. From this time operator, a tunneling time is obtained that is equal to the average over momentum components of the phase time.

Although the majority of the discussions about tunneling times have been based on the nonrelativistic Schrödinger equation, some recent works have attended to particular features of relativistic tunneling particles, such as superluminality [19–22] and existence of the Hartman effect for relativistic potentials [23–26]. The solutions of the Dirac equation are oscillatory in the diffusion $E > V_0 + m$ [27,28] and Klein $E < V_0 - m$ [29] energy zones, where V_0 is the barrier height and m is the particle mass. The phenomenon of multiple peaks occurs in the diffusion zone [30]. The Klein region, in its turn, involves pair production and dynamical localized states [31]. The tunneling energy zone $\max\{m, V_0 - m\} < E < V_0 + m$ is characterized by evanescent solutions [26]. In this paper, we aim to present an analytic and numerical study of the phase time for one-dimensional relativistic tunneling in the opaque barrier limit. We choose an incoming spectrum and vary the barrier height in such a way that the momentum distribution remains at the evanescent zone with above potential energies, i.e.,

$$\max\{m, V_0\} < E < V_0 + m, \tag{1}$$

from which we observe that energy components with $E < V_0$ correspond to below potential states, the interpretation of which is still subject to discussions [26,27,31].

In Sec. II, we review some of the standard calculations on Dirac tunneling and find an approximation for the transmission coefficient in the opaque limit. This approximation is then used in Sec. III to obtain a closed formula for the phase time in relativistic tunneling. Such an analytic expression corrects the well-known formula obtained by the SPM. The tunneling time is proportional to the barrier width for an incoming momentum distribution, the upper limit of which is very close to the barrier height. This result clearly contradicts the Hartman effect. Our analytical expression and the SPM formula coincide for higher potentials. To support our analytical study, we also present numerical calculations. Our conclusions are drawn in Sec. IV.

II. OPAQUE BARRIERS AND FILTER EFFECT

Consider a relativistic spin one-half particle of mass m moving along the z axis in the presence of a one-dimensional electrostatic potential with height V_0 in the region $0 < z < L$ and zero elsewhere. The particle dynamics is described by [$\hbar = c = 1$]

$$i\partial_t \mathbf{\Psi}(z,t) = \gamma_0[m - i\gamma_3\partial_z]\mathbf{\Psi}(z,t) + V(z)\mathbf{\Psi}(z,t), \tag{2}$$

where γ_0 and γ_3 are the Pauli-Dirac matrices defined by

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}.$$

Let us consider as the incident wave packet

$$\mathbf{\Psi}_{\text{inc}}(z,t) = N \int_{p_{\text{m}}}^{p_{\text{M}}} dp\, g(p) u(p) e^{i(pz - Et)}, \tag{3}$$

with

$$g(p) = \exp[-(p - p_0)^2 a^2/4] \quad \text{and} \quad u(p) = \left(1 \;\; 0 \;\; \frac{p}{E+m} \;\; 0\right)^t.$$

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N is a normalization constant and a is related to the spatial localization of the incoming particle. In this paper, we choose

$$\max\{0,\ \sqrt{V_0^2 - m^2}\} \leqslant p_{\rm m} \quad \text{and} \quad p_{\rm M} \leqslant \sqrt{V_0^2 + 2mV_0}$$

to guarantee that the energy spectrum remains in the tunneling zone [Eq. (1)]. This avoids Klein and diffusion phenomena, as well the presence of below potential evanescent solutions. By solving the Dirac equation (2) and imposing the continuity of the wave function at 0 and L, we find, for the transmitted wave packet [26–28],

$$\mathbf{\Psi}_{\rm tra}(z,t) = N \int_{p_{\rm m}}^{p_{\rm M}} dp\, g(p) T(p,L) u(p) e^{i(pz-Et)}, \tag{4}$$

with

$$T(p,L) = e^{-ipL}\left[\cosh(qL) - i\,\frac{p^2 - EV_0}{qp}\sinh(qL)\right]^{-1}, \tag{5}$$

where $q = \sqrt{m^2 - (E - V_0)^2}$. To summarize our notation, we shall denote by $q_{\rm m,M}$ and $E_{\rm m,M}$ the q and E functions, respectively, calculated in $p_{\rm m}$ and $p_{\rm M}$.

For thin barriers, the transmitted momentum distribution is very similar to the incoming distribution. Consequently, the transmitted momentum is still centered in p_0. For much thicker barriers, the mean value of the transmitted momentum $\langle p \rangle_{\rm T}$ tends to $p_{\rm M}$ (see Fig. 1). By increasing the barrier width, the potential acts as a momentum filter. Observe that the filter effect is more evident for momentum distributions, the upper limit of which coincides with the upper limit of the tunneling zone [see Fig. 1(b)]. In the opaque limit $mL \gg 1$, we can approximate the expression for T given in Eq. (5) by

$$T(p,L) \approx \frac{2}{mV_0}\, qp\, e^{-qL} e^{-ipL + i\varphi}, \tag{6}$$

where $\varphi = \arctan[(p^2 - EV_0)/qp]$.

Before beginning our discussion on the phase-time formula obtained by using the stationary phase method, let us calculate, in the opaque limit, the average momentum

$$\langle p \rangle_{\rm T} = \int_{p_{\rm m}}^{p_{\rm M}} dp\, p \|g(p)T(p,L)u(p)\|^2 \Bigg/ \int_{p_{\rm m}}^{p_{\rm M}} dp \|g(p)T(p,L)u(p)\|^2. \tag{7}$$

By using the approximation (6), we find

$$\langle p \rangle_{\rm T} \approx \int_{p_{\rm m}}^{p_{\rm M}} dp\, p^3 q^2 g^2(p) \|u(p)\|^2 e^{-2qL} \Bigg/ \int_{p_{\rm m}}^{p_{\rm M}} dp\, p^2 q^2 g^2(p) \|u(p)\|^2 e^{-2qL}. \tag{8}$$

Observing that $p\,dp = Eq\,dq/(V_0 - E)$, we can change the variable of integration from p to q:

$$\langle p \rangle_{\rm T} \approx \int_{q_{\rm m}}^{q_{\rm M}} dq\, p^2 q^3 \underbrace{[E/(V_0 - E)] g^2(p) \|u(p)\|^2}_{f(p)} e^{-2qL} \Bigg/ \int_{p_{\rm m}}^{p_{\rm M}} dp\, p q^3 f(p) e^{-2qL}. \tag{9}$$

The filter effect suggests expanding the factors that appear in the integrands around $q_{\rm M}$:

$$p^2q^3 = (p^2q^3)_{\rm M} + (p^2q^3)_{q,{\rm M}}(q - q_{\rm M}) + O((q - q_{\rm M})^2),$$
$$pq^3 = (pq^3)_{\rm M} + (pq^3)_{q,{\rm M}}(q - q_{\rm M}) + O((q - q_{\rm M})^2),$$
$$f(p) = f_{\rm M} + f_{q,{\rm M}}(q - q_{\rm M}) + O((q - q_{\rm M})^2).$$

After algebraic manipulations, we find

$$\begin{aligned}\langle p \rangle_{\rm T} &\approx p_{\rm M}\left\{1 + \frac{1}{2L}\left[\frac{(p^2q^3)_{q,{\rm M}} - p_{\rm M}(pq^3)_{q,{\rm M}}}{(p^2q^3)_{\rm M}}\right]\right\} \\ &= p_{\rm M}\left\{1 + \frac{1}{2L}\left[\frac{p_{q,{\rm M}}(pq^3)_{\rm M} + p_{\rm M}(pq^3)_{q,{\rm M}} - p_{\rm M}(pq^3)_{q,{\rm M}}}{(p^2q^3)_{\rm M}}\right]\right\} \\ &= p_{\rm M} - \frac{q_{\rm M}E_{\rm M}}{2p_{\rm M}(E_{\rm M} - V_0)L}. \end{aligned} \tag{10}$$

Thus, for opaque barriers, the momentum distribution is sharply peaked in the neighborhood of $p_{\rm M}$.

The standard phase-time formula is obtained by calculating the space-time points in which the phase φ is stationary. The maximum of the wave packet is found by imposing that the derivative of the phase calculated in $\langle p \rangle_{\rm T}$ is equal to zero. For opaque barriers,

$$(pz - Et - pL + \varphi)_{p,{\rm M}} = 0.$$

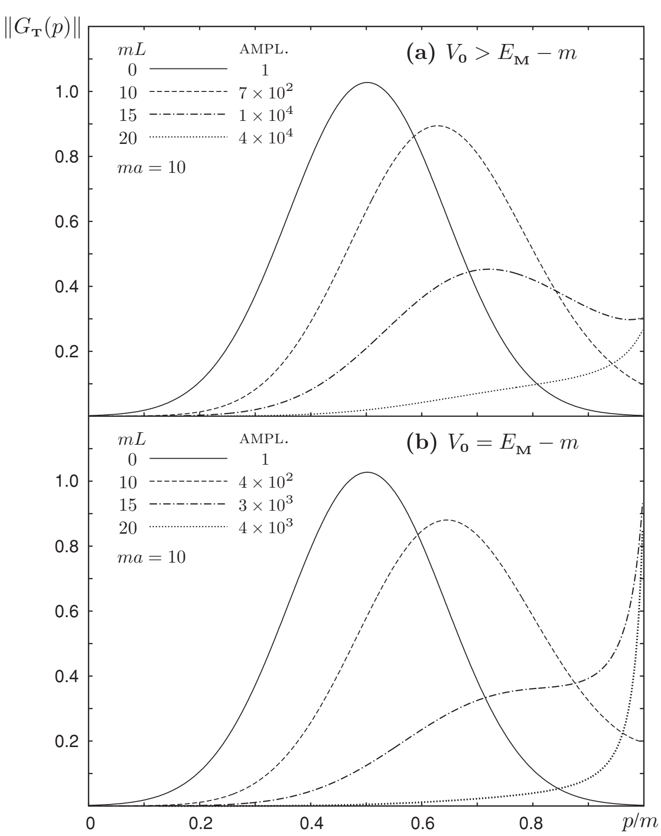

FIG. 1. Transmitted momentum distributions $G_{\rm T}(p) = g(p)T(p,L)u(p)$ are plotted for different barrier widths. The continuous line represents the incoming Gaussian spectrum with $p_{\rm m} = 0$, $p_0 = m/2$, $p_{\rm M} = m$, and $ma - 10$. The barrier heights are chosen equal to $V_0 = E_{\rm M} - m \approx 0.414m$ in (b) and $V_0 = 0.46m$ in (a). Filter effect is evident in both cases.

Consequently, at the edge of the barrier $z = L$, we have $E_{p,\mathrm{M}}\,\tau_{\mathrm{SPM}} = \varphi_{p,\mathrm{M}}$. Finally,

$$\tau_{\mathrm{SPM}} = \frac{2E_{\mathrm{M}} - V_0}{p_{\mathrm{M}} q_{\mathrm{M}}}. \tag{11}$$

We observe that, for $E_{\mathrm{M}} \to V_0 + m$, $\tau_{\mathrm{SPM}} \to \infty$. In the next section, we shall overcome this ambiguity by proposing a different method to calculate the phase time. It is based on the analytical calculation of the probability density by means of the approximation (6). We then compare our phase-time formula with Eq. (11) and discuss in details the validity of the Hartman effect.

III. REVISING THE PHASE-TIME FORMULA

In this section, we propose an analytic method to obtain the phase time. This method is essentially based on the search of the time for which the electronic density is greater at the far edge of the barrier $z = L$. This means taking the derivative of the electronic density with respect to time and finding, when it is equal to zero,

$$(\|\Psi(L,t)\|^2)_t = 0. \tag{12}$$

The time solution of the previous equation will be indicated by τ. Using this time, it is possible to introduce a tunneling (or transit) velocity defined by $v_{\mathrm{tun}} = L/\tau$.

Numerical solutions τ_{num} of Eq. (12) were calculated to an incoming Gaussian distribution $g(p)$ characterized by

$$ma = 10 \quad \text{and} \quad (p_{\mathrm{m}}, p_0, p_{\mathrm{M}}) = \left(0, \tfrac{1}{2}, 1\right)m.$$

For a potential barrier with height $V_0 = E_{\mathrm{M}} - m(\approx 0.414m)$, we find that the tunneling velocity tends to a constant value when we increase the barrier width L (see Table I and Fig. 2). We can also see that, for increasing potentials [for example, $V_0 = 0.46m(> E_{\mathrm{M}} - m)$], the transit time tends to a constant value. This time is in agreement with the phase time obtained from the stationary phase method (11) (see the last row in Table I).

Let us now begin our analytical discussion of Eq. (12). For opaque barriers, we can use the approximation given in Eq. (6) and, due to the filter effect, develop the phase around k_{M} (or q_{M} if we intend to change the variable of integration from k to q). Using the expansions up to the second order,

$$\varphi = \varphi_{\mathrm{M}} + \varphi_{q,\mathrm{M}}(q - q_{\mathrm{M}}) + \varphi_{qq,\mathrm{M}}(q - q_{\mathrm{M}})^2/2 + O((q - q_{\mathrm{M}})^3), \tag{13}$$

$$E = E_{\mathrm{M}} + E_{q,\mathrm{M}}(q - q_{\mathrm{M}}) + E_{qq,\mathrm{M}}(q - q_{\mathrm{M}})^2/2 + O((q - q_{\mathrm{M}})^3),$$

and changing the variable of integration into $\rho = q - q_{\mathrm{M}}$, we obtain

$$\Psi_{\mathrm{tra}}(L,t) \approx \frac{2N g_{\mathrm{M}} E_{\mathrm{M}} u_{\mathrm{M}}}{m V_0 (E_{\mathrm{M}} - V_0)} e^{i\varphi_{\mathrm{M}} - iE_{\mathrm{M}}t - q_{\mathrm{M}}L} S(t), \tag{14}$$

with

$$S(t) := \int_0^{q_{\mathrm{m}} - q_{\mathrm{M}}} d\rho (\rho + q_{\mathrm{M}})^2 e^{-\rho L} \exp\left\{ i\left[(\varphi_{q,\mathrm{M}} - E_{q,\mathrm{M}}t)\rho + \frac{\varphi_{qq,\mathrm{M}} - E_{qq,\mathrm{M}}t}{2}\rho^2 \right] \right\}. \tag{15}$$

TABLE I. Tunneling times are listed for an incoming Gaussian distribution with $ma = 10$, $p_{\mathrm{m}} = 0$, $p_0 = m/2$, and $p_{\mathrm{M}} = m$ varying the barrier width L. The barrier height plays a fundamental role in their characterization. When $V_0 + m$ is greater than E_{M}, for $mL \gg 1$ the tunneling time tends to a constant value. In this case, the standard formula obtained by the SPM [Eq. (11)] represents a good approximation. On the other hand, if $V_0 + m = E_{\mathrm{M}}$, i.e., $q_{\mathrm{M}} = 0$, the tunneling time is proportional to L. The agreement between analytic [Eq. (19)] and numerical data is impressive.

	$V_0 = E_{\mathrm{M}} - m$		$V_0 = 0.46m$	
mL	$m\tau_{\mathrm{num}}$	$\tau_{\mathrm{num}}/\tau_{\mathrm{ana}}$	$m\tau_{\mathrm{num}}$	$\tau_{\mathrm{num}}/\tau_{\mathrm{ana}}$
50	23.62	0.8807	6.520	0.9448
100	51.25	0.9553	7.339	0.9906
150	77.87	0.9677	7.552	0.9963
200	104.3	0.9719	7.650	0.9981
250	130.6	0.9739	7.706	0.9988
300	156.9	0.9749	7.743	0.9992
350	183.2	0.9755	7.769	0.9994
400	209.4	0.9759	7.788	0.9996
450	235.7	0.9762	7.803	0.9997
500	261.9	0.9764	7.815	0.9997
$m\tau_{\mathrm{spm}}$	∞		7.918	

At this point, it is convenient to introduce the quantities α and β defined by

$$\alpha = \alpha_1 - \alpha_2 t = \varphi_{q,\mathrm{M}} - E_{q,\mathrm{M}}t = \frac{V_0 - 2E_{\mathrm{M}}}{p_{\mathrm{M}}(E_{\mathrm{M}} - V_0)} - \frac{q_{\mathrm{M}}}{V_0 - E_{\mathrm{M}}}t,$$

$$\beta = \beta_1 - \beta_2 t = \varphi_{qq,\mathrm{M}} - E_{qq,\mathrm{M}}t = \frac{q_{\mathrm{M}}\left[m^2 V_0 + E_{\mathrm{M}}\left(2E_{\mathrm{M}}V_0 - 2E_{\mathrm{M}}^2 - V_0^2\right)\right]}{2p_{\mathrm{M}}^3(E_{\mathrm{M}} - V_0)^3} - \frac{m^2}{2(V_0 - E_{\mathrm{M}})^3}t. \tag{16}$$

The observation that, for $q_{\mathrm{M}} \to 0$, the main time contribution comes from the beta term and that, for increasing time, this term (which is coupled to ρ^2) becomes comparable to the α term (proportional to ρ), suggests to consider the following approximation for the exponential that appears in Eq. (15),

$$\exp\{i[\alpha\rho + \beta\rho^2]\} \simeq 1 + i(\beta\rho^2 + \alpha\rho) - \tfrac{1}{2}\alpha^2\rho^2 - \tfrac{1}{2}\beta^2\rho^4 - \alpha\beta\rho^3.$$

This means that, in the calculation of the integral Eq. (15), we shall find integrals of the form

$$s(n) := \int_0^{q_{\mathrm{m}} - q_{\mathrm{M}}} d\rho \left(\rho^{n+2} + 2q_{\mathrm{M}}\rho^{n+1} + q_{\mathrm{M}}^2\rho^{n}\right) e^{-\rho L} \simeq \frac{(n+2)!}{L^{n+3}} \times \left[1 + \frac{2q_{\mathrm{M}}L}{n+2} + \frac{q_{\mathrm{M}}^2 L^2}{(n+2)(n+1)}\right]. \tag{17}$$

Finally,

$$|S(t)|^2 \approx \left|s(0) - \tfrac{1}{2}\alpha^2\rho^2 - \tfrac{1}{2}\beta^2\rho^4 - \alpha\beta\rho^3 + i(\beta\rho^2 + \alpha\rho)\right|^2 \approx s^2(0) + \alpha^2[s^2(1) - s(0)s(2)] + 2\alpha\beta[s(1)s(2) - s(0)s(3)] + \beta^2[s^2(2) - s(0)s(4)]. \tag{18}$$

Our phase-time formula is obtained by taking the derivative with respect to time and imposing that it is equal to zero, i.e.,

$$\alpha\alpha_t[s^2(1) - s(0)s(2)] + (\alpha\beta_t + \alpha_t\beta)[s(1)s(2) - s(0)s(3)] + \beta\beta_t[s^2(2) - s(0)s(4)] = 0.$$

Observing that $\beta_1\beta_2[s^2(2) - s(0)s(4)] \ll \alpha_1\alpha_2[s^2(1) - s(0)s(2)]$, we find the following analytical expression for the transit time:

$$\tau_{\text{ana}} = \frac{\alpha_1\alpha_2[s^2(1) - s(0)s(2)] + (\alpha_1\beta_2 + \beta_1\alpha_2)[s(1)s(2) - s(0)s(3)]}{\alpha_2^2[s^2(1) - s(0)s(2)] + 2\alpha_2\beta_2[s(1)s(2) - s(0)s(3)] + \beta_2^2[s^2(2) - s(0)s(4)]}. \tag{19}$$

In the limit $q_\text{M} \to 0$ (which implies $\alpha_2 \to 0$), the previous expression simplifies into

$$\tau_{\text{ana}} \to \frac{\alpha_1}{\beta_2}\frac{[s(1)s(2) - s(0)s(3)]}{[s^2(2) - s(0)s(4)]} = \frac{2(2m + V_0)}{\sqrt{V_0^2 + 2mV_0}} \times \frac{(3!4! - 2!5!)/L^9}{(4!4! - 2!6!)/L^{10}} = \frac{2}{9}\sqrt{1 + \frac{2m}{V_0}}\,L. \tag{20}$$

The SPM result is recovered for q_M values for which the time dependence in α is not more negligible. In such a limit, the β term plays no role in the calculation and

$$\tau_{\text{ana}} \to \frac{\alpha_1}{\alpha_2} = \frac{2E_\text{M} - V_0}{p_\text{M}q_\text{M}}. \tag{21}$$

The phase-time formula (19), the formula obtained by the stationary phase method (11), and the numerical simulations are shown in Fig. 3. Our phase-time formula is in excellent agreement with the numerical analysis.

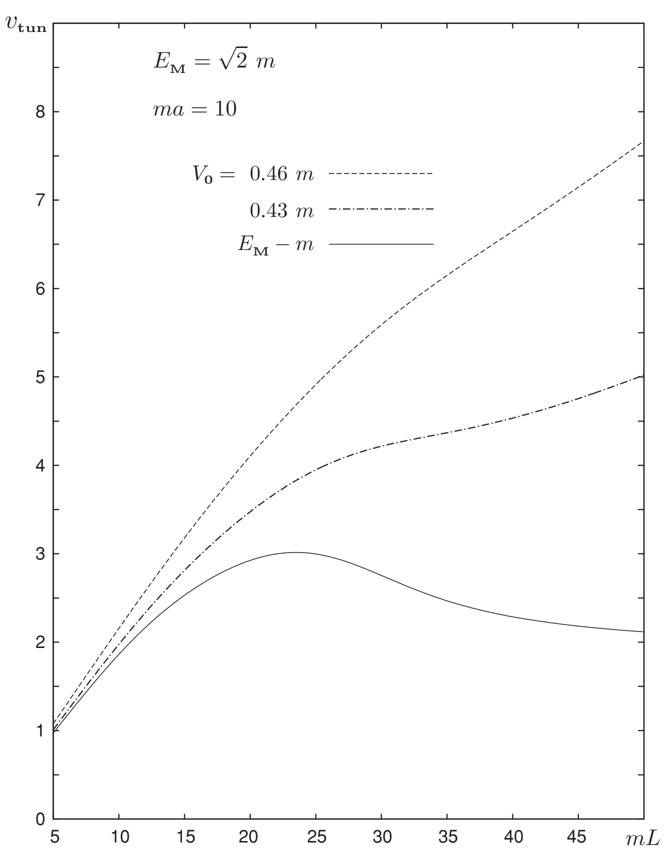

FIG. 2. Tunneling velocities are plotted as a function of mL. They have been numerically obtained by the expression (12), with $p_\text{m} = 0$, $p_0 = m/2$, $p_\text{M} = m$, and $ma = 10$. Actually, if $V_0 + m$ is greater than E_M, then v_{tun} increases linearly with the barrier width (Hartman effect). However, if the incoming energy distributions reach the upper tunneling zone $E_\text{M} = V_0 + m$, the velocity tends to a constant value. Numerical tunneling times for $L > 50/m$ are given in Table I.

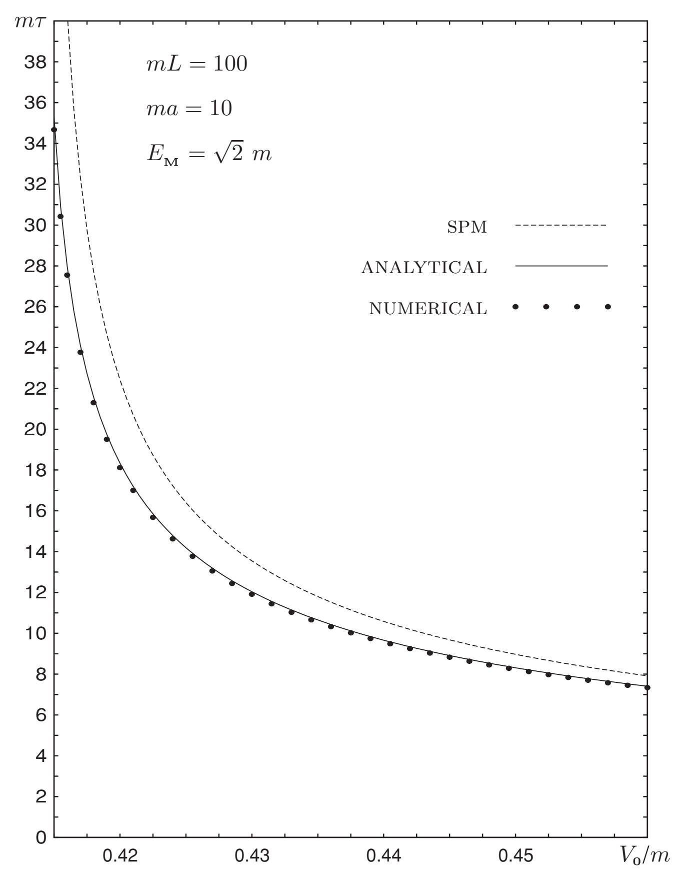

FIG. 3. Analytical, numerical, and SPM tunneling times are plotted as a function of V_0/m. The incoming momentum spectrum is characterized by $p_\text{m} = 0$, $p_0 = m/2$, $p_\text{M} = m$, and $ma = 10$. The agreement between τ_{ana} and τ_{num} is excellent. The SPM represents a good approximation for higher barriers.

Before concluding this section, let us make some observations. In the case $q_M = 0$, the spatial phase $\exp[ik(x-L)]$ can be approximated by using

$$k = \sqrt{V_0(V_0+m)} - \frac{V_0+m}{2m\sqrt{V_0(V_0+m)}}q^2 + O(q^3).$$

This implies that the spatial dependence has to be included in the β term (which contains the terms in q^2). To reduce the maximum of $S(t)$ [see Eq. (18)] of a factor $1/e$, observing that the time (20) guarantees $\alpha^2 \sim \alpha\beta \sim \beta^2$, we have to consider $x - L \sim \tau_{ana}$. Consequently, the spreading of the wave packet in configuration space is *proportional* to the barrier width

It is also interesting to note that, in the nonrelativistic limit, i.e., $E \ll m$ and $V_0 \ll m$, the transit velocity L/τ_{ana} tends to $4.5\sqrt{V_0/2m}$, which is clearly subluminal.

IV. CONCLUSIONS

In this paper, we have investigated the tunneling through opaque barriers for relativistic particles. By an analytical study, we have found a closed formula for the phase time in agreement with numerical simulations. In order to avoid "negative energies" in the potential region, the incoming momentum spectrum has been restricted to the evanescent zone with $E > V_0$. We have assumed that tunneling time τ is defined as the instant of maximum probability of finding the transmitted particle at the barrier edge, that is to say, the peak of the wave packet along the time for fixed $z = L$. Due to wave-packet spreading, this measurement differs from mapping of the peak dynamics along the z axis. Nevertheless, numerical calculations, which can be readily performed, guarantee that such variation is not relevant. An approximation on the transmission coefficient has allowed us to obtain a closed expression for τ. This formula has exhibited excellent agreement with the numerical calculations and has shown cases in which the stationary phase method gives a satisfactory approximation. The most important prediction of our analytical formula concerns the validity of the Hartman effect. It does not hold for incoming distributions, the maximum energy value of which coincides with the upper limit of the evanescent zone. In this case, the phase time increases linearly as a function of the barrier width. In this limit, the result obtained by the SPM becomes meaningless since its formula gives an infinity. For incident wave packets with momentum integration truncated before $V_0 + m$, we have found a tunneling time that is independent of L in the opaque limit, as predicted by the Hartman effect.

Superluminal transit velocities appear for small barriers (see Fig. 3). This phenomenon surely deserves further investigations. Nevertheless, it has to be underlined that there is *no* causal connection between the peak of the incoming wave packet and the peak of the transmitted wave packet. One can consider energy components and barrier heights such that new tunneling features are introduced, namely, negative $E - V_0$ values and a Klein zone that borders the evanescent one.

The numerical simulations presented in this paper are based on the numerical calculations of the Gaussian convolution of the barrier stationary solutions. Braun, Su, and Grobe [32] have proposed a numerical approach, based on the split-operator technique, to solve the time-dependent three-dimensional Dirac equation. In view of the results presented in this paper, it could be interesting to revise tunneling phenomena by using the BSG approach.

ACKNOWLEDGMENTS

The authors thank the referee for his observations and his very useful suggestions. One of the authors (S.D.L) also thanks the Department of Physics, University of Salento (Lecce, Italy), for the hospitality and the FAPESP (Brazil) for financial support by Grant No. 10/02216-2.

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